Orthonormal bases of regular wavelets in metric and quasi-metric spaces

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joint work with Tuomas Hytönen (ACHA, 2012)
Let $X$ be a quasi-metric space with a quasi-distance $d$ on $X \times X$:

- $0 \leq d(x, y) = d(y, x) < \infty$
- $d(x, y) = 0$ iff $x = y$
- $d(x, y) \leq A_0(d(x, z) + d(z, y))$

with $A_0 \geq 1$ (the quasi-metric constant).

Open balls $B(x, r) = \{y \mid d(x, y) < r\}$ form a basis of the topology. If $A_0 > 1$, they may not be open sets, nor Borel sets, but $B(x, r) \subset B(x, 2A_0 r)$ and $B(x, r/A_0) \subset \text{Int } B(x, r)$.

$(X, d)$ called **geometrically doubling** (GD) with doubling constant $N$ if every open ball $B(x, 2r)$ can be covered by at most $N$ balls $B(x_i, r)$. We always assume (GD).
Nested sets

Assume \((X, d)\) is (GD). Let \(0 < \delta < 10^{-3}\).

One can find nested meshes: these are numerable sets \(X^k\), \(k \in \mathbb{Z}, k \geq k_0\), \(\delta^k\)-separated and \(2A_0\delta^k\)-dense, with the nested property \(X^k \subset X^{k+1}\).

\(k\) is the generation number and \(\delta^k\) the scale at that generation.

\[ Y^k := X^{k+1} \setminus X^k. \]

This will be the label set for wavelets. The distance to the set \(Y^k\) will play an important role: it measures the holes in \(X\) at scale \(\delta^k\).
(X, d) is quasi-metric and \( \mu \) a Borel measure. \( \mu \) is **doubling** if

\[
0 < \mu(B(x, 2r)) \leq D\mu(B(x, r)) < \infty.
\]

[If \( B(x, r) \) not Borel set, replace it by its closure. We assume \( B(x, r) \) Borel set to simplify] \((X, d, \mu)\) called **space of homogeneous type** in the sense of Coifman-Weiss.

The best \( D \) is the doubling constant of \( \mu \).

If \((X, d)\) admits a doubling measure, it is geometrically doubling.
Main result

Theorem

Let \((X, d, \mu)\) be any space of homogeneous type with quasi-triangle constant \(A_0\), and \(a := (1 + 2 \log_2 A_0)^{-1}\) or \(a := 1\) if \(d\) is Lipschitz-continuous. There exist \(C < \infty, \eta > 0, \gamma > 0\) and an orthonormal basis \(\psi^k_{\beta}, k \in \mathbb{Z}\) (and \(k \geq k_0\) if \(X\) is bounded), localised at \(y^k_{\beta} \in \mathcal{Y}^k\), of \(L^2(\mu)\) (or the orthogonal space to constants if \(X\) is bounded) with

\[
|\psi^k_{\beta}(x)| \leq C \frac{\exp \left( - \gamma(\delta^{-k} d(y^k_{\beta}, x))^a \right)}{\sqrt{\mu(B(y^k_{\beta}, \delta^k))}} := CG^k_{\beta}(x),
\]

\[
|\psi^k_{\beta}(x) - \psi^k_{\beta}(y)| \leq C \left( \frac{d(x, y)}{\delta^k} \right)^{\eta} (G^k_{\beta}(x) + G^k_{\beta}(y)),
\]

\[
\int_X \psi^k_{\beta}(x) \, d\mu(x) = 0.
\]
Wavelet frames

Representations with no orthogonality: \( \sum |\langle f, \varphi^k \rangle|^2 \sim \| f \|^2 \).

Theory of Han-Sawyer (1995) on Ahlfors-David sets: 
\( \mu(B(x, r)) \sim r \) [One can change \( d \) to a topologically equivalent quasi-distance with this property: this changes the balls].

Theory of Han-Müller-Yang (2008) with reverse doubling: 
\( \mu(B(x, 2r)) > (1 + \varepsilon) \mu(B(x, r)) \).
Such spaces have no holes.

