Two-sided bounds for L_p-norms of combinations of products of independent random variables

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Wojciechowski Question

Let $X, X_1, X_2, \ldots$ be i.i.d. nonnegative r.v.'s such that $\mathbb{E}X = 1$ and $\mathbb{P}(X = 1) < 1$. Define

$$R_0 := 1 \quad \text{and} \quad R_k := \prod_{j=1}^{k} X_j \ \text{ for } k = 1, 2, \ldots.$$

Obviously $\mathbb{E}R_k = 1$ and therefore for any $a_0, a_1, \ldots, a_n$,

$$\mathbb{E}\Big|\sum_{k=0}^{n} a_k R_k\Big| \le \sum_{k=0}^{n} |a_k|.$$

Question. (M. Wojciechowski) Is it true that for any i.i.d. sequence the above estimate may be reversed, i.e. there exists a constant $c > 0$ that depends only on the distribution of X such that

$$\mathbb{E}\Big|\sum_{k=0}^{n} a_k R_k\Big| \ge c \sum_{k=0}^{n} |a_k| \quad \text{for any } a_0, \ldots, a_n?$$

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L_1 bound for products of i.i.d. nonnegative r.v's

$R_k := \prod_{j=1}^k X_j$
The answer to Wojciechowski's question is positive even in the more general case of vector coefficients.

Theorem (Latała 2013)

Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence of nonnegative nondegenerate r.v's such that $\mathbb{E}X = 1$. Then there exists a constant c that depends only on the distribution of X such that for any $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$,

$$\mathbb{E}\Big\| \sum_{k=0}^{n} v_k R_k \Big\| \geq c \sum_{k=0}^{n} \|v_k\|.$$

L_1 bound in the non i.i.d. case

Consider sequence (X_i) satisfying the following assumptions:

$$X_1, X_2, \dots \text{ are independent, nonnegative mean one r.v's }, \quad (1)$$

$$\mathbb{E}\sqrt{X_l} \le \lambda < 1 \quad \text{and} \quad \mathbb{E}|X_l - 1| \ge \mu > 0 \quad \text{for all } l, \quad (2)$$

$$\mathbb{E}|X_l - 1|\mathbf{1}_{\{X_l \ge A\}} \le \frac{1}{4}\mu \quad \text{for all } l. \quad (3)$$

Theorem (Latała 2013)

Let $X_1, X_2, \dots$ satisfy assumptions (1), (2) and (3). Then for any vectors $v_0, v_1, \dots, v_n$ in a normed space $(F, \|\ \|)$, we have

$$\mathbb{E}\Big\|\sum_{k=0}^{n} v_k R_k\Big\| \ge \frac{1}{512r}\mu^3 \sum_{k=0}^{n} \|v_k\|,$$

where r is a positive integer such that

$$\frac{2^{17}}{(1-\lambda)^2} r\lambda^{2r-2} A \le \mu^3. \quad (4)$$

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L_p-bounds for products of i.i.d. r.v's

It turns out that L_1-bounds may be extended to L_p for $p > 0$. Positivity of X is not needed. Namely we have

Theorem (Latała, Nayar, Tkocz, Damek)

Let $p > 0$ and $X, X_1, X_2, \ldots$ be an i.i.d. sequence of r.v.'s $\mathbb{P}(|X| = t) < 1$ for all t. Then there exist constants $0 < c_{p,X} \leq C_{p,X} < \infty$ which depend only on p and the distribution of X such that for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \|)$,

$$c_{p,X} \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}|R_i|^p \leq \mathbb{E} \left\| \sum_{i=0}^{n} v_i R_i \right\|^p \leq C_{p,X} \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}|R_i|^p.$$

$R_k := \prod_{j=1}^{k} X_j$, $\mathbb{E}|X|^p$ not necessarily equal 1.

