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Spectral decay of the sinc kernel operator and approximations by Prolate Spheroidal Wave Functions.

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In collaboration with Aline Bonami)

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Outline

1. PSWFs and Properties
   - Historical origins of the PSWFs
   - Some Properties of the PSWFs
   - A General Framework of the PSWFs (the DOS)
2. Spectral behaviour and decay rate of the eigenvalues $\lambda_n(c)$
   - Some classical results
   - Uniform estimate of the PSWFs by WKB method
   - New sharp decay rate of the eigenvalues $\lambda_n(c)$.
3. Some applications of the PSWFs
   - Approximation of almost bandlimited and and almost timelimited functions
   - PSWFs based spectral approximation in Sobolev spaces.
   - Exact reconstruction of band-limited functions with missing data

The origins of the PSWFs go back to the 1880's [Niven (1880)]. The Spheroidal coordinates are given by

$$\begin{aligned} x &= a\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \quad y = a\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \\ z &= a\xi\eta, \quad \xi > 1 \quad \eta \in [-1,1], \quad \phi \in [0,2\pi]. \end{aligned}$$

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The Helmotz Wave equation $\Delta\Phi + k^2\Phi = 0$ in spheroidal coordinates with a solution of the form

$$\Phi(\xi,\eta,\phi) = R_{mn}(c,\xi)S_{mn}(c,\eta) \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi, \quad c = \frac{1}{2}ak$$

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$$(1-x^2)\frac{d^2\psi_{n,c}(x)}{dx^2} - 2x\frac{d\psi_{n,c}(x)}{dx} + (\chi_n(c) - c^2x^2)\psi_{n,c}(x) = 0.$$

D. Slepian and H. Pollack uncertainty principle

In 1960's, a breakthrough in the area of the PSWFs has been made by Slepian, Pollack and landau. They have shown that if $\tau, \omega \in R_+^*$ and

$$B_\omega = \{f \in L^2(R); Supp^t \widehat{f} \subseteq [-\omega, \omega]\},$$

and if a practical measure of a signal concentration in B_ω is given by:

$$\alpha^2(\tau) = \frac{\|f\|_{2,\tau}^2}{\|f\|_2^2} \qquad \|f\|_{2,\tau}^2 = \int_{-\tau}^{\tau} |f(t)|^2 \, dt.$$

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$$\alpha^2(\tau) \text{ is maximum} \iff \int_{-\omega}^{\omega} \frac{\sin 2\pi\tau(x-y)}{\pi(x-y)} \widehat{f}(y)\, dy = \alpha^2(\tau)\widehat{f}(x),\ |x| \leq \omega.$$

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Properties of the PSWFs and their eigenvalues

$$\rho(\mathcal{Q}_c) = \{\lambda_n(c),\, n \in N;\; 1 > \lambda_0(c) > \lambda_1(c) > \cdots \lambda_n(c) > \cdots\}.$$

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If $\psi_{n,c}$ denotes the eigenfunction associated with $\lambda_n(c)$, then $\{\psi_{n,c},\ n \in \mathbf{N}\}$ is an orthogonal basis of $L^2[-1,1]$, an orthonormal basis of B_c. Thus an orthonormal system of $L^2(\mathbf{R})$.

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$$\widehat{\psi}_{n,c}(\xi) = (-i)^n \sqrt{\frac{2\pi}{c\,\lambda_n}}\, \psi_{n,c}\left(\frac{\xi}{c}\right)\ 1_{[-c,c]}(\xi).$$

The General Framework of the PSWFs

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Theorem (Seip (1991))

Let $(f_k)_{k\in\mathbf{N}}$ be an orthonormal basis of V. Then $(f_k)_{k\in\mathbf{N}}$ is furthermore orthogonal for the induced scalar product $< \cdot, \cdot >_A$ if and only if f_k are singular function of $\mathcal{P}\mathcal{T}$.

For the special case $\mathcal{H} = L^2(\mathbf{R})$, $A = [-1, 1]$, $V = B_c$, the Paley-Wiener space of $c-$band-limited functions. Then, V is a RKHS with kernel

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Remark

If $Lf(x) = \frac{d}{dx}[P(x)f'(x)] + \gamma(x)f(x),\ x \in [-1, 1]$, *with* $P(\pm 1) = 0$, *then* $F_c L = L F_c$ *if and only if* $P(x) = 1 - x^2$ *and* $\gamma(x) = -c^2x^2$.

