JOURNAL OF COMBINATORIAL THEORY (B) 22, 227–232 (1977)

Every Connected Regular Graph of Even Degree is a Schreier Coset Graph

JONATHAN L. GROSS*

Department of Mathematical Statistics, Columbia University, New York, New York 10027

Received August 18, 1975

Using Petersen's theorem, that every regular graph of even degree is 2-factorable, it is proved that every connected regular graph of even degree is isomorphic to a Schreier coset graph. The method used is a special application of the permutation voltage graph construction developed by the author and Tucker. This work is related to graph imbedding theory, because a Schreier coset graph is a covering space of a bouquet of circles.

1. INTRODUCTION

This paper proves that every connected regular graph of even degree is isomorphic to a Schreier coset graph and indicates the importance of this result to the study of surface imbeddings of graphs.

A Schreier coset graph is a generalization of a Cayley "color" graph using cosets of some specified subgroup as vertices instead of group elements. Appropriate definitions are provided in Section 2, along with some examples of Schreier coset graphs to show how much more general they are than Cayley graphs. Section 3 employs the theorem of Petersen [6], which states that every regular graph is 2-factorable, to prove the title theorem. The method used is based on the permutation voltage graph construction of Gross and Tucker [4], which the reader need not now know or learn. Although Sections 2 and 3 are entirely self-contained, the reader may want to read more about graph factoring, for which Harary [5] is recommended, or more about graphs and groups, for which White [8] is suggested. Both these sources cite other works of possible interest.

Understanding of Section 4 depends on prior knowledge of some graph imbedding theory, covering spaces, and voltage graphs (or current graphs) in particular.

* The author is an Alfred P. Sloan Fellow. This research was partially supported by NSF Contract MPS74-05481-A01 at Columbia University.

2. On Schreier Coset Graphs

In this paper, a *graph* has finitely many vertices and edges. Sometimes, an edge might adjoin a vertex to itself, or two or more edges might adjoin the same pair of vertices.

The *degree* of a vertex is the number of edges incident on that vertex. A graph is called *regular* if all vertices have the same degree.

Let G be a group, let H be a subgroup of G, and let $g_1, ..., g_r$ be a sequence in G whose members generate G. The (right) *Schreier coset graph* for that group, subgroup, and generating sequence is defined as follows. Its vertex set is the set of right cosets of H in G. For each right coset H_i and each generator g_j there is an edge from H_i to the right coset $H_i g_j$. It is sometimes convenient, as when stating the title theorem, to ignore the implicit edge directions.

When r generators are specified, the in-degree and out-degree of each vertex of a Schreier coset graph are both r, so the graph is regular of degree $2r$. The definition of edges via right multiplication by generators assures that a Schreier coset graph is connected.

Let G be a group and $g_1, ..., g_r$ a sequence in G whose members generate G. The (right) *Cayley graph* for that group and generating sequence has as vertices the elements of G. For each $g \in G$ and each generator g_j there is an edge from the vertex g to the vertex gg_j. This construction corresponds to the special case when the identity subgroup $\{e\}$ is used for H in our definition of a Schreier coset graph.

Figures 1 and 2 show Schreier coset graphs that cannot be isomorphic to any Cayley graphs. Both figures are later used to illustrate the method devised for proving the title theorem.

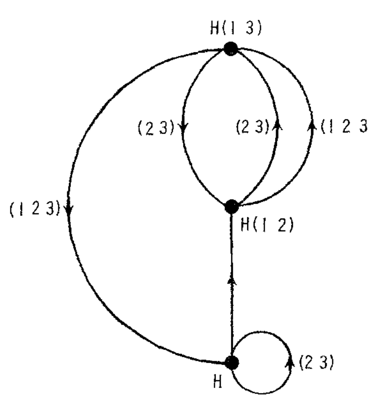

Fig. 1. A Schreier graph for the subgroup H of permutations in the symmetric group Σ_3 that fix the symbol 1. The generators for Σ_3 are the cyclic permutations (2 3) and (1 2 3).