L_p-bounds $p > 0$

$R_k := \prod_{j=1}^{k} X_j$

Theorem (Latała, Nayar, Tkocz, Damek)

Let $p > 0$ and $X_1, X_2, \ldots$ be independent r.v's with finite p-th moments, $|X_i|$ non degenerate, satisfying some "uniform behavior" assumptions. Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \,\|)$ we have

$$c(X_1, X_2, \ldots) \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}|R_i|^p \le \mathbb{E} \left\| \sum_{i=0}^{n} v_i R_i \right\|^p \le C(X_1, X_2, \ldots) \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}|R_i|^p,$$

where $c(X_1, X_2, \ldots)$, $C(X_1, X_2, \ldots)$ are positive constants that depend only on the "uniform behavior" and they are quite explicit.

Example

Assumption $\mathbb{P}(|X| = t) < 1$ for all t is crucial since for any $p > 0$ by the Khintchine inequality,

$$\mathbb{E}\Big|\sum_{k=1}^{n}\prod_{l=1}^{k}\varepsilon_l\Big|^p = \mathbb{E}\Big|\sum_{k=1}^{n}\varepsilon_k\Big|^p \sim_p \Big(\mathbb{E}\Big|\sum_{k=1}^{n}\varepsilon_k\Big|^2\Big)^{p/2} = n^{p/2}.$$

Here ε_l are i.i.d symmetric random variables taking values $1, -1$. $\mathbb{E}|\varepsilon_l|^p = 1$, $v_i = 1$, .

$$nc_{p,X} \leq \mathbb{E}\Big\|\sum_{i=0}^{n} R_i\Big\|^p \leq nC_{p,X}$$

Riesz products

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one-dimensional torus and m be the normalized Haar measure on $\mathbb{T}$. Riesz products are defined on $\mathbb{T}$ by the formula

$$\bar{R}_i(t) = \prod_{j=1}^{i}(1 + \cos(n_j t)), \quad i = 1, 2, \ldots, \tag{5}$$

where $(n_k)_{k\geq 1}$ is a lacunary increasing sequence of positive integers.

The result of Y. Meyer gives that if $n_{k+1}/n_k \geq 3$ and $\sum_k \frac{n_k}{n_{k+1}} < \infty$ then

$$\|\sum_{k=0}^{n} a_k \bar{R}_k\|_{L_p(\mathbb{T})} \sim (\mathbb{E}|\sum_{k=0}^{n} a_k R_k|^p)^{1/p} \text{ for } p \geq 1,$$

where R_k are products of independent random variables distributed as $\bar{R}_1$. Therefore main Theorem yields an estimate for $\|\sum_{i=0}^{n} a_i \bar{R}_i\|_{L_p(\mathbb{T})}$.

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Riesz products

Corollary

Suppose that $(n_k)_{k\geq 1}$ is an increasing sequence of positive integers such that $n_{k+1}/n_k \geq 3$ and $\sum_{k=1}^{\infty} \frac{n_k}{n_{k+1}} < \infty$. Then for any coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $p \geq 1$,

$$c_p \sum_{k=0}^{n} |a_k|^p \int_{\mathbb{T}} |\bar{R}_k(t)|^p \mathrm{d}m(t) \leq \int_{\mathbb{T}} \left| \sum_{k=0}^{n} a_k \bar{R}_k(t) \right|^p \mathrm{d}m(t)$$

$$\leq C_p \sum_{k=0}^{n} |a_k|^p \int_{\mathbb{T}} |\bar{R}_k(t)|^p \mathrm{d}m(t),$$

where $0 < c_p \leq C_p < \infty$ are constants depending only on p and the sequence (n_k).

Dechamps condition $\sum_{k=1}^{\infty} (\frac{n_k}{n_{k+1}})^2 < \infty$. Anyway $n_k \asymp (k!)^{\alpha}$

Riesz products

We expect that the assumptions on the growths of n_k may be weakened to $n_{k+1}/n_k \geq C_p$, but we are able to show it only for $p = 1$.