Petrushev-Kerkyacharian (2014): doubling metric spaces equipped with a diffusive self-adjoint operator built from a Dirichlet form [Analog of the \( \varphi \)-transform of Frazier-Jawerth]

This excludes point masses situations. No Littlewood-Paley type analysis available on arbitrary SHT until now.
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OK with no regularity: Haar bases using the dyadic “cubes” of M. Christ.

Regular wavelet bases on $\mathbb{R}^n$, open sets, on certain Lie groups, discrete groups: symmetries and group representations.

Regular wavelet bases on spaces of homogeneous types (even under AD or RD)?
\[ G^k_\beta(x) = \frac{\exp\left(-\gamma(\delta^{-k} d(y^k_\beta, x))^a\right)}{\sqrt{\mu(B(y^k_\beta, \delta^k))}} \]

\[ \sum_{k, y^k_\beta \in \mathcal{Y}^k} |G^k_\beta(x) G^k_\beta(y)| \leq C \mu(B(x, d(x, y)))^{-1}. \quad (1) \]

Does not work for summation with \( y^k_\alpha \in \mathcal{X}^k \) if \( X \) has holes so that growth of volume of balls is too slow:

\[ \sum_{\delta^k \geq r} \mu(B(x, \delta^k))^{-1} \lesssim \mu(B(x, r))^{-1} \]

may be false. But holes imply relative growth of distance of \( x, y \) to \( \mathcal{Y}^k \), forcing convergence of (1).
The linear spline function on $\mathbb{R}$:

$$s(x) = 1_{[0,1)} * 1_{[0,1)}(x) = \int_0^1 1_{[0,1)}(x - u)\,du = \int_0^1 1_{[u,u+1)}(x)\,du$$

Random dyadic intervals of sidelength 1, in the sense of Nazarov, Treil and Volberg: translate the standard intervals $[k, k+1)$ by a random number $u \in [0, 1)$. Splines are expected values of random indicators:

$$s(x) = \mathbb{E}_u(1_{[u,u+1)}(x)) = \mathbb{P}_u(x \in [u, u+1)).$$

Need random dyadic systems on spaces of homogeneous type (Hytönen-Martikainen, Hytönen-Kairema)
Assume (GD). A system of dyadic cubes follows from a partial order \((\ell, \beta) \leq (k, \alpha)\) (descendant of) on some meshes at different scales, constructed with constraints on distances (M. Christ). Any \((k, \alpha)\) has 1 parent \((k - 1, \beta)\) and boundedly many children \((k + 1, \beta)\).

From meshes \(X^k = \{x^k_\alpha\}\), get preliminary dyadic cubes \(\hat{Q}^k_\alpha\) from the order:

\[
\hat{Q}^k_\alpha = \{x^\ell_\beta : (\ell, \beta) \leq (k, \alpha)\}.
\]

One has

\[
B(x^k_\alpha, c_0 \delta^k) \subset \hat{Q}^k_\alpha \subset B(x^k_\alpha, c_1 \delta^k),
\]

No need for nestedness of meshes at this stage.
Random dyadic structures I

- Randomness is adapted to design splines with hierarchical properties. Identify $x^k_\alpha$ and $(k, \alpha)$.
- $x^k_\alpha$ has at most $M$ children.
- $x^k_\alpha$ has at most $L$ neighbours: two mesh points of $k$th generation are neighbours if they have children within distance $c_3 \delta^k$ for some $c_3 > 0$.
- Assign to $x^k_\alpha$ two labels:

\[ \ell_1(x^k_\alpha) \in \{0, 1, \ldots, L\}, \quad \ell_2(x^k_\alpha) \in \{1, \ldots, M\}, \]

in such a way that two neighbours have different label 1 and two children of $x^k_\alpha$ have different label 2.
Random dyadic structures II

- Probability space:

\[ \Omega = \left( \{0, 1, \ldots, L\} \times \{1, \ldots, M\} \right)^Z, \]

\[ \omega = (\omega_k), \omega_k = (\ell_k, m_k) \in \{0, 1, \ldots, L\} \times \{1, \ldots, M\}. \] Natural uniform probability on each level.

