Ham Sandwich Theorem and Other Adventures in Topology

by

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An academic exercise in partial fulfillment for the Degree of Bachelor of Science with Honours in Mathematics

Contents

Introduction

As suggested by the title, this paper is about Ham Sandwich Theorem; whose original statement is: given any 3 bounded objects in 3-dimensional space, they can be cut into half simultaneously with one cut only. Although this can be proved by elementary method, it is hard to generalize to higher dimension. So we choose to use Borsuk-Ulam Theorem. Borsuk-Ulam Theorem is an interesting theorem on its own, because of its numerous applications and admits many kinds of proof. Here we choose to appeal to 2 big machinery in algebraic topology, namely: *covering space* and *homology theory*. Starting from a cute little theorem, we end out with some big tools, and so it justifies the term "adventure".

It is not necessary to use all these big tools, however, we use it because of 2 reasons:

1. These tools are interesting on its own.

2. They give a more conceptual proof.

These tools are introduced in the general forms, which is in algebraic nature. The topology will only return to our sight when we try to apply these tools to our problems. We make some efforts in clearly distinguishing the roles played by algebra and topology, not always successful. Perhaps this is a good place to quote G.H.Hardy on what is a good proof, one of my favourite mathematician,

> ...there is a very high degree of unexpectedness, combined with inevitability and economy. The arguments take so odd and suprising form; the weapons used seem to be childishly simple when compared with the far-reaching results; but there is no escape from the conclution.

The proof here is not that good by this very standard, however I think that this is not an uncommon situation where the weapon is more useful than the theorem.

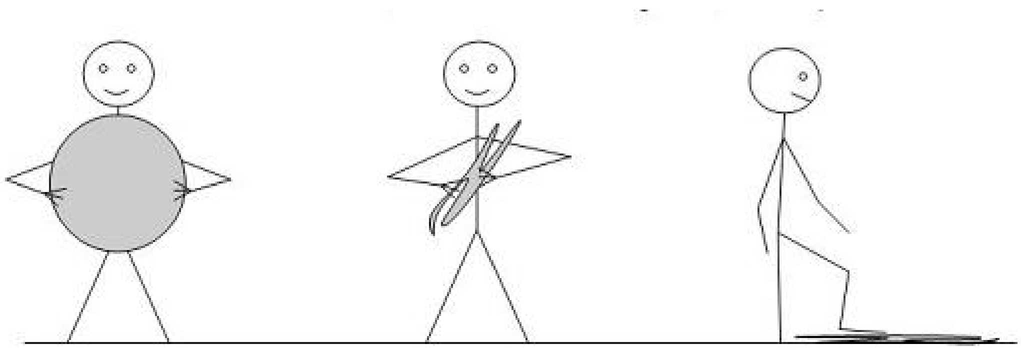
Chapter 1

Borsuk-Ulam Theorem

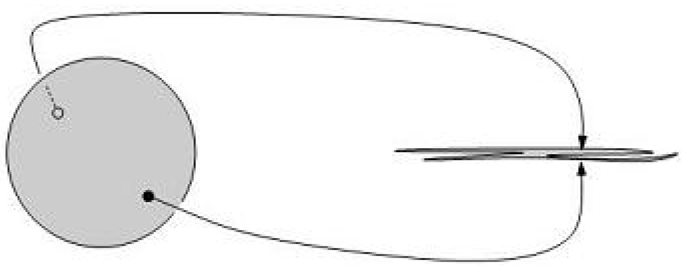
Like many great theorems, Borsuk-Ulam Theorem comes in a simple form: *For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exist a point $x \in S^n$ with $f(x) = f(-x)$*, understandable to even the high school students, and yet it admits many non-trivial generalizations and inspires numerous other results. This paper is a justification to the claim above.

It is always a good idea to get some intuition about the theorem, and the Borsuk-Ulam Theorem has this interesting interpretation[1]: given a ball, deflate, crumple, and lay it flat,

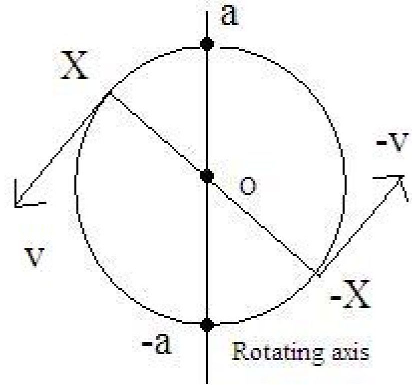
[1] This example is taken from the book [Mat03]



then there is a pair of antipodal points that lies on top of another.



Notice that the same result hold for maps $S^n \xrightarrow{f} \mathbb{R}^m$ where $m \leq n$ because we can extend the range of the function, $f = (f_1, ..., f_m)$, to $\mathbb{R}^n$ by letting $f = (f_1, ..., f_m, c_{m+1}, ..., c_n)$ where c_i are constants. However, the result is false if $m > n$. To see this, consider a rotating sphere, its velocity field is simply a continuous function $S^2 \xrightarrow{f} \mathbb{R}^2$ Obviously all the antipodal points, $x, -x$, are heading opposite direction, except a and (-a). Now let us colour the sphere continuously, says, black at the north pole and gradually tone down to the white at the south pole. Then the result obviously doesn't hold for function $f(x) = (velocity, colour)$


a
X
-v
o
v
-X
-a
Rotating axis

1.1 Equivalent Formulations

This section is devoted to the following theorem:

Theorem For every n, the followings are equivalent:

1. (Borsuk Theorem) For every continuous mapping $f : S^n \longrightarrow \mathbb{R}^n$ there exist a point $x \in S^n$ with $f(x) = f(-x)$
2. For every *antipodal mapping* $f : S^n \longrightarrow \mathbb{R}^n$, ie: f is continuous and $f(-x) = -f(x)$, there exist a point $x \in S^n$ satisfying $f(x) = 0$
3. There is no antipodal mapping $f : S^n \longrightarrow S^{n-1}$
4. There is no continuous mapping $f : B^n \longrightarrow S^{n-1}$ that is antipodal on the boundary.

Proof

- (1) $\Rightarrow$ (2) Apply (1) to the antipodal map in (2). Then we have a a such that $f(-a) = -f(a) = -f(-a)$, which implies $f(a) = 0$

- (2) $\Rightarrow$ (1) Apply (2) to $g(x) = f(x) - f(-x)$

- (2) $\Rightarrow$ (3) Suppose there is an antipodal mapping $S^n \longrightarrow S^{n-1}$, then it is also an antipodal mapping $S^n \longrightarrow \mathbb{R}^n$. Apply (2), the mapping has a zero point, which is impossible, since the codomain of the mapping is a sphere.