Theorem (Latała, Nayar, Tkocz)

There exist constants $C_1 < 1.2 \cdot 10^9$ and $c_1 > 2 \cdot 10^{-7}$ such that if $n_{k+1}/n_k \geq C_1$ then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$,

$$\sum_{k=0}^{n} \|v_k\| \geq \int_{\mathbb{T}} \Big\| \sum_{k=0}^{n} v_k \bar{R}_k \Big\| \mathrm{d}m \geq c_1 \sum_{k=0}^{n} \|v_k\|.$$

Perpetuities

Let us consider the random difference equation

$$S \stackrel{d}{=} XS + B, \tag{6}$$

where the equality is meant in law, (X, B) is a random variable with values in $[0, \infty) \times \mathbb{R}$ independent of S.
Let (X_i, B_i) be i.i.d. copies of (X, B). It is known that (under some mild integrability assumptions) the infinite series

$$\sum_{i=1}^{\infty} R_{i-1} B_i = B_1 + X_1 \sum_{i=2}^{\infty} R_{i-1} B_i, \quad R_{i-1} := \prod_{j=1}^{i-1} X_j$$

converges almost everywhere to a solution of (6). It is called perpetuity.
The conditions for convergence are

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Over the last 40 years equation

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and its various modifications (in particular multidimensional analogues) have attracted a lot of attention. It has a rather wide spectrum of applications including random walks in random environment, branching processes, fractals, finance, insurance, telecommunications, various physical and biological models.

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More can be said if we assume additionally that for some $p > 0$,

$$\mathbb{P}(X = 1) < 1, \quad \mathbb{E}X^p = 1, \quad \mathbb{E}\|B\|^p < \infty. \tag{8}$$

Then for every $q < p$

$$\mathbb{E}X^q < 1$$

because the function $q \mapsto \mathbb{E}X^q$ is convex and equal 1 at $0, p$. In particular, for $q < p$

$$\mathbb{E}\|\sum_{i=1}^{\infty} R_{i-1}B_i\|^q < \infty.$$

No moment of order p.

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Perpetuities

But how fast does

$$\mathbb{E}\|\sum_{i=1}^{n} R_{i-1}B_i\|^p$$

grow with n?

Moreover if $\mathbb{E}X^p = 1$, $\mathbb{E}X^p \log X < \infty$, $\log X$ has nonlattice distribution, $\mathbb{E}\|B\|^p < \infty$ and $\mathbb{P}(Xv + B = v) < 1$ for any v then

$$\lim_{t\to\infty} t^p \mathbb{P}(\|\sum_{i=1}^{\infty} R_{i-1}B_i\| > t) = c_\infty(X,B) = \frac{1}{\alpha\rho}\mathbb{E}(|S|^p - |S-B|^p)$$

and $c_\infty(X,B)$ is a finite positive constant.

The latter was proved by Kesten, then the proof was simplified by Goldie. The proof goes via the renewal theorem and there is no good expression for the constant $c_\infty(X,B)$ called nowadays the Goldie constant or the Goldie-Kesten constant. Although Goldie provided a formula for $c_\infty(X,B)$ but positivity of the latter could not be derived from it. A new idea was needed. It was provided by Grincevicius in the seventies and further developed by Goldie.

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Goldie-Kesten constant and finite sums of perpetuities

There is an expression for $c_\infty(X, B)$ in a paper by N.Enriquez, C.Sabot, O.Zindy 2009 but it is very complicated and only for positive B independent of X. There is another one in a paper by J. Collamore, A. Vidyashankar 2013 again for positive B and the law of X being non singular. There is something simpler in a paper by K. Bartkowiak, A.Jakubowski, T.Mikosch and O.Wintenberger 2011 but only for $B = 1$

$$\lim_{n\to\infty} \frac{1}{np\rho}\mathbb{E}\|\sum_{i=1}^{n} R_{i-1}\|^p = c_\infty(X, 1) > 0.$$

The assumptions are as in Goldie. In particular, $\rho = \mathbb{E}X^p \log X < \infty$. The latter was an inspiration for my team.