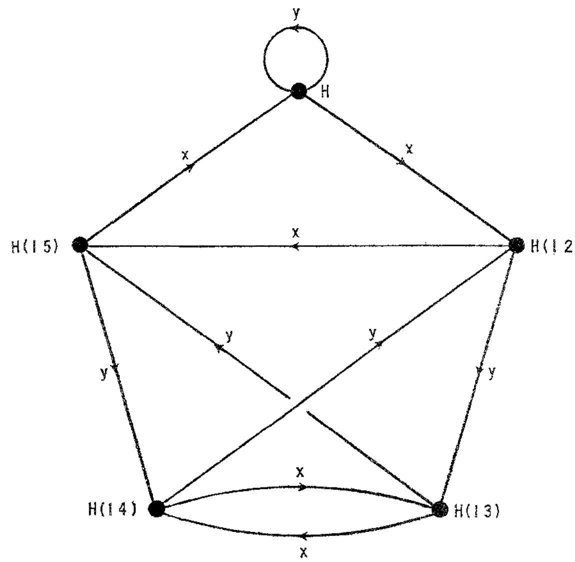

FIG. 2. The group G is the subgroup of the symmetric group Σ_5 generated by the permutations $x = (1\ 2\ 5)(3\ 4)$ and $y = (1)(2\ 3\ 5\ 4)$. Take H to be the subgroup of G that fixes the symbol 1.

In our definition of a Cayley graph, it is clear that if b and c are any two elements of the group G, then the permutation $g \to cb^{-1}g$ on the elements of G induces a Cayley graph automorphism taking the vertex b into the vertex c. The underlying graph of Fig. 1 cannot possibly be a Cayley graph because it has no automorphism taking the vertex labeled H into the vertex labeled $H(1\ 2)$.

3. On 2-Factors and Permutations

An *s-factor* of a graph K is a subgraph of K which is regular of degree s and which contains every vertex of K. For instance, a Hamiltonian circuit is a 2-factor, but in general, a 2-factor (or s-factor) need not be connected. When the edges of K can be partitioned into s-factors, we say that K is *s-factorable.*

THEOREM 1. *Every regular graph (connected or not) of even degree is 2-factorable.*

Proof. This is Petersen's theorem [6]. His proof is elegant and elementary.

THEOREM 2. *Every connected regular graph of even degree is isomorphic to a Schreier coset graph.*

Proof. Let K be a connected regular graph of degree $2r$ with p vertices, labeled $1,\dots,p$. By Theorem 1 the edges of K can be partitioned into r 2-factors $F_1,\dots,F_r$. Assign an arbitrary orientation to each cycle of the 2-factors (thereby making K a directed graph). To each oriented 2-factor F_i, we associate a permutation π_i on the symbols $1,\dots,p$ as follows.

Suppose the 2-factor F has m component circuits $C_1,\dots,C_m$. Let v_1 be a vertex on circuit C_j, let v_2 be the next vertex, and so on, so that the cyclic permutation $(v_1 \cdots v_d) = \pi_{i,j}$ corresponds to the cyclic order in which the oriented circuit C_j passes through vertices of the graph K. Then the permutation π_i corresponding to the 2-factor F_i is the product of the (disjoint) cyclic permutations $\pi_{i,1},\dots,\pi_{i,m}$.

Let G be the group of permutations on the symbols $1,\dots,p$ generated by the permutations $\pi_1,\dots,\pi_r$ corresponding to the 2-factors $F_1,\dots,F_r$. Let H be the subgroup of G which fixes the symbol 1. Then the graph K is isomorphic to the Schreier coset graph for the group G, subgroup H, and generators $\pi_1,\dots,\pi_r$.

To see the isomorphism, let f_j be a permutation in G such that $f_j(1) = j$, for $j = 2,\dots,p$. (Group G acts transitively because graph K is connected.) Relabel vertices $1,\dots,p$ by the right cosets $H, Hf_2,\dots,Hf_p$, respectively. Then for $i = 1,\dots,r$ label every edge of the 2-factor F_i by the permutation π_i.

