# Orthonormal bases of regular wavelets in metric and quasi-metric spaces

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Conference in honor of Aline Bonami

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- joint work with Tuomas Hytönen (ACHA, 2012)

# quasi-metric space

$X$ set, $d$ quasi-distance on $X \times X$:

- $0 \leq d(x,y) = d(y,x) < \infty$
- $d(x,y) = 0$ iff $x = y$
- $d(x,y) \leq A_0(d(x,z) + d(z,y))$

with $A_0 \geq 1$ (the quasi-metric constant).

Open balls $B(x,r) = \{y\,;\ d(x,y) < r\}$ form a basis of the topology. If $A_0 > 1$, they may not be open sets, nor Borel sets, but $\overline{B(x,r)} \subset B(x, 2A_0 r)$ and $B(x, r/A_0) \subset \operatorname{Int} B(x,r)$.

(X,d) called **geometrically doubling** (GD) with doubling constant $N$ if every open ball $B(x,2r)$ can be covered by at most $N$ balls $B(x_i, r)$. We always assume (GD).

# Nested sets

Assume $(X, d)$ is (GD). Let $0 < \delta < 10^{-3}$.

One can find nested meshes: these are numerable sets $\mathcal{X}^k$, $k \in \mathbf{Z}$, $k \geq k_0$, $\delta^k$-separated and $2A_0\delta^k$-dense, with the nested property $\mathcal{X}^k \subset \mathcal{X}^{k+1}$.

$k$ is the generation number and $\delta^k$ the scale at that generation.

$$\mathcal{Y}^k := \mathcal{X}^{k+1} \setminus \mathcal{X}^k.$$

This will be the label set for wavelets. The distance to the set $\mathcal{Y}^k$ will play an important role: it measures the holes in $X$ at scale $\delta^k$.

# doubling measure

$(X, d)$ is quasi-metric and $\mu$ a Borel measure.
$\mu$ is **doubling** if

$$0 < \mu(B(x, 2r)) \leq D\mu(B(x, r)) < \infty.$$

[If $B(x, r)$ not Borel set , replace it by its closure. We assume $B(x, r)$ Borel set to simplify] $(X, d, \mu)$ called **space of homogeneous type in the sense of Coifman-Weiss.**

The best $D$ is the doubling constant of $\mu$.

If $(X, d)$ admits a doubling measure, it is geometrically doubling.

# Main result

## Theorem

*Let $(X, d, \mu)$ be any space of homogeneous type with quasi-triangle constant $A_0$, and $a := (1 + 2\log_2 A_0)^{-1}$ or $a := 1$ if $d$ is Lipschitz-continuous. There exist $C < \infty$, $\eta > 0$, $\gamma > 0$ and an orthonormal basis $\psi_\beta^k$, $k \in \mathbf{Z}$ (and $k \geq k_0$ if $X$ is bounded), localised at $y_\beta^k \in \mathcal{Y}^k$, of $L^2(\mu)$ (or the orthogonal space to constants if $X$ is bounded) with*

$$|\psi_\beta^k(x)| \leq C\, \frac{\exp\big(-\gamma(\delta^{-k} d(y_\beta^k, x))^a\big)}{\sqrt{\mu(B(y_\beta^k, \delta^k))}} := C G_\beta^k(x),$$

$$|\psi_\beta^k(x) - \psi_\beta^k(y)| \leq C\Big(\frac{d(x,y)}{\delta^k}\Big)^\eta (G_\beta^k(x) + G_\beta^k(y)),$$

$$\int_X \psi_\beta^k(x)\, d\mu(x) = 0.$$

# Wavelet frames

Representations with no orthogonality: $\sum |\langle f, \varphi_\alpha^k \rangle|^2 \sim \|f\|^2$.

Theory of Han-Sawyer (1995) on Ahlfors-David sets:
$\mu(B(x,r)) \sim r$ [One can change $d$ to a topologically equivalent quasi-distance with this property: this changes the balls].

Theory of Han-Müller-Yang (2008) with reverse doubling :
$\mu(B(x,2r)) > (1+\varepsilon)\mu(B(x,r))$.
Such spaces have no holes.

