Heat Kernel Estimates, Riesz Transform and Sobolev Algebra Property

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joint with Thierry Coulhon and Dorothee Frey.
Aim: Let \((M, d, \mu)\) be a space of homogeneous type and \(L\) a “second order” operator, nonnegative, self-adjoint and generating a heat semigroup \((e^{-Lt})_{t \geq 0}\). We may consider the Sobolev spaces

\[
L^p_{\alpha} := \{ f \in \mathcal{D}(L), L^{\alpha/2}(f) \in L^p \}.
\]

Understand which assumptions on the semigroup, would imply the Algebra property of \(L^\infty \cap L^p_{\alpha}\)? With the inequality

\[
\|fg\|_{L^p_{\alpha}} \lesssim \|f\|_{L^p_{\alpha}} \|g\|_{\infty} + \|f\|_{\infty} \|g\|_{L^p_{\alpha}}.
\]

In the Euclidean situations: Strichartz [1967] (with quadratic functionals), Bony-Coifman-Meyer [1980] (via paraproducts), ...
Later, Coulhon-Russ-Tardivel [2001] (via quadratic functionals on Riemannian manifolds with bounded geometry)
Badr-Bernicot-Russ [2012] (via quadratic functionals and paraproducts (Frey,Sire, ...) on Riemannian manifolds)
Objective:

- Understand some regularity properties on such heat semigroup; It appears that regularity are closely related to Poincaré inequalities, lower Gaussian estimates, Riesz transform ...

- Apply them to prove Sobolev Algebra through two approaches (paraproducts and quadratic functionals).
1. **Regularity Estimates on the Heat Semigroup**
   - Gradient and Hölder estimates
   - $L^p$ De Giorgi property
   - Connection with Poincaré inequalities and lower Gaussian estimates

2. **Riesz Transform without $(P_2)$**

3. **Sobolev Algebra Property**
   - The Algebra property via Paraproducts and quadratic functionals
   - Chain rule and Paralinearization
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2. **Riesz Transform Without $(P_2)$**

3. **Sobolev Algebra Property**
Context: \((M, d, \mu)\) a unbounded doubling space (of homogeneous dimension \(\nu\)) and \(L\) a “second order” operator, nonnegative, self-adjoint and generating a heat semigroup \((e^{-tL})_{t>0}\). We assume that the associated quadratic form \(((f, g) \rightarrow \int fL(g)d\mu)\) defines and is a strongly local and regular Dirichlet form with a “carré du champ” \(\nabla\).

Examples:

- Doubling Riemannian manifold with \(L\) its nonnegative Beltrami Laplacian and then \(\nabla\) is the Riemannian gradient;
- Discrete situation (Graphs, Trees, ...).

Typical upper estimates of the heat kernel:

\[
p_t(x, y) \lesssim \frac{1}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}}, \quad \forall \ t > 0, \ \text{a.e.} \ x, y \in M. \quad (DUE)
\]

which self-improves into a Gaussian upper estimate

\[
p_t(x, y) \lesssim \frac{1}{V(x, \sqrt{t})} \exp \left( -\frac{d(x, y)^2}{Ct} \right), \quad \forall \ t > 0, \ \text{a.e.} \ x, y \in M. \quad (UE)
\]
Heat Kernel estimates and Sobolev Algebra property

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3. Sobolev Algebra property
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Gradient estimates (Auscher-Coulhon-Duong-Hofmann [2004]) : for $p \in (1, \infty]$

$$\sup_{t>0} \| \sqrt{t} |\nabla e^{-tL}| \|_{p \to p} < +\infty, \quad (G_p)$$

Under $(DUE)$, for $2 < p \leq +\infty$, $(G_p)$ is almost equivalent to

$$\| \nabla_x p_t(\cdot, y) \|_p \leq \frac{C_p}{\sqrt{t} \left[ V(y, \sqrt{t}) \right]^{1 - \frac{1}{p}}}, \quad \forall \, y, \, t > 0, \quad (1)$$

Under $(DUE)$ :

- $(G_p)$ can be seen as a $L^p$-norm of the gradient of the heat kernel ;
- $(G_p)$ holds for every $p \in (1, 2]$ (Coulhon-Duong [1999]) ;
- for $p > 2$, $(G_p)$ is closely related to $(R_p)$, the boundedness of the Riesz transform in $L^p$ ([ACDH]) ;
We aim now to define similar quantities, replacing a gradient (regularity of order 1) by a Hölder quantity (regularity of order $\eta \in (0, 1)$).

