Interpolating sequences and Carleson measures in the Hardy-Sobolev spaces of the ball in $\mathbb{C}^n$.

E. Amar

En l'honneur de Aline Bonami, Orléans, Juin 2014.

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Definition

The **sequence** S **is Carleson, CS**,*in* $H_s^p(\mathbb{B})$, *if the associated measure*

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- S *is Carleson in* $H^q_s(\mathbb{B})$.

Then S *is* H^r_s *interpolating with the bounded linear extension property, provided that* $p\leq 2$.

AnOunceOfProbability

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$$\forall \lambda \in \ell^\infty(S),\ \exists m \in \mathcal{M}_s^p :: \forall a \in S,\ m(a) = \lambda_a \text{ and } \|m\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_\infty.$$

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Definition

Let S *be an interpolating sequence in* $\mathcal{M}_s^p$ *; we say that* S *has a* **bounded linear extension operator, BLEO**, *if there is a a bounded linear operator* $E\ :\ \ell^\infty(S) \to \mathcal{M}_s^p$ *and a* $C > 0$ *such that*

$$\forall \lambda \in \ell^\infty(S),\ E(\lambda) \in \mathcal{M}_s^p,\ \|E(\lambda)\|_{\mathcal{M}_s^p} \leq\ C\|\lambda\|_\infty\ :\ \forall a \in S,\ E(\lambda)(a) = \lambda_a.$$

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Theorem

If S is interpolating for $\mathcal{M}_s^p$ and $p \geq 2$, then S has a bounded linear extension operator.

Definition

The **sequence** S *of points in* $\mathbb{B}$ *is* **dual bounded** *(or minimal, or weakly interpolating) in the multipliers algebra* $\mathcal{M}_s^p$ *of* $H_s^p(\mathbb{B})$ *if there is a bounded sequence* $\{\rho_a\}_{a\in S}\subset\mathcal{M}_s^p$ *such that*

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Definition

The **sequence** S *of points in* $\mathbb{B}$ *is* δ **separated** *in* H_s^p *if*

$$\forall a, b \in S,\ a \neq b,\ \exists f \in H_s^p :: f(a) = 0,\ f(b) = \|k_a\|_{s,p'},\ \|f\|_{s,p} \leq \delta^{-1}.$$

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Let S be an interpolating sequence for $\mathcal{M}_s^p$ then S is Carleson $H_s^p(\mathbb{B})$.

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Corollary

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HarmonicAnalysis

Thank you !

Typeset by the TeX preprocessor jPreTeX

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with the reproducing kernel :

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Set $N = \#S \in \mathbb{N},\ S := \{a_1, ..., a_N\} \subset \mathbb{B}$ and $\theta := \exp \frac{2i\pi}{N}$.
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and $\forall j = 1, ..., N,\ \|\beta(j, \cdot)\|_{\mathcal{M}_s^p} \leq C(S)$.
Let $\gamma(l, \cdot) := \frac{1}{N} \sum_{j=1}^{N} \theta^{-jl} \beta(j, \cdot) \in \mathcal{M}_s^p \Rightarrow\ \|\gamma(l, \cdot)\|_{\mathcal{M}_s^p} \leq C(S)$.

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Moreover we have

$$\gamma(l, a_k) = \frac{1}{N} \sum_{j=1}^{N} \theta^{-jl} \beta(j, a_k) = \delta_{lk}.$$

We have by Plancherel on this group

$$\forall z \in \mathbb{B},\ \sum_{l=1}^{N} \left|\gamma^k(l,z)\right|^2 = \frac{1}{N}\sum_{j=1}^{N}\left|\underbrace{\beta * \cdot * \beta}_{k\ times}(j,z)\right|^2.$$

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And this is the "miracle lemma" we use to get our results.

Thank you !

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