Dedekind Cuts

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1 Decimal Expansions

How would you define a real number? It would seem that the easiest way is to say that a real number is a decimal expansion of the form

$$N.d_1d_2d_3...,$$

where N is one of $0, 1, 2, 3, ...$ and each *digit* d_k is one of $0, ..., 9$. You could then say that the above decimal expansion represents the number

$$N + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + ...$$

However, this is circular! What does this infinite sum mean in terms of the supposed definition? You have to define the reals before you can make sense of these kinds of sums. Alternatively, you could define the reals in terms of decimal expansions, forget about the sum, and just work out all their properties from the crazy algorithms you've already learned about adding and multiplying decimal expansions. This is kind of a nightmare.

A better approach is to define the reals in some other way and then use the decimal expansion as a way of *naming* one of the real numbers. The sum then explains which real number the decimal expansion names.

The question remains: how do you define a real number? That is what these notes are about. In the first part of the notes, I'll explain what we'd want out of real numbers, and then in the second part, I'll explain how you get what you want using something called Dedekind cuts.

The basic problem with the rational numbers is that the rational number system has holes in it – missing numbers. The beauty of Dedekind cuts is that it gives a formal way to talk about these holes purely in terms of rational numbers.

2 What do We Want

First of all, we want the real numbers to retain all the basic arithmetic operations defined on rational numbers. Precisely, the rationals satisfy the following properties.

- There are operations $+$ and $\times$ defined on the set of rational numbers.
- $a + b = b + a$ and $a \times b = b \times a$.
- $a + (b + c) = (a + b) + c$ and $(a \times b) \times c = a \times (b \times c)$.
- $a \times (b + c) = a \times b + a \times c$.
- The 0 element has the property that $a + 0 = a$ for all a.
- The 1 element has the property that $a \times 1 = a$ for all a.
- For any a there is some b such that $a + b = 0$.
- For any nonzero a there is some b such that $a \times b = 1$.

We could also list properties of subtraction and division, but these are consequences of what we've already listed. For instance, you could define $a - b$ to be $a + b'$ where b' is such that $b + b' = 0$.

A set which satisfies the above properties is called a *field*. So, basically everything you learn in math until middle school, more or less, can be summarized in the sentence that *the rational numbers form a field*. We want the real numbers to form a field.

The rational numbers have their usual ordering on them, and it satisfies two properties.

- For any rationals a and b we have either $a < b$ or $a = b$ or $b < a$.
- If $a < b$ and $b < c$ then $a < c$.

The ordering on the rationals is compatible with the field operations. For instance if $a < b$ then $a + c < b + c$. Likewise if $a < b$ and $c > 0$ then $ac < bc$. All these properties make the rationals into an *ordered field*. We want the reals to be an ordered field.

The one additional property that we want out of the real numbers is that the real number system should not have any holes in it. Below I'll define what this means, in terms of something called the *least upper bound property*.

3 A First Pass

Suppose that we already know about some real number x. When we could define a pair of sets (A_x, B_x), where

- A_x is the set of all rational numbers y such that $y < x$, and
- B_x is the set of all rational numbers y such that $y > x$.

The way to think about this is that you are cutting the number line by an infinitely thin knife, at x, and A_x is all the numbers to the left of the knife and B_x is all the numbers to the right.

Part of this definition is very good: The sets A_x and B_x are both subsets of rationals, so they do not directly refer to real numbers. However, the problem with this definition is that it depends on us already knowing about the number x.

The idea behind Dedekind cuts is to just work with the pairs (A, B), without direct reference to any real number. Basically, we just look at all the properties that (A_x, B_x) has and then make these "axioms" for what we mean by a Dedekind cut.

4 The Main Definition

A Dedekind cut is a pair (A, B), where A and B are both subsets of rationals. This pair has to satisfy the following properties.

1. A is nonempty.
2. B is nonempty.
3. If $a \in A$ and $c < a$ then $c \in A$.
4. If $b \in B$ and $c > b$ then $c \in B$.
5. If $b \notin B$ and $a < b$, then $a \in A$.
6. If $a \notin A$ and $b > a$, then $b \in B$.
7. For each $a \in A$ there is some $b > a$ so that $b \in A$.
8. For each $b \in B$ there is some $a < b$ so that $a \in B$.

That's the definition. A real number is defined to be a Dedekind cut.

5 Commentary

Here is some commentary on the definition: Property 3 informally says that A is a "ray" pointing to the left. Property 4 informally says that B is a "ray" poonting to the right. Properties 5 and 6 informally say that the the two rays "come together" in some sense. Proerties 7 and 8 say that the rays do not include their "endpoints". This commentary is just meant to give you an intuitive feel for what is going on. Notice that the 8 properties above do not mention real numbers.