Petrushev-Kerkyacharian (2014): doubling metric spaces equipped with a diffusive self-adjoint operator built from a Dirichlet form [Analog of the $\varphi$-transform of Frazier-Jawerth]

This excludes point masses situations. No Littlewood-Paley type analysis available on arbitrary SHT until now.

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# Wavelet bases

OK with no regularity: Haar bases using the dyadic “cubes” of M. Christ.

Regular wavelet bases on $\mathbf{R}^n$, open sets, on certain Lie groups, discrete groups: symmetries and group representations.

Regular wavelet bases on spaces of homogeneous types (even under AD or RD)?

# Label set

$$G^k_\beta(x) = \frac{\exp\left(-\gamma(\delta^{-k} d(y^k_\beta, x))^a\right)}{\sqrt{\mu(B(y^k_\beta, \delta^k))}}$$

$$\sum_{k, y^k_\beta \in \mathcal{Y}^k} |G^k_\beta(x) G^k_\beta(y)| \leq C\mu(B(x, d(x,y)))^{-1}. \qquad (1)$$

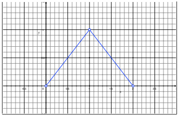
Does not work for summation with $y^k_\alpha \in \mathcal{X}^k$ if $X$ has holes so that growth of volume of balls is too slow:

$$\sum_{\delta^k \geq r} \mu(B(x, \delta^k))^{-1} \lesssim \mu(B(x, r))^{-1}$$

may be false. But holes imply relative growth of distance of $x, y$ to $\mathcal{Y}^k$, forcing convergence of (1).

# The idea

The linear spline function on **R**:



$$s(x) = 1_{[0,1)} * 1_{[0,1)}(x) = \int_0^1 1_{[0,1)}(x-u)du = \int_0^1 1_{[u,u+1)}(x)du$$

Random dyadic intervals of sidelength 1, in the sense of Nazarov, Treil and Volberg : translate the standard intervals $[k, k+1)$ by a random number $u \in [0, 1)$. Splines are expected values of random indicators:

$$s(x) = \mathbb{E}_u(1_{[u,u+1)}(x)) = \mathbb{P}_u(x \in [u, u+1)).$$

Need **random dyadic systems on spaces of homogeneous type** (Hytönen-Martikainen, Hytönen-Kairema)

Assume (GD). A system of dyadic cubes follows from a partial order $(\ell, \beta) \le (k, \alpha)$ (descendant of) on some meshes at different scales, constructed with constraints on distances (M. Christ). Any $(k, \alpha)$ has 1 parent $(k-1, \beta)$ and boundedly many children $(k+1, \beta)$.
From meshes $\mathcal{X}^k = \{x_\alpha^k\}$, get preliminary dyadic cubes $\hat{Q}_\alpha^k$ from the order:

$$\hat{Q}_\alpha^k = \{x_\beta^\ell : (\ell, \beta) \le (k, \alpha)\}.$$

One has

$$B(x_\alpha^k, c_0\delta^k) \subset \bar{\hat{Q}}_\alpha^k \subset B(x_\alpha^k, c_1\delta^k),$$

No need for nestedness of meshes at this stage.

# Random dyadic structures I

• Randomness is adapted to design splines with hierarchical properties. Identify $x_\alpha^k$ and $(k, \alpha)$.
• $x_\alpha^k$ has at most $M$ children.
• $x_\alpha^k$ has at most $L$ neighbours: two mesh points of $k$th generation are neighbours if they have children within distance $c_3\delta^k$ for some $c_3 > 0$.
• Assign to $x_\alpha^k$ two labels:

$$\ell_1(x_\alpha^k) \in \{0, 1, \dots, L\}, \quad \ell_2(x_\alpha^k) \in \{1, \dots, M\},$$

in such a way that two neighbours have different label 1 and two children of $x_\alpha^k$ have different label 2.

# Random dyadic structures II

- Probability space:

$$\Omega = \Big(\{0,1,\dots,L\} \times \{1,\dots,M\}\Big)^{\mathbf{Z}},$$

$\omega = (\omega_k)$, $\omega_k = (\ell_k, m_k) \in \{0,1,\dots,L\} \times \{1,\dots,M\}$. Natural uniform probability on each level.

