Particle systems as solutions of SDEs systems

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12 June 2014

Conference in honor of Aline Bonami
Harmonic Analysis, Probability and Applications
Orléans, June 2014
P. Graczyk, J. Malecki
Multidimensional Yamada-Watanabe theorem and its applications to particle systems

P. Graczyk, J. Malecki
Strong solutions of non-colliding particle systems,
preprint (2014)

P. Graczyk, J. Malecki
Generalized Squared Bessel particle systems and Wallach set,
Motivation: processes conditioned to non-colliding

Consider a system of $p$ Brownian particles $(B_1, \ldots, B_p)$, i.e. independent BM on $\mathbb{R}$.
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For each starting point $(B_1(0), \ldots, B_p(0)) \in \mathbb{R}^p$, the first collision time

$$T_B = \inf\{t > 0 : B_i(t) = B_j(t) \text{ for some } i \neq j\}$$

is finite with probability 1.
Consider a system of $p$ Brownian particles $(B_1, \ldots, B_p)$, i.e. independent BM on $\mathbb{R}$.

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is finite with probability 1.

We condition $(B_1, \ldots, B_p)$ to non-colliding
Consider Vandermonde determinant

\[ V(x_1, \ldots, x_p) = \prod_{i<j} (x_j - x_i), \]

- \( V = 0 \) iff some \( x_i = x_j \) collide (\( i \neq j \)),
- \( V > 0 \) when \( x_1 < \ldots < x_p \),
- \( V \) is \( \Delta \)-harmonic
Conditioning to non-colliding

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- Denote by \((\lambda_1, \ldots, \lambda_p)\) the process \((B_1, \ldots, B_p)\) starting from

\[ B_1(0) < \ldots < B_p(0), \]

conditioned using the Doob \( h \)-transform with \( h = V \)
The system \((\lambda_1, \ldots, \lambda_p)\) starts from \(\lambda_1(0) < \ldots \lambda_p(0)\).

The first collision time

\[ T_\Lambda = \inf \{ t > 0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j \} \]

is infinite with prob. 1.

The particles remain ordered

\[ \lambda_1(t) < \ldots < \lambda_p(t) \]

i.e. the particles belong to the positive Weyl chamber
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The system \((\lambda_1, \ldots, \lambda_p)\) satisfies

\[ d\lambda_i(t) = dB_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt \]

The repulsive drift terms \(\frac{1}{\lambda_i - \lambda_j}\) prevent collisions, to which the martingale parts tend
\( \beta \)-Dyson BM is described for \( \beta > 0 \) by

\[
d\lambda_i = dB_i + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt.
\]

- \( p \)-dim. BM conditioned not to collide is a Dyson BM with \( \beta = 2 \).
$\beta$-Dyson BM is described for $\beta > 0$ by

$$d\lambda_i = dB_i + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt.$$ 

- $p$-dim. BM conditioned not to collide is a Dyson BM with $\beta = 2$.
- A $\beta$-Dyson BM is non-colliding iff $\beta \geq 1$ (Rogers, Shi, 1993)
- For $\beta < 1$ the repulsion force $\frac{\beta}{\lambda_i - \lambda_j}$ is too little w.r. to the colliding martingales $dB_i$. 

Let \((X_1, \ldots, X_p)\) be a system of independent BESQ processes on \(\mathbb{R}^+\) with dimension \(\alpha > 0\)

\[
dX_i = 2\sqrt{X_i}dB_i + \alpha dt, \quad i = 1, \ldots, p, \quad \alpha > 0.
\]

starting from \(X_i(0) > 0\).
Let \((X_1, \ldots, X_p)\) be a system of independent BESQ processes on \(\mathbb{R}^+\) with dimension \(\alpha > 0\)

\[dX_i = 2\sqrt{X_i}dB_i + \alpha dt, \quad i = 1, \ldots, p, \quad \alpha > 0.\]

starting from \(X_i(0) > 0\).

In such a system collisions happen with probability 1. The function

\[V(x_1, \ldots, x_p) = \prod_{i<j}(x_j - x_i)\]

is harmonic for the generator of \((X_1, \ldots, X_p)\)

By \(h\)-Doob transform \((h = V)\) we obtain a non-colliding squared Bessel particle system
Non-colliding squared Bessel particles

Process \((\lambda_1, \ldots, \lambda_p)\) satisfies the following system of SDEs:

\[
    d\lambda_i = 2\sqrt{\lambda_i} dB_i + \left(\alpha + 2(p-1) + 2 \sum_{j \neq i} \lambda_i + \lambda_j \lambda_i - \lambda_j\right) dt,
\]

where \(\lambda_1(0) < \ldots < \lambda_p(0)\).