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Recently Buraczewski, Damek and Zienkiewicz showed that if additionally $\mathbb{E}(X^{p+\varepsilon} + \|B\|^{p+\varepsilon}) < \infty$ for some $\varepsilon > 0$ then

$$\lim_{n\to\infty} \frac{1}{np\rho} \mathbb{E} \| \sum_{i=1}^{n} R_{i-1} B_i \|^p = c_\infty(X, B) > 0,$$

where $\rho := \mathbb{E} X^p \log X, \ \mathbb{E} X^p = 1.$ The same for some multidimensional models.

$$S = XS + B, \quad B, S \in \mathbb{R}^d, \text{ or similarities in place of } X$$

The first observation is that this finite sums grow like n. Secondly they give an expression for the Goldie-Kesten constant. Is it good? It is simple, but not necessarily good for simulations.

Goldie-Kesten constant and finite sums of perpetuities

Recently Buraczewski, Damek and Zienkiewicz showed that if additionally $\mathbb{E}(X^{p+\varepsilon} + \|B\|^{p+\varepsilon}) < \infty$ for some $\varepsilon > 0$ then

$$\lim_{n\to\infty} \frac{1}{np\rho}\mathbb{E}\|\sum_{i=1}^{n} R_{i-1}B_i\|^p = c_\infty(X, B) > 0,$$

where $\rho := \mathbb{E}X^p \log X,\ \mathbb{E}X^p = 1.$ The same for some multidimensional models.

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$$c_{p,X}\sum_{i=0}^{n}\|v_i\|^p \le \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \le C_{p,X}\sum_{i=0}^{n}\|v_i\|^p.$$

Notice that our L_p bounds on moments yield that if X, B are independent, $\mathbb{E}X^p = 1$ then for every n,

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via conditioning on B. There is no limit, only bounds but besides independence the assumptions are much weaker, even weaker then in the Goldie theorem. We need only $\mathbb{E}X^p = 1$, $\mathbb{E}\|B\|^p < \infty$, $\mathbb{P}(X = 1) < 1$. In fact we may get rid of the independence assumption.

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A family of perpetuities

Before we proceed further let us consider the family of random equations

$$S^{(d)} = XS^{(d)} + B^{(d)}$$

$B^{(d)}$ being a random vector in $\mathbb{R}^d$. So $S^{(d)} \in \mathbb{R}^d$. Suppose for a moment that X and $B^{(d)}$ are independent,

$$S^{(d)} = \sum_{i=1}^{\infty} R_{i-1} B_i^{(d)}$$

and if $\mathbb{E}X^p = 1$, $\mathbb{E}\|B^{(d)}\|^p < \infty$, $\mathbb{P}(X = 1) < 1$ then

$$c_{p,X}\mathbb{E}\|B^{(d)}\|^p \leq \frac{1}{n}\mathbb{E}\left\|\sum_{i=0}^{n} R_i B_i^{(d)}\right\|^p \leq C_{p,X}\mathbb{E}\|B^{(d)}\|^p$$

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Together with Buraczewski, Zienkiewicz, Damek result (that requires more moments)

$$\lim_{n\to\infty} \frac{1}{np\rho}\mathbb{E}\left\|\sum_{i=0}^{n} R_i B_i^{(d)}\right\|^p = C_\infty(X, B^{(d)})$$

this gives uniform bounds for the Goldie constant

$$C_\infty(X, B^{(d)}) = \lim_{t\to\infty} t^p \mathbb{P}(\|\sum_{i=0}^{\infty} R_i B_i^{(d)}\| > t)$$

(independent of the dimension)

$$c_{p,X}\mathbb{E}\|B^{(d)}\|^p \le C_\infty(X, B^{(d)}) \le C_{p,X}\mathbb{E}\|B^{(d)}\|^p$$

L_p bound for finite sums of perpetuities

Theorem (Latała, Nayar, Tkocz, Damek)

