Particle systems as solutions of SDEs systems

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Articles

- P. Graczyk, J. Malecki
  Multidimensional Yamada-Watanabe theorem and its applications to particle systems
  J. Math. Phys. 54(2013)
- P. Graczyk, J. Malecki
  Strong solutions of non-colliding particle systems,
  preprint (2014)
- P. Graczyk, J. Malecki
  Generalized Squared Bessel particle systems and Wallach set,
  preprint (2014).

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- We condition $(B_1, \ldots, B_p)$ to non-colliding

Conditioning to non-colliding

- Consider Vandermonde determinant

$$V(x_1, \ldots, x_p) = \Pi_{i<j}(x_j - x_i),$$

  - $V = 0$ iff some $x_i = x_j$ collide ($i \neq j$),
  - $V > 0$ when $x_1 < \ldots < x_p$,
  - V is Δ-harmonic

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  - $V > 0$ when $x_1 < \ldots < x_p$,
  - V is Δ-harmonic
- Denote by $(\lambda_1, \ldots, \lambda_p)$ the process $(B_1, \ldots, B_p)$ starting from

$$B_1(0) < \ldots < B_p(0),$$

conditioned using the Doob h-transform with $h = V$

Dyson Brownian Motion (1962)

- The system $(\lambda_1, \ldots, \lambda_p)$ starts from $\lambda_1(0) < \ldots \lambda_p(0)$.
- The first collision time

$$T_\Lambda = \inf\{t > 0 : \lambda_i(t) = \lambda_j(t) \text{ for some i} \neq j\}$$

  is infinite with prob. 1.
- The particles remain ordered

$$\lambda_1(t) < \ldots < \lambda_p(t)$$

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  i.e. the particles belong to the positive Weyl chamber
- The system $(\lambda_1, \dots, \lambda_p)$ satisfies

$$d\lambda_i(t) = dB_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt$$

- The repulsive drift terms $\frac{1}{\lambda_i - \lambda_j}$ prevent collisions, to which the martingale parts tend

β-Dyson Brownian Motion

β-Dyson BM is described for $\beta > 0$ by

$$d\lambda_i = dB_i + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt.$$

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- p-dim. BM conditioned not to collide is a Dyson BM with $\beta = 2$.
- A β-Dyson BM is non-colliding iff $\beta \geq 1$ (Rogers,Shi, 1993)
- For $\beta < 1$ the repulsion force $\frac{\beta}{\lambda_i - \lambda_j}$ is too little w.r. to the colliding martingales dB_i.

Non-colliding squared Bessel particles (Koenig, O'Connell, 2001)

- Let $(X_1, \ldots, X_p)$ be a system of independent BESQ processes on $\mathbb{R}^+$ with dimension $\alpha > 0$

$$dX_i = 2\sqrt{X_i}dB_i + \alpha dt, \quad i = 1, \ldots, p, \quad \alpha > 0.$$

starting from $X_i(0) > 0$.

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  starting from $X_i(0) > 0$.

- In such a system collisions happen with probability 1. The function

$$V(x_1, \ldots, x_p) = \Pi_{i<j}(x_j - x_i)$$

  is harmonic for the generator of $(X_1, \ldots, X_p)$

- By h-Doob transform $(h = V)$ we obtain a non-colliding squared Bessel particle system

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$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \left(\alpha + 2(p-1) + 2 \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt,$$

where $\lambda_1(0) < \ldots < \lambda_p(0)$.

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- It is a special case of a β-BESQ particle system:

$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \beta \left(\alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt.$$

Further motivation - Eigenvalues of matrix processes

- Let X_t be a Brownian Motion on the space of symmetric matrices Sym_p (stochastic Gaussian Orthogonal Ensemble)

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- Let X_t be a Brownian Motion on the space of symmetric matrices Sym_p (stochastic Gaussian Orthogonal Ensemble)
- The process X_t satisfies a matrix SDE

$$dX_t = \frac{1}{2}dW_t + \frac{1}{2}dW_t^T$$

where W_t is a $p \times p$ Brownian square matrix

Proposition

Let X_t be a stochastic matrix process on Sym_p and Λ_t its ordered eigenvalues, $\lambda_1(t) \leq \ldots \leq \lambda_p(t)$.
Suppose that X_t satisfies the SDE

$$dX_t = h(X_t)dW_t g(X_t) + g(X_t)dW_t^T h(X_t) + b(X_t)dt$$

where the functions $g, h, b : \mathbb{R} \to \mathbb{R}$ act spectrally on Sym_p.
If $\lambda_1(0) \leq \ldots \leq \lambda_p(0)$, then the process Λ_t is a semimartingale, satisfying for $t < T$=first collision time the SDEs system:

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dB_i + \left(b(\lambda_i) + \sum_{j \neq i} \frac{G(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt,$$

where $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$.

