Tilings and Spectra

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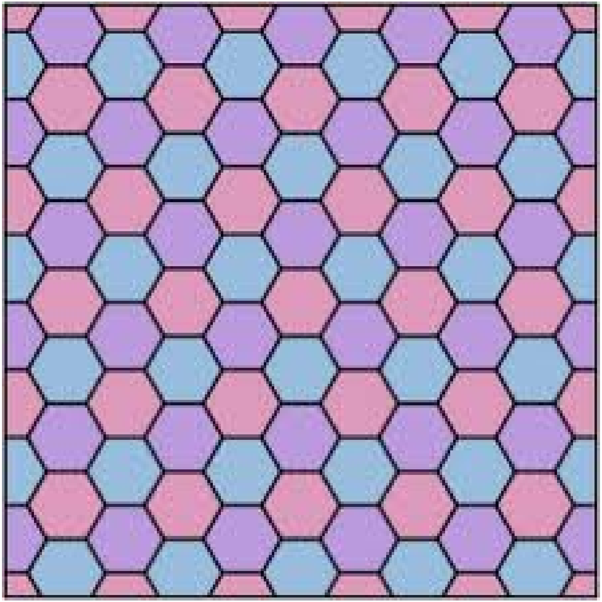

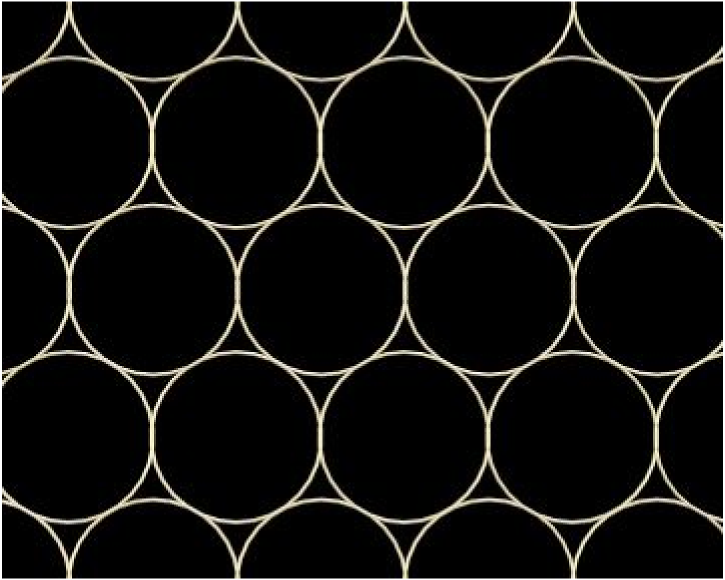

The Setting

Let Ω be a bounded measurable subset of $\mathbb{R}^n$, with a *nice* boundary.

Ω is not necessarily a connected set.

We will assume that $|\Omega| = 1$.

Definitions

- For an element $x \in \mathbb{R}^n$, we let $\Omega + x$ denote the translate of Ω by x,

$$\Omega + x = \{y \in \mathbb{R}^n : y - x \in \Omega\}$$

- A set $\mathcal{T} \subset \mathbb{R}^n$ is called a Tiling set for a set Ω, if $\{\Omega + t : t \in \mathcal{T}\}$ forms a partition a.e of $\mathbb{R}^n$. Then Ω is called a **prototile**, and $(\Omega, \mathcal{T})$ is called a **Tiling pair**.

  Equivalently, $(\Omega, \mathcal{T})$ is a Tiling pair iff

$$\sum_{t \in \mathcal{T}} \chi_\Omega(x + t) = 1 \ a.e$$

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Definitions contd.

- A set $\Lambda \subset \mathbb{R}^n$ is called a spectrum for Ω if the set of exponentials

$$E_\Lambda = \{e_\lambda(x) = e^{2\pi i\lambda.x}\chi_\Omega(x); \lambda \in \Lambda\}$$

  is an orthonormal basis for $L^2(\Omega)$. If a spectrum exists for a set Ω, then Ω is called a Spectral set.

- Equivalently, Λ is a spectrum for Ω iff

$$\sum_{\lambda \in \Lambda} |\widehat{\chi_\Omega}|^2(\xi - \lambda) = 1 \ a.e.$$

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The conditions for Tiling and Spectra, namely

$$\sum_{t \in \mathcal{T}} \chi_\Omega(x+t) = 1 \; a.e$$

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can both be viewed as tiling by non-negative functions, respectively χ_Ω and $|\widehat{\chi_\Omega}|^2$.