Suppose that F is a separable Banach space. Let $p > 0$ and let an i.i.d. sequence $(X, B), (X_1, B_1), \ldots$ with values in $[0, \infty) \times F$ be such that X is nondegenerate and $\mathbb{E}\|B\|^p$, $\mathbb{E}X^p = 1$. Assume additionally that

$$\mathbb{P}(Xv + B = v) < 1 \text{ for every } v \in F.$$

Then there are constants $0 < c_p(X, B) \le C_p(X) < \infty$ which depend on p and the distribution of (X, B) such that for every n,

$$c_p(X, B) n \mathbb{E}\|B\|^p \le \mathbb{E}\left\| \sum_{i=1}^{n} R_{i-1} B_i \right\|^p \le C_p(X) n \mathbb{E}\|B\|^p.$$

Finite sums of perpetuities

There are quite explicit formulae for the constants $0 < c_p(X,B) \leq C_p(X) < \infty$ and although they are not close it seems that in some particular cases of perpetuities one can elaborate.

The sum $\sum_{i=1}^{\infty} R_{i-1}B_i$ is much easier to study then partial sums $\sum_{i=1}^{n} R_{i-1}B_i$ due to the renewal theorem.

Nobody looked at perpetuities in this way yet. Recently D.Buraczewski, J. Collamore, J.Zienkiewicz and myself have developed methods to study tails partial sums i.e.

$$\mathbb{P}(\sum_{i=1}^{n} R_{i-1}B_i > t) \asymp \frac{1}{\sqrt{n}} t^{-\alpha(n,t)} \text{ or } t^{-\alpha(n,t)}$$

without using Latała, Nayar, Tkocz, Damek result.

I have a dream to combine both. In particular both approaches work for (X_i, B_i) being not necessarily i.i.d just independent provided some uniform behavior is guaranteed.

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A lemma

Lemma

Let $p > 0$. Suppose that $\mathbb{P}(|X| = 1) < 1$ and $\mathbb{E}|X|^p = 1$. There is δ such that for any vectors in a normed space we have

$$\mathbb{E}\|Xu + v\|^p \geq \delta(\|u\|^p + \|v\|^p)$$

Then one does induction

$$\begin{aligned} \mathbb{E}\|\sum_{i=0}^{n} v_i R_i\|^p =& \mathbb{E}\|v_0 + X_1 \sum_{i=1}^{n} v_i X_2 \dots X_i\|^p \\ \geq& ?\|v_0\|^p + ?\mathbb{E}\|\sum_{i=1}^{n} v_i X_2 \dots X_i\|^p \end{aligned}$$

Another lemma

Lemma

Let $0 < p \leq 1$ *and* Y, Z *be random vectors such that*

$$\mathbb{E}\|Z\|^p \mathbf{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} \leq \frac{1}{8}\mathbb{E}\|Z\|^p.$$

Then

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.$$

Proof. We have for any $u, v \in F$,
$\|u+v\|^p \geq \big|\|u\| - \|v\|\big|^p \geq \|u\|^p - \|v\|^p$, therefore

$$\begin{aligned}\mathbb{E}\|Y+Z\|^p &\geq \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Z\|^p)\mathbf{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\ &\quad + \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Y\|^p)\mathbf{1}_{\{\|Y\|^p < \frac{1}{8}\mathbb{E}\|Z\|^p\}}\end{aligned}$$

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$R_k := \prod_{j=1}^{k} X_j$

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Then there are constants $0 < c_p(X, B) \leq C_p(X) < \infty$ which depend on p and the distribution of (X, B) such that for every n,

$$c_p(X, B)\mathbb{E}\|B\|^p \sum_{i=1}^{n} \mathbb{E}R_{i-1}^p \leq \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p$$

$$\leq C_p(X)\mathbb{E}\|B\|^p \sum_{i=1}^{n} \mathbb{E}R_{i-1}^p.$$

Symmetric i.i.d sequences

Corollary

Let $p > 0$ and $X, X_1, X_2, \ldots$ be an i.i.d. sequence of symmetric r.v.'s such that $\mathbb{E}|X|^p < \infty$ and $\mathbb{P}(|X| = t) < 1$ for all t. Then there exist constants $0 < c_{p,X} \leq C_{p,X} < \infty$ which depend only on p and the distribution of X such that for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \|)$,

