HEAT KERNEL ESTIMATES, RIESZ TRANSFORM AND SOBOLEV ALGEBRA PROPERTY

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June 2014, 10–13

Analyse Harmonique, Probabilités et Applications
Conférence en l'honneur d'Aline Bonami

joint with Thierry Coulhon and Dorothee Frey.

Aim : Let (M, d, μ) be a space of homogeneous type and L a "second order" operator, nonnegative, self-adjoint and generating a heat semigroup $(e^{-tL})_{t>0}$. We may consider the Sobolev spaces

$$L^p_\alpha := \overline{\left\{f \in \mathcal{D}(L), L^{\alpha/2}(f) \in L^p\right\}}.$$

Understand which assumptions on the semigroup, would imply the Algebra property of $L^\infty \cap L^p_\alpha$? With the inequality

$$\|fg\|_{L^p_\alpha} \lesssim \|f\|_{L^p_\alpha} \|g\|_\infty + \|f\|_\infty \|g\|_{L^p_\alpha}.$$

In the Euclidean situations : Strichartz [1967] (with quadratic functionals), Bony-Coifman-Meyer [1980] (via paraproducts), ...
Later, Coulhon-Russ-Tardivel [2001] (via quadratic functionals on Riemannian manifolds with bounded geometry)
Badr-Bernicot-Russ [2012] (via quadratic functionals and paraproducts (Frey,Sire, ...) on Riemannian manifolds)

Objective :

- Understand some regularity properties on such heat semigroup ; It appears that regularity are closely related to Poincaré inequalities, lower Gaussian estimates, Riesz transform ...
- Apply them to prove Sobolev Algebra through two approaches (paraproducts and quadratic functionals).

1. REGULARITY ESTIMATES ON THE HEAT SEMIGROUP
   - Gradient and Hölder estimates
   - L^p De Giorgi property
   - Connection with Poincaré inequalities and lower Gaussian estimates
2. RIESZ TRANSFORM WITHOUT (P_2)
3. SOBOLEV ALGEBRA PROPERTY
   - The Algebra property via Paraproducts and quadratic functionals
   - Chain rule and Paralinearization

Heat Kernel estimates and Sobolev Algebra property

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Context : (M, d, μ) a unbounded doubling space (of homogeneous dimension ν) and L a "second order" operator, nonnegative, self-adjoint and generating a heat semigroup $(e^{-tL})_{t>0}$. We assume that the associated quadratic form $((f,g) \rightarrow \int fL(g)d\mu)$ defines and is a strongly local and regular Dirichlet form with a "carré du champ" ∇.

Examples :

- Doubling Riemannian manifold with L its nonnegative Beltrami Laplacian and then ∇ is the Riemannian gradient ;
- Discrete situation (Graphs, Trees, ...).

Typical upper estimates of the heat kernel :

$$p_t(x,y) \lesssim \frac{1}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}, \quad \forall\, t > 0, \text{ a.e. } x, y \in M. \qquad (DUE)$$

which self-improves into a Gaussian upper estimate

$$p_t(x,y) \lesssim \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{Ct}\right), \quad \forall\, t > 0, \text{ a.e. } x, y \in M. \qquad (UE)$$

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Gradient estimates (Auscher-Coulhon-Duong-Hofmann [2004]) : for $p \in (1, \infty]$

$$\sup_{t>0} \| \sqrt{t} |\nabla e^{-tL}| \|_{p \to p} < +\infty, \qquad (G_p)$$

Under (DUE), for $2 < p \leq +\infty$, (G_p) is almost equivalent to

$$\|\nabla_x p_t(., y)\|_p \leq \frac{C_p}{\sqrt{t}\left[V(y, \sqrt{t})\right]^{1-\frac{1}{p}}}, \quad \forall\, y,\ t > 0, \qquad (1)$$

Under (DUE) :

- (G_p) can be seen as a L^p-norm of the gradient of the heat kernel ;
- (G_p) holds for every $p \in (1, 2]$ (Coulhon-Duong [1999]) ;
- for $p > 2$, (G_p) is closely related to (R_p), the boundedness of the Riesz transform in L^p ([ACDH]) ;

We aim now to define similar quantities, replacing a gradient (regularity of order 1) by a Hölder quantity (regularity of order $\eta \in (0,1)$).