Back to Brownian Motion on Sym_p

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- If X_0 has no multiple eigenvalues:

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then the eigenvalue process Λ_t satisfies

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- If X_t is a BM on $Herm_p$(Stochastic UGE)
(W_t is a complex matrix BM, $dX_t = \frac{1}{2}dW_t + \frac{1}{2}dW_t^*$)
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- In both cases Λ_t is a Dyson Brownian Motion

Squared Bessel Matrix processes on Sym_p^+

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Open problems we solve

Consider a system of SDEs on the cone
$\overline{C_+} = \{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 \le x_2 \le \ldots \le x_p\}$

$$d\lambda_i = \sigma_i(\lambda_i) dB_i + \left(b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt$$
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We prove, when starting from $\lambda_1(0) \le \ldots \le \lambda_p(0)$
and under natural conditions on the coefficients σ_i, H_{ij}, b_i

- strong existence and pathwise unicity
- non-colliding of solutions of this system
- by methods of classical Itô calculus

Motivation for different H_{ij}

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- Important example when different H_{ij} appear:

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- Important example when different H_{ij} appear:

  Brownian particles with nearest neighbour repulsion
  $\sigma_i = 1, b_i = 0,$
  $H_{ij} = \gamma$ when $|i - j| = 1$ and zero otherwise

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- D. Lépingle(2010):
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- using the techniques of Multivalued SDEs

Main difficulty: singularities in SDEs

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- Recall the SDE for a Bessel process of dimension $\alpha > 0$ (index $\mu = \alpha/2 - 1$)

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- The singular drift $\frac{\alpha-1}{2X_t}$ is problematic, when $X_t = 0$.
- Multiplying by the indicator $1_{\{X_t \neq 0\}}$ practised in the literature

$$dX_t = dB_t + \frac{\alpha - 1}{2X_t} 1_{\{X_t \neq 0\}} dt$$

does not help!

1-dimensional Bessel processes

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- When $X_0 = 0$ uniqueness of solutions does not hold
- By Tanaka formula, pathwise uniqueness holds if we consider only non-negative $X_t \geq 0$

1-dimensional Squared Bessel processes

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- No more singularity in the drift part
- A non-Lipschitz function $\sqrt{x}$ in the martingale part
- The equation is solved by the Yamada-Watanabe theorem, allowing 1/2-Hölder coefficients in the martingale part

SDEs for non-colliding Squared Bessel processes

In equations for non-colliding BESQ particles

$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \beta \left(\alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt$$

both problems appear

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both problems appear

- non-Lipschitz functions $\sqrt{x}$ in martingale parts (Yamada-Watanabe th. is 1-dimensional!)
- The drift part contains singularities $(\lambda_i - \lambda_j)^{-1}$ (physicists want to start from $(0, \ldots, 0)$!)

Back to general problem

Solve the system of SDEs

$$d\lambda_i = \sigma_i(\lambda_i)dB_i + \left(b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt$$

$$i = 1, \ldots, p$$

on the cone

$$\overline{C_+} = \{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 \leq x_2 \leq \ldots \leq x_p\}$$

- the functions σ_i, b_i, H_{ij} are continuous

- the functions H_{ij} are non-negative and

$$H_{ij}(x,y) = H_{ji}(y,x), \quad x,y \in \mathbb{R}.$$

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$$H_{ij}(x, y) = H_{ji}(y, x), \quad x, y \in \mathbb{R}.$$

i.e. the particles push away one another with the same forces

$$\frac{H_{ij}(x, y)}{y - x}$$

(C1) there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_{0^+} \rho^{-1}(x)dx = \infty$ and that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho(|x-y|), \quad x, y \in \mathbb{R},\ i = 1, \ldots, p$$

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- the functions b_i are Lipschitz continuous

(C2) There exists $c > 0$ such that

$$\begin{aligned} \sigma_i^2(x) + b_i(x)x &\leq c(1 + |x|^2), \quad x \in \mathbb{R}, \\ H_{ij}(x, y) &\leq c(1 + |xy|), \quad x, y \in \mathbb{R} \end{aligned}$$

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(these are standard conditions which give finiteness of the solutions for every $t \geq 0$;
the sublinear growth of b_i can be replaced by non-positivity of $b_i(x)x$ for large x)

(A1) For every $i \neq j$ and $w < x < y < z$

$$\frac{H_{ij}(w,z)}{z-w} \leq \frac{H_{ij}(x,y)}{y-x}$$

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This is a crucial condition to prove the pathwise uniqueness of solutions by Tanaka formula