For $p \in [1, \infty]$ and $\eta \in (0, 1]$, we shall say that property $(H^\eta_{p,p})$ holds if for every $0 < r \leq \sqrt{t}$, every pair of concentric balls $B_r, B_{\sqrt{t}}$ with respective radii $r$ and $\sqrt{t}$, and every function $f \in L^p(M, \mu)$,

$$
\left( \frac{\|e^{-tL}f - \int_{B_r} e^{-tL}f d\mu\|_p}{d\mu} \right)^{1/p} \lesssim \left( \frac{r}{\sqrt{t}} \right)^{\eta} \|B_{\sqrt{t}}\|^{-1/p} \|f\|_p,
$$

$(H^\eta_{p,p})$ may be thought as a $L^p \rightarrow L^\infty$ version of a $\eta$-Hölder regularity (related to $\|\sqrt{t}\nabla e^{-tL}\|_{p \rightarrow \infty}$).
We say that $(HH^m_{p,p})$ holds if the following is satisfied: for every $0 < r \leq \sqrt{t}$, every ball $B_{\sqrt{t}}$ with radius $\sqrt{t}$ and every function $f \in L^p(M, \mu)$,

\[
\left( \int_{B_{\sqrt{t}}} \left( \int_{B(x,2r)} \left| e^{-tL}f(y) - \int_{B(x,2r)} e^{-tL}fd\mu \right|^p \right)^{1/p}
\right) \lesssim \left( \frac{r}{\sqrt{t}} \right)^{\eta} |B_{\sqrt{t}}|^{-1/p} \|f\|_p,
\]

$(HH^m_{p,p})$ may be thought as a $L^p \rightarrow L^p$ version of a $\eta$-Hölder regularity related to $\|\sqrt{t} \nabla e^{-tL}\|_{p \rightarrow p}$.
**Proposition**

Under (DUE),

- The lower Gaussian estimates for the heat kernel (LE) (which are equivalent to $(P_2)$) are equivalent to the existence of some $p \in (1, \infty)$ and some $\eta > 0$ such that $(H^n_{p,p})$ holds;
- Moreover, for every $\lambda \in (0, 1]$ the property $\bigcap_{\eta < \lambda} (H^n_{p,p})$ is independent on $p \in [1, \infty]$ and will be called

$$
(H^\lambda) = \bigcup_{p \in [1, \infty]} \bigcap_{\eta < \lambda} (H^n_{p,p}) = \bigcap_{\eta < \lambda} \bigcup_{p \in [1, \infty]} (H^n_{p,p}).
$$

Proof: Using (DUE), $(H^n_{p,p})$ self-improves into $(H^n_{p,\infty})$ (using the $L^p - L^\infty$ off-diagonal estimates of the semigroup);
Up to a small loss on $\eta$, $(H^n_{p,\infty})$ implies $(H^n_{1,\infty})$ and so every $(H^n_{q,q})$. Moreover, $(H^n_{1,\infty})$ yields a Hölder regularity of the heat kernel which implies (LE).
Reciprocally, under (LE) (or equivalently $(P_2)$) we know that the heat kernel satisfies a Hölder regularity.
To obtain \((H^\lambda)\) for some \(\lambda\), we have to control oscillations, which is easy under some Poincaré inequality.