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Proof. Let (ε_i) be a sequence of independent symmetric ± 1 r.v.'s, independent of (X_i). Then

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Ideas of the proof - upper bound

We have $\| \sum_{i=0}^n v_i R_i \| \leq \sum_{i=0}^n \|v_i\| |R_i|$, so it is enough to consider the case when $F = \mathbb{R}$ and $v_k \geq 0$. Since it is only a matter of normalization we may also assume that $\mathbb{E} X_i^p = 1$ for all i.

We proceed by an induction on $m = \lceil p \rceil$. For $m = 1$ i.e. $p \leq 1$ there is nothing to prove, since $(x+y)^p \leq x^p + y^p$ for $x, y > 0$. For $p > 1$ we use the inequality

$$(x+y)^p \leq x^p + 2^p (y x^{p-1} + y^p) \quad \text{for } x, y \geq 0. \tag{9}$$

We have by (**??**)

$$\mathbb{E} \left| \sum_{i=0}^n v_i R_i \right|^p \leq \mathbb{E} \left| \sum_{i=1}^n v_i R_i \right|^p + 2^p \left(v_0 \mathbb{E} \left| \sum_{i=1}^n v_i R_i \right|^{p-1} + v_0^p \right).$$

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$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \leq v_n^p \mathbb{E} R_n^p + 2^p\left(\sum_{k=0}^{n-1} v_k \mathbb{E} R_k \left(\sum_{i=k+1}^n v_i R_i\right)^{p-1} + \sum_{i=0}^{n-1} v_i^p \mathbb{E} R_i^p\right).$$

However, $\mathbb{E}R_k(\sum_{i=k+1}^n v_i R_i)^{p-1} = \mathbb{E}R_k^p \mathbb{E}(\sum_{i=k+1}^n v_i R_{k+1,i})^{p-1}$ and $\mathbb{E}R_k^p = \prod_{j=1}^k \mathbb{E}X_j^p = 1$. Hence

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \leq 2^p \sum_{i=0}^n v_i^p + 2^p \sum_{k=0}^{n-1} v_k \mathbb{E}\left(\sum_{i=k+1}^n v_i R_{k+1,i}\right)^{p-1}.$$

The induction assumption yields

$$\mathbb{E}\left(\sum_{i=k+1}^n v_i R_{k+1,i}\right)^{p-1} \leq C(p-1) \sum_{i=k+1}^n v_i^{p-1} \mathbb{E}R_{k+1,i}^{p-1}$$

$$= C(p-1) \sum_{i=k+1}^n v_i^{p-1} \prod_{j=k+1}^i \mathbb{E}X_j^{p-1} \leq C(p-1) \sum_{i=k+1}^n v_i^{p-1} \lambda_1^{(p-1)(i-k)}.$$

To finish the proof we observe that

$$\sum_{k=0}^{n-1} v_k \sum_{i=k+1}^n v_i^{p-1} \lambda_1^{(p-1)(i-k)} \leq \sum_{0 \leq k < i \leq n} \left(\frac{1}{p} v_k^p + \frac{p-1}{p} v_i^p\right) \lambda_1^{(p-1)(i-k)}$$

$$\leq \sum_{i=0}^n v_i^p \sum_{j=1}^\infty \lambda_1^{(p-1)j} = \frac{\lambda_1^{p-1}}{1-\lambda_1^{p-1}} \sum_{i=0}^n v_i^p.$$

However, $\mathbb{E}R_k(\sum_{i=k+1}^n v_i R_i)^{p-1} = \mathbb{E}R_k^p \mathbb{E}(\sum_{i=k+1}^n v_i R_{k+1,i})^{p-1}$ and $\mathbb{E}R_k^p = \prod_{j=1}^k \mathbb{E}X_j^p = 1$. Hence