It is a special case of a \(\beta\)-BESQ particle system:

\[
    d\lambda_i = 2\sqrt{\lambda_i} dB_i + \beta \left(\alpha + \sum_{j \neq i} \lambda_i + \lambda_j \lambda_i - \lambda_j\right) dt.
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Process \((\lambda_1, \ldots, \lambda_p)\) satisfies the following system of SDEs:

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d\lambda_i = 2\sqrt{\lambda_i} dB_i + \beta \left( \alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt.
\]
Let $X_t$ be a Brownian Motion on the space of symmetric matrices $\text{Sym}_p$ (stochastic Gaussian Orthogonal Ensemble).
Further motivation - Eigenvalues of matrix processes

- Let $X_t$ be a Brownian Motion on the space of symmetric matrices $\text{Sym}_p$ (stochastic Gaussian Orthogonal Ensemble)
- The process $X_t$ satisfies a matrix SDE

$$dX_t = \frac{1}{2}dW_t + \frac{1}{2}dW_t^T$$

where $W_t$ is a $p \times p$ Brownian square matrix
Proposition

Let $X_t$ be a stochastic matrix process on $Sym_p$ and $\Lambda_t$ its ordered eigenvalues, $\lambda_1(t) \leq \ldots \leq \lambda_p(t)$. Suppose that $X_t$ satisfies the SDE

$$dX_t = h(X_t)dW_t g(X_t) + g(X_t)dW_t^T h(X_t) + b(X_t)dt$$

where the functions $g, h, b : \mathbb{R} \to \mathbb{R}$ act spectrally on $Sym_p$. If $\lambda_1(0) \leq \ldots \leq \lambda_p(0)$, then the process $\Lambda_t$ is a semimartingale, satisfying for $t < T=\text{first collision time}$ the SDEs system:

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)dB_i + \left( b(\lambda_i) + \sum_{j \neq i} \frac{G(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt,$$

where $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. 

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Particle systems as solutions of SDEs systems
Back to Brownian Motion on $\text{Sym}_p$

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Back to Brownian Motion on $\text{Sym}_p$

$$dX_t = \frac{1}{2} dW_t + \frac{1}{2} dW_t^T$$

- If $X_0$ has no multiple eigenvalues:
  
  $$\lambda_1(0) < \ldots < \lambda_p(0),$$

  then the eigenvalue process $\Lambda_t$ satisfies

  $$d\lambda_i = dB_i + \frac{1}{2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt, \quad i = 1, \ldots, p,$$
Back to Brownian Motion on $\text{Sym}_p$

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- If $X_t$ is a BM on $\text{Herm}_p$(Stochastic UGE)

  \( (W_t \text{ is a complex matrix BM, } dX_t = \frac{1}{2} dW_t + \frac{1}{2} dW_t^* ) \)

  we obtain

  \[ d\lambda_i = dB_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt, \quad i = 1, \ldots, p, \]

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Particle systems as solutions of SDEs systems
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  we obtain
  \[ d\lambda_i = dB_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt, \quad i = 1, \ldots, p, \]

- In both cases $\Lambda_t$ is a Dyson Brownian Motion
Squared Bessel Matrix processes on $\text{Sym}_p^+$

$$dX_t = \sqrt{X_t}dW_t + dW_t^T \sqrt{X_t} + \alpha ld t, \quad \alpha \geq p - 1.$$
Squared Bessel Matrix processes on $\text{Sym}_p^+$

$$dX_t = \sqrt{X_t}dW_t + dW_t^T \sqrt{X_t} + \alpha ld\tau, \quad \alpha \geq p - 1.$$  

- When $\alpha \in \mathbb{N}$, the process $X_t = N_t N_t^T$ with $N_t=$ Brownian Motion on $p \times \alpha$ matrices
Squared Bessel Matrix processes on $Sym_p^+$

$$dX_t = \sqrt{X_t} dW_t + dW_t^T \sqrt{X_t} + \alpha ld t, \quad \alpha \geq p - 1.$$ 

- When $\alpha \in \mathbb{N}$, the process $X_t = N_t N_t^T$ with $N_t =$ Brownian Motion on $p \times \alpha$ matrices
- Process $X_t$ is also called Wishart (Laguerre) process (Bru(1991), Koenig, O’Connell(2001), Matsumoto, Yor, Donati-Martin(2004) )
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- When $\alpha \in \mathbb{N}$, the process $X_t = N_t N_t^T$ with $N_t=$ Brownian Motion on $p \times \alpha$ matrices
- Process $X_t$ is also called Wishart (Laguerre) process (Bru(1991), Koenig, O’Connell(2001), Matsumoto, Yor, Donati-Martin(2004))
- If $X_0$ has no multiple eigenvalues,

$$d\lambda_i = 2\sqrt{\lambda_i}dB_i + \left(\alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}\right) dt$$
Consider a system of SDEs on the cone
\[ \overline{C_+} = \left\{ (x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 \leq x_2 \leq \ldots \leq x_p \right\} \]

\[
d\lambda_i = \sigma_i(\lambda_i) dB_i + \left( b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt \]

\[ i = 1, \ldots, p \]