Some motivations of this decay rate study

Many Applications of the PSWFs heavily rely on the decay rate of the $\lambda_n(c)$. For example

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• Performance of MIMO Systems in wireless network and under a Line-of-Sight Environment, [Desgroseilliers, Lévèque, Preissmann (2013)].

Behavior of the $\lambda_n(c)$

Theorem (Landau, Widom (1980))

$\forall c > 0,\ \forall\, 0 < \alpha < 1,\ N(\alpha) = \#\{\lambda_i(c);\ \lambda_i(c) > \alpha\}$ *is given by*

$$N(\alpha) = \frac{2c}{\pi} + \left[\frac{1}{\pi^2}\log\left(\frac{1-\alpha}{\alpha}\right)\right]\log(c) + o(\log(c)),$$

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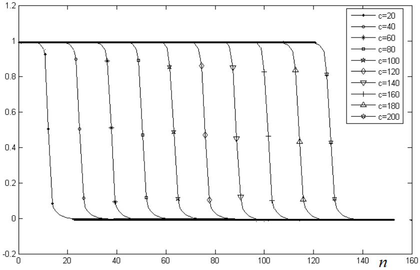

Figure : Graph of the $\lambda_n(c)$ for different values of c and n

D. Slepian decay rate of the $\lambda_n(c)$

From the Slepian's equality [Slepian (1964)], $\lambda_n(c) = \lambda'_n \times \lambda''_n$, with

$$\lambda'_n = \frac{c^{2n+1}(n!)^4}{2((2n)!)^2(\Gamma(n+3/2))^2} \tag{1}$$

$$\lambda''_n = \exp\left(2\int_0^c \frac{(\psi_{n,\tau}(1))^2 - (n+1/2)}{\tau}\, d\tau\right). \tag{2}$$

one gets for $q = c^2/\chi_n \leq \alpha < 1$, and a constant M_α

$$\lambda'_n \leq \frac{Kc}{n}\left(\frac{ec}{4n}\right)^{2n}, \quad \lambda''_n \leq e^{2M_\alpha(1+c^2/n)},$$

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$$K \sim \frac{8e^{-\gamma-1}}{3\sqrt{\pi}} e^{7\pi^2/72}, \quad \gamma \quad \text{is the Euler constant.}$$

H. Widom decay rate of the $\lambda_n(c)$ [Widom (1964)]

If $q_n = \frac{c^2}{\chi_n} < 1$, then

$$\lambda_n(c) = e^{-2\sqrt{\chi_n}\log\left(\frac{4\sqrt{\chi_n}}{ec}\right) + O\left(\frac{c^2}{n}\log(n/c)\right)}\left(1 + O(n^{-1}\log n)\right).$$

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The above estimate of the $\lambda_n(c)$ is a consequence of an involved asymptotic behaviour of the function $f(x) = xe^{-cx}\psi_{n,c}(x)$ with $\psi_{n,c}(1) = 1$, combined with the equality

$$\lim_{x\to+\infty} xe^{-cx}\psi_{n,c}(x) = \frac{1}{c\mu_n(c)}.$$

Uniform estimates of the PSWFs.

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Suppose that $n \geq 2$ is a non-negative integer.

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- *If $n > (2c/\pi) + 1$, then $\chi_n < c^2$.*
- *If $(2c/\pi) - 1 < n < (2c/\pi)$, then either inequality is possible.*

Uniform estimate of the PSWFs by WKB method

Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\psi'(x)\right]+\chi_n(1-qx^2)\psi(x)=0, \quad x\in[-1,1].$

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Lemma (Bonami, K. (2014))

For $q < 1$, there exists a function $F(\cdot)$ that is continuous on $[0, S(0)]$, satisfying $|F(S(x))| \leq \frac{3+2q}{4}\frac{1}{(1-qx^2)^2}, \quad x \in [0,1]$ and such that U is a solution of the equation

$$U''(s) + \left(\chi_n + \frac{1}{4s^2}\right)U(s) = F(s)U(s), \quad s \in [0, S(0)]. \qquad (3)$$