- (3) $\Rightarrow$ (2) Suppose there is no x such that $f(x) = 0$, then the function $g = \frac{f}{|f|} : S^n \longrightarrow S^{n-1}$ is well defined. This contradicts (3).

- (4) $\Rightarrow$ (3) Let π be the homeomorphism of the upper hemisphere of S^n with B^n. So any antipodal map $f : S^n \longrightarrow S^{n-1}$ induces the antipodal map $f\pi^{-1} : B^n \longrightarrow S^{n-1}$.

- (3) $\Rightarrow$ (4) Conversely, given any $f : B^n \longrightarrow S^{n-1}$ that is antipodal on the boundary, we can define $g : S^n \longrightarrow S^{n-1}$ by $g(x) = f\pi(X)$ and $g(-x) = -f\pi(x)$.

$\square$

1.2 Types of Proofs

There are many proofs of Borsuk-Ulam Theorem. For completeness, we summarize those proofs that are known to us and provide the references:

1. In Bourgin's book [Bou63], Borsuk-Ulam Theorem is a particular application of Smith Theory.

2. The most common proof uses the notion of degree, see Hatcher [Hat02].

3. A more advance proof using cohomology ring is given by J.P.May [May99].

4. An elementary proof using Tucker Lemma can be found in [GD03].

5. Another elementary proof using homotopy extension argument can be found in [Mat03].

The proof that we are going to show here is essentially the second kind of proof, but we have discard the idea of degree of map.

Chapter 2

Covering Space

2.1 Definitions

Definition A map $p : E \longrightarrow B$ is a *covering* if (i) p is surjective, (ii) for each $b \in B$, there is a neighborhood N of b such that $p^{-1}(N)$ is the disjoint union of open sets U_i such that p maps each U_i homeomorphically into N. U_i is called fundamental neighborhood if it is a path connected open set. We call E the *total space*, B the *base space*, and $F_b := p^{-1}(b)$ a *fiber* of the covering where $b \in B$.

Lemma **(Uniqueness of Lifting)**

Let $p : E \longrightarrow B$ be a covering, X a connected space, $f : X \longrightarrow B$ a continuous mapping, $g, g' : X \longrightarrow E$ such that $pg = pg' = f$. If g, g' agree on one point then they agree on the whole X.

Proof Let W be the subset of X where g, g' agree on, we just have to prove that W is both open and close for the only not empty subset that is both close and open in a connected space is the whole space itself[Mun75]. Let $w \in W$, N a fundamental neighborhood of $f(w)$. By continuity, g, g' must map some neighborhood of w into the same open set of $p^{-1}(f(w))$. Thus g, g' must agree on that neighborhood. Similarly, if g, g' disagree on a point, they must disagree on some neighborhood of that point. So W is both open and close. □

Lemma Let $p : E \longrightarrow B$ be a covering, $b \in B$, and $e \in F_b$

1. (**Path Lifting Lemma**) A path $f : I \longrightarrow B$ with $f(0) = b$ lift uniquely to a path $g : I \longrightarrow E$ such that $g(0) = e$ and $pg = f$.

2. (**Homotopy Lifting Lemma**) Homotopical paths $f \cong f'$ that start at b lift up homotopical path that start at e, hence $g(1) = g'(1)$

Proof

1. Since I is compact, by Lebesgue Lemma, we can subdivide I into subintervals such that each is map into fundamental neighborhood of B. Then we can lift the path inductively by using the homeomorphisms between the open sets of E and the fundamental neighborhood of B.

2. Let $h : I \times I \longrightarrow B$ be the homotopy of paths f, f' and divide the square $I \times I$ into subsquares that are mapped into fundamental neighborhood of B. Then we can lift h to $H : I \times I \longrightarrow E$. By considering g as the lifting of h along the bottom of the square, and the uniqueness of lifting, H is the homotopy of g, g'

□

2.2 Fundamental Theorem of covering space

Definition Given a space E and a point $e \in E$, the *star of* e, denoted by $St(e)$, is the set of all paths that start at e, up to homotopical equivalence.

With the definiton above, we can rephrase the unique path lifting property as: the covering $p : E \longrightarrow B$ induces an isomorphism $St(e) \longrightarrow St(p(e))$ for all $e \in E$. Now we are ready to state the main theorem in this chapter:

Theorem (**Fundamental Theorem of Covering Spaces** [May99]) Given a covering $p : E \longrightarrow B$ and a mapping $f : X \longrightarrow B$, where X is path-connected, and choose a base point $x_0 \in X$. Let $b_0 = f(x_0)$ and choose $e_0 \in F_{b_0}$. Then there exist a mapping $g : X \longrightarrow E$ such that $g(x_0) = e_0$ and $pg = f$ if and only if

$$f_*(\pi(X, x_0)) \subset p_*(\pi(E, e_0))$$

in $\pi(B, b_0)$, where f_*, g_* are the induced mapping of f, g by the homotopy group functor. When this condition holds, the mapping g is unique.

Proof (Necessity)The lifting g ensures that $imf \subset imp$, so we have $f_*(\pi(X, x_0)) \subset p_*(\pi(E, e_0))$ by functoriality.
(Sufficiency) Let $x \in X$ be an arbitrary point, $[\alpha] : x_0 \longrightarrow x$ be an equivalence class of path from x_0 to x. Let $[\beta]$ be the unique element in $St(e_0)$ such that $p([\beta]) = f([\alpha])$. We define $g(x)$ to be the endpoint of path β. To show that g is well defined, let $[\alpha']$ be a distinct equivalence class of paths from x_0 to x. We want to show that the corresponding $[\beta']$ has the same end point as $[\beta]$. Since $[\alpha^{-1}\alpha'] \in \pi(X, x_0)$ we have some $l \in \pi(E, e_0)$ such that $p(l) = f(\alpha^{-1}\alpha') = f^{-1}(\alpha)f(\alpha')$. So we have $p(\beta l) = f(\alpha)f^{-1}(\alpha)f(\alpha') = f(\alpha')$. That is, $\beta' = \beta l$, and its end point is same as β, as required.
To show that g is continuous. let $U \subset E$ be an open set. We can find a smaller open set $U' \subset U$ such that $p(U')$ is some open set in B for p is a local homeomorphism. Then $g^{-1}(U') = f^{-1}(p(U'))$ is open, hence g is continuous. □

To justify the effort we have make, we prove the following:

Theorem (2-dimensional Borsuk Ulam Theorem) There is no antipodal mapping $f : S^2 \longrightarrow S^1$

Proof Suppose there is an antipodal map $f : S^2 \longrightarrow S^1$. Let q be the canonical projection map, and $p : S^1 \longrightarrow S^1$, the map defined by $z \mapsto z^2$. Then we can define g by $g(\pm x) := f^2(x)$ such that the diagram below commutes.