EXAMPLE 1. Returning to Fig. 1, we observe that every edge of one 2-factor of the graph shown is labeled by the permutation (2 3), while every edge of the other 2-factor is labeled (1 2 3).

EXAMPLE 2. In Fig. 2, every edge of one 2-factor is labeled by the permutation $x = (1\ 2\ 5)(3\ 4)$, while every edge of the other 2-factor is labeled $y = (1)(2\ 3\ 5\ 4)$.

Remark. A given connected regular graph of even degree may have several different 2-factorizations, so the group G for which it is a Schreier graph is not unique.

4. On Graph Imbeddings

The proof of Theorem 2 is based on the permutation voltage graph construction of Gross and Tucker [4], where it is proved that every covering space of a graph is obtainable as the derived graph from a permutation voltage assignment.

Gross and Alpert [2] have established that the most common present means of building a surface imbedding of a large graph, the so-called current graph techniques, are really covering space constructions. It is slightly easier to explain such methods in their dual formulation, the ordinary voltage graphs of Gross [1], of which permutation voltage graphs are a generalization.

In voltage graph theory, one builds an imbedding of a given large graph K by finding a smaller graph L of which K is a covering space, by imbedding the smaller graph L in a surface, and by then lifting the imbedding of L so that its covering graph K is imbedded in a surface (see [3]).

A *bouquet of n circles* is a graph with one vertex and n edges.

THEOREM 3. *A connected regular p vertex graph of degree $2r$ is a p-sheeted covering space of the bouquet of r circles.*

Proof. This is an immediate corollary of Theorem 2 and the fact (see [4, Theorem 6]) that a Schreier coset graph covers a bouquet of circles.

Part of the significance of Theorem 3 is that it suggests that voltage graph or current graph techniques might be used to find the genus or solve other imbedding problems for a connected regular graph of even degree or, as we now observe, for certain other graphs as well.

Suppose that the edges of a graph K can be partitioned into 1-factors and 2-factors. Take K' to be the supergraph of K obtained by doubling all the 1-factors. ("Doubling" an edge between vertices u and v means adding an extra edge between them.) Doubling a 1-factor transforms it into a special kind of 2-factor, whose associated permutation in the method of Section 3 is the product of disjoint transpositions. If an imbedding of K' is constructed (e.g., by covering space methods) so that each edge from one of the 1-factors of K and its double bound a digon, then we can obtain an imbedding of K by discarding such digons and sewing up the holes, matching an edge to its double.

In this connection we notice the following result of Tutte [7] and state an obvious corollary.

THEOREM 4. *Let K be an $(r - 1)$-connected regular graph of degree r with an even number of vertices. Then K has a 1-factor.*

COROLLARY. *Let K be a $2r$-connected regular graph of degree $2r + 1$ with an even number of vertices. Then the edges of K can be partitioned into a 1-factor and r 2-factors.*

REFERENCES

1. J. L. GROSS, Voltage graphs, *Discrete Math.* **9** (1974), 239–246.
2. J. L. GROSS AND S. R. ALPERT, The topological theory of current graphs, *J. Combinatorial Theory Ser. B* **17** (1974), 218–233.
3. J. L. GROSS AND T. W. TUCKER, Quotients of complete graphs: Revisiting the Heawood map-coloring problem, *Pacific J. Math.* **55** (1974), 391–402.

4. J. L. GROSS AND T. W. TUCKER, Generating all graph coverings by permutation voltage assignments, *Discrete Math.*, to appear.
5. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass., 1969.
6. J. PETERSEN, Die Theorie de regulären Graphen, *Acta Math.* **15** (1891), 193–220.
7. W. TUTTE, The factorizations of linear graphs, *J. London Math. Soc.* **22** (1947), 107–111.
8. A. T. WHITE, "Graphs, Groups and Surfaces," North-Holland, Amsterdam, 1973.