Main Estimation Theorem

Let $\mathbf{K}(\eta) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\eta^2 t^2)}}$, $\quad \mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt, 0 \leq k \leq 1.$

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Theorem (Bonami, K. (2014))

There exists a constant C_1 such that, when n, c are such that $(1-q)\sqrt{\chi_n(c)} > 3.5E(\sqrt{q})$, we have, for $0 \leq x \leq 1$

$$\psi_{n,c}(x) = \sqrt{\frac{\pi}{2\mathbf{K}(\sqrt{q})}} \frac{\chi_n(c)^{1/4}\sqrt{S_q(x)}J_0(\sqrt{\chi_n(c)}S_q(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} + \widetilde{R}_{n,c}(x) \qquad (4)$$

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$$|\widetilde{R}_{n,c}(x)| \leq \frac{C_1}{(1-q)\sqrt{\chi_n}} \sqrt{\frac{1}{K(\sqrt{q})}} \min\left(\chi_n^{1/4}, (1-x^2)^{-1/4}(1-qx^2)^{-1/4}\right). \quad (5)$$

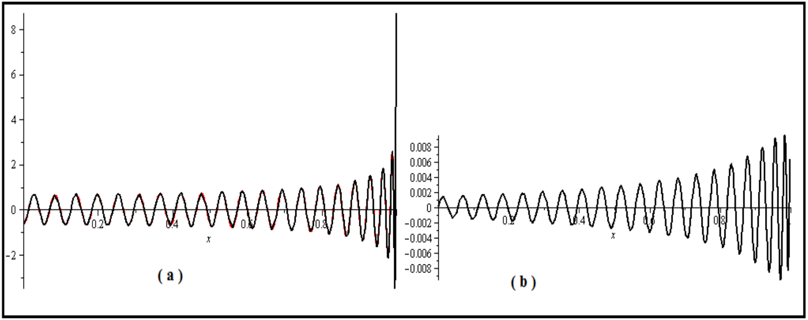

Figure : (a) Graphs of the ψ_n (black), and its WKB approximant (Red), $c = 100$, $n = 80$. (b) Graph the corresponding approximation errors.

Useful bounds of the χ_n

Lemma (Bonami, K. (2014))

For all $c > 0$ *and* $n \geq 2$ *we have*

$$\Phi\left(\frac{2c}{\pi(n+1)}\right) < \frac{c}{\sqrt{\chi_n}} < \Phi\left(\frac{2c}{\pi n}\right), \tag{6}$$

where Φ *is the inverse of the function* $k \mapsto \frac{k}{\mathbf{E}(k)} = \Psi(k),\ 0 \leq k \leq 1.$

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$$\Phi'(x) \geq 0, \quad x \leq \Phi(x) \leq \frac{\pi}{2}x, \quad 0 \leq x \leq 1.$$

As a consequence of the previous lemma

$$\frac{\pi n}{2\mathbf{E}(\sqrt{q})} < \sqrt{\chi_n} < \frac{\pi(n+1)}{2\mathbf{E}(\sqrt{q})}.$$

For $n \geq 2$ and $q < 1$, we have

$$(1-q)\sqrt{\chi_n} \geq \frac{(n - \frac{2c}{\pi}) - e^{-1}}{\log n + 5},$$

A further improvement of the previous inequality is given by the following lemma:

Lemma

Let $n \geq 3$, $q < 1$ and $\kappa \geq 4$. Then one of the following conditions ,

$$c \leq n - \kappa, \tag{7}$$

$$\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 9), \tag{8}$$

implies the inequality

$$(1-q)\sqrt{\chi_n(c)} > \kappa.$$

Moreover, if we assume already that $c > \frac{n+1}{2}$, then the condition $\frac{\pi n}{2} - c > \frac{\kappa}{4}(\ln(n) + 6)$ is sufficient.