- The *new dyadic points* $z_\alpha^k = z_\alpha^k(\omega) = z_\alpha^k(\omega_k)$:

$$z_\alpha^k := \begin{cases} x_\beta^{k+1} & \text{if } \ell_1(x_\alpha^k) = \ell_k, \text{ and } \exists \text{ child of } x_\alpha^k \text{ with } \ell_2(x_\beta^{k+1}) = m_k, \\ x_\alpha^k & \text{if } \ell_1(x_\alpha^k) \neq \ell_k, \text{ or } \nexists \text{ child of } x_\alpha^k \text{ with } \ell_2(x_\beta^{k+1}) = m_k. \end{cases}$$

- New points are $c_4\delta^k$-separated and $c_5\delta^k$-dense.
- Let $x_\beta^{k+1}$ be a fixed child of $x_\alpha^k$. Then

$$\mathbb{P}_\omega(z_\alpha^k(\omega) = x_\beta^{k+1}) \geq \frac{1}{(L+1)M}.$$

# Random dyadic structures III

From new points $z_\alpha^k(\omega)$, get a partial order $\leq_\omega$ in such a way that truth or falsity of "$(k+1,\beta) \leq_\omega (k,\alpha)$" depends only on $\omega_k$: If $x_\beta^{k+1}$ is close to some new dyadic point $z_\alpha^k(\omega_k)$, then set $(k,\alpha)$ to be the new parent of $(k+1,\beta)$. If no such close point exists, use parent for $\leq$.

$$\hat{Q}_\alpha^k(\omega) = \{z_\beta^\ell(\omega) : (\ell,\beta) \leq_\omega (k,\alpha)\}.$$

$$B(x_\alpha^k, c_6\delta^k) \subset \bar{\hat{Q}}_\alpha^k(\omega) \subset B(x_\alpha^k, c_7\delta^k).$$

## Proposition

*Small boundaries in probability for fixed generation $k$:*

$$\mathbb{P}_\omega\Big(x \in \bigcup_\alpha \partial_\epsilon Q_\alpha^k(\omega)\Big) \leq C\epsilon^\eta$$

$$\partial_\epsilon Q_\alpha^k(\omega) := \{y \in \bar{\hat{Q}}_\alpha^k(\omega) : d(y, {}^c\tilde{Q}_\alpha^k(\omega)) < \epsilon\delta^k\}$$

# Splines

$$s_\alpha^k(x) := \mathbb{P}_\omega\Big(x \in \bar{\hat{Q}}_\alpha^k(\omega)\Big).$$

Bounded support

$$1_{B(x_\alpha^k, \frac{1}{8}A_0^{-3}\delta^k)}(x) \leq s_\alpha^k(x) \leq 1_{B(x_\alpha^k, 8A_0^5\delta^k)}(x);$$

Interpolation and reproducing properties

$$s_\alpha^k(x_\beta^k) = \delta_{\alpha\beta}, \quad \sum_\alpha s_\alpha^k(x) = 1, \quad s_\alpha^k(x) = \sum_\beta p_{\alpha\beta}^k \cdot s_\beta^{k+1}(x)$$

where $\{p_{\alpha\beta}^k\}_\beta$ is a finitely nonzero set of nonnegative coefficients with $\sum_\beta p_{\alpha\beta}^k = 1$; and Hölder-continuity

$$|s_\alpha^k(x) - s_\alpha^k(y)| \leq C\Big(\frac{d(x,y)}{\delta^k}\Big)^\eta.$$

$$
\begin{aligned}
s_\alpha^k(x) &= \mathbb{P}_\omega\Big(x \in \bigcup_{\beta:(k+1,\beta)\leq_\omega(k,\alpha)} \bar{\hat{Q}}_\beta^{k+1}(\omega)\Big) \\
&= \sum_\beta \mathbb{P}_\omega\Big(\Big\{(k+1,\beta) \leq_\omega (k,\alpha)\Big\} \cap \Big\{x \in \bar{\hat{Q}}_\beta^{k+1}(\omega)\Big\}\Big) \\
&= \sum_\beta \mathbb{P}_\omega\Big((k+1,\beta) \leq_\omega (k,\alpha)\Big)\mathbb{P}_\omega\Big(x \in \bar{\hat{Q}}_\beta^{k+1}(\omega)\Big) \\
&= \sum_\beta \mathbb{P}_\omega\Big((k+1,\beta) \leq_\omega (k,\alpha)\Big) s_\beta^{k+1}(x) =: \sum_\beta p_{\alpha\beta}^k \cdot s_\beta^{k+1}(x),
\end{aligned}
$$

