On isomorphisms of Hardy spaces for certain Schrödinger operators*

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Conference in Honor of Aline Bonami, Orleans, 10-13.06.2014.

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^d$, $V \geq 0$, $V \not\equiv 0$.
$K_t(x, y)$ be the integral kernels of $\{K_t\}_{t>0} = \{e^{-tL}\}_{t>0}$.
The Feynman-Kac formula implies that

$$0 \leq K_t(x, y) \leq (4\pi t)^{-d/2} \exp\big(-|x-y|^2/4t\big) = P_t(x-y).$$

One possible definition of the Hardy space associated with L is by means of the maximal function

$$\mathcal{M}_L f(x) = \sup_{t>0} |K_t f(x)|.$$

We say that an $L^1(\mathbb{R}^d)$-function f belongs to H^1_L if $\mathcal{M}_L f \in L^1(\mathbb{R}^d)$. Then we set $\|f\|_{H^1_{L,\max}} = \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}$.

These spaces are generalizations of the classical Hardy spaces, which could be thought as the spaces associated with the classical heat semigroups, $H^1(\mathbb{R}^d) = H^1_\Delta$.
The very original works in this field are due to C. Fefferman, E. Stein, G.Weiss, R. Coifman, R. Latter, also D. Burkholder, R. Gundy, M. Silverstein, then L. Carleson, R. Macias, C. Segovia, P.W. Jones, Y. Meyer, R. Rochberg, S. Semmes, A. Uchiyama, J.M. Wilson, G. Folland, M. Christ, D. Geller, D. Goldberg, ...
There are many characterizations of the Hardy spaces. Among them one plays an important role, namely characterization by atoms.

A function a is an $(1,\infty)$-atom for the Hardy space $H^1(\mathbb{R}^d)$ if there is a ball B such that
supp $a \subset B$ (support condition);
$\|a\|_{L^\infty} \le |B|^{-1}$ (size condition);
$\int a = 0$ (cancellation condition).
Then $\|a\|_{H^1} \le C$ for every atom a.
The atomic norm $\|f\|_{H^1_{atom}(\mathbb{R}^d)}$ is defined as

$$\|f\|_{H^1_{atom}(\mathbb{R}^d)} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$.

Theorem (Coifman, Latter)

The spaces $H^1(\mathbb{R}^d)$ and $H^1_{atom}(\mathbb{R}^d)$ coincide and the norms $\|f\|_{H^1(\mathbb{R}^d)}$ and $\|f\|_{H^1_{atom}(\mathbb{R}^d)}$ are equivalent.

Question: Let (X, d, μ) be the metric space and let e^{-tA} be a semigroup of linear operators acting on functions on X. What can be said about the Hardy space H^1 if we replace the classical heat kernel in the definition of the Hardy space by e^{-tA}:

$$H^1_A = \left\{ f \in L^1(X) : \ \left\| \sup_{t>0} |e^{-tA} f| \right\|_{L^1} = \|f\|_{H^1_A} \right\} ?$$

Hardy spaces associated with certain operators attracted attention of many authors (J-Ph. Anker, P. Auscher, N. Ben Salem, F. Bernicot, J. Betancor, W. Czaja, X. Duong, G. Garrigós, N. Hamda, E. Harboure, S. Hofmann, T. Hytönen, Z.J. Lou, G. Lu, T. Martínez, G. Mauceri, S. Mayboroda, A. McIntosh, S. Meda, D. Mitrea, M. Mitrea, E. Ouhabaz, M. Picardello, P. Portal, F. Ricci, E. Russ, A. Sikora, P. Sjögren, X. Tolsa, J.L. Torrea, D. Yang, L. Yan, J. M. Vallarino, J. Zhao, ...

In a project with Jacek Zienkiewicz we have been studying Hardy spaces associated with semigroup generated by Schrödinger operators on $\mathbb{R}^d$,

$$-L = \Delta - V(x), \quad V(x) \geq 0.$$

Our task was/is: How does the space H^1_L look like for some classes of potentials?
The best way to see the space is by their atomic characterization.
It turns out that the space H^1_L and its properties depend on the potential V and on the dimension d.

If $d = 1$ or if V is "large", then the kernel $K_t(x,y)$ of the semigroup e^{-tL} has faster decay comparing to the classical heat kernel, (see the Faynman-Kac formula)

$$\int K_t(x,y)f(y)\,dy = E^x\left(e^{-\int_0^t V(b_s)\,ds} f(b_t)\right).$$

This reflects, that the Hardy space H^1_L may admit local atoms, that is, some atoms may not satisfy any cancellation condition.

Our task is to study the case of "small" potentials V.

Example and motivation for today's talk.

Assume that supp $V \subset B(0,1)$, $V \in L^{d/2+\varepsilon}(\mathbb{R}^d)$ and $d \geq 3$.
Define $w(x) = \lim_{t\to\infty} \int K_t(x,y)\,dy$.
The limit exists and defines bounded from below and above L-harmonic function w, that is, $K_t w = w$, such that $0 < \delta \leq w(x) \leq 1$.
Then in the definition of atoms for the space H^1_L the classical cancellation condition is replaced by

$$\int a(x)w(x)\,dx = 0.$$

In other words the mapping

$$H^1_L \ni f(x) \mapsto w(x)f(x) \in H^1(\mathbb{R}^d) \quad \text{is an isomorphism.}$$

This result was slightly generalized with M. Preisner.