For $p \in [1,\infty]$ and $\eta \in (0,1]$, we shall say that property $(H^{\eta}_{p,p})$ holds if for every $0 < r \leq \sqrt{t}$, every pair of concentric balls B_r, $B_{\sqrt{t}}$ with respective radii r and $\sqrt{t}$, and every function $f \in L^p(M,\mu)$,

$$\left(\fint_{B_r} \left| e^{-tL} f - \fint_{B_r} e^{-tL} f d\mu \right|^p d\mu \right)^{1/p} \lesssim \left(\frac{r}{\sqrt{t}} \right)^{\eta} \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p, \qquad (H^{\eta}_{p,p})$$

- $(H^{\eta}_{p,p})$ may be thought as a $L^p \to L^{\infty}$ version of a η-Hölder regularity (related to $\|\sqrt{t}\nabla e^{-tL}\|_{p\to\infty}$).

We say that $(HH^{\eta}_{p,p})$ holds if the following is satisfied : for every $0 < r \leq \sqrt{t}$, every ball $B_{\sqrt{t}}$ with radius $\sqrt{t}$ and every function $f \in L^p(M,\mu)$,

$$\left(\fint_{B_{\sqrt{t}}} \fint_{B(x,2r)} \left| e^{-tL} f(y) - \fint_{B(x,2r)} e^{-tL} f d\mu \right|^p d\mu(y) d\mu(x) \right)^{1/p} \lesssim \left(\frac{r}{\sqrt{t}} \right)^{\eta} \left| B_{\sqrt{t}} \right|^{-1/p} \|f\|_p, \qquad (HH^{\eta}_{p,p})$$

- $(HH^{\eta}_{p,p})$ may be thought as a $L^p \to L^p$ version of a η-Hölder regularity related to $\|\sqrt{t} \nabla e^{-tL}\|_{p \to p}$.

Proposition

Under (DUE),

- *The lower Gaussian estimates for the heat kernel (LE) (which are equivalent to (P_2)) are equivalent to the existence of some $p \in (1, \infty)$ and some $\eta > 0$ such that $(H^\eta_{p,p})$ holds ;*
- *Moreover, for every $\lambda \in (0, 1]$ the property $\bigcap_{\eta<\lambda}(H^\eta_{p,p})$ is independent on $p \in [1, \infty]$ and will be called*

$$(H^\lambda) = \bigcup_{p\in[1,\infty]} \bigcap_{\eta<\lambda} (H^\eta_{p,p}) = \bigcap_{\eta<\lambda} \bigcup_{p\in[1,\infty]} (H^\eta_{p,p}).$$

Proof : Using (DUE), $(H^\eta_{p,p})$ self-improves into $(H^\eta_{p,\infty})$ (using the $L^p - L^\infty$ off-diagonal estimates of the semigroup) ;
Up to a small loss on η, $(H^\eta_{p,\infty})$ implies $(H^\eta_{1,\infty})$ and so every $(H^\eta_{q,q})$. Moreover, $(H^\eta_{1,\infty})$ yields a Hölder regularity of the heat kernel which implies (LE).
Reciprocally, under (LE) (or equivalently (P_2)) we know that the heat kernel satisfies a Hölder regularity.

To obtain (H^λ) for some λ, we have to control oscillations, which is easy under some Poincaré inequality.
We reobtain the following result (Coulhon [2003] and Boutayeb [2009]) :

PROPOSITION

For $p \in (\nu, \infty)$,

$$(G_p) + (P_p) \Longrightarrow (DUE) + (H^{1-\frac{\nu}{p}}) \Longrightarrow (DUE) + (LE).$$

Proof : $(G_p) + (P_p) \Longrightarrow (DUE)$, indeed using a self-improvement of Poincaré inequality we then obtain $L^{p-\epsilon} - L^p$ boundedness of the semigroup. By iteration/extrapolation, we obtain $L^1 - L^\infty$ estimates. Difficulties for the non-polynomial situations (weighted estimates ... Boutayeb-Coulhon-Sikora [2014]).

Question : for $p \in (2, \nu)$?

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The L^2-situation

Introduced by De Giorgi [1957] (then Moser, Giaquinta, Auscher, Hofmann, Kim ...), the L^2 De Giorgi property : there exists $\epsilon \in (0,1)$ such that for every pair of balls B_r, B_R with radii r, R with $B_r \subset B_R$, and all functions $u \in W^{1,2}$ harmonic in B_R, i.e. $Lu = 0$ in B_R, one has

$$\left(\fint_{B_r} |\nabla u|^2 d\mu \right)^{\frac{1}{2}} \lesssim \left(\frac{R}{r} \right)^{\epsilon} \left(\fint_{2B_R} |\nabla u|^2 d\mu \right)^{\frac{1}{2}}. \qquad (DG_2)$$

The Dirichlet energy integral of harmonic functions (solutions of the elliptic problem) have a growth at most linear on the radius.