We prove, when starting from \( \lambda_1(0) \leq \ldots \leq \lambda_p(0) \)
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We prove, when starting from \( \lambda_1(0) \leq \ldots \leq \lambda_p(0) \) and under natural conditions on the coefficients \( \sigma_i, H_{ij}, b_i \)
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We prove, when starting from \( \lambda_1(0) \leq \ldots \leq \lambda_p(0) \)

and under natural conditions on the coefficients \( \sigma_i, H_{ij}, b_i \)

- strong existence and pathwise unicity
Open problems we solve

Consider a system of SDEs on the cone
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- non-colliding of solutions of this system
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We prove, when starting from \(\lambda_1(0) \leq \ldots \leq \lambda_p(0)\) and under natural conditions on the coefficients \(\sigma_i, H_{ij}, b_i\)

- strong existence and pathwise unicity
- non-colliding of solutions of this system
- by methods of classical Itô calculus
Motivation for different $H_{ij}$

Important example when different $H_{ij}$ appear:

Brownian particles with nearest neighbour repulsion $\sigma_i = 1$, $b_i = 0$, $H_{ij} = \gamma$ when $|i - j| = 1$ and zero otherwise.
Motivation for different $H_{ij}$

- Important example when different $H_{ij}$ appear:

Brownian particles with nearest neighbour repulsion

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Motivation for different $H_{ij}$

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$\sigma_i = 1$, $b_i = 0$,

$H_{ij} = \gamma$ when $|i - j| = 1$ and zero otherwise
What was known on the existence of pathwise unique strong non-colliding solutions

WORLD CENTER OF THIS KNOWLEDGE: ORLEANS!
What was known on the existence of pathwise unique strong non-colliding solutions

E. Cépa and D. Lépingle (1997):
for Dyson Brownian Motions with linear drift
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- B. Schapira (2007), N. Demni (2009): for radial Dunkl and Heckman-Opdam SDEs, with more general singularities
- D. Lépingle (2010): for Squared Bessel particle systems with $\alpha > \frac{p}{2}$ and some other SDE systems using the techniques of Multivalued SDEs
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- using the techniques of Multivalued SDEs
Main difficulty: singularities in SDEs

Recall the SDE for a Bessel process of dimension $\alpha > 0$ (index $\mu = \alpha/2 - 1$)

$$dX_t = dB_t + \frac{\alpha - 1}{2} X_t^\mu dt.$$ 

The singular drift $\frac{\alpha - 1}{2} X_t^\mu$ is problematic, when $X_t = 0$.

Multiplying by the indicator $1_{\{X_t \neq 0\}}$ practised in the literature
does not help!

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Main difficulty: singularities in SDEs

- Recall the SDE for a Bessel process of dimension $\alpha > 0$ (index $\mu = \alpha/2 - 1$)

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- The singular drift $\frac{\alpha - 1}{2X_t}$ is problematic, when $X_t = 0$.
- Multiplying by the indicator $1_{\{X_t \neq 0\}}$ practised in the literature

$$dX_t = dB_t + \frac{\alpha - 1}{2X_t} 1_{\{X_t \neq 0\}} dt$$

does not help!
1-dimensional Bessel processes

\[ dX_t = dB_t + \frac{\alpha - 1}{2X_t} 1_{\{X_t \neq 0\}} dt \]

When \( X_0 = 0 \) uniqueness of solutions does not hold.

By Tanaka formula, pathwise uniqueness holds if we consider only non-negative \( X_t \geq 0 \).
1-dimensional Bessel processes

\[ dX_t = dB_t + \frac{\alpha - 1}{2X_t} \mathbf{1}_{\{X_t \neq 0\}} \, dt \]

- When \( X_0 = 0 \) uniqueness of solutions does not hold
1-dimensional Bessel processes

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1-dimensional Squared Bessel processes

\[ dX_t = 2\sqrt{X_t} dB_t + \alpha dt \]

- No more singularity in the drift part
1-dimensional Squared Bessel processes

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- No more singularity in the drift part
- A non-Lipschitz function \( \sqrt{x} \) in the martingale part
1-dimensional Squared Bessel processes

\[ dX_t = 2\sqrt{X_t} dB_t + \alpha dt \]

- No more singularity in the drift part
- A non-Lipschitz function \(\sqrt{x}\) in the martingale part
- The equation is solved by the Yamada-Watanabe theorem, allowing 1/2-Hölder coefficients in the martingale part