Tools for the proof of the decay rate

Note that $\partial_t \ln \lambda_n(t) = \frac{2|\psi_{n,t}(1)|^2}{t}$

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$$\frac{\pi\sqrt{\chi_n}}{2\mathbf{K}(\sqrt{q})}(1-\delta(k)\varepsilon_n) \leq \psi_{n,\tau}^2(1) \leq \frac{\pi\sqrt{\chi_n}}{2\mathbf{K}(\sqrt{q})}(1+\delta(k)\varepsilon_n).$$

Let $I(a,b)=\int_a^b \frac{(\psi_{n,\tau}(1))^2}{\tau}\,d\tau, \quad \mathcal{J}(x)=\frac{\pi^2}{4}\int_{\Phi\left(\frac{2x}{\pi}\right)}^1 \frac{1}{t(\mathbf{E}(t))^2}\,dt.$

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If $c<c_n^\kappa$, then $I(c,c_n^\kappa)\approx\frac{\pi}{2}\int_c^{c_n^\kappa}\frac{d\tau}{2\sqrt{q(\tau)}\mathbf{K}(\sqrt{q(\tau)})}$.

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It remains to bound $I(c,c_n^*)-I(c,c_n^\kappa)$ which is possible since c_n^κ and c_n^* are sufficiently close.

Main decay results of the $\lambda_n(c)$.

Theorem (Bonami, K. (2014))

There exist three non negative constants $\delta_1, \delta_2, \delta_3$ *such that, for* $n \geq 3$ *and* $c \leq \frac{\pi n}{2}$, *we have*

$$\int_c^{c_n^*} \frac{(\psi_{n,\tau}(1))^2}{\tau}\, d\tau = \frac{\pi^2(n+\frac{1}{2})}{4} \int_{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right)}^1 \frac{1}{t(\mathbf{E}(t))^2}\, dt + \mathcal{E}, \qquad (9)$$

with

$$|\mathcal{E}| \leq \delta_1 + \delta_2 \ln(n) + \delta_3 \ln^+(1/c). \qquad (10)$$

Theorem (Bonami, K. (2014))

There exist three constants $\delta_1 \geq 1, \delta_2, \delta_3, \geq 0$ *such that, for* $n \geq 3$ *and* $c \leq \frac{\pi n}{2}$,

$$\delta_1^{-1} n^{-\delta_2} \left(\frac{c}{c+1}\right)^{\delta_3} \leq \frac{\widetilde{\lambda_n(c)}}{\lambda_n(c)} \leq \delta_1 n^{\delta_2} \left(\frac{c}{c+1}\right)^{-\delta_3}, \tag{11}$$

where

$$\widetilde{\lambda_n(c)} = \frac{1}{2} \exp\left(-\frac{\pi^2(n+\frac{1}{2})}{2} \int_{\Phi\left(\frac{2c}{\pi(n+\frac{1}{2})}\right)}^{1} \frac{1}{t(\mathbf{E}(t))^2}\, dt\right). \tag{12}$$

We have the double inequality,

$$\frac{1}{2}\left(\frac{ec}{4(n+\frac{1}{2})}\right)^{2n+1} e^{-\frac{\pi^2}{4}\frac{c^2}{n+\frac{1}{2}}} \leq \widetilde{\lambda_n(c)} \leq \frac{1}{2}\left(\frac{ec}{4(n+\frac{1}{2})}\right)^{2n+1} e^{+\frac{\pi^2}{4}\frac{c^2}{n+\frac{1}{2}}}.$$

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For any $\delta > 0$,

$$\lambda_n(c) \leq e^{-\delta n}, \quad \forall\, n \geq N_{\delta,c}.$$

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$$\lambda_n(c) > e^{-2n\log\left(\frac{bn}{c}\right)}, \quad \forall\, n \geq N_{c,b}.$$