A non-homogeneous equivalent (under some Poincaré inequality) version (that is, not restricted to harmonic functions) :

$$\left(\fint_{B_r} |\nabla f|^2 d\mu \right)^{1/2} \lesssim \left(\frac{R}{r} \right)^{\epsilon} \left[\left(\fint_{2B_R} |\nabla f|^2 d\mu \right)^{1/2} + R \|Lf\|_{L^\infty(2B_R)} \right].$$

THEOREM

Under Poincaré (P_2) then (DG_2) holds. Indeed, a Faber-Krahn inequality and elliptic regularity property implies (DG_2).

The proof relies on a suitable iteration argument with Caccioppoli inequality (for harmonic and subharmonic functions) to compare oscillations and the gradient. To get the elliptic regularity property : elliptic Moser iteration argument (Moser [1961]).

THEOREM

Under Poincaré (P_2) then $(H^{1-\epsilon}_{2,2})$ holds (with $\epsilon > 0$ given by (DG_2)) and so (LE), and a parabolic Harnack inequality.

This allows us to get around a parabolic iteration argument : we get the parabolic property directly from a elliptic regularity property (Hebisch Saloff-Coste [2001]).

Extension to L^p for $p > 2$

Definition (L^p De Giorgi property)

For $p \in [1,\infty)$ and $\epsilon \in (0,1)$, we say that $(DG_{p,\epsilon})$ holds if : for every pair of balls B_r, B_R with $B_r \subset B_R$ and respective radii r and R, and for every function $f \in \mathcal{D}$, one has

$$\left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{R}{r}\right)^{\epsilon} \left[\left(\fint_{2B_R} |\nabla f|^p d\mu\right)^{1/p} + R\|Lf\|_{L^\infty(2B_R)}\right]. \qquad (DG_{p,\epsilon})$$

We write (DG_p) if $(DG_{p,\epsilon})$ is satisfied for some $\epsilon \in (0,1)$.

A non-local (and weaker) version will be sufficient :

$$\left(\fint_{B_r} |\nabla f|^p d\mu\right)^{1/p} \lesssim \left(\frac{R}{r}\right)^{\epsilon} \left[|B_R|^{-1/p}\left(\||\nabla f|\|_p + R\|Lf\|_p\right) + R\|Lf\|_{L^\infty(B_R)}\right]. \qquad (\overline{DG}_{p,\epsilon})$$

Important : $(DG_{p,\epsilon})$ and $(\overline{DG}_{p,\epsilon})$ are always satisfied for $p > \nu$ with $\epsilon = \nu/p \in (0,1)$!

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THEOREM

For any $p \geq 2$, (G_p), (P_p) *and* $(\overline{DG}_p)$ *imply* (LE) *(which is equivalent to* (P_2)*).*
It's almost optimal since if (G_p) *holds for some* $p \in (2, \infty)$*, then for every* $q \in [2, p]$

$$(P_q) + (\overline{DG}_q) \iff (P_2).$$

Proof : From (G_p), (P_p) and $(\overline{DG}_p)$, we get some oscillation estimate (H^λ) and so (P_2). Starting from (P_2), we know that we have (DG_2). Try to interpolate between (DG_2) and (DG_{ν^+}) to get (DG_p) for $p \in (2, \nu]$, not directly but going through oscillation estimates, characterizing De Giorgi inequalities.

Condition (G_p) is stronger and stronger for $p \geq 2$ increasing
Condition (P_p) is weaker and weaker for $p \geq 2$ increasing.

COROLLARY

For $2 \leq q < p < \infty$*, we have*

$$(G_p) + (P_p) + (\overline{DG}_p) \Longrightarrow (G_q) + (P_q) + (\overline{DG}_q).$$

The proof relies on a non-local L^p Cacciopoli inequality :

Proposition (L^p Caccioppoli inequality)

Assume (G_p) and (DUE) for some $p \in [2, \infty]$. Then for every $q \in (1, p]$

$$\left(\fint_B |r \nabla f|^q d\mu \right)^{1/q} \lesssim \left[\left(\fint_{2B} |f|^q d\mu \right)^{1/q} + \left(\fint_{2B} |r^2 L f|^q d\mu \right)^{1/q} \right]$$

for all $f \in \mathcal{D}$ and all balls B of radius r.

Proof : Only using (G_p), we get a non-local inequality with fast decaying off-diagonal quantities. Improvment using simultaneously the finite speed propagation property to keep the information of supports.

Proposition

Under (P_2), there exists some $\kappa > 0$ such that (G_p) and (R_p) hold for $p \in (2, 2+\kappa)$.