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "q"'] & S^1 \arrow[d, "p"'] \\ \mathbb{R}P^2 \arrow[r, "g"] & S^1 \end{tikzcd} \tag{2.1}$$

Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, and $\pi_1(S^1) = \mathbb{Z}$ [1], $g_*(\mathbb{Z}/2) = 0$ for it is a torsion subgroup of $\mathbb{Z}$.By the Fundamental Theorem of Covering Spaces above there is a lifting $\tilde{g} : \mathbb{R}P^2 \longrightarrow S^1$. By construction, $\tilde{g}q$ and f agree on either x or $-x$ and they are both lifting of $\tilde{g}$ hence by the uniqueness of lifting lemma above $\tilde{g}q = f$. It follows that

$$f(x) = \tilde{g}q(x) = \tilde{g}q(-x) = f(-x) = -f(x)$$

This will force $f(x)$ to be zero, which is imposible since the codomain of f is a sphere. $\square$

[1] For the calculation of homotopy group, see [DP97]

Chapter 3

Homological Algebra

This chapter provides us with the elementary algebraic tools to be applied to topological problems later. Although homological algebra is itself an interesting topic, with the goal to introduce homology functor[1] and long exact sequence in mind, it's presentation here is not only incomplete but also highly compact. To separate the algebra from topology does not diminish their intimate connection, but enables us to see clearly in which parts they play in this vast subject: Algebraic topology.

3.1 Definitions

To define the homology functor, we introduce the categories of graded modules and chain complexes in this section. We assume familiarity

[1]For reference to category theory , see [BK00a]

with rings and modules[2] here and let Λ denote a fixed, but otherwise arbitrary, ring through out this chapter.

Definition The category of *graded Λ-modules*, $\mathcal{MOD}_{\Lambda}^{\mathbb{Z}}$, is given by the following data:

- An object in the category, C, is a family Λ-module, $\{C_n\}$, where $n \in \mathbb{Z}$.
- A morphism of degree p, $C \xrightarrow{\phi} D$, is a family of Λ-homomorphisms, $\{\phi_n : C_n \longrightarrow D_{n+p}\}$

Definition The category of *Chain Complexes over Λ* is given by the following data:

- An object in the category is a pair (C, ∂), where C is a graded Λ -modules and $\partial : C \longrightarrow C$ is an endomorphism of degree -1 with $\partial_n \partial_{n+1} = 0$
- A morphism $\Phi : (C, \partial) \longrightarrow (\widetilde{C}, \widetilde{\partial})$ is a graded morphism of degree 0 such that $\partial_n \Phi_{n-1} = \Phi_n \widetilde{\partial}_n$, ie: the following diagram commutes, for every n :

$$\begin{tikzcd}
\dots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "\Phi_n"] & C_{n-1} \arrow[r] \arrow[d, "\Phi_{n-1}"] & \dots \\
\dots \arrow[r] & \widetilde{C}_n \arrow[r, "\widetilde{\partial}_n"] & \widetilde{C}_{n-1} \arrow[r] & \dots
\end{tikzcd} \tag{3.1}$$

[2]For reference to ring theory, see [BK00b]

Terminology and convention:

- The morphism ∂ (as well as its components) is called a differential operator, or boundary operator in most topological settings.
- For simplicity, we follow the standard practice of dropping the subscripts in module homomorphisms ∂_n and Φ_n whenever the context makes it clear. For example, we shall only write $\partial\Phi = \Phi\partial$ to express the commutativity of the diagram above.
- Furthermore, we shall write the boundary operator of all chain complexes as ∂

To facilitate the definition of the homology functor, we need the following easy and yet important lemma.

Lemma (Chain Map Lemma): Let $\psi : (C, \partial) \longrightarrow (D, \partial')$ be a chain map. Then ψ_n induces:

1. $\psi' : Im\partial_{n+1} \longrightarrow Im\partial'_{n+1}$
2. $\psi'' : Ker\partial_n \longrightarrow Ker\partial'_n$
3. $\psi_* : Ker\partial_n/Im\partial_{n+1} \longrightarrow Ker\partial'_n/Im\partial'_{n+1}$

Proof $Ker\partial_n$ and $Im\partial_{n+1}$ are both submodule of C_n. Thus $\psi_n|_{Ker\partial_n}$ and $\psi_n|_{Im\partial_{n+1}}$ are module homomorphisms to D_n. All we have to prove is the the codomain of the 2 homomorphism are $Ker\partial'_n$ and $Im\partial'_{n+1}$ respectively.

1. Let $x \in Im\partial_{n+1}$, ie: $x = \partial_{n+1}(y)$ for some $y \in C_{n+1}$.Then $\psi_n(x) = \partial'_{n+1}\psi_{n+1}(y) \in Im\partial'_{n+1}$ by commutativity of the diagram 1.1.

2. Let $x \in Ker\partial_n$. Then $\partial'_n\phi_n(x) = \phi_{n+1}\partial_n(x) = 0$ by commutativity of the diagram 1.1.

3. We have $Ker\partial_n \xrightarrow{\psi''} Ker\partial'_n \xrightarrow{\epsilon} Ker\partial'_n/Im\partial_{n+1}$. And since $\psi''(Im\partial_{n+1}) = 0$, by the universal property of quotient map, there is a unique map $Ker\partial_n/Im\partial \xrightarrow{\Psi} Ker\partial'_n$. Then the composition map, $\epsilon\Psi$ is the required homomorphism.

$\square$

Definition Given a chain complex $C = (C_n, \partial_n)$, the condition $\partial^2 = 0$ ensures that $H_n(C) := ker\partial_n/im\partial_{n+1}$ is well defined for all $n \in \mathbb{Z}$. Therefore, we can associate with chain complex C the graded module $H(C) := \{H_n(C)\}$. Furthermore a chain map $\phi : C \longrightarrow D$ induces a graded morphism of degree 0, $H(\phi) = \phi_* : H(C) \longrightarrow H(D)$ by chain map lemma. $H(-)$ can now be precisely called a functor from the category of chain complexes over Λ to the category of graded Λ-modules.