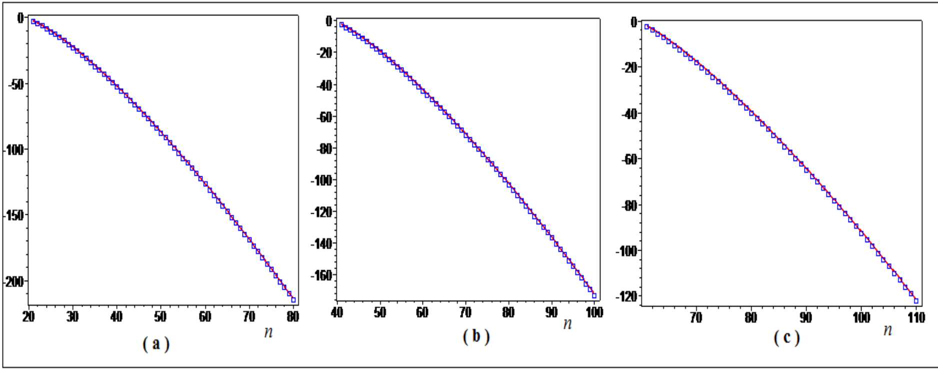

Figure : Graphs of $\ln(\widetilde{\lambda_n(c)})$ (boxes) and $\ln(\lambda_n(c))$ (red) with $c = 10\pi$ for (a), $c = 20\pi$ for (b) and $c = 30\pi$ for (c).

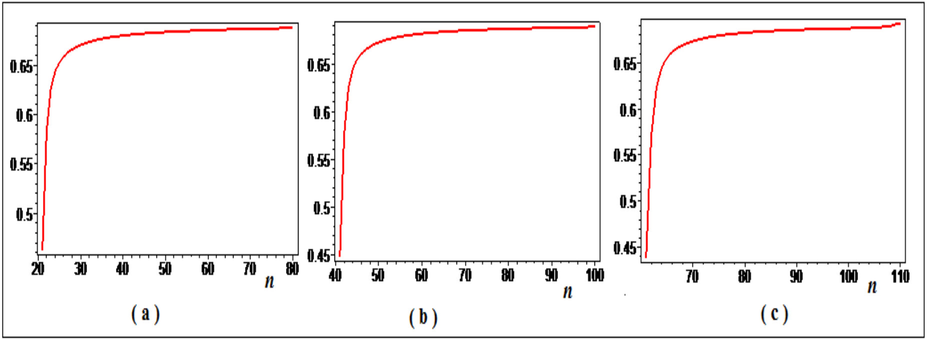

Figure : Graphs of $\ln\left(\frac{\lambda_n(c)}{\widetilde{\lambda_n}(c)}\right)$ with $c = 10\pi$ for (a), $c = 20\pi$ for (b) and $c = 30\pi$ for (c).

Approximation of band-limited functions

Lemma

Let $f \in B_c$ be an L^2 normalized function. Then

$$\int_{-1}^{+1} |f - S_N f|^2 dt \leq \lambda_N(c). \tag{13}$$

Approximation of almost band-limited functions

Let T and Ω de two measurable sets. A function pair $(f, \widehat{f})$ is said to be ϵ_T-concentrated in T and $\epsilon_\Omega-$concentrated in Ω if

$$\int_{T^c} |f(t)|^2\, dt \leq \epsilon_T^2, \qquad \int_{\Omega^c} |\widehat{f}(\omega)|^2\, d\omega \leq \epsilon_\Omega^2.$$

Next we define the time-limiting operator P_T and the band-limiting operator Π_Ω by:

$$P_T(f)(x) = \chi_T(x) f(x), \qquad \Pi_\Omega(f)(x) = \frac{1}{2\pi} \int_\Omega e^{ix\omega} \widehat{f}(\omega)\, d\omega.$$

Approximation of almost band-limited functions

Proposition

If f is an L^2 normalized function that is ϵ_T-concentrated in $T = [-1, +1]$ and $\epsilon_\Omega-$band concentrated in $\Omega = [-c, +c]$, then for any positive integer N, we have

$$\left(\int_{-1}^{+1} |f - S_N f|^2 dt \right)^{1/2} \leq \epsilon_\Omega + \sqrt{\lambda_N(c)} \tag{14}$$

and, as a consequence,

$$\|f - P_T S_N f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)}. \tag{15}$$

Approximation by the PSWFs in the Sobolev spaces

Theorem

Let $c \geq 0$ be a positive real number and let $I = [-1, 1]$. Assume that $f \in H^s(I)$,, for some positive real number $s > 0$. Then for any integer $N \geq 1$, we have

$$\|f - S_N f\|_2 \leq K(1 + c^2)^{-s/2}\|f\|_{H^s} + K\sqrt{\lambda_N(c)}\|f\|_2. \qquad (16)$$

Here, the constant K depends only on s. Moreover it can be taken equal to 1 when f belongs to the space $H_0^s(I)$.