(A2) There exists $c_1 \geq 0$ such that for every $i \neq j$

$$\sigma_i^2(x) + \sigma_j^2(y) \leq c_1(x-y)^2 + 4H_{ij}(x,y)$$

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(A3) There exists $c_2 \geq 0$ such that for every $x < y < z$ and $i < j < k$

$$\begin{aligned} H_{ij}(x,y)(y-x) \quad + \quad & H_{jk}(y,z)(z-y) \leq \\ & c_2(z-y)(z-x)(y-x) + H_{ik}(x,z)(z-x) \end{aligned}$$

(A2) There exists $c_1 \geq 0$ such that for every $i \neq j$

$$\sigma_i^2(x) + \sigma_j^2(y) \leq c_1 (x-y)^2 + 4H_{ij}(x,y)$$

(drift part is appropriately bigger than the martingale part, to prevent collisions)

(A3) There exists $c_2 \geq 0$ such that for every $x < y < z$ and $i < j < k$

$$H_{ij}(x,y)(y-x) \quad + \quad H_{jk}(y,z)(z-y) \leq c_2 (z-y)(z-x)(y-x) + H_{ik}(x,z)(z-x)$$

(repulsion by exterior particles does not make collide interior particles)

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or, otherwise, such points x are isolated and

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(if $b_i(x) > b_j(x)$ then the particle x_i could catch up with the particle x_j thanks to the bigger drift force.)

Main result

Theorem (PG, J. Malecki, 2014)

If the conditions (C1), (C2) and (A1)-(A5) hold, then there exists a unique strong non-exploding solution $[\Lambda(t)]_{t\geq 0}$. The first collision time

$$T = \inf\{t > 0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j,\ i,j = 1,\ldots,p\}$$

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End the existence of a unique strong solution follows

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- The symmetry of $H_{ij}(x, y)$ implies that the singularities $\frac{1}{\lambda_i - \lambda_j}$ cancel!
- Analogous phenomenon occurs for other basic symmetric polynomials of $(\lambda_1, \ldots, \lambda_p)$

$$e_2 = \sum_{j>i} \lambda_j \lambda_i,$$
$$\ldots$$
$$e_p = \lambda_1 \cdot \ldots \cdot \lambda_p$$

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then the smooth function

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By the continuity of roots of a polynomial as functions of its coefficients, f extends to a continuous function

$$f : \overline{e(C_+)} \xrightarrow{1-1} \overline{C_+}$$

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where ($\bar{i}$ means that the i-th variable is omitted)

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and U_n are BMs such that
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- We show that λ_i never collide for $t > 0$

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We compute the SDEs for the semimartingales

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– the fact that (A4) guarantees the instant exit from a collision in a "degenerate point" x, $\sigma_k^2(x) + \sigma_l^2(x) + H_{kl}(x,x) = 0$.

End of Step 1: limit passage $s \to 0$

For every $t > s > 0$, by Itô formula

$$\lambda_i(t) - \lambda_i(s) = \int_s^t \sigma_i(\lambda_i(u))dB_i(u) + \int_s^t \left(b_i(\lambda_i(u)) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i(u), \lambda_j(u))}{\lambda_i(u) - \lambda_j(u)} \right) du$$

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When $s \to 0$, we have $\lambda_i(s) \to \lambda_i(0)$ and

$$\int_s^t \sigma_i(\lambda_i(u)) dB_i(u) \to \int_0^t \sigma_i(\lambda_i(u)) dB_i(u)$$

in L^2, so almost surely for a subsequence $s_k \to 0$.

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Gronwall Lemma ends the proof.

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We know (Bru, 1991) that for $\alpha \geq p-1$ the BESQ matrix(Wishart) processes exist on $\overline{\mathcal{S}ym_p^+}$.

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Intuitively, X_0 cannot be of rank superior to α: the process (X_t) evolves in rank α

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Theorem

(1) When $\alpha < p - 1$ and α is not integer, the BESQ matrix process cannot exist on $\overline{\mathcal{S}ym_p^+}$.
(2) When $\alpha \in \{0, 1, 2, \ldots, p-2\}$ is integer, and X_0 is of rank greater than α then the BESQ matrix process cannot exist on $\overline{\mathcal{S}ym_p^+}$.
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Comments:
– (1) gives a simple stochastic proof of the classical Wallach set
– (2) gives a simple stochastic proof of a result of Letac-Massam (based on ideas of J. Faraut), on non-central Wishart laws (unpublished yet)

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Look at e_4. This is a $BESQ(\alpha - 3)$ process starting from $\mathbb{R}^+$, with a time change $e_3(t)$.
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We repeat this argument and deduce that $e_1 = 0$ for t near 0.
This is however impossible because of the SDE for e_1. Its drift part $4\alpha dt$ is not 0.

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$e_2 = \sum_{1 \leq i < j \leq 4} \lambda_i \lambda_j = 0$ implies that $\lambda_2 = \lambda_3 = 0$, contradiction with $\mathrm{rank}(X_0) = 2$ or 3.