where the key third step used the independence of the two events; namely, the event $(k+1,\beta) \leq_\omega (k,\alpha)$ depends only on $\omega_k$, while the cube $\bar{\hat{Q}}_\beta^{k+1}(\omega)$ depends on $\omega_\ell$ for $\ell \geq k+1$.

Introduce now doubling measure $\mu$.
Set $V_k$ be the closed linear span of $\{s_\alpha^k\}_\alpha$ in $L^2(\mu)$.
Then $V_k \subseteq V_{k+1}$, and

$$\overline{\bigcup_{k\in\mathbf{Z}} V_k} = L^2(\mu), \qquad \bigcap_{k\in\mathbf{Z}} V_k = \begin{cases} \{0\}, & \text{if } X \text{ is unbounded,} \\ V_{k_0} = \{\text{constants}\}, & \text{if } X \text{ is bounded,} \end{cases}$$

where $k_0$ is some integer. Moreover, the functions $s_\alpha^k/\sqrt{\mu_\alpha^k}$ form a Riesz basis of $V_k$: for all sequences of numbers $\lambda_\alpha$,

$$\Big\|\sum_\alpha \lambda_\alpha s_\alpha^k\Big\|_{L^2(\mu)} \approx \Big(\sum_\alpha |\lambda_\alpha|^2 \mu_\alpha^k\Big)^{1/2},$$

with $\mu_\alpha^k := \mu(B(x_\alpha^k, \delta^k))$.

$W_k =$ orthogonal complement of $V_k$ in $V_{k+1}$. Follow an algorithm of Y. Meyer to get the wavelets: this is where we use the nestedness to have a label set $\mathcal{Y}^k$. Project the linearly independent $\{s_\beta^{k+1} : x_\beta^{k+1} \in \mathcal{X}^{k+1} \setminus \mathcal{X}^k\}$ orthogonally onto $W_k$. Use orthogonalisation method (inverse square roots of Gram matrices) and that the inverse square roots of band-matrices have exponential decay by adapting results of Demko to 1-separated sets for a quasi-distance (get $\exp(-\gamma d(x,y)^a)$ for specified $a > 0$ of statement).

Remark: in metric case, Hytönen-Tapiola (ArXiv) propose a different random construction to get arbitrary regularity $\eta < 1$, but not $\eta = 1$.

Questions:
- can one get $\eta = 1$ (could be no, related to Coifman's talk)?
- can one get bounded support?

# Main result

## Theorem

*Let $(X, d, \mu)$ be any space of homogeneous type with quasi-triangle constant $A_0$, and $a := (1 + 2\log_2 A_0)^{-1}$ or $a := 1$ if $d$ is Lipschitz-continuous. There exist $C < \infty$, $\eta > 0$, $\gamma > 0$ and an orthonormal basis $\psi_\beta^k$, $k \in \mathbf{Z}$ (and $k \geq k_0$ if $X$ is bounded), localised at $y_\beta^k \in \mathcal{Y}^k$, of $L^2(\mu)$ (or the orthogonal space to constants if $X$ is bounded) with*

$$|\psi_\beta^k(x)| \leq C \frac{\exp\left(-\gamma(\delta^{-k} d(y_\beta^k, x))^a\right)}{\sqrt{\mu(B(y_\beta^k, \delta^k))}} := CG_\beta^k(x),$$

$$|\psi_\beta^k(x) - \psi_\beta^k(y)| \leq C\left(\frac{d(x,y)}{\delta^k}\right)^\eta (G_\beta^k(x) + G_\beta^k(y)),$$

$$\int_X \psi_\beta^k(x)\, d\mu(x) = 0.$$

Thank you

Best wishes to Aline