Legendre expansion of the PSWFs,

$$\psi_n(x) = \sum_{k\geq 0} \beta_k^n \overline{P_k}(x). \tag{17}$$

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Lemma

Let $c > 0$, be a fixed positive real number. Then, for all positive integers k, n such that $k(k-1) + 1.13\,c^2 \leq \chi_n(c)$, we have $\beta_k^n \geq 0$.

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Proposition

Let $c > 0$, be a fixed positive real number. Then, for all positive integers n, k such that $k(k-1) + 1.13\, c^2 \leq \chi_n(c)$, we have

$$|\beta_0^n| \leq \frac{1}{\sqrt{2}} |\mu_n(c)| \quad \text{and} \quad |\beta_k^n| \leq \sqrt{\frac{5}{4\pi}} \left(\frac{2}{\sqrt{q}}\right)^k |\mu_n(c)|. \tag{18}$$

Lemma

Let $c \geq 1$*, then there exist constants* $M > 1.40$ *and* M'*,* $a > 0$ *such that, when* $n \geq \max(cM, 3)$ *and* $f(x) = e^{ik\pi x}$ *with* $|k| \leq n/M$*, we have*

$$|\langle f, \psi_n \rangle| \leq M' e^{-an}. \tag{19}$$

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Theorem (Bonami, K. (2014))

Let $c \geq 1$*, then there exist constants* $M > 1.40$ *and* $M', a > 0$ *such that, when* $N \geq \max(cM, 3)$ *and* $f \in H^s_{per}, s > 0$*, we have the inequality*

$$\|f - S_N(f)\|_{L^2(I)} \leq M'(1 + (\pi N)^2)^{-s/2} \|f\|_{H^s_{per}} + M' e^{-aN} \|f\|_{L^2}. \tag{20}$$

Corollary

Let $c \geq 1$, *and let* $s > 0$ *with* $[s] = m \in \mathbb{N}$, *and* $s \notin \frac{1}{2} + \mathbb{N}$. *Let* $f \in H^s(I)$, *then there exist constants* $M \geq 1.40$ *and* $M', M'_s > 0$ *such that, when* $N \geq \max(cM, 3)$, *we have the inequality*

$$\|f - S_N(f)\|_{L^2(I)} \leq M'_s(1 + N^2)^{-s/2}\|f\|_{H^s([-1,1])} + M' e^{-aN}\|f\|_{L^2([-1,1])}. \quad (21)$$

Exact reconstruction of band-limited functions with missing data

From [Donoho,Stark (1989)], if $\|f\|_2 = \|\widehat{f}\|_2 = 1$ and $(f, \widehat{f})$ is ϵ_T-concentrated on T and $\epsilon_\Omega-$concentrated on Ω, then

$$|\Omega||T| \geq (1 - (\epsilon_T + \epsilon_\Omega))^2 .$$

Hence, if $|\Omega||T| < 1$, then the following band-limited reconstruction problem has a unique solution in B_Ω.

Find $S \in B_\Omega$ such that $r(t) = \chi_{T^c}(t)\,(S(t) + \eta(t))\,,\ \eta(\cdot) \in L^2$.

The solution S is given by

$$S(t) = Qr(t) = \sum_{n\geq 0} (P_T P_\Omega)^n \, r(t), \quad t \in \mathbf{R}.$$

$$\|S - Qr\| \leq C\|\eta\|, \quad C \leq (1 - \sqrt{|T||\Omega|})^{-1}.$$

If $T = [-\tau, \tau]$, $\Omega = [-c, c]$, then

$$P_\Omega P_T(f)(x) = \int_{-\tau}^{\tau} \frac{\sin 2\pi c(x-y)}{\pi(x-y)} \, f(y)\, dy, \quad x \in \mathbf{R}.$$

Hence

$$\|P_T P_\Omega\| \leq \lambda_0(c) < 1.$$

Consequently, the band-limited reconstruction problem has a band-limited solution no matter how large are T and Ω.

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Thank You