Particle systems as solutions of SDEs systems
In equations for non-colliding BESQ particles

\[ d\lambda_i = 2\sqrt{\lambda_i}dB_i + \beta \left( \alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt \]

both problems appear
In equations for non-colliding BESQ particles

\[ d\lambda_i = 2\sqrt{\lambda_i} dB_i + \beta \left( \alpha + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt \]

both problems appear

- non-Lipschitz functions \( \sqrt{x} \) in martingale parts
  (Yamada-Watanabe th. is 1-dimensional!)
- The drift part contains singularities \( (\lambda_i - \lambda_j)^{-1} \)
  (physicists want to start from \((0, \ldots, 0)!\) )
Solve the system of SDEs

\[ d\lambda_i = \sigma_i(\lambda_i)dB_i + \left( b_i(\lambda_i) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} \right) dt \]

\[ i = 1, \ldots, p \]

on the cone

\[ \overline{C_+} = \{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 \leq x_2 \leq \ldots \leq x_p\} \]
Assumptions on coefficients

General conditions

- the functions \( \sigma_i, b_i, H_{ij} \) are continuous

- the functions \( H_{ij} \) are non-negative and

\[
H_{ij}(x, y) = H_{ji}(y, x), \quad x, y \in \mathbb{R}.
\]
the functions $\sigma_i, b_i, H_{ij}$ are continuous

the functions $H_{ij}$ are non-negative and

$$H_{ij}(x, y) = H_{ji}(y, x), \quad x, y \in \mathbb{R}.$$  

i.e. the particles push away one another with the same forces

$$\frac{H_{ij}(x, y)}{y - x}$$
Assumptions on coefficients

Regularity conditions

\( \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that
\[
\int_0^\infty \rho^{-1}(x) \, dx = \infty
\]
and that
\[
|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho(|x - y|), \quad x, y \in \mathbb{R}, \quad i = 1, \ldots, p
\]
(the functions \( \sigma_i \) are at least \( \frac{1}{2} \) Hölder)

The functions \( b_i \) are Lipschitz continuous.
(C1) there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that
\[
\int_{0^+} \rho^{-1}(x)dx = \infty
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(C1) there exists a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

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and that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho(|x - y|), \quad x, y \in \mathbb{R}, \; i = 1, \ldots, p$$

(the functions $\sigma_i$ are at least $\frac{1}{2}$-Hölder)

- the functions $b_i$ are Lipschitz continuous
Assumptions on coefficients

Non-explosion conditions

\[ \sigma_i^2(x) + b_i(x)x \leq c(1 + |x|^2), \quad x \in \mathbb{R}, \]

\[ H_{ij}(x, y) \leq c(1 + |xy|), \quad x, y \in \mathbb{R}, \]

(These are standard conditions which give finiteness of the solutions for every \( t \geq 0; \) the sublinear growth of \( b_i \) can be replaced by non-positivity of \( b_i(x)x \) for large \( x \)).
Assumptions on coefficients
Non-explosion conditions

(C2) There exists $c > 0$ such that

$$\sigma_i^2(x) + b_i(x)x \leq c(1 + |x|^2), \quad x \in \mathbb{R},$$

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(these are standard conditions which give finiteness of the solutions for every $t \geq 0$; the sublinear growth of $b_i$ can be replaced by non-positivity of $b_i(x)x$ for large $x$).
(A1) For every $i \neq j$ and $w < x < y < z$

\[
\frac{H_{ij}(w, z)}{z - w} \leq \frac{H_{ij}(x, y)}{y - x}
\]
(A1) For every $i \neq j$ and $w < x < y < z$

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(Exterior particles interact less than the interior ones)
Assumptions on coefficients

A physical condition

(A1) For every \( i \neq j \) and \( w < x < y < z \)

\[
\frac{H_{ij}(w, z)}{z - w} \leq \frac{H_{ij}(x, y)}{y - x}
\]

(Exterior particles interact less than the interior ones)

This is a crucial condition to prove the pathwise uniqueness of solutions by Tanaka formula
(A2) There exists $c_1 \geq 0$ such that for every $i \neq j$

$$\sigma_i^2(x) + \sigma_j^2(y) \leq c_1(x - y)^2 + 4H_{ij}(x, y)$$
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\sigma_i^2(x) + \sigma_j^2(y) \leq c_1(x - y)^2 + 4H_{ij}(x, y)
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(drift part is appropriately bigger than the martingale part, to prevent collisions)
Assumptions on coefficients
Conditions for non-collisions

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(drift part is appropriately bigger than the martingale part, to prevent collisions)

(A3) There exists $c_2 \geq 0$ such that for every $x < y < z$ and $i < j < k$

$$H_{ij}(x, y)(y - x) + H_{jk}(y, z)(z - y) \leq c_2(z - y)(z - x)(y - x) + H_{ik}(x, z)(z - x)$$
Assumptions on coefficients