Notation and Terminology Often in applications to topology,

- an element in C_n is called *n-chain*
- $Ker\partial_n$ is denoted as $Z_n = Z_n(C)$, whose element is called *n-cycle*
- $Im\partial_{n+1}$ is denoted as $B_n = B_n(C)$, whose element is called *n-boundary*
- Two $n - cycles$ which determine the same element in $H_n(C)$ are called *homologous*
- The element in $H_n(C)$ determined by a $n - cycle$, c, is called the homology class of c, denoted by $[c]$.

3.2 Some Lemmas

Lemma $\partial_n : C_n \longrightarrow C_{n-1}$ induces $\widetilde{\partial_n} : coker\partial_{n+1} \longrightarrow ker\partial_{n-1}$ with $ker\tilde{\partial}_n = H_n(C)$ and $coker\tilde{\partial}_n = H_{n-1}(C)$

Proof Since $im\partial_{n+1} \subset ker\partial_n$ and $im\partial_n \subset ker\partial_{n-1}, \partial$ induces $\tilde{\partial}_n$ as follows:

$$coker\partial_{n+1} = C_n/im\partial_{n+1} \longrightarrow C_n/ker\partial_n \cong im\partial_n \subset ker\partial_{n-1}$$

On the other hand, $ker\tilde{\partial}_n = ker\partial_n/im\partial_{n+1} = H_n(C)$,and $coker\tilde{\partial}_n = ker\partial_{n-1}/im\partial_n = H_{n-1}(C)$ □

Lemma Let

$$
\begin{tikzcd}
A' \arrow[r, "\alpha_1"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\Sigma_1"] & A \arrow[r, "\alpha_2"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Sigma_2"] & A'' \arrow[d, "\theta"'] \\
B' \arrow[r, "\beta_1"] & B \arrow[r, "\beta_2"] & A''
\end{tikzcd}
\tag{3.2}
$$

be commutative diagram with exact rows. Then ϕ induces an isomorphism

$$\Phi : ker\theta\alpha_2/(ker\alpha_2 + ker\phi) \longrightarrow (Im\phi \cap Im\beta_1)/Im\phi\alpha_1$$

Proof To show that Φ is an isomorphism,

- Let $x \in ker\theta\alpha_2$, obviously $\phi(x) \in im\phi$. Since $\beta_2\phi(x) = \theta\alpha_2(x) = 0$, we have $\phi(x) \in ker\beta_2 = im\beta_1$. Thus we have induced homomorphism $ker\phi\alpha_2 \longrightarrow (Im\phi \cap Im\beta_1)$

- If $x \in ker\alpha_2$, then $x \in im\alpha_1$, and hence $\phi(x) \in im\phi\alpha_1$. If $x \in ker\phi$, then $\phi(x) = 0 \in im\phi\alpha_1$. Thus Φ is well defined.

- To show that Φ is surjetive, let $y \in im\phi \cap im\beta_1$, then there exist $x \in A$ with $\phi(x) = y$. Since $\theta\alpha_2(x) = \beta_2\phi(x) = \beta_2(y) = (y \in im\beta_1 = ker\beta_2)0$, so $x \in ker\theta\alpha_2$.

- To show that Φ is injective, let $x \in ker\theta\alpha_2$ such that $\phi(x) \in im\phi\alpha_1$, ie $\phi(x) = \phi\alpha_1 y$ for some $y \in A'$. Then $x = \alpha_1(y) + m$, where $m \in ker\phi$. That is, $x \in ker\alpha_2 + ker\phi$.

$\square$

Notation

$$\begin{tikzcd} A' \arrow[r, "\alpha_1"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\Sigma"] & A \arrow[d, "\phi"'] \\ B' \arrow[r, "\beta_1"] & B \end{tikzcd} \tag{3.3}$$

Given the diagram above, we denote

$$im\Sigma = (im\phi \cap im\beta)/im\phi\beta$$

$$ker\Sigma = ker\phi\alpha/ker\alpha + ker\psi$$

Thus we can express the conclusion of the lemma above as $ker\Sigma_2 = im\Sigma_1$.

Lemma (Snake Lemma) Let the following commutative diagram have exacts rows [3]

$$\begin{tikzcd} & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\mu'"] & B' \arrow[r, "\epsilon'"] & C' & \end{tikzcd} \tag{3.4}$$

Then there is a homomorphism ω that makes the following sequence exact:

$$ker\alpha \xrightarrow{\mu_*} ker\beta \xrightarrow{\epsilon_*} ker\gamma \xrightarrow{\omega} coker\alpha \xrightarrow{\mu'_*} coker\beta \xrightarrow{\epsilon'_x} coker\gamma$$

[3]The Snake Lemma is introduced in all homological algebra books, we give one here [HS97]

Proof By Chain Map Lemma, the sequences $ker\alpha \xrightarrow{\mu_*} ker\beta \xrightarrow{\epsilon_*} ker\gamma$ and $coker\alpha \xrightarrow{\mu'_*} coker\beta \xrightarrow{\epsilon'_x} coker\gamma$ are exact. So we just have to construct the connecting homomorphism ω and prove that $ker\gamma \xrightarrow{\omega} coker\alpha$ is exact. Now consider the diagram below:

$$
\begin{tikzcd}
 & & ker\beta \arrow[r, "\epsilon_*"] \arrow[d] \arrow[dr, phantom, "\Sigma_1"] & ker\gamma \arrow[d] & \\
 & A \arrow[r, "\mu"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Sigma_3"] & B \arrow[r, "\epsilon"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\Sigma_2"] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\mu'"] \arrow[d] \arrow[dr, phantom, "\Sigma_4"] & B' \arrow[r, "\epsilon'"] \arrow[d] & C' & \\
 & coker\alpha \arrow[r, "\mu'_*"] & coker\beta & &
\end{tikzcd}
\tag{3.5}
$$