We reobtain the following result (Coulhon [2003] and Boutayeb [2009]) :

**Proposition**

For \(p \in (\nu, \infty)\),

\[
(G_p) + (P_p) \implies (DUE) + (H^{1-\frac{\nu}{p}}) \implies (DUE) + (LE).
\]

Proof: \((G_p) + (P_p) \implies (DUE)\), indeed using a self-improvement of Poincaré inequality we then obtain \(L^{p-\epsilon} - L^p\) boundedness of the semigroup. By iteration/extrapolation, we obtain \(L^1 - L^\infty\) estimates. Difficulties for the non-polynomial situations (weighted estimates ... Boutayeb-Coulhon-Sikora [2014]).

**Question** : for \(p \in (2, \nu)\) ?
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THE $L^2$-situation

Introduced by De Giorgi [1957] (then Moser, Giaquinta, Auscher, Hofmann, Kim ...), the $L^2$ De Giorgi property: there exists $\epsilon \in (0, 1)$ such that for every pair of balls $B_r, B_R$ with radii $r, R$ with $B_r \subset B_R$, and all functions $u \in W^{1,2}$ harmonic in $B_R$, i.e. $Lu = 0$ in $B_R$, one has

$$\left( \int_{B_r} |\nabla u|^2 d\mu \right)^{1/2} \lesssim \left( \frac{R}{r} \right)^\epsilon \left( \int_{2B_R} |\nabla u|^2 d\mu \right)^{1/2}. \quad (DG_2)$$

The Dirichlet energy integral of harmonic functions (solutions of the elliptic problem) have a growth at most linear on the radius.

A non-homogeneous equivalent (under some Poincaré inequality) version (that is, not restricted to harmonic functions):

$$\left( \int_{B_r} |\nabla f|^2 d\mu \right)^{1/2} \lesssim \left( \frac{R}{r} \right)^\epsilon \left[ \left( \int_{2B_R} |\nabla f|^2 d\mu \right)^{1/2} + R \|Lf\|_{L^\infty(2B_R)} \right].$$
\textbf{Theorem}

Under Poincaré ($P_2$) then ($DG_2$) holds. Indeed, a Faber-Krahn inequality and elliptic regularity property implies ($DG_2$).

The proof relies on a suitable iteration argument with Caccioppoli inequality (for harmonic and subharmonic functions) to compare oscillations and the gradient. To get the elliptic regularity property: elliptic Moser iteration argument (Moser [1961]).

\textbf{Theorem}

Under Poincaré ($P_2$) then ($H_{2,2}^{1-\epsilon}$) holds (with $\epsilon > 0$ given by ($DG_2$)) and so ($LE$), and a parabolic Harnack inequality.

This allows us to get around a parabolic iteration argument: we get the parabolic property directly from a elliptic regularity property (Hebisch Saloff-Coste [2001]).
**Definition (L^p De Giorgi property)**

For \( p \in [1, \infty) \) and \( \epsilon \in (0, 1) \), we say that \((DG_{p,\epsilon})\) holds if: for every pair of balls \( B_r, B_R \) with \( B_r \subset B_R \) and respective radii \( r \) and \( R \), and for every function \( f \in \mathcal{D} \), one has

\[
\left( \int_{B_r} |\nabla f|^p \, d\mu \right)^{1/p} \lesssim \left( \frac{R}{r} \right)^\epsilon \left[ \int_{2B_R} |\nabla f|^p \, d\mu \right]^{1/p} + R \| Lf \|_{L^\infty(2B_R)} . \]  

\((DG_{p,\epsilon})\)

We write \((DG_p)\) if \((DG_{p,\epsilon})\) is satisfied for some \( \epsilon \in (0, 1) \).

A non-local (and weaker) version will be sufficient:

\[
\left( \int_{B_r} |\nabla f|^p \, d\mu \right)^{1/p} \lesssim \left( \frac{R}{r} \right)^\epsilon \left[ |B_R|^{-1/p} \left( \| \nabla f \|_p + R \| Lf \|_p \right) + R \| Lf \|_{L^\infty(B_R)} \right] . \]  

\((DG_{p,\epsilon})\)

Important: \((DG_{p,\epsilon})\) and \((DG_{p,\epsilon})\) are always satisfied for \( p > \nu \) with \( \epsilon = \nu / p \in (0, 1) \)!
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**THEOREM**

For any \( p \geq 2 \), \((G_p), (P_p)\) and \((\overline{DG}_p)\) imply \((LE)\) (which is equivalent to \((P_2)\)).

It’s almost optimal since if \((G_p)\) holds for some \( p \in (2, \infty) \), then for every \( q \in [2, p] \)

\[
(P_q) + (\overline{DG}_q) \iff (P_2).
\]

Proof: From \((G_p), (P_p)\) and \((\overline{DG}_p)\), we get some oscillation estimate \((H^\lambda)\) and so \((P_2)\). Starting from \((P_2)\), we know that we have \((DG_2)\). Try to interpolate between \((DG_2)\) and \((DG_{\nu^+})\) to get \((DG_p)\) for \( p \in (2, \nu] \), not directly but going through oscillation estimates, characterizing De Giorgi inequalities.

Condition \((G_p)\) is stronger and stronger for \( p \geq 2 \) increasing

Condition \((P_p)\) is weaker and weaker for \( p \geq 2 \) increasing.

**COROLLARY**

For \( 2 \leq q < p < \infty \), we have

\[
(G_p) + (P_p) + (\overline{DG}_p) \implies (G_q) + (P_q) + (\overline{DG}_q).
\]
The proof relies on a non-local $L^p$ Cacciopoli inequality:

**Proposition ($L^p$ Cacciopoli Inequality)**

Assume $(G_p)$ and (DUE) for some $p \in [2, \infty]$. Then for every $q \in (1, p]$  

$$
\left( \int_{B} |r \nabla f|^q \, d\mu \right)^{1/q} \lesssim \left[ \left( \int_{2B} |f|^q \, d\mu \right)^{1/q} + \left( \int_{2B} |r^2 Lf|^q \, d\mu \right)^{1/q} \right]
$$

for all $f \in \mathcal{D}$ and all balls $B$ of radius $r$.

Proof: Only using $(G_p)$, we get a non-local inequality with fast decaying off-diagonal quantities. Improvement using simultaneously the finite speed propagation property to keep the information of supports.
Under \( (P_2) \), there exists some \( \kappa > 0 \) such that \((G_p)\) and \((R_p)\) hold for \( p \in (2, 2 + \kappa) \).

[Auscher-Coulhon, 2005] (using Riemannian structure and differential forms)

Extension to this more general setting (relying on a Gehring’s argument on reverse Hölder inequalities).

**Proposition**

Under \( (P_2) \), there exists some \( \kappa > 0 \) such that \((\overline{D}G_p)\) holds for \( p \in (2, 2 + \kappa) \).

Open question : Can we directly get \((\overline{D}G_p)\) for some \( p > 2 \) through \((G_p)\) and \((P_p)\)?
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**Proposition**

For some $p_0 \in (2, \infty)$, under $(P_{p_0})$ and $(G_{p_0})$, there exists some $\kappa > 0$ such that $(G_p)$ holds for $p \in (p_0, p_0 + \kappa)$.

Proof: we already know that we have (DUE) then we adapt to a $L^{p_0}$-version of the previous proposition.

Riesz transform: $|\nabla L^{-1/2}|$

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**Proposition**

For some $p_0 \in (2, \infty)$, under $(P_{p_0})$ and $(G_{p_0})$, then $(R_p)$ holds for $p \in (1, p_0)$.

Proof: Improvement of [ACDH] where $(P_2)$ was required. It relies on the fact that $(P_{p_0})$ and $(G_{p_0})$ implies a $L^2$-Poincaré inequality for harmonic functions and with an extra term (involving the laplacian) for non harmonic functions. Application of a self-improving property of reverse Hölder inequalities.
**Proposition**

Let \( \omega \in L^1_{loc} \) be a non-negative function such that for some \( 1 < p < q \leq \infty \) and every ball \( B \subset M \),

\[
\left( \int_B \omega^q \, d\mu \right)^{1/q} \lesssim \left( \int_{2B} \omega^p \, d\mu \right)^{1/p}.
\]

Then for every \( \eta \in (0, 1) \) there is an implicit constant such that for every ball \( B \)

\[
\left( \int_B \omega^q \, d\mu \right)^{1/q} \lesssim \left( \int_{2B} \omega^{\eta p} \, d\mu \right)^{1/(\eta p)}.
\]

So the RHS exponent of a reverse Hölder inequality always self-improves.

Combining all these results:

**Theorem**

If for some \( p_0 \in [2, \infty] \), the combination \((P_{p_0})\) with \((G_{p_0})\) holds then there exists \( p(L) \in (p_0, \infty) \) such that

\[
(1, p(L)) = \{ p \in (1, \infty), \ (G_p) \text{ holds} \} = \{ p \in (1, \infty), \ (R_p) \text{ holds} \}.
\]
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**Definition**

For $\alpha \in (0, 1]$ and $p \in (1, \infty)$ we say that Property $A(\alpha, p)$ holds if:

- the space $\dot{L}^p_\alpha(M) \cap L^\infty(M)$ is an algebra for the pointwise product;
- and the Leibniz rule inequality is valid:

$$\|fg\|_{\dot{L}^p_\alpha(M)} \lesssim \|f\|_{\dot{L}^p_\alpha(M)} \|g\|_\infty + \|f\|_\infty \|g\|_{\dot{L}^p_\alpha(M)}, \quad \forall f, g \in \dot{L}^p_\alpha(M) \cap L^\infty(M).$$

Two methods: paraproducts or characterization of Sobolev spaces via quadratic functionals.
Idea : Use the spectral decomposition

\[ f = c \int_{0}^{\infty} \psi(tL)f \frac{dt}{t} \]

where \( \psi(tL) = (tL)^N e^{-tL} \). Let \( \phi \) such that \( \phi(0) = 1 \) and \( \phi'(x) = \psi(x)/x \). Then \( \psi(tL) \) can be thought as a regular version of \( 1_{[t^{-1},2t^{-1}]}(L) \) and is a smooth restriction operator to "frequencies at the scale \( t^{-1/2} \)."

By analogy with the Euclidean paraproducts,

\[ \Pi_g(f) = \int_{0}^{\infty} \psi(tL)f \cdot \phi(tL)g \frac{dt}{t} \cdot \]

Product decomposition : \( fg = \Pi_g(f) + \Pi_f(g) \).
In [B.-Sire] and [Badr-B.-Russ], other paraproducts are defined:

\[ \tilde{\Pi}_g(f) = \int_0^\infty \phi(tL) [\psi(tL)f \cdot \phi(tL)g] \frac{dt}{t}. \]

Boundedness in Lebesgue spaces [B., Frey], using Poincaré inequality \((P_2)\) and a \(T(1)\)-theorem adapted to semigroup.

**Question**: When for \( g \in L^\infty \), is \( \Pi_g \) bounded on \( \dot{L}_\alpha^p \)? In this case, we have \( A(\alpha, p) \)!

Is such a boundedness equivalent to \( A(\alpha, p) \)?
**The Quadratic Functionals**

Idea: Use a semigroup version of Strichartz functional. Let $\rho > 0$ be an exponent. For $\alpha > 0$ and $x \in M$, we define

$$S_\alpha^\rho f(x) = \left( \int_0^\infty \left[ \frac{1}{r^\alpha} \text{Osc}_{\rho,B(x,r)}(f) \right]^2 \frac{dr}{r} \right)^{1/2}$$

where for $B$ a ball $\text{Osc}_{\rho,B}$ denotes the $L^\rho$-oscillation:

$$\text{Osc}_{\rho,B}(f) = \left( \int_B \left| f - \int_B f d\mu \right|^\rho d\mu \right)^{1/\rho}.$$

Question: When do we have $\| \cdot \|_{L_\alpha^\rho} \simeq \| S_\alpha^\rho(\cdot) \|_\rho$? In this case, we have $A(\alpha, p)$ since

$$\text{Osc}_{\rho,B}(fg) \leq \|g\|_\infty \text{Osc}_{\rho,B}(f) + \|f\|_\infty \text{Osc}_{\rho,B}(g).$$

Is such a property equivalent to $A(\alpha, p)$?
THE CASE FOR $p \in (1, 2]$ 

**Theorem**

Under (DUE) then for every $\alpha \in (0, 1)$ and $p \in (1, 2]$ the paraproduct is bounded on $\dot{L}^p_\alpha$. Consequently, $A(\alpha, p)$ holds.

Proof: First obtain the $p = 2$ case, using orthogonality and duality in tent spaces (or Carleson measure). Then use an extrapolation argument to get the other boundedness for $p \in (1, 2)$. It suffices to get $L^2 - L^2$ off-diagonal estimates.
THE CASE FOR $p > 2$

THEOREM

Then

(A) Under (DUE), $(G_{q_0})$ with $(\overline{DG}_{q_1,\kappa})$ for some $2 \leq q_1 < q_0 \leq \infty$ and $\kappa \in (0, 1)$, $A(\alpha, p)$ holds for every $p \leq [2, \infty)$ and $\alpha < 1 - \kappa \left(1 - \frac{2}{p}\right)$;

(B) Under (DUE) and $(H^\eta)$ for some $\eta \in (0, 1]$, $A(\alpha, p)$ holds for every $\alpha \in (0, \eta)$ and $p \in (1, \infty)$;

(C) under the combination $(G_{q_0}), (P_{q_0})$ with $(\overline{DG}_{q_0,\kappa})$ for some $\kappa \in (0, 1)$ and $q_0 > 2$, $A(\alpha, p)$ holds for every $p \leq [2, q_0]$ and $\alpha < 1$;

(D) under the combination $(G_{q_0}), (P_{q_0})$ with $(\overline{DG}_{q_0,\kappa})$ for some $\kappa \in (0, 1)$ and $q_0 > 2$, $A(\alpha, p)$ holds for every $p > q_0$ with $\alpha < 1 - \kappa \left(1 - \frac{q_0}{p}\right)$.

(E) under $(R_{q_0})$ for some $q_0 > 2$, $A(\alpha, p)$ holds for every $p \in [2, q_0)$ with $\alpha < 1$.

Moreover, in the four first situations the paraproduct is bounded in the corresponding Sobolev spaces. And we have a characterization by quadratic functional of the Sobolev norm in cases (B) and (C).
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If we have a characterization via quadratic functionals, then optimal chain rule: the composition through a Lipschitz function preserves the Sobolev space.

For a Ahlfors regular space (i.e. \( \mu(B(x, r)) \approx r^\nu \)) we can describe a Paralinearization result (which requires more regularity):

**Theorem (Chain rule)**

Let \( F \in C^3(\mathbb{R}) \) a nonlinearity with \( F(0) = 0 \). For \( \alpha \in (0, 1) \), \( p \in (1, \infty] \) with \( \frac{\nu}{p} < \alpha < 1 \), consider a fixed function \( f \in L^\infty \cap \dot{L}^p_\alpha \). Then

\[
F(f) \in L^\infty \cap \dot{L}^p_\alpha.
\]

If \( F \in C^2(\mathbb{R}) \), then we have the paralinearization:

\[
F(f) - \Pi_{F'(f)}(f) \in L^\infty \cap \dot{L}^p_\alpha \cap \dot{L}^p_{\alpha+\rho},
\]

for some \( 0 < \rho \approx \alpha - \frac{\nu}{p} \).