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Ideas of the proof - lower bound

Proofs of lower bounds are much more involved. They are also based on some induction. For $p \le 1$ we have

Proposition

Let $0 < p \le 1$ *and independent nonnegative r.v.'s* $X_1, X_2, \ldots$ *satisfy* $\mathbb{E}X_i^p = 1$, $\mathbb{E}X_i^{p/2} \le \lambda < 1$ *and* $\mathbb{E}(X_i^p - 1)\mathbf{1}_{\{1 \le X_i^p \le A\}} \ge \delta$. *Then for any vectors* $v_0, v_1, \ldots, v_n$ *in a normed space* $(F, \| \ \|)$ *and any integer* $k \ge 1$ *we have*

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \ge \varepsilon_0 \|v_0\|^p + \sum_{i=1}^{n}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,$$

where $\varepsilon_0 = \delta/8$, $\varepsilon_1 = \delta^3/8$, $c_i = 0$ *for* $1 \le i \le k-1$,

$$c_i = \Phi \sum_{j=k}^{i} \lambda^j \ \textit{for}\ i \ge k \quad \textit{and} \quad \Phi = \frac{2^8 A}{1-\lambda}\lambda^{k-2}.$$

L_p-bounds in the non iid case $p \le 1$

In the non iid case for $p \in (0,1]$ we assume that

$$X_1, X_2, \ldots \text{ are independent, nonnegative r.v.'s, } \mathbb{E}X_i^p < \infty, \quad (10)$$

$$\exists_{\lambda<1}\ \forall_i\ \mathbb{E}X_i^{p/2} \le \lambda(\mathbb{E}X_i^p)^{1/2}, \quad (11)$$

$$\exists_{0<\delta<1, A>1}\ \forall_i\ \mathbb{E}(X_i^p - \mathbb{E}X_i^p)\mathbf{1}_{\{\mathbb{E}X_i^p \le X_i^p \le A\mathbb{E}X_i^p\}} \ge \delta\mathbb{E}X_i^p. \quad (12)$$

Theorem

*Let $0 < p \le 1$ and $X_1, X_2, \ldots$ satisfy assumptions (**??**)-(**??**). Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ we have*

$$c(p,\lambda,\delta,A)\sum_{i=0}^{n}\|v_i\|^p\mathbb{E}R_i^p \le \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \le \sum_{i=0}^{n}\|v_i\|^p\mathbb{E}R_i^p,$$

where $c(p,\lambda,\delta,A)$ is a constant which depends only on p, λ, δ and A.

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where $c(p,\lambda,\delta,A)$ is a constant which depends only on p, λ, δ and A.

L_p-bounds in the non iid case $p > 1$

For $p > 1$ to get the lower bound we assume

$$\exists_{\mu>0,A<\infty}\ \forall_i\ \mathbb{E}|X_i - \mathbb{E}X_i| \geq \mu(\mathbb{E}X_i^p)^{1/p}$$

$$\mathbb{E}|X_i - \mathbb{E}X_i|\mathbf{1}_{\{X_i > A(\mathbb{E}X_i^p)^{1/p}\}} \leq \frac{1}{4}\mu(\mathbb{E}X_i^p)^{1/p}, \quad (13)$$

$$\exists_{q>\max\{p-1,1\}}\ \exists_{\lambda<1}\ \forall_i\ (\mathbb{E}X_i^q)^{1/q} \leq \lambda(\mathbb{E}X_i^p)^{1/p}. \quad (14)$$

To derive the upper L_p-bounds we need

$$\forall_{k=1,2,\ldots,\lceil p\rceil-1}\ \exists_{\lambda_k<1}\ \forall_i\ (\mathbb{E}X_i^{p-k})^{1/(p-k)} \leq \lambda_k(\mathbb{E}X_i^{p-k+1})^{1/(p-k+1)}. \quad (15)$$

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