Elias Stein asked the question: Is it possible to characterize all nonnegative potentials V in $\mathbb{R}^d$, $d \geq 3$, such that the Hardy space H^1_L for the semigroup e^{-tL}, $L = -\Delta + V$, is isomorphic to the classical Hardy space $H^1(\mathbb{R}^d)$ by multiplication by a function w, $0 < c \leq w(x) \leq C$?
In other words:

For which $V \geq 0$ there is $0 < \delta \leq w(x) \leq C$ such that

$$H^1_L \ni f \mapsto wf \in H^1(\mathbb{R}^d) \quad \text{isomorphism?}$$

Our purpose is to give a complete answer to his question.

Step 1. If such isomorphism exists,

$$H^1_L \ni f \mapsto wf \in H^1(\mathbb{R}^d),$$

then w is L-harmonic ($K_t w = w$).
For the proof we use the following lemma.

Lemma

$K_t f - f \in H^1_L$ for every $f \in C_c(\mathbb{R}^d)$.

$$\begin{aligned} 0 &= \int w(x)(K_t f(x) - f(x))\, dx \\ &= \int w(x)\Big(\int K_t(x,y) f(y)\, dy\Big)\, dx - \int w(x) f(x)\, dx \\ &= \int \Big(\int K_t(x,y) w(x)\, dx\Big) f(y)\, dy - \int w(y) f(y)\, dy. \end{aligned}$$

Hence, $K_t w(y) = w(y)$.

To see the lemma take $f \in C_c$ and let $\operatorname{supp} f \subset B(0, R)$.
Note that $\mathcal{M}_L f(x) \leq \|f\|_{L^\infty}$,
$\sup_{0<s<t+R^2} |K_s(K_t(f) - f)| \leq 2 \sup_{0<s<2t+R^2} |K_s f| \in L^1(\mathbb{R}^d)$.
To evaluate $\sup_{s>t+R^2} |K_s(K_t(f) - f)|$ on $B(0, 2R)^c$ we w use the estimate

$$\left| \frac{d}{dt} K_t(x, y) \right| \lesssim t^{-1} t^{-d/2} \exp(-c|x - y|^2/t).$$

$$\begin{aligned}
|K_s(K_t - f)(x)| &= \left| \int_s^{t+s} \int \frac{d}{du} K_u(x, y) f(y)\, dy\, du \right| \\
&\lesssim ts^{-1} s^{-d/2} \int_{|y|<R} \exp(-c|x - y|^2/s)|f(y)|\, dy \\
&\lesssim ts^{-1} s^{-d/2} \exp(-c'|x|^2/s)\|f\|_{L^1} \lesssim t|x|^{-d-2}\|f\|_{L^1}.
\end{aligned}$$

So $\mathcal{M}_L(K_t f - f) \in L^1$.

Step 2. For which V there is L-harmonic w: $0 < c \leq w(x) \leq C$?

Recall that $d \geq 3$. The following are equivalent:
(a) There is an L-harmonic function w: $0 < c \leq w(x) \leq C$;
(b) $\int K_t(x,y)\,dy \geq \delta > 0$
(c) The global Kato norm

$$\|V\|_K = c \sup_{x\in\mathbb{R}^d} \int \frac{V(y)}{|x-y|^{d-2}}\,dy = \|\Delta^{-1}V\|_{L^\infty} \quad \text{is bounded;}$$

(d) $ct^{-d/2}e^{-C|x-y|^2/t} \leq K_t(x,y)$ (Gaussian lower bounds).

Proof of $(b) \Rightarrow (a)$. Assume (b):
$\delta \leq \int K_{t+s}(x,y)dy = \iint K_t(x,z)K_s(z,y)dzdy \leq \int K_t(x,z)dz \;\leq$ 1.
Hence, $\lim_{t\to\infty} \int K_t(x,y)dy = w(x)$ exists and defines an L-harmonic function.

Proof of $(a) \Rightarrow (b)$. Assume (a). Then
$1 \geq \int K_t(x,y)\,dy \geq \int K_t(x,y)w(y)\,dy = w(x) \geq \delta > 0.$
Proof of $(b) \Rightarrow (c)$. Assume (a) or equivalently (b). By the perturbation formula,
$1 = \int P_t(x-y)dy =$
$\int K_t(x,y)dy + \int \int_0^t \int P_s(x,z)V(z)K_{t-s}(z,y)dz\,ds\,dy$
$\geq \int_0^t \int P_s(x,z)V(z)\delta\,dz\,ds$
Letting $t \to \infty$ we get $1 \geq \delta \cdot (-\Delta)^{-1}V(x) \geq 0$, which gives (c).
Proof of $(c) \Rightarrow (b)$. Assume $\|V\|_K = \|\Delta^{-1}V\|_{L^\infty} < \infty$.
Assume additionally that $\|V\|_K < 1$. Then
$1 = \int K_t(x,y)dy + \int \int_0^t \int P_s(x,z)V(z)K_{t-s}(z,y)dz\,ds\,dy$
$\leq \int K_t(x,y)dy + \int \int_0^t \int P_s(x,z)V(z)\,dz\,ds \leq$
$\int K_t(x,y)\,dy + \|V\|_K$, which implies (b).
The case $\|V\|_K \geq 1$ follows from the previous one by using the Feynman-Kac formula and the Hölder inequality. We omit details here.