- The new dyadic points \( z^k_\alpha = z^k_\alpha(\omega) = z^k_\alpha(\omega_k): \)

\[ z^k_\alpha := \begin{cases} x^{k+1}_\beta & \text{if } \ell_1(x^k_\alpha) = \ell_k, \text{ and } \exists \text{ child of } x^k_\alpha \text{ with } \ell_2(x^{k+1}_\beta) = m_k, \\ x^k_\alpha & \text{if } \ell_1(x^k_\alpha) \neq \ell_k, \text{ or } \nexists \text{ child of } x^k_\alpha \text{ with } \ell_2(x^{k+1}_\beta) = m_k. \end{cases} \]

- New points are \( c_4 \delta^k \)-separated and \( c_5 \delta^k \)-dense.
- Let \( x^{k+1}_\beta \) be a fixed child of \( x^k_\alpha \). Then

\[ \mathbb{P}_{\omega}(z^k_\alpha(\omega) = x^{k+1}_\beta) \geq \frac{1}{(L + 1)M}. \]
From new points $z^k_{\alpha}(\omega)$, get a partial order $\leq_\omega$ in such a way that truth or falsity of “$(k + 1, \beta) \leq_\omega (k, \alpha)$” depends only on $\omega_k$: If $x^k_{\beta} + 1$ is close to some new dyadic point $z^k_{\alpha}(\omega_k)$, then set $(k, \alpha)$ to be the new parent of $(k + 1, \beta)$. If no such close point exists, use parent for $\leq$.

$$\hat{Q}^k_{\alpha}(\omega) = \{ z^\ell_{\beta}(\omega) : (\ell, \beta) \leq_\omega (k, \alpha) \}.$$  

$$B(x^k_{\alpha}, c_6 \delta^k) \subset \tilde{\hat{Q}}^k_{\alpha}(\omega) \subset B(x^k_{\alpha}, c_7 \delta^k).$$

**Proposition**

*Small boundaries in probability for fixed generation $k$:*  

$$\mathbb{P}_{\omega}(x \in \bigcup_{\alpha} \partial_\epsilon Q^k_{\alpha}(\omega)) \leq C \epsilon^\eta$$

$$\partial_\epsilon Q^k_{\alpha}(\omega) := \{ y \in \tilde{\hat{Q}}^k_{\alpha}(\omega) : d(y, c \tilde{\hat{Q}}^k_{\alpha}(\omega)) < \epsilon \delta^k \}$$
Splines

\[ s^k_\alpha(x) := \mathbb{P}_\omega(x \in \bar{Q}^k_\alpha(\omega)). \]

Bounded support

\[ 1_{B(x^k_\alpha, \frac{1}{8} A_0^{-3} \delta^k)}(x) \leq s^k_\alpha(x) \leq 1_{B(x^k_\alpha, 8 A_0^5 \delta^k)}(x); \]

Interpolation and reproducing properties

\[ s^k_\alpha(x^k_\beta) = \delta_{\alpha \beta}, \quad \sum_\alpha s^k_\alpha(x) = 1, \quad s^k_\alpha(x) = \sum_\beta p^{k}_{\alpha \beta} \cdot s^{k+1}_\beta(x) \]

where \( \{p^k_{\alpha \beta}\}_\beta \) is a finitely nonzero set of nonnegative coefficients with \( \sum_\beta p^k_{\alpha \beta} = 1 \); and Hölder-continuity

\[ |s^k_\alpha(x) - s^k_\alpha(y)| \leq C \left( \frac{d(x, y)}{\delta^k} \right)^\eta. \]
$s^k_\alpha(x) = \mathbb{P}_\omega \left( x \in \bigcup_{\beta: (k+1, \beta) \leq \omega(k, \alpha)} \tilde{Q}^k_\beta(\omega) \right)$

$= \sum_{\beta} \mathbb{P}_\omega \left( \{(k+1, \beta) \leq \omega (k, \alpha)\} \cap \{x \in \tilde{Q}^k_\beta(\omega)\} \right)$