By the lemma above, we have $im\Sigma_1 = ker\Sigma_2 = im\Sigma_3 = ker\Sigma_4$. $im\Sigma_1 = (ker\gamma \cap \epsilon(B))/\epsilon_*(ker\beta) = ker\gamma/\epsilon_*(ker\beta) = coker\epsilon_*$ ($\epsilon(B) = C$ as ϵ is surjective). Similarly, we have $ker\Sigma_4 = ker\mu'_*$. Hence we have $ker\gamma \longrightarrow coker\epsilon_* = ker\mu'_* \longrightarrow coker\alpha$ an exact sequence and define the composite homomorphism as ω. □

Theorem (Long Exact Sequence) [4]

Given a short exact sequence $A \xhookrightarrow{\phi} B \overset{\psi}{\twoheadrightarrow} C$ of chain complexes, then there is long exact sequence as follow:

$$\ldots \xrightarrow{\omega_{n+1}} H_n(A) \xrightarrow{\phi_*} H_n(B) \xrightarrow{\psi_*} H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \overset{\phi_*}{\cdots\rightarrow} \tag{3.6}$$

[4] This proof is suggested by the exercise in [HS97]

Proof Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & ker\partial_n \arrow[r] \arrow[d] & ker\partial_n \arrow[r] \arrow[d] & ker\partial_n \arrow[d] & \\
& A_n \arrow[r, "\phi"] \arrow[d, "\partial_n"'] & B_n \arrow[r, "\psi"] \arrow[d, "\partial_n"'] & C_n \arrow[d, "\partial_n"'] & \\
& A_{n-1} \arrow[r, "\phi"] \arrow[d] & B_{n-1} \arrow[r, "\psi"] \arrow[d] & C_{n-1} \arrow[d] & \\
& coker\partial_n \arrow[r] & coker\partial_n \arrow[r] & coker\partial_n \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

By snake lemma, the top and bottom sequences are exact for all n.Then by the first lemma in this section, we have the following diagram:

$$\begin{tikzcd}
& H_n(A) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[d] & \\
& coker\partial_{n+1} \arrow[r, "\phi"] \arrow[d, "\widetilde{\partial_n}"'] & coker\partial_{n+1} \arrow[r, "\psi"] \arrow[d, "\widetilde{\partial_n}"'] & coker\partial_{n+1} \arrow[r] \arrow[d, "\widetilde{\partial_n}"'] & 0 \\
0 \arrow[r] & ker\partial_{n-1} \arrow[r] \arrow[d] & ker\partial_{n-1} \arrow[r] \arrow[d] & ker\partial_{n-1} \arrow[d] & \\
& H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) &
\end{tikzcd} \tag{3.8}$$

Applying the snake lemma again, we get the connecting morphism

$$\omega : H_n(C) \longrightarrow H_{n-1}(A)$$

such that the sequence(3.6) is exact. $\square$

Chapter 4

Homology Theory

4.1 Axiomatic Homology theory

Definition Given an abelian group, π, a homology theory consists of a functor $H_*(X, A; \pi)$ from the category of pairs of space to the category of graded abelian group together with the natural transformations $\partial : H_q(X, A; \pi) \longrightarrow H_{q-1}(A; \pi)$. This functor and natural transformations satisfy and are characterized by the following axioms (see [May99],[ES52] for other axioms.)

DIMENSION If p is a point, then $H_0(\{p\}; \pi) = \pi$ and $H_q(\{p\}; \pi) = 0$ for all integers q.

EXACTNESS The following sequence is exact:

$$\ldots \longrightarrow H_q(A;\pi) \longrightarrow H_q(X;\pi) \longrightarrow H_q(X,A;\pi) \xrightarrow{\partial} H_{q-1}(A;\pi) \cdots\cdots\to$$

EXCISION If $(X;A,B)$ is an excisive triad, ie: X is the union of the interior of A and B, then the inclusion $(A, A\cap B) \longrightarrow (X,B)$ induces an isomorphism

$$H_*(A, A\cap B;\pi) \longrightarrow H_*(X,B;\pi)$$

.

ADDITIVITY If (X,A) is the disjoint union of a set of pairs (X_i,A_i)m then the inclusions $(X_i,A_i) \longrightarrow (X,A)$ induce an isomorphism

$$\Sigma_i H_*(X_i,A_i;\pi) \longrightarrow H_*(X,A;\pi)$$

.

HOMOTOPY If f is homotopy to g then $H_*(f) = H_*(g)$. We often denote $H_*(f)$ as f_*.

Definition A *reduced homology group* $\widetilde{H_q}(X,A;\pi)$ is defined to be the kernel of $H_q(X,A;\pi) \longrightarrow H_q(point;\pi)$. Thus we have, by definition, $H_q(X,A;\pi) = \widetilde{H_q}(X,A;\pi)$ for $q > 0$

4.2 Singular Homology

We need to construct a concrete functor that satisfies the axioms. There are many of them, which can be proved are essentially the same[May99]. However, in this paper, we are going to introduce just one of them, namely singular homology.

Let Δ^n be an n-simplex, ie: $\Delta^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | \Sigma x_i = 1\}$, X be an arbitrary space and G some abelian group. Now we define $C_n(X; G)$ be the free abelian group generated by the mapping from $\delta : \Delta^n \longrightarrow X$ tensoring with G. We shall omit the G from our notation by fixing it. To make $\{C_i(X)\}$ a chain complex, we just have to define $\partial : C_k(X) \longrightarrow C_{k-1}(X)$ such that $\partial\partial = 0$. This can be done by define ∂ on the generator of the group since the group is free:

$$\partial(\delta) := \sum_i (-1)^i \delta|\Delta^i$$

Δ^i is defined to be the subspace of Δ such that its i-coordinate is 0. If Δ is an n-simplex then Δ^i can be regarded as an $(n-1)$-simplex, thus ∂ really maps into $C_{n-1}(X)$. And we calculate as follow:

$$\partial\partial(\delta) = \sum_{j<i} (-1)^i (-1)^j \delta|\Delta^{ji} + \sum_{j>i} (-1)^i (-1)^{j-1} \delta|\Delta^{ij} = 0$$

So we have show that for any space X we have a chain complex $\{C_i(X)\}$ and hence a homology functor by previous chapter. We will not attempt to show that this functor does satisfy the axioms for the task is tedious and we need not use many of the axioms in the following proof. But we do need the homology groups of some space to proceed, so we record down the results that we want. For the detailed calculation, please refer to [DP97],[Mun75]

- $H_*(B^n; \pi) = 0$ for all n and all ablelian group π since B^n is contractible .
- $H_n(S^n; \pi) = \pi$
- $H_i(S^n; \pi) = 0$ for $0 < i < n$
- $H_n((\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2$