A crucial difference is that while the first has support of finite measure, the latter cannot have support with finite measure (by Benedicks' Uncertainty Principle).

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Facts

- Both $\mathcal{T}$ and Λ (when they exist), are discrete sets, in fact, they are uniformly separated.
- With $|\Omega| = 1$, both Λ and $\mathcal{T}$ have upper asymptotic density 1, where the upper asymptotic density of a set S is defined as:

$$\rho(S) = \limsup_{R\to\infty} \frac{card([-R, R]^d \cap S)}{(2R)^d}$$

- Neither the Spectrum, nor the Tiling set are unique.
- If Λ is a spectrum, and $\lambda_0 \in \Lambda$, then it is easy to see that the set $\Lambda - \lambda_0$ is also a spectrum for the same set. Henceforth, we will assume that $0 \in \Lambda$. From this and orthogonality, it follows that

$$0 \in \Lambda \subset \Lambda - \Lambda \subset \{\xi : \widehat{\chi_\Omega}(\xi) = 0\} \cup \{0\}$$

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Fuglede's Conjecture

A set $\Omega \subset \mathbb{R}^n$ is a spectral set if and only if Ω tiles $\mathbb{R}^n$ by translations.
(Fuglede, B. J. Funct. Anal, 1974)

- As stated above, the conjecture is far too general. First, there is no assumption on the structure of the set Ω (diameter, connectedness etc). Further, neither the Tiling set not the Spectrum need be unique, so what relation can be expected between a $\mathcal{T}$ and a Λ?
- Fuglede's conjecture arose from his investigation into the problem of existence of commuting self-adjoint extensions of the operators $-i(\partial/\partial x_j)$, $j = 1, ..., n$ defined on $C_0^\infty(\Omega)$ to a dense subspace of $L^2(\Omega)$.

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Fuglede himself proved the following Theorem:

Theorem

Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^n$ and let $\mathcal{L}^*$ be its dual lattice. Then $(\Omega, \mathcal{L})$ is a Spectral pair iff $(\Omega, \mathcal{L}^*)$ is a Tiling pair.

This result is essentially Fourier Analysis for the n-torus in $\mathbb{R}^n$, upto affine transformations. (Poisson Summation Formula).

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Important Recent Results

- For $d \geq 3$, the conjecture is not true in the generality in which it is stated, in either direction.
- For $d = 5$, Terence Tao gave a counterexample in 2004. For $d = 3, 4$, Matolsci and Kolountzakis (2006) were able to use Tao's idea to give counterexamples and showed that both implications of Fuglede's fail.
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The case of convex sets in dimension 2

The case where Ω is a convex planar set received a lot of attention recently.

- Theorem(Venkov; McMullen,1980) A convex body K which tiles $\mathbb{R}^n$ by translations is a symmetric polytope.
- It is known that whenever a convex polytope tiles $\mathbb{R}^n$, there exists a lattice tiling. Thus "Tiling implies Spectral" holds for convex sets in any dimension.
- Theorem (Kolountzakis, 2000) A convex planar set which is spectral has to be symmetric.
- Theorem (Iosevich, Katz and Tao, 2001) A convex planar spectral set cannot have a point of curvature.
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The Structure of Tiling sets in dimension 1

- In one dimension, the conjecture is trivial if the set is an interval.
- The conjecture is known to hold in particular cases, with additional hypotheses.
- If Ω is assumed to be a finite union of intervals, then the only case for which it is known to hold is when Ω is a union of two intervals, Laba (2001).
- If Ω is a union of three intervals, it is known that Tiling implies Spectral; and Spectral implies Tiling holds with "one additional hypothesis" (BCKM 2010, BM 2013)

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Lagarias and Wang [1996] gave a complete characterization of the structure of a tiling set in one dimension for any compact set Ω.

Theorem (Periodicity)

Suppose that Ω tiles $\mathbb{R}$ by translation. Then every tiling $\mathcal{T}$ by translations of Ω is a periodic tiling with an integer period.

The analogue of this theorem is false in higher dimensions, e.g. the unit square Q in R^2 gives infinitely many nonperiodic tilings of R^2 (which are translation inequivalent).