Conditions for non-collisions

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(repulsion by exterior particles does not make collide interior particles)
(A4) \( \sigma_k^2(x) + \sigma_i^2(x) + H_{kl}(x, x) \neq 0 \)
(A4) \[ \sigma_k^2(x) + \sigma_i^2(x) + H_{kl}(x, x) \neq 0 \]

or, otherwise, such points \( x \) are isolated and

\[
\sum_{i=k}^{l} \left( b_i(x) + \sum_{j=1}^{p-2} \frac{H_{ij}(x, y_j)}{x - y_j} 1_{\mathbb{R}\setminus\{x\}}(y_j) \right) \neq 0,
\]

for every \( y_1, \ldots, y_{p-2} \in \mathbb{R} \).
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(in each collision point $x$ there is a force making the particles leave from it).
Assumptions on coefficients

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(in each collision point \(x\) there is a force making the particles leave from it).

\[(A5) \quad \text{If } i < j \text{ then } b_i(x) \leq b_j(x) \text{ for all } x \in \mathbb{R}.
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Conditions for non-collisions

\[(A4) \quad \sigma^2_k(x) + \sigma^2_i(x) + H_{kl}(x, x) \neq 0\]

or, otherwise, such points \(x\) are isolated and

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\]

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(in each collision point \(x\) there is a force making the particles leave from it).

\[(A5) \quad \text{If } i < j \text{ then } b_i(x) \leq b_j(x) \text{ for all } x \in \mathbb{R}.\]

(if \(b_i(x) > b_j(x)\) then the particle \(x_i\) could catch up with the particle \(x_j\) thanks to the bigger drift force.)
Main result

Theorem (PG, J. Malecki, 2014)

If the conditions (C1), (C2) and (A1)-(A5) hold, then there exists a unique strong non-exploding solution $[\Lambda(t)]_{t \geq 0}$. The first collision time

$$T = \inf\{t > 0 : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j, i, j = 1, \ldots, p\}$$

is infinite almost surely.

Applications:
- General Brownian NC (non-colliding) particle systems (e.g. neighbor interaction)
- BESQ NC-particle systems for $\alpha \geq p-1$
- Generalized BESQ NC-particle systems (for $\alpha < p-1$)
- General trigonometric and hyperbolic particle systems

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Particle systems as solutions of SDEs systems
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- General trigonometric and hyperbolic particle systems
Scheme of the proof

Step 1 Assuming (A2)-(A5) we show that there exists a weak solution \( \Lambda(t) \), having no collisions after the start.

Tools for existence: SDEs for basic symmetric polynomials \( e_1, \ldots, e_p \), via Itô formula. Solving them; defining \( \Lambda_t = \Lambda(\mathcal{e}_t) \).

Proving non-collisions of \( \Lambda_t \).

Limit passage \( \Lambda_s \to \Lambda_0 \).

Tools for non-collisions: symmetric polynomials in \( (\lambda_i - \lambda_j)^2 \).

McKean argument (non-explosion of \( U = \ln \mathcal{V}, \mathcal{V} = \mathcal{V} \) Vandermond determinant).

Step 2 Assuming conditions (C1) and (A1) we show the pathwise uniqueness of the solutions.

Tool: Tanaka formula.

We get much more: Yamada-Watanabe theorem in dimension \( p \).

The existence of a unique strong solution follows.
Scheme of the proof

Step 1 Assuming (A2)-(A5) we show that there exists a weak solution $\Lambda(t)$, having no collisions after the start.
Scheme of the proof

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We get much more:
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End  the existence of a unique strong solution follows
Consider $e_1 = \lambda_1 + \ldots + \lambda_p$. 
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$$de_1 = \left( \sum_i \sigma_i (\lambda_i)^2 \right)^{\frac{1}{2}} dW_t + \sum_i b_i (\lambda_i) dt$$

for a 1-dimensional BM $W_t$. 

\[ e_2 = \sum_{i > j} \lambda_j \lambda_i, \quad \ldots \quad e_p = \lambda_1 \cdots \lambda_p. \]
Consider $e_1 = \lambda_1 + \ldots + \lambda_p$.

It is easy to see that

$$de_1 = \left( \sum_i \sigma_i(\lambda_i)^2 \right)^{\frac{1}{2}} dW_t + \sum_i b_i(\lambda_i) dt$$

for a 1-dimensional BM $W_t$.

- The symmetry of $H_{ij}(x, y)$ implies that the singularities $\frac{1}{\lambda_i - \lambda_j}$ cancel!
Idea of the proof of weak existence (Step 1)

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- The symmetry of $H_{ij}(x, y)$ implies that the singularities $\frac{1}{\lambda_i - \lambda_j}$ cancel!
- Analogous phenomenon occurs for other basic symmetric polynomials of $(\lambda_1, \ldots, \lambda_p)$

$$e_2 = \sum_{j > i} \lambda_j \lambda_i,$$

$$e_p = \lambda_1 \ldots \lambda_p$$
Idea of the proof of weak existence (Step 1)

If we restrict the arguments to the open set $C^+ = \{(x_1,...,x_p) \in \mathbb{R}^p : x_1 < x_2 < ... < x_p}\}$ then the smooth function $e = (e_1,...,e_p) : C^+ \rightarrow \mathbb{R}^p$ is one-to-one. Thus $e$ is a diffeomorphism between $C^+$ and $e(C^+)$ which is open.