[Auscher-Coulhon, 2005] (using Riemannian structure and differential forms)
Extension to this more general setting (relying on a Gehring's argument on reverse Hölder inequalities).

Proposition

Under (P_2), there exists some $\kappa > 0$ such that $(\overline{DG}_p)$ holds for $p \in (2, 2+\kappa)$.

Open question : Can we directly get $(\overline{DG}_p)$ for some $p > 2$ through (G_p) and (P_p) ?

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Proposition

For some $p_0 \in (2, \infty)$, under (P_{p_0}) and (G_{p_0}), there exists some $\kappa > 0$ such that (G_p) holds for $p \in (p_0, p_0 + \kappa)$.

Proof : we already know that we have (DUE) then we adapt to a L^{p_0}-version of the previous proposition.

Riesz transform : $|\nabla L^{-1/2}|$

Proposition

For some $p_0 \in (2, \infty)$, under (P_{p_0}) and (G_{p_0}), then (R_p) holds for $p \in (1, p_0)$.

Proof : Improvment of [ACDH] where (P_2) was required. It relies on the fact that (P_{p_0}) and (G_{p_0}) implies a L^2-Poincaré inequality for harmonic functions and with an extra term (involving the laplacian) for non harmonic functions. Application of a self-improving property of reverse Hölder inequalities.

Proposition

Let $\omega \in L^1_{loc}$ be a non-negative function such that for some $1 < p < q \leq \infty$ and every ball $B \subset M$,

$$\left(\fint_B \omega^q \, d\mu \right)^{1/q} \lesssim \left(\fint_{2B} \omega^p \, d\mu \right)^{1/p}.$$

Then for every $\eta \in (0,1)$ there is an implicit constant such that for every ball B

$$\left(\fint_B \omega^q \, d\mu \right)^{1/q} \lesssim \left(\fint_{2B} \omega^{\eta p} \, d\mu \right)^{1/(\eta p)}.$$

So the RHS exponent of a reverse Hölder inequality always self-improves.

Combining all these results :

Theorem

If for some $p_0 \in [2, \infty]$, the combination (P_{p_0}) with (G_{p_0}) holds then there exists $p(L) \in (p_0, \infty]$ such that

$$(1, p(L)) = \{p \in (1, \infty),\ (G_p) \textit{ holds}\} = \{p \in (1,\infty),\ (R_p) \textit{ holds}\}.$$

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Definition

For $\alpha \in (0,1]$ and $p \in (1,\infty)$ we say that Property $A(\alpha,p)$ holds if :

- the space $\dot{L}^p_\alpha(M) \cap L^\infty(M)$ is an algebra for the pointwise product ;
- and the Leibniz rule inequality is valid :

$$\|fg\|_{\dot{L}^p_\alpha(M)} \lesssim \|f\|_{\dot{L}^p_\alpha(M)} \|g\|_\infty + \|f\|_\infty \|g\|_{\dot{L}^p_\alpha(M)}, \quad \forall f,g \in \dot{L}^p_\alpha(M) \cap L^\infty(M).$$

Two methods : paraproducts or characterization of Sobolev spaces via quadratic functionals.

The paraproducts

Idea : Use the spectral decomposition

$$f = c \int_0^\infty \psi(tL) f \frac{dt}{t}$$

where $\psi(tL) = (tL)^N e^{-tL}$. Let ϕ such that $\phi(0) = 1$ and $\phi'(x) = \psi(x)/x$. Then $\psi(tL)$ can be thought as a regular version of $\mathbf{1}_{[t^{-1}, 2t^{-1}]}(L)$ and is a smooth restriction operator to "frequencies at the scale $t^{-1/2}$".
By analogy with the Euclidean paraproducts,

$$\Pi_g(f) = \int_0^\infty \psi(tL) f \cdot \phi(tL) g \, \frac{dt}{t}.$$

Product decomposition : $fg = \Pi_g(f) + \Pi_f(g)$.

In [B.-Sire] and [Badr-B.-Russ], other paraproducts are defined :

$$\tilde{\Pi}_g(f) = \int_0^\infty \phi(tL)\left[\psi(tL)f \cdot \phi(tL)g\right] \frac{dt}{t}.$$

Boundedness in Lebesgue spaces [B., Frey], using Poincaré inequality (P_2) and a $T(1)$-theorem adapted to semigroup.

Question : When for $g \in L^\infty$, is Π_g bounded on $\dot{L}^p_\alpha$? In this case, we have $A(\alpha, p)$!
Is such a boundedness equivalent to $A(\alpha, p)$?