4.3 Proof of Borsuk-Ulam Theorem

In this section, we will use singular homology and fixed π to be $\mathbb{Z}/2$ and omit it in our homology group notation. For convenience, we restate the theorem here, in the version that we want:

Theorem There is no *antipode-preserving map* $f : S^n \longrightarrow S^{n-1}$. [1]

Proof If such mapping exist, then by restricting the mapping to S^{n-1}, we will have the restriction map $f' : S^{n-1} \longrightarrow S^{n-1}$. We claim that f' induced an isomorphism $f'_* : H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(S^{n-1})$ on

[1] An antipode-preserving map f is a map that satisfies the equation $fT = fT$, where $T : S^n \longrightarrow S^n$ is the antipodal map, ie: $T(x) = -x$

the homology group. But since f' can be extended to $f'' : B^n \longrightarrow S^{n-1}$ and the homology group of B^n is zero, the induced isomorphism f'_* can be factored through a zero group. This implies that f'_* is a zero map. But this is impossible since f'_* is an isomorphism between 2 non-trivial group. So what left is to prove that $f'_* : H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(S^{n-1})$ is an isomorphism if f is antipodal. $\square$

Lemma Let $T : S^n \longrightarrow S^n$ be antipode-preserving map. Then f induces an isomorphism $f_* : H_n(S^n) \longrightarrow H_n(S^n)$

Proof First, we construct the following short exact sequence of chain complex:

$$0 \longrightarrow C(\mathbb{R}P^n) \xrightarrow{t_*} C(S^n) \xrightarrow{\pi_*} C(\mathbb{R}P^n) \longrightarrow 0$$

Let k-chain, $\Gamma : \Delta^k \longrightarrow \mathbb{R}P^n$ be in $\mathbb{R}P^n$. Since Δ^k is simply connected, it has trivial fundamental homotopy group. Then by the Fundamental Theorem of Covering Spaces[2], there are exactly 2 lifting, say : $\Lambda_1, T(\Lambda_1) := \Lambda_2 : \Delta^k \longrightarrow S^n$, ie:$C(S^n) \xrightarrow{\pi_*} C(\mathbb{R}P^n)$ is surjective, where $\pi : S^n \longrightarrow \mathbb{R}P^n$ is the covering map. Now we define the map $C(\mathbb{R}P^n) \xrightarrow{t_*} C(S^n)$ on the generators since it is a free ggroup by $t_*(\Gamma) = \Lambda_1 + \Lambda_2$, where Γ, Λ_i are defined as above. t_* is injective because the sum of 2 distinct liftings of non-trivial chain is never 0. Since we are using coefficient $\mathbb{Z}/2$, the kernel of π_* is generated by

[2] see chapter 3

$\{\Lambda+T(\Lambda)\}$, which is exactly the image if t_*.(Given $\pi_*(\Lambda_1+...+\Lambda_n) = \pi_*(\Lambda_1)+...+\pi_*(\Lambda_n) = 0$ where Λ_i are the generators of $C(S^n)$, to be zero in $C(\mathbb{R}P^n)$ each $\pi_*(\Lambda_i)$ must occur twice since they are non-zero and $\Lambda_i, T(\Lambda_i)$ must pair up to get the double occurance since the Λ_i are distinct.) Hence the above sequence is exact.

Next we prove that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & C(\mathbb{R}P^n) \arrow[r, "t_*"] \arrow[d, "f'_*"] & C(S^n) \arrow[r, "\pi_*"] \arrow[d, "f_*"] & C(\mathbb{R}P^n) \arrow[r] \arrow[d, "f'_*"] & 0 \\ 0 \arrow[r] & C(\mathbb{R}P^n) \arrow[r, "t_*"] & C(S^n) \arrow[r, "\pi_*"] & C(\mathbb{R}P^n) \arrow[r] & 0 \end{tikzcd} \tag{4.1}$$

Since f is antipodal, we can define f' as the induced map satisfying the diagram below:

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "\pi"'] & S^n \arrow[d, "\pi"'] \\ \mathbb{R}P^n \arrow[r, "f'"] & \mathbb{R}P^n \end{tikzcd} \tag{4.2}$$

Then the right hand side square commutes by functoriality. Let Γ be a chain in $C(\mathbb{R}P^n)$, and $\Lambda_1, T(\Lambda_1) := \Lambda_2$ are its 2 lifting. Then we have $f_*t_*(\Gamma) = f_*(\Lambda_1)+f_*(\Lambda_2)$. On the other hand, $f\Lambda_1$ and $f\Lambda_2$ are the 2 lifting of $f'_*\Gamma$, so $t_*f'_*(\Gamma) = f(\Lambda_1)+f(\Lambda_2)$ as well. Hence the left hand site square commutes as well

We then have the following long exact sequence:[cf. chapter3 the section about long exact sequence]

$$\begin{aligned}
&0 \longrightarrow H_n(\mathbb{R}P^n) \xrightarrow{t_*} H_n(S^n) \xrightarrow{\pi_*} H_n(\mathbb{R}P^n) \xrightarrow{\partial} H_{n-1}(\mathbb{R}P^n) \longrightarrow 0 \\
&\longrightarrow 0 \longrightarrow H_{n-1}(\mathbb{R}P^n) \xrightarrow{\partial} H_{n-2}(\mathbb{R}P^n) \longrightarrow 0 \longrightarrow \\
&\dots \longrightarrow H_1(\mathbb{R}P^n) \xrightarrow{\partial} H_0(\mathbb{R}P^n) \xrightarrow{t_*} H_0(S^n) \xrightarrow{\pi_*} H_0(\mathbb{R}P^n) \longrightarrow 0
\end{aligned} \tag{4.3}$$