6 Examples

We already know what rational numbers are. If x is any rational number, then the sets A_x and B_x defined above make sense.

Problem 1: Prove that the pair (A_x, B_x) is a Dedekind cut.

I'll start you off with a proof of the first property. Certainly there is some $a < x$. By definition $a \in A_x$. Therefore A_x is nonempty. That takes care of the first property. The other 7 properties are similar.

You might wonder if there are any other examples of Dedekind cuts. Here's an example. Let A denote the set of rational numbers x such that either $x < 0$ or $x^2 < 2$. Let B denote the set of rational numbers x such that $x > 0$ and $x^2 > 2$. Here's a partial check that (A, B) is a Dedekind cut.

Property 1: $0 \in A$, so A is nonempty.

Property 2: $2 \in B$, because $2^2 = 4 > 2$. So B is nonempty.

Property 3: Suppose that $a \in A$ and $c < a$. There are two cases:

- If $c < 0$ then $c \in A$ by definition.
- If $c \geq 0$ then $a \geq 0$ as well, and $c^2 < a^2 < 2$. So, again $c \in A$.

Property 7: Suppose that $a \in A$. If $a < 0$ then we could take $b = 0$ and get $b \in A$. If $a \geq 0$ then we have $a^2 < 2$. Let's write $a = p/q$. We can take $p > 0$ and $q > 0$. We know that

$$p^2 < 2q^2.$$

Consider the new rational number

$$b = \frac{Np + 1}{Nq} = \frac{p}{q} + \frac{1}{Nq}.$$

Here N is some positive integer. Note that $a < b$. (Intuitively, b is just a little larger than a.) We win the game if $b^2 < 2$. This is the same as

$$(Np + 1)^2 - 2(Nq)^2 < 0$$

Let's call this number Z_N. So, we want to choose N so that $Z_N < 0$. Since N is positive, the numbers Z_N and Z_N/N have the same sign. So, we want to choose N so that $Z_N/N < 0$.

We compute

$$\frac{Z_N}{N} = N(p^2 - 2q^2) + 2p + 1/N.$$

The number $p_2 - 2q^2$ is an integer less than 0. Therefore

$$N(p^2 - 2q^2) \leq -N.$$

Also $1/N \leq 1$. Therefore

$$\frac{Z_N}{N} < -N + 2p + 1.$$

If we choose $N = 2p + 2$ then we get $Z_N/N < 0$. That's it.

Problem 2: Check the rest of the properties for the pair (A, B).

Property 7 turned out to be something of a nightmare, but I hope that you can use this as a template for establishing the other properties. Since you've already got Properties 1,2,3,7, you can get Properties 4 and 8 pretty easily. Then you just have to get Properties 5 and 6. These two properties have the same kind of proof, so really the main new input is that you need an idea for Property 5.

7 Arithmetic with Dedekind Cuts

The Dedekind cut from Problem 2 is the real number $\sqrt{2}$. Notice that we managed to define $\sqrt{2}$ without reference to any real numbers. The number $\sqrt{2}$ is really just a certain pair of subsets of rational numbers. To really make sense of the statement that our pair (A, B) from problem has the property that $(A, B) \times (A, B) = 2$ (the Dedekind cut representing 2) we have to define the basic arithmetic operations with Dedekind cuts.

Addition: Suppose that (A_1, B_1) and (A_2, B_2) are both Dedekind cuts. Then $(A_1, B_1) + (A_2, B_2)$ is defined to be the pair (A_3, B_3) where A_3 is the set of all rationals of the form $a_1 + a_2$ where $a_1 \in A_1$ and $a_2 \in A_2$. Likewise B_3 is the set of all rationals of the form $b_1 + b_2$ with $b_1 \in B_1$ and $b_2 \in B_2$.

Problem 3: Prove that $(A_1, B_1) + (A_2, B_2)$ is a Dedekind cut.

Negation: Given any set X of rational numbers, let $-X$ denote the set of the negatives of those rational numbers. That is $x \in X$ if and only if $-x \in -X$. If (A, B) is a Dedekind cut, then $-(A, B)$ is defined to be $(-B, -A)$. This is pretty clearly a Dedekind cut.

The Sign: A Dedekind cut (A, B) is called *positive* if $0 \in A$ and *negative* if $0 \in B$. If (A, B) is neither positive nor negative, then (A, B) is the cut representing 0. If (A, B) is positive, then $-(A, B)$ is negative. Likewise, if (A, B) is negative, then $-(A, B)$ is positive. The cut (A, B) is *non-negative* if it is either positive or 0.