THE QUADRATIC FUNCTIONALS

Idea : Use a semigroup version of Strichartz functional. Let $\rho > 0$ be an exponent. For $\alpha > 0$ and $x \in M$, we define

$$S_\alpha^\rho f(x) = \left(\int_0^\infty \left[\frac{1}{r^\alpha} \mathrm{Osc}_{\rho, B(x,r)}(f) \right]^2 \frac{dr}{r} \right)^{1/2}$$

where for B a ball $\mathrm{Osc}_{\rho,B}$ denotes the L^ρ-oscillation :

$$\mathrm{Osc}_{\rho,B}(f) = \left(\fint_B \left| f - \fint_B f d\mu \right|^\rho d\mu \right)^{\frac{1}{\rho}}.$$

Question : When do we have $\| \cdot \|_{\dot{L}^p_\alpha} \simeq \| S_\alpha^\rho(\cdot) \|_p$? In this case, we have $A(\alpha, p)$ since

$$\mathrm{Osc}_{\rho,B}(fg) \leq \|g\|_\infty \mathrm{Osc}_{\rho,B}(f) + \|f\|_\infty \mathrm{Osc}_{\rho,B}(g).$$

Is such a property equivalent to $A(\alpha, p)$?

THE CASE FOR $p \in (1, 2]$

THEOREM

Under (DUE) then for every $\alpha \in (0, 1)$ and $p \in (1, 2]$ the paraproduct is bounded on $\dot{L}^p_\alpha$. Consequently, $A(\alpha, p)$ holds.

Proof : First obtain the $p = 2$ case, using orthogonality and duality in tent spaces (or Carleson measure). Then use an extrapolation argument to get the other boundedness for $p \in (1, 2)$. It suffices to get $L^2 - L^2$ off-diagonal estimates.

THEOREM

Then

(A) *Under* (DUE), (G_{q_0}) *with* ($\overline{DG}_{q_1,\kappa}$) *for some* $2 \leq q_1 < q_0 \leq \infty$ *and* $\kappa \in (0,1)$, $A(\alpha, p)$ *holds for every* $p \leq [2, \infty)$ *and* $\alpha < 1 - \kappa\left(1 - \frac{2}{p}\right)$ *;*

(B) *Under* (DUE) *and* (H^η) *for some* $\eta \in (0,1]$, $A(\alpha, p)$ *holds for every* $\alpha \in (0, \eta)$ *and* $p \in (1, \infty)$ *;*

(C) *under the combination* (G_{q_0}), (P_{q_0}) *with* ($\overline{DG}_{q_0,\kappa}$) *for some* $\kappa \in (0,1)$ *and* $q_0 > 2$, $A(\alpha, p)$ *holds for every* $p \leq [2, q_0]$ *and* $\alpha < 1$ *;*

(D) *under the combination* (G_{q_0}), (P_{q_0}) *with* ($\overline{DG}_{q_0,\kappa}$) *for some* $\kappa \in (0,1)$ *and* $q_0 > 2$, $A(\alpha, p)$ *holds for every* $p > q_0$ *with* $\alpha < 1 - \kappa(1 - \frac{q_0}{p})$.

(E) *under* (R_{q_0}) *for some* $q_0 > 2$, $A(\alpha, p)$ *holds for every* $p \in [2, q_0)$ *with* $\alpha < 1$.

Moreover, in the four first situations the paraproduct is bounded in the corresponding Sobolev spaces. And we have a characterization by quadratic functional of the Sobolev norm in cases (B) and (C).

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Paralinearization

If we have a characterization via quadratic functionals, then optimal chain rule : the composition through a Lipschitz function preserves the Sobolev space.

For a Ahlfors regular space (i.e. $\mu(B(x,r)) \simeq r^\nu$) we can describe a Paralinearization result (which requires more regularity) :

Theorem (Chain rule)

Let $F \in C^3(\mathbb{R})$ a nonlinearity with $F(0) = 0$. For $\alpha \in (0,1)$, $p \in (1,\infty]$ with $\frac{\nu}{p} < \alpha < 1$, consider a fixed function $f \in L^\infty \cap \dot{L}^p_\alpha$. Then

$$F(f) \in L^\infty \cap \dot{L}^p_\alpha.$$

If $F \in C^2(\mathbb{R})$, then we have the paralinearization :

$$F(f) - \Pi_{F'(f)}(f) \in L^\infty \cap \dot{L}^p_\alpha \cap \dot{L}^p_{\alpha+\rho},$$

for some $0 < \rho \simeq \alpha - \frac{\nu}{p}$.