Denote the inverse diffeomorphism by $f = (f_1,...,f_p) : e(C^+) \rightarrow C^+$. By the continuity of roots of a polynomial as functions of its coefficients, $f$ extends to a continuous function $f : e(C^+) \rightarrow C^+$. 

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Particle systems as solutions of SDEs systems
Idea of the proof of weak existence (Step 1)

If we restrict the arguments to the open set

\[ C_+ = \{(x_1, \ldots, x_p) \in \mathbb{R}^p : x_1 < x_2 < \ldots < x_p\} \]

then the smooth function

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\((-1)^k e_k(X)\) is the coefficient of \(x^{p-k}\) in \(P(x) = \prod_{i=1}^{p} (x - x_i)\)

(\(e\) is a diffeomorphism between \(C_+\) and \(e(C_+)\) which is open)

Denote the inverse diffeomorphism by \(f = (f_1, \ldots, f_p) : e(C_+) \rightarrow C_+\)

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By the continuity of roots of a polynomial as functions of its coefficients, \(f\) extends to a continuous function

\[ f : \overline{e(C_+)} \xrightarrow{1-1} \overline{C_+} \]
Using Itô formula and the diffeomorphism $f$, we compute SDEs

\[ y_n = e_n(\Lambda_t) \]
\[ dy_n = a_n(y_1, \ldots, y_p) dU_n + q_n(y_1, \ldots, y_p) dt, \]

where ($i$ means that the $i$-th variable is omitted)

\[ a_n(y) = \left( \sum_{i=1}^{p} \sigma_i^2(f_i(y)) \right)^{1/2}, y \in C^+, \]
\[ q_n(y) = \sum_{i=1}^{p} b_i(f_i(y)) e_i n - 2(e_i, f_j(y)) H_{ij}(f_i(y), f_j(y)). \]

and $U_n$ are BMs such that

\[ \langle a_n dU_n, a_m dU_m \rangle = \sum_{i=1}^{p} \sigma_i^2(f_i(y)) e_i n - 1(f_i(y)) e_j m - 1(f_j(y)). \]
Using Itô formula and the diffeomorphism $f$, we compute SDEs for $y_n = e_n(\Lambda_t)$
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dy_n = a_n(y_1, \ldots, y_p)du_n + q_n(y_1, \ldots, y_p)dt, \quad n = 1, \ldots, p,
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Idea of the proof of weak existence (Step 1)

Using Itô formula and the diffeomorphism $f$, we compute SDEs for $y_n = e_n(\Lambda_t)$

$$dy_n = a_n(y_1, \ldots, y_p) dU_n + q_n(y_1, \ldots, y_p) dt, \quad n = 1, \ldots, p,$$

where ($\bar{i}$ means that the $i$-th variable is omitted)

$$a_n(y) = \left( \sum_{i=1}^{p} \sigma_i^2(f_i(y))(e_{n-1}^\bar{i}(f(y)))^2 \right)^{1/2}, \quad y \in \overline{C_+},$$

$$q_n(y) = \sum_{i=1}^{p} b_i(f_i(y)) e_{n-1}^\bar{i}(f(y)) - \sum_{i<j} e_{n-2}^{\bar{i}, \bar{j}}(f(y)) H_{ij}(f_i(y), f_j(y))$$
Idea of the proof of weak existence (Step 1)

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$$

$$
q_n(y) = \sum_{i=1}^p b_i(f_i(y))e_{n-1}^i(f(y)) - \sum_{i<j} e_{n-2}^{i,j}(f(y))H_{ij}(f_i(y), f_j(y))
$$

and $U_n$ are BMs such that

$$
\langle a_n dU_n, a_m dU_m \rangle = \sum_{i=1}^p \sigma_i^2(f_i(y))(e_{n-1}^i(f(y)))e_{m-1}^j(f(y)).
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Example: BESQ particle systems, $p = 4$

\[
\begin{align*}
\text{de}_1 &= 2 \sqrt{e_1} \text{d}U_1 + 4 \alpha \text{d}t \\
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Back to the general proof:

The SDEs for $e_n$ are not sensible to the start from a collision (they do not have singularities in the drift term). We solve them on $e(C + )$. We define $\Lambda = f(e_1, \ldots, e_p)$ and show that $\lambda_i$ never collide for $t > 0$. 