The initial term is zero because the higher dimensional homology group of S^n and $\mathbb{R}P^n$ vanish as they are n-dimensional CW-complex[May99]. It is well-known that $H_0(S^n) = H_n(S^n) = \mathbb{Z}/2$, $H_i(S^n) = 0$ otherwise, and $H_i(\mathbb{R}P^n) = \mathbb{Z}/2$ for all i. Since $H_n(\mathbb{R}P^n) \xrightarrow{t_*} H_n(S^n)$ is injective by exactness, therefore it is also surjective as an endomorphism of finite set. It implies that $H_n(S^n) \xrightarrow{\pi_*} H_n(\mathbb{R}P^n)$ is zero and hence $H_n(\mathbb{R}P^n) \xrightarrow{\partial} H_{n-1}(\mathbb{R}P^n)$ is an isomorphism. In conclusion, we have $\partial : H_i(\mathbb{R}P^n) \longrightarrow H_{i-1}(\mathbb{R}P^n)$ isomorphism for i =1,2,3,...,n
Now by naturality of the long exact sequence, we have the following diagrams:

$$\begin{tikzcd}
H_n(\mathbb{R}P^n) \arrow[r] \arrow[d, "f'_*"'] & H_n(S^n) \arrow[d, "f_*"'] \\
H_n(\mathbb{R}P^n) \arrow[r] & H_n(S^n)
\end{tikzcd} \tag{4.4}$$

$$\begin{tikzcd}
H_i(\mathbb{R}P^n) \arrow[r, "\partial"] \arrow[d, "f'_*"'] & H_{i-1}(\mathbb{R}P^n) \arrow[d, "f'_*"'] \\
H_i(\mathbb{R}P^n) \arrow[r, "\partial"] & H_{i-1}(\mathbb{R}P^n)
\end{tikzcd} \tag{4.5}$$

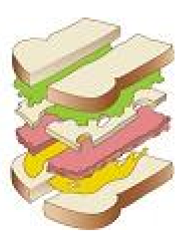
Since $\boldsymbol{f_*} : \boldsymbol{H_0(S^n)} \longrightarrow \boldsymbol{H_0(S^n)}$ is clearly an isomorphism , and hence the induced $\boldsymbol{f_*} : \boldsymbol{H_0}(\mathbb{R}\boldsymbol{P^n}) \longrightarrow \boldsymbol{H_0}(\mathbb{R}\boldsymbol{P^n})$ is also an isomorphism. By induction and the diagram (4.4), we see that $\boldsymbol{f_*} : \boldsymbol{H_i}(\mathbb{R}\boldsymbol{P^n}) \longrightarrow \boldsymbol{H_i}(\mathbb{R}\boldsymbol{P^n})$ is an isomorphism for $\boldsymbol{i}$ =0,1,2,...,n Then by the diagram(4.3) we have $\boldsymbol{f_*} : \boldsymbol{H_n(S^n)} \longrightarrow \boldsymbol{H_n(S^n)}$ an isomorphism. $\square$

Chapter 5

Applications

5.1 Ham Sandwich Theorem

Ham Sandwich Theorem is one of the classical applications of Borsuk Ulam Theorem. It take its name from the case when $n = 3$(see below), the 3 objects are suppose to be a ham and 2 breads. According to Beyer and Zardecki[BZ04], the earliest known paper about the ham sandwich theorem is by Steinhaus. It attributes the posing of the problem to Hugo Steinhaus, and credits Stefan Banach as the first to solve the problem, by a reduction to the BorsukUlam theorem



Theorem Let $M_1, M_2, ..., M_n$ be n bounded measurable[1] subsets of $\mathbb{R}^n$, then there exist an $n-1$-dimensional hyperplane that cut them into half of their measures.

Definition Inside affine space $\mathbb{R}^n$, a hyperplane, P, is the zero set of a non-trivial linear polynomial $F_p = \sum_{i=1}^n a_i x_i + a_0$ where $a_i \in \mathbb{R}$ for $i = 0, 1, ..., n$. All linear polynomials in $\mathbb{R}[x_1, ..., x_n]$ forms an $n+1$-dimensional vector space. Call $\mathbb{H}$ the set of all hyperplane.

Lemma Let S^n be the n-sphere, then there exist a surjective set map $T : S^n \longrightarrow \mathbb{H} \cup \{\emptyset\}$

Proof Let $P = \{(x_i) \in \mathbb{R}^n | \sum_{i=1}^n a_i x_i = a_{n+1}\}$. Since not all coefficients are zero, so $P = \{(x_i) \in \mathbb{R}^n | \sum_{i=1}^n \hat{a}_i x_i = \hat{a}_{n+1}\}$ where

$$\widehat{a_i} = \frac{a_i}{\sqrt{\sum_1^{n+1} a_i^2}} \quad i = 1, 2, ..., n+1$$

Observed that $(\widehat{a_1}, \widehat{a_2}, ..., \widehat{a_n}) \in S^n$, so the map, $T : S^n \longrightarrow \mathbb{H}$ defined by $(a_1, ...a_{n+1}) \mapsto \sum_{i=1}^n a_i x_i = a_{n+1}$ is surjective. Because this map is not defined on the point $(0, ..., 1)$, we shall extend the definition by assigning value to this point, which could be anything, here we choose the empty set. $\square$

N.B. Since S^n is a topological space, T induces a topology on $\mathbb{H}$

[1] For reference of measure theory, see [Rud21]

Definition Let $p \in S^n$ and $P = T(p)$, we define the **positive side of P, P^+,** as the set $\{(x_i) \in \mathbb{R}^n | \sum_{i=1}^n a_i x_i > a_{n+1}\}$.

Definition Let χ_S be the characteristic function of the set S, ie: $\chi_S(x)$ is 1 if $x \in S$, 0, otherwise. Given a measurable set, $M \subset \mathbb{R}^n$, the measure of M on the positive side of a hyperplane, P, is given by the integral $\int_{\mathbb{R}^n} \chi_M \chi_{P^+} d\mu$

Lemma Let $p \in S^n, P = T(p)$, and $M \subset \mathbb{R}^n$ a measurable set, The function $f : S^n \longrightarrow \mathbb{R}$ given by $p \mapsto \int_{\mathbb{R}^n} \chi_M \chi_{P^+} d\mu$ is continuous.

Proof Let $\{p_i\}$ be a sequence converging to p in S^n, then $\{\chi_{P_i^+}\}$ converges to χ_P^+. Since $|\chi_M \chi_{P_i^+}| < \chi_M$ for all i, by Lebesgue's dominated convergence theorem[Rud21], $\int_{\mathbb{R}^n} \chi_M \chi_{P_i^+} d\mu$ converges to $\int_{\mathbb{R}^n} \chi_M \chi_{P^+} d\mu$

$\square$

Now we can prove our main theorem,

Theorem (Ham Sandwich Theorem)

Let $M_1, M_2, ..., M_n$ be n bounded measurable subsets in $\mathbb{R}^n$, then there exist an $n-1$-dimensional hyperplane that cuts them into half of their measures.

Proof Let $f_i : S^n \to \mathbb{R}$ be the function in the lemma above, applying to M_i, and $F := (f_1, f_2, ..., f_n) : S^n \to \mathbb{R}^n$. F is continuous since all its component functions are. So by Borsuk Ulam theorem, there is a point $p \in S^n$ such that $F(p) = F(-p)$. Since $(-P)^+$ is just the other side of P, this asserts that each M_i has equal measure on both side of P. So P is the hyperplane that cut all n sets into half of their measures. One technical point here is that if the point p here is the point $(0, ...1)$, then we could just change the coordinates, so that the argument make sence. (Since we didn't define what is the hyperplane represented by empty set.) $\square$

We give an interesting generalization, which is due to Gromov [Gro03]:

Theorem (**Polynomial Ham Sandwich Theorem**)
Let $d \geq 1$, and let $M_1, ..., M_{C_d^{n+d}-1}$[2] be bounded measurable set in $\mathbb{R}^n$. Then there exist a non-trivial polynomial,$P \in \mathbb{R}[x_1, ..., x_n]$ of at most degree d such that the set $P = 0$ cuts them into half of their measures.

Proof Observe that the polynomial of at most d degree forms a real vector space of dimension C_d^{n+d}.By the same argument of the first lemma, the zero set of these polynomials can be parameterized by the points of $S^{C_d^{n+d}} - 1$. So again we can define a function f_i that maps a zero set to the measure of the part of M_i that intersects with the positive side of the zero set. This function is continuous by the same reason. Thus the function $F := (f_1, ..., f_{C_d^{n+d}} - 1) : S^{C_d^{n+d}-1} \longrightarrow \mathbb{R}^{C_d^{n+d}-1}$ is continuous and the result follows as before. $\square$

[2] C_n^m denotes the number of ways of choosing n items from m items

5.2 Lyusternik Shnirelmann Theorem

In this section, we give another equivalent form of Borsuk-Ulam Theorem, which is part of a more general theory called *Lyusternik Shnirelmann Category*(for detail see [H.F41]

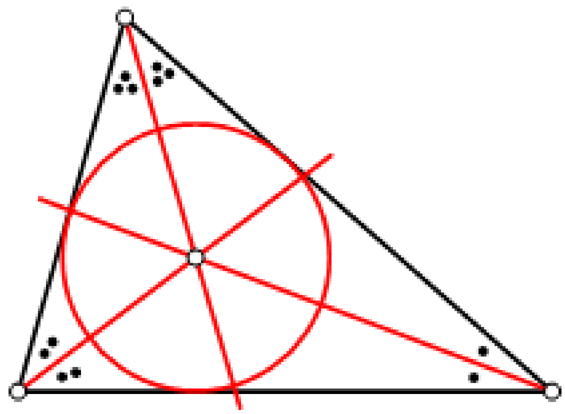
Theorem Let $A_1, ..., A_{n+1}$ be covering of S^n by closed sets A_i. Then there exists i such that $A_i \cap (-A_i) \neq \emptyset$

Proof Define the continuous function $f : S^n \longrightarrow \mathbb{R}^n : x \longrightarrow (d(x, A_1), ..., d(x, A_n))$, where $d(x, A_i)$ is the distance from the point x to the A_i.(For continuity of d, see [Mun75]). According to Borsuk-Ulam Theorem, there exists a pair of antipodal points, $x_0, (-x_0)$, that are identified by f. If $d(x_0, A_i) = 0$ for some i, then $x_0, (-x_0) \in A_i$ for A_i is closed. On the other hand, if $d(x_0, A_i) > 0$ for all i, then $x_0, (-x_0) \in A_{n+1}$ for A_i form a cover. $\square$

To prove the converse, we need the following result:

Lemma There exists a covering of S^{n-1} by closed sets $A_1, ..., A_{n+1}$ such that $A_i \cap (-A_i) = \emptyset$ for all i.

Proof Simply consider the projection of the faces of n-simplex to the sphere. Geometrically,



□

Proof (Lyusternik Shnirelmann Theorem implies Borsuk Ulam Theorem)

Suppose there exists an antipodal map $f : S^n \longrightarrow S^{n-1}$, then $f^{-1}(A_1), ..., f^{-1}(A_{n+1})$ is a closed covering of S^n such that $f^{-1}(A_i) \cap (-f^{-1}(A_i) = \emptyset$ for all i.

This is contradicted to Lyusternik Shnirelmann Theorem. □

For a direct proof of Lyusternik Shnirelmann Theorem, please refer to [Mat03].

5.3 Further Development

We would like to end this paper by presenting to which direction the material here can be further developed.

1. A natural generalization of Borsuk-Ulam Theorem is Knaster Conjecture: Given a continuous map $f : \mathbb{S}^{n-1} \to \mathbb{R}^m$ and $n - m + 1$

points $p_1, \ldots, p_{n-m+1} \in \mathbb{S}^{n-1}$, does there exist a rotation $\varrho \in SO(n)$ such that $f(\varrho(p_1)) = \ldots = f(\varrho(p_{n-m+1}))$?[Jer72].

2. Using Ham Sandwich Theorem in Kakeya Set Conjecture. [Gro03]

   Theorem Let $n > 1$, and let$E \in \mathbb{R}^n$ contain a unit line segment in every direction (such sets are known as Kakeya sets or Besicovitch sets). Then E has Hausdorff dimension and Minkowski dimension equal to n.

3. Borsuk-Ulam Theorem can be putted into a larger context, Lyusternik Shnirelmann Category. [H.F41] The LyusternikSchnirelmann category (or, LusternikSchnirelmann category, LS-category, or simply, category) of a topological space X is the topological invariant defined to be the smallest cardinality of an index set I such that there is an open covering U_i of X with the property that for each i , the inclusion map $U_i \longrightarrow X$ is nullhomotopic. For example, if X is the circle, this takes the value two. And Borsuk-Ilam Theorem is just saying that the category of the projective space is 3.

4. Using Borsuk-Ulam Theorem in various combinatorial problems like Kneser's Conjecture. [Mat03] It states that whenever the n-subsets of a $(2n + k)$-set are divided into $k + 1$ classes, then two disjoint subsets end up in the same class.

These list is by no mean complete, but a few directions that I feel particularly inerested in.

I would like to close this paper with another quotation from G.H.Hardy,

> The "seriousness" of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects.

I do hope that the ideas I present here is serious enough.

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