Positive Multiplication: If (A_1, B_1) and (A_2, B_2) are both non-negative, then we define $(A_1, B_1) \times (A_2, B_2)$ to be the pair (A_3, B_3) where

- A_3 is the set of all products $a_1 a_2$ where $a_1 \in A_1$ and $a_2 \in A_2$ and at least one of the two numbers is non-negative.
- B_3 is the set of all products of the form $b_1 b_2$ where $b_1 \in B_1$ and $b_2 \in B_2$.

Let's take an example to see why we need the more complicated definition for A_3. Suppose that (A_1, B_1) and (A_2, B_2) both represent the number 1. Then we have $-2 \in A_1$ and $-2 \in A_2$. But $(-2)(-2) = 4$ and we don't want $4 \in A_3$.

General Multiplication: This is a bit painful.

- If (A_1, B_1) is the 0 cut, then $(A_1, B_1) \times (A_2, B_2) = (A_1, B_1)$.
- If (A_2, B_2) is the 0 cut, then $(A_1, B_1) \times (A_2, B_2) = (A_2, B_2)$.
- If (A_1, B_1) is negative and (A_2, B_2) is positive then

$$(A_1, B_1) \times (A_2, B_2) = -\Big(- (A_1, B_1) \times (A_2, B_2)\Big).$$

- If (A_1, B_1) is positive and (A_2, B_2) is negative then

$$(A_1, B_1) \times (A_2, B_2) = -\Big((A_1, B_1) \times -(A_2, B_2)\Big).$$

- If (A_1, B_1) is negative and (A_2, B_2) is negative then

$$(A_1, B_1) \times (A_2, B_2) = \Big(- (A_1, B_1) \times -(A_2, B_2)\Big).$$

That takes care of all the cases. This is kind of a crazy scheme, and I guess that there is probably a more efficient definition. But, this does the job.

Field Axioms: Now that we've defined addition and multiplication of Dedekind cuts (i.e., real numbers), it is a tedious but routine matter to check that these things satisfy all the field axioms. I'm not going to work this out in these notes, but if you have a day to kill you can probably do it yourself.

The Ordering: Now we're doing to change notation. Since real numbers are Dedekind cuts, we're going to denote them with variables in the usual way. As I mentioned above, once you have the field axioms, you can define subtraction and negation. So, we write $a < b$ if and only if $a - b$ is negative. It is again a routine but tedious matter to check that this makes the reals into an ordered field.

Notation: The set of real numbers is usually denoted by $\boldsymbol{R}$. In summary, $\boldsymbol{R}$ is the ordered field of Dedekind cuts, and $\boldsymbol{R}$ contains $\boldsymbol{Q}$ (the rationals) as a subset.

8 Least Upper Bound Property

Now I'm going to explain why $\boldsymbol{R}$ doesn't have any *holes* in it. Call a set $S \subset \boldsymbol{R}$ *bounded from above* if there is some N such that $s < N$ for all $s \in S$. For instance, the set of numbers less than 25 is bounded from above. In any Dedekind cut (A, B), the set A is bounded from above.

Supposing that S is a set which is bounded from above, an *upper bound* for S is some real number x such that $s \leq x$ for all $x \in S$. For instance 17 is an upper bound for the set of negative numbers. But 0 is also an upper bound for the set of negative numbers.

A number x is called a *least upper bound* for S if x is an upper bound for S but no $y < x$ is an upper bound for S. You might say that the set S "creeps right up to" its least upper bound.

Now let's prove that every set $S \subset \boldsymbol{R}$ which is bounded from above has a least upper bound. Define a Dedekind cut (A, B) like this:

- A consists of all rationals x such that $x < s$ for some $s \in S$.
- B consits of all rationals of the form $x + y$ where $s < x$ for all $s \in S$ and $y > 0$. This definition is designed to "cut the endpoint off" B.

It is a tedious but not difficult exercise to check that (A, B) really is a Dedekind cut, and that (A, B) is the least upper bound for S. This is the sense in which $\boldsymbol{R}$ has no holes:

So, if you define a real number to be a Dedekind cut then the set $\boldsymbol{R}$ of real numbers is a complete ordered field with the least upper bound property.

9 Final Word

Why are the reals better than the rationals. They are both ordered fields and we can make the least upper bound definition for the rationals. However, there are sets of rationals which are bounded from above but do not have a least upper bound (in the rationals).

Let's take the set S of rationals x such that $x^2 < 2$. We'd like that $\sqrt{2}$ is the least upper bound for S but $\sqrt{2}$ is not a rational number. So, if we choose a *rational* upper bound, it must be greater than $\sqrt{2}$. But then we can find some smaller upper bound. So, in the rationals, there is no least upper bound to S.