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Idea of the proof of non-collisions (Step 1)

We compute the SDEs for the semimartingales

\[ V_1 = \sum_{j > i} (\lambda_i - \lambda_j)^2 \]
\[ \vdots \]
\[ V_N = \prod_{j > i} (\lambda_j - \lambda_i)^2, \quad N = p(p+1)/2 \]

they are symmetric polynomials of

\[ (\lambda_i - \lambda_j)^2 \]

We show that even if \( V_N(0) = 0 \), if (A4) holds then

\[ \tau_N = \inf \{ t > 0 : V_N(t) > 0 \} = 0 \]

almost surely (instant diffraction)

The proof is based on:

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For every $t > s > 0$, by Itô formula

\[
\lambda_i(t) - \lambda_i(s) = \int_s^t \sigma_i(\lambda_i(u)) dB_i(u) + \int_s^t \left( b_i(\lambda_i(u)) + \sum_{j \neq i} \frac{H_{ij}(\lambda_i(u), \lambda_j(u))}{\lambda_i(u) - \lambda_j(u)} \right) du
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When $s \to 0$, we have $\lambda_i(s) \to \lambda_i(0)$ and

$$
\int_s^t \sigma_i(\lambda_i(u))dB_i(u) \to \int_0^t \sigma_i(\lambda_i(u))dB_i(u)
$$

in $L^2$, so almost surely for a subsequence $s_k \to 0$. 
Step 2: pathwise uniqueness

Let $(\Lambda, B)$ and $(\tilde{\Lambda}, B)$ be two solutions with $\Lambda(0) = \tilde{\Lambda}(0)$. Local time of $\lambda_i - \tilde{\lambda}_i$ at 0 is 0. By Tanaka formula

$$p\sum_{i=1} E|\lambda_i(t) - \tilde{\lambda}_i(t)| = E\int_0^t p\sum_{i=1} \text{sgn}(\lambda_i - \tilde{\lambda}_i) \sum_{j \neq i} (H_{ij}(\lambda_i, \lambda_j) \lambda_i - H_{ij}(\tilde{\lambda}_i, \tilde{\lambda}_j) \tilde{\lambda}_i - \tilde{\lambda}_j) du + E\int_0^t p\sum_{i=1} \text{sgn}(\lambda_i - \tilde{\lambda}_i) (b_i(\lambda_i) - b_i(\tilde{\lambda}_i)) du.$$ 

Lipschitz condition on $b_i(x)$ implies that the second term $\leq c_3 E\int_0^t \sum_{i=1} |\lambda_i - \tilde{\lambda}_i| du$. Assumption (A1) ensures that the first term is non-positive!!! Gronwall Lemma ends the proof.
Let $(\Lambda, B)$ and $(\tilde{\Lambda}, B)$ be two solutions with $\Lambda(0) = \tilde{\Lambda}(0)$. Local time of $\lambda_i - \tilde{\lambda}_i$ at 0 is 0. By Tanaka formula

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dX_t = \sqrt{X_t} dB_t + dB_T_t \sqrt{X_t + \alpha I} dt, \quad \alpha \geq p - 1.
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By quadratic construction it is straightforward to see that they exist also for $\alpha = 1, 2, \ldots, p - 2$.

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Particle systems as solutions of SDEs systems
Theorem

(1) When $\alpha < p - 1$ and $\alpha$ is not integer, the BESQ matrix process cannot exist on $\text{Sym}_p^+$. 
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Comments:
– (1) gives a simple stochastic proof of the classical Wallach set
– (2) gives a simple stochastic proof of a result of Letac-Massam (based on ideas of J. Faraut), on non-central Wishart laws (unpublished yet).

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Proof of (1), Example $p = 4$

Suppose a "true" BESQ matrix process exists for $\alpha < 3, \alpha \not\in \mathbb{N}$.

de_1 = 2 \sqrt{e_1} dU_1 + 4 \alpha dt
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Look at $e_4$. This is a BESQ $(\alpha - 3)$ process starting from $\mathbb{R}^+$. 

If $e_3(t) \neq 0$, then $e_4$ cannot live on $\mathbb{R}^+$ as a BESQ $(\alpha - 3)$ process with $\alpha - 3 < 0$. Thus $e_3(t) = 0$ for $t$ near 0.

Look at the SDE for $e_3$. We infer that $e_2 = 0$ for $t$ near 0, otherwise the drift part of $e_3$ would be equal to its martingale part.

We repeat this argument and deduce that $e_1 = 0$ for $t$ near 0.

This is however impossible because of the SDE for $e_1$. Its drift part $4\alpha dt$ is not 0.
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Look at $e_4$. This is a $BESQ(\alpha - 3)$ process starting from $\mathbb{R}^+$, with a time change $e_3(t)$.
If $e_3(t) \neq 0$, then $e_4$ cannot live on $\mathbb{R}^+$ as a $BESQ(\alpha - 3)$ process with $\alpha - 3 < 0$. Thus $e_3(t) = 0$ for $t$ near 0.