$= \sum_{\beta} \mathbb{P}_\omega \left( (k+1, \beta) \leq \omega (k, \alpha) \right) \mathbb{P}_\omega \left( x \in \tilde{Q}^k_\beta(\omega) \right)$

$= \sum_{\beta} \mathbb{P}_\omega \left( (k+1, \beta) \leq \omega (k, \alpha) \right) s^{k+1}_\beta(x) =: \sum_{\beta} p^k_{\alpha \beta} \cdot s^{k+1}_\beta(x),$

where the key third step used the independence of the two events; namely, the event $(k+1, \beta) \leq \omega (k, \alpha)$ depends only on $\omega_k$, while the cube $\tilde{Q}^k_\beta(\omega)$ depends on $\omega_\ell$ for $\ell \geq k + 1$. 
Introduce now doubling measure $\mu$. Set $V_k$ be the closed linear span of $\{s^k_\alpha\}_\alpha$ in $L^2(\mu)$. Then $V_k \subseteq V_{k+1}$, and

$$
\bigcup_{k \in \mathbb{Z}} V_k = L^2(\mu), \quad \bigcap_{k \in \mathbb{Z}} V_k = \begin{cases} 
\{0\}, & \text{if } X \text{ is unbounded}, \\
V_{k_0} = \{\text{constants}\}, & \text{if } X \text{ is bounded},
\end{cases}
$$

where $k_0$ is some integer. Moreover, the functions $s^k_\alpha/\sqrt{\mu^k_\alpha}$ form a Riesz basis of $V_k$: for all sequences of numbers $\lambda_\alpha$,

$$
\left\| \sum_\alpha \lambda_\alpha s^k_\alpha \right\|_{L^2(\mu)} \asymp \left( \sum_\alpha |\lambda_\alpha|^2 \mu^k_\alpha \right)^{1/2},
$$

with $\mu^k_\alpha := \mu(B(x^k_\alpha, \delta^k))$. 

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Regular wavelets on spaces of homogeneous type
$W_k = \text{orthogonal complement of } V_k \text{ in } V_{k+1}$. Follow an algorithm of Y. Meyer to get the wavelets: this is where we use the nestedness to have a label set $\mathcal{Y}^k$. Project the linearly independent $\{s^{k+1}_\beta : x^{k+1}_\beta \in \mathcal{X}^{k+1} \setminus \mathcal{X}^k\}$ orthogonally onto $W_k$. Use orthogonalisation method (inverse square roots of Gram matrices) and that the inverse square roots of band-matrices have exponential decay by adapting results of Demko to $1$-separated sets for a quasi-distance (get $\exp(-\gamma d(x, y)^a)$ for specified $a > 0$ of statement).

Remark: in metric case, Hytönen-Tapiola (ArXiv) propose a different random construction to get arbitrary regularity $\eta < 1$, but not $\eta = 1$.

Questions:
- can one get $\eta = 1$ (could be no, related to Coifman’s talk)?
- can one get bounded support?
Let \((X, d, \mu)\) be any space of homogeneous type with quasi-triangle constant \(A_0\), and \(a := (1 + 2 \log_2 A_0)^{-1}\) or \(a := 1\) if \(d\) is Lipschitz-continuous. There exist \(C < \infty, \eta > 0, \gamma > 0\) and an orthonormal basis \(\psi^k_\beta, k \in \mathbb{Z}\) (and \(k \geq k_0\) if \(X\) is bounded), localised at \(y^k_\beta \in \gamma^k\), of \(L^2(\mu)\) (or the orthogonal space to constants if \(X\) is bounded) with

\[
|\psi^k_\beta(x)| \leq C \exp\left(-\gamma(\delta^{-k}d(y^k_\beta, x))^a\right) \frac{1}{\sqrt{\mu(B(y^k_\beta, \delta^k))}} := CG^k_\beta(x),
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|\psi^k_\beta(x) - \psi^k_\beta(y)| \leq C \left(\frac{d(x, y)}{\delta^k}\right)^\eta (G^k_\beta(x) + G^k_\beta(y)),
\]

\[
\int_X \psi^k_\beta(x) \, d\mu(x) = 0.
\]
Thank you

Best wishes to Aline