Clearly (d) implies (b).
The proof that (b) implies (d) is more complex, however it uses only the perturbation formula and the Feynman-Kac formula and an iteration argument.

Remark. The equivalence (c) $\Leftrightarrow$ (d) is due to Yu.A. Semenov, *Stability of L^p-spectrum of generalized Schrödinger operators and equivalence of Green's functions*, IMRN 12 (1997), 573–593.

Step 3.
We are in a position to state our first main result:

Theorem (J.D. & J.Z.)

The following are equivalent
(i) the operator L admits an L-harmonic function
$0 < \delta \leq w(x) \leq 1$
(ii) $\|V\|_K = \sup_{x\in\mathbb{R}^d} \int V(y)|x-y|^{2-d}\,dy < \infty$
(iii) there is a function $0 < \delta \leq w(x) \leq 1$ *such that*
$H^1_L \ni f \mapsto w(x)f(x) \in H^1(\mathbb{R}^d)$ *is an isomorphism.*

Recent results.

Theorem (J.D. & J.Z.)

Assume that $\|V\|_K < \infty$, $d \geq 3$.
Then the operators

$$L^{1/2}(-\Delta)^{-1/2} \quad \textit{and} \quad (-\Delta)^{1/2}L^{-1/2}$$

are bounded on L^1.
Moreover,

$$H^1_L \ni f \mapsto (-\Delta)^{1/2}L^{-1/2}f \in H^1(\mathbb{R}^d)$$

is isomorphism of the Hardy spaces with the inverse

$$H^1(\mathbb{R}^d) \ni f \mapsto L^{1/2}(-\Delta)^{-1/2}f \in H^1_L.$$

Corollary (J.D. & J.Z.)

The Hardy space H^1_L admits characterization by the Riesz transforms $R_j = \partial_{x_j} L^{-1/2}$, that is, an L^1 function f belongs to the Hardy space H^1_L if and only if $R_j f \in L^1$.

Proof. Based on:

$$\underbrace{(\partial_{x_j}(-\Delta)^{-1/2})}_{(classical\ \ Riesz\ \ transform)} \underbrace{(-\Delta)^{1/2} L^{-1/2}}_{(isomorphism\ \ from\ \ H^1_L\ \ to\ \ H^1(\mathbb{R}^d))} = \underbrace{\partial_{x_j} L^{-1/2}}_{(Riesz\ \ transform\ \ for\ \ L)}$$

$$\underbrace{(\partial_{x_j} L^{-1/2})}_{(Riesz\ \ transform\ \ for\ \ L)} \underbrace{(L^{1/2}(-\Delta)^{-1/2})}_{(isomorphism\ \ from\ \ H^1(\mathbb{R}^d)\ \ to\ \ H^1_L)} = \underbrace{\partial_{x_j}(-\Delta)^{-1/2}}_{(classical\ \ Riesz\ \ transform)}$$

Assume that $R_j f \in L^1$. Then, by the first part of the theorem, there is $g \in L^1$ such that $f = L^{1/2}(-\Delta)^{-1/2} g$.
Hence, $L^1 \ni R_j f = (\partial_{x_j} L^{-1/2})(L^{1/2}(-\Delta)^{-1/2}) g = \partial_{x_j}(-\Delta)^{-1/2} g$.
Thus $g \in H^1(\mathbb{R}^d)$. By the second part of the theorem $f = L^{1/2}(-\Delta)^{-1/2} g \in H^1_L$.

Conversely, assume that $f \in H^1_L$. Then, by the second part of the theorem, $g = (-\Delta)^{1/2} L^{-1/2} f \in H^1(\mathbb{R}^d)$.
So, the Riesz transform characterization of the classical H^1 space

$$L^1 \ni \partial_{x_j} (-\Delta)^{-1/2} g$$

$$= \partial_{x_j} (-\Delta)^{-1/2} (-\Delta)^{1/2} L^{-1/2} f$$

$$= \partial_{x_j} L^{-1/2} f = R_j f$$

The case of compactly supported potentials in dimension 2.

Assume that V is a nonnegative nonzero compactly supported C^2-function in $\mathbb{R}^2$.

Theorem (J.D. & J.Z.)

There exists a regular L-harmonic weight w such that

$$C^{-1}\ln(2+|x|) \le w(x) \le C\ln(2+|x|),$$

$$|\nabla w(x)| \le C(1+|x|)^{-1},$$

and the space $H^1_{L,max}$ admits atomic decomposition with atoms satisfying: $\operatorname{supp} a \subset B$, $\|a\|_\infty \le |B|^{-1}$, $\int a(x)w(x)\,dx = 0$.

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Thank you

Aline – all the best to you!