Look at the SDE for $e_3$. We infer that $e_2 = 0$ for $t$ near 0, otherwise the drift part of $e_3$ would be equal to its martingale part.
Proof of (1), Example $p = 4$

Suppose a "true" BESQ matrix process exists for $\alpha < 3$, $\alpha \notin \mathbb{N}$.

$$de_1 = 2\sqrt{e_1} dU_1 + 4\alpha dt$$
$$de_2 = 2\sqrt{e_1 e_2} + 3e_3 dU_2 + 3(\alpha - 1)e_1 dt$$
$$de_3 = 2\sqrt{e_3 e_2} + 6e_1 e_4 dU_3 + 2(\alpha - 2)e_2 dt$$
$$de_4 = 2\sqrt{e_4 e_3} dU_4 + (\alpha - 3)e_3 dt$$

Look at $e_4$. This is a $BESQ(\alpha - 3)$ process starting from $\mathbb{R}^+$, with a time change $e_3(t)$.

If $e_3(t) \neq 0$, then $e_4$ cannot live on $\mathbb{R}^+$ as a $BESQ(\alpha - 3)$ process with $\alpha - 3 < 0$. Thus $e_3(t) = 0$ for $t$ near 0.

Look at the SDE for $e_3$. We infer that $e_2 = 0$ for $t$ near 0, otherwise the drift part of $e_3$ would be equal to its martingale part.

We repeat this argument and deduce that $e_1 = 0$ for $t$ near 0.
Proof of (1), Example \( p = 4 \)

Suppose a ”true” BESQ matrix process exists for \( \alpha < 3, \alpha \not\in \mathbb{N} \).

\[
\begin{align*}
  de_1 &= 2\sqrt{e_1}dU_1 + 4\alpha dt \\
  de_2 &= 2\sqrt{e_1e_2 + 3e_3}dU_2 + 3(\alpha - 1)e_1 dt \\
  de_3 &= 2\sqrt{e_3e_2 + 6e_1e_4}dU_3 + 2(\alpha - 2)e_2 dt \\
  de_4 &= 2\sqrt{e_4e_3}dU_4 + (\alpha - 3)e_3 dt
\end{align*}
\]

Look at \( e_4 \). This is a \( BESQ(\alpha - 3) \) process starting from \( \mathbb{R}^+ \), with a time change \( e_3(t) \).
If \( e_3(t) \neq 0 \), then \( e_4 \) cannot live on \( \mathbb{R}^+ \) as a \( BESQ(\alpha - 3) \) process with \( \alpha - 3 < 0 \). Thus \( e_3(t) = 0 \) for \( t \) near 0.

Look at the SDE for \( e_3 \). We infer that \( e_2 = 0 \) for \( t \) near 0, otherwise the drift part of \( e_3 \) would be equal to its martingale part.

We repeat this argument and deduce that \( e_1 = 0 \) for \( t \) near 0.
This is however impossible because of the SDE for \( e_1 \). Its drift part \( 4\alpha dt \) is not 0.
Proof of (2), Example $p = 4, \alpha = 1,$

$\Lambda_0 = \text{diag}(0, 0, \lambda_3 > 0, \lambda_4)$ or $(0, \lambda_2 > 0, \lambda_3, \lambda_4)$
Proof of (2), Example $p = 4, \alpha = 1,$
$\Lambda_0 = diag(0, 0, \lambda_3 > 0, \lambda_4)$ or $(0, \lambda_2 > 0, \lambda_3, \lambda_4)$

The argument is identical as in the proof of (1), but stops on the level

$$e_2 = 0$$
Proof of (2), Example $p = 4, \alpha = 1,$
$\Lambda_0 = \text{diag}(0, 0, \lambda_3 > 0, \lambda_4)$ or $(0, \lambda_2 > 0, \lambda_3, \lambda_4)$

The argument is identical as in the proof of (1), but stops on the level

$$e_2 = 0$$

(The argument cannot go down to $e_1$ because the drift $(\alpha - 1)dt$ of $e_2$ is 0 for $\alpha = 1$.)
Proof of (2), Example $p = 4, \alpha = 1,$
$\Lambda_0 = diag(0,0,\lambda_3 > 0, \lambda_4)$ or $(0, \lambda_2 > 0, \lambda_3, \lambda_4)$

The argument is identical as in the proof of (1), but stops on the level

$$e_2 = 0$$

(The argument cannot go down to $e_1$ because the drift $(\alpha - 1) dt$ of $e_2$ is 0 for $\alpha = 1$.)

$e_2 = \sum_{1\leq i<j\leq 4} \lambda_i \lambda_j = 0$ implies that $\lambda_2 = \lambda_3 = 0$, contradiction with $\text{rank}(X_0) = 2$ or 3.