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Proof Outline

There are essentially three steps in the proof of the theorem:

- Step 1: Any tiling has the local finiteness property, i.e. for every closed interval J, there are only finitely many ways to tile J.
- Step 2: If $\Omega \subset [-N, N]$, and a patch $\mathcal{P}$ covers $[-N, N]$, and if this patch can be extended to a tiling of $\mathbb{R}$, then this extension is unique.
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Tilings

Theorem (Rationality)

Suppose that a bounded region Ω tiles $\mathbb{R}$ by translation, using a d-periodic tiling set $\mathcal{T}$ given by

$$\mathcal{T} = \cup_1^d (r_j + dZ)$$

Then all differences $r_j - r_k$ are rational.

Outline of the proof

The proof of this theorem uses Fourier Analysis, and Szemeredi's theorem (or the Skolem-Mahler-Lech Theorem) on the zeros of exponential polynomials.

Let $f \in L^1(\mathbb{R})$. The integer zero set of f is given by

$$\mathbb{Z}(f) = \{n \in \mathbb{Z}; \hat{f}(n) = 0\}$$

- **Lemma:** If f is a compactly supported non-negative function, such that $0 < |supp(f)| < 1$, then the

$$\rho(\mathbb{Z}(f)) < 1$$

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- By periodicity we can write the tiling set as

$$\mathcal{T} = \bigcup_{1}^{d} (r_j + d\mathbb{Z})$$

Let $\{\mathcal{R} = r_j : 1 \leq j \leq J\}$.

- Partition $\mathcal{R}$ by an equivalence relation $r_j \equiv r_k \iff r_j - r_k \in \mathbb{Q}$, and write

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- Consider the distribution $\delta_{\mathcal{R}}$. Its Fourier transform is the exponential polynomial $f(\lambda) = \sum \exp 2\pi i \lambda r_j$, and for each eqivalence class R_k^*, we let

$$f_k(\lambda) = \sum_{r_j \in \mathcal{R}_k^*} \exp 2\pi i \lambda r_j$$

- We then show that

$$\mathbb{Z}(f) = X \cup Y,$$

where X is the common integer zero set of the f_k's which is a union of complete arithmetic progressions, and Y is a set of density zero.

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where X is the common integer zero set of the f_k's which is a union of complete arithmetic progressions, and Y is a set of density zero.

- The final step in the proof is to show that there is only one equivalence class by showing that

$$\text{if } \Omega_k = \cup_{r_j \in \mathcal{R}_k^*}(\Omega + r_j), \text{ then } \rho(\mathbb{Z}(\chi_{\Omega_k})) \geq 1,$$

and so by the lemma $|\Omega_k| = 1$.

Theorem (LW (Structure))

A d-periodic tiling $\mathcal{T}$ as above is also a Tiling set for a set Ω_1 which is a union of d equal intervals (each of length $1/d$), with endpoints lying in $\mathbb{Z}/d$.

- In fact for

$$\mathcal{T} = \bigcup_{1}^{d}(a_j/N + d\mathbb{Z}),$$

with $A = \{a_j : j = 1, 2, ...d\}$, the set Ω_1 is of the form

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Results on the structure of Spectra are very recent.
First

Theorem (BM, 2011)

Let $\Omega = \cup_{j=1}^{n} I_j$, $|\Omega| = 1$. If (Ω, Λ) is a spectral pair, then Λ is a d-periodic set with $d \in \mathbb{N}$. Hence Λ has the form $\Lambda = \cup_{j=1}^{d} \{\lambda_j + d\mathbb{Z}\}$.

In 2012, Iosevich and Kolountzakis extended the above periodicity result to general compact sets Ω.

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Let $\Omega = \cup_{j=1}^{n} I_j$, $|\Omega| = 1$. If (Ω, Λ) is a spectral pair, then Λ is a d-periodic set with $d \in \mathbb{N}$. Hence Λ has the form $\Lambda = \cup_{j=1}^{d} \{\lambda_j + d\mathbb{Z}\}$.

In 2012, Iosevich and Kolountzakis extended the above periodicity result to general compact sets Ω.

Outline of proof

- The proof of this theorem for the spectrum has the same kind of strategy as in the corresponding result for tiling: Local finiteness, unique extension, and pigeonhole principle.
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An Embedding of the spectrum

We embed the spectrum in a vector space as follows: Consider the $2n$-dimensional vector space

$$\mathbb{C}^n \times \mathbb{C}^n = \{\underline{v} = (v_1, v_2) : \; v_1, v_2 \in \mathbb{C}^n\}.$$

Define a conjugate linear form $\odot$ on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$\underline{v} \odot \underline{