6. Uniform distribution mod 1

§6.1 Uniform distribution and Weyl's criterion

Let x_n be a sequence of real numbers. We may decompose x_n as the sum of its *integer part* $[x_n] = \sup\{m \in \mathbb{Z} \mid m \leq x_n\}$ (i.e. the largest integer which is less than or equal to x_n) and its *fractional part* $\{x_n\} = x_n - [x_n]$. Clearly, $0 \leq \{x_n\} < 1$. The study of x_n mod 1 is the study of the sequence $\{x_n\}$ in $[0, 1)$.

Definition. We say that the sequence x_n is *uniformly distributed mod 1* if for every a, b with $0 \leq a < b < 1$, we have that

$$\frac{1}{n}\mathrm{card}\{j \mid 0 \leq j \leq n-1,\ \{x_j\} \in [a, b]\} \to b - a, \quad \text{as } n \to \infty.$$

(The condition is saying that the proportion of the sequence $\{x_n\}$ lying in $[a, b]$ converges to $b - a$, the length of the interval.)

Remark. We can replace $[a, b]$ by $[a, b)$, $(a, b]$ or (a, b) with the same result.

Exercise 6.1
Show that if x_n is uniformly distributed mod 1 then $\{x_n\}$ is dense in $[0, 1)$.

The following result gives a necessary and sufficient condition for x_n to be uniformly distributed mod 1.

Theorem 6.1 (Weyl's Criterion)
The following are equivalent:

(i) the sequence x_n is uniformly distributed mod 1;

(ii) for each $\ell \in \mathbb{Z} \setminus \{0\}$, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \to 0$$

as $n \to \infty$.

§6.2 The sequence $\mathbf{x_n = n\alpha}$

The behaviour of the sequence $x_n = n\alpha$ depends on whether α is rational or irrational. If $\alpha \in \mathbb{Q}$, it is easy to see that $\{n\alpha\}$ can take on only finitely many values in $[0, 1)$: if $\alpha = p/q$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$, $\mathrm{hcf}(p, q) = 1$) then $\{n\alpha\}$ takes the q values

$$0, \left\{\frac{p}{q}\right\}, \left\{\frac{2p}{q}\right\}, \ldots, \left\{\frac{(q-1)p}{q}\right\}.$$

In particular, $\{n\alpha\}$ is *not* uniformly distributed mod 1.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then the situation is completely different. We shall apply Weyl's Criterion. For $l \in \mathbb{Z} \setminus \{0\}$, $e^{2\pi i\ell\alpha} \neq 1$, so we have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i\ell j\alpha} = \frac{1}{n}\frac{e^{2\pi i\ell n\alpha} - 1}{e^{2\pi i\ell\alpha} - 1}.$$

Hence

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i\ell j\alpha}\right| \leq \frac{1}{n}\frac{2}{|e^{2\pi i\ell\alpha} - 1|} \to 0, \quad \text{as } n \to \infty.$$

Hence $n\alpha$ is uniformly distributed mod 1.

Remarks.

1. More generally, we could consider the sequence $x_n = n\alpha + \beta$. It is easy to see by modifying the above arguments that x_n is uniformly distributed mod 1 if and only if α is irrational.

2. Fix $\alpha > 1$ and consider the sequence $x_n = \alpha^n x$, for some $x \in (0, 1)$. Then it is possible to show that for almost every x, the sequence x_n is uniformly distributed mod 1. We will prove this later in the course, at least in the cases when $\alpha = 2, 3, 4, \ldots$.

3. Suppose in the above remark we fix $x = 1$ and consider the sequence $x_n = \alpha^n$. Then one can show that x_n is uniformly distributed mod 1 for almost all $\alpha > 1$. However, not a single example of such an α is known! In fact, not a single example of an α for which $\alpha^n \bmod 1$ is dense is known. (Even $(3/2)^n \bmod 1$ is not known to be dense.)

Exercise 6.2

Calculate the frequency with which 2^n has r ($r = 1, \ldots, 9$) as the leading digit of its base 10 representation. (You may assume that $\log_{10} 2$ is irrational.)

(Hint: first show that 2^n has leading digit r if and only if

$$r\,10^\ell \leq 2^n < (r+1)10^\ell$$

for some $\ell \in \mathbb{Z}^+$.)

Exercise 6.3
Calculate the frequency with which 2^n has r ($r = 0, 1, \ldots, 9$) as the second digit of its base 10 representation.

§6.3 Proof of Weyl's criterion

Remark. In the following proof of Weyl's criterion, we assume some familiarity with properties of the Riemann integral. This was discussed in, for example, MT2222 Real Analysis.

Proof. Since $e^{2\pi i x_j} = e^{2\pi i \{x_j\}}$, we may suppose, without loss of generality, that $x_j = \{x_j\}$.

(i) $\Rightarrow$ (ii): Suppose that x_j is uniformly distributed mod 1. If $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, then we may rewrite the definition of uniform distribution in the form

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) \to \int_0^1 \chi_{[a,b]}(x)\,dx, \quad \text{as } n \to \infty.$$

From this we deduce that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int_0^1 f(x)\,dx, \quad \text{as } n \to \infty,$$

whenever f is a *step function*, i.e., a linear combination of characteristic functions of intervals.

Now let g be a continuous function on $[0, 1]$ (with $g(0) = g(1)$). Then, given $\varepsilon > 0$, we can find a step function f with $\|g - f\|_\infty \leq \varepsilon$. We have the estimate

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) - \int_0^1 g(x)\,dx\right| \\
&\leq \left|\frac{1}{n}\sum_{j=0}^{n-1}(g(x_j) - f(x_j))\right| + \left|\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) - \int_0^1 f(x)\,dx\right| \\
&\qquad + \left|\int_0^1 f(x)\,dx - \int_0^1 g(x)\,dx\right| \\
&\leq 2\varepsilon + \left|\frac{1}{n}\sum_{i=0}^{n-1} f(x_j) - \int_0^1 f(x)\,dx\right|.
\end{aligned}$$

Since the last term converges to zero, we thus obtain

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) - \int_0^1 g(x)\,dx\right| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this gives us that

$$\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) \to \int_0^1 g(x)\,dx,$$

as $n \to \infty$, and this holds, in particular, for $g(x) = e^{2\pi i \ell x}$. If $\ell \neq 0$ then

$$\int_0^1 e^{2\pi i \ell x}\,dx = 0,$$

so the first implication is proved.

(ii) $\Rightarrow$ (i): Suppose now that Weyl's Criterion holds. Then

$$\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) \to \int_0^1 g(x)\,dx, \quad \text{as } n \to \infty,$$

whenever $g(x) = \sum_{k=1}^m \alpha_k e^{2\pi i \ell_k x}$ is a trigonometric polynomial.

Let f be any continuous function on $[0,1]$ with $f(0) = f(1)$. Given $\varepsilon > 0$ we can find a trigonometric polynomial g such that $\|f - g\|_\infty \leq \varepsilon$. As in the first part of the proof, we can conclude that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int_0^1 f(x)\,dx, \quad \text{as } n \to \infty.$$

Now consider the interval $[a,b] \subset [0,1)$. Given $\varepsilon > 0$, we can find continuous functions f_1, f_2 (with $f_1(0) = f_1(1)$, $f_2(0) = f_2(1)$) such that

$$f_1 \leq \chi_{[a,b]} \leq f_2$$

and

$$\int_0^1 f_2(x) - f_1(x)\,dx \leq \varepsilon.$$

We then have that

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) &\geq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_1(x_j) = \int_0^1 f_1(x)\,dx \\ &\geq \int_0^1 f_2(x)\,dx - \varepsilon \geq \int_0^1 \chi_{[a,b]}(x)\,dx - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) &\leq \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_2(x_j) = \int_0^1 f_2(x)\,dx \\ &\leq \int_0^1 f_1(x)\,dx + \varepsilon \leq \int_0^1 \chi_{[a,b]}(x)\,dx + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) = \int_0^1 \chi_{[a,b]}(x)\,dx = b - a,$$

so that x_i is uniformly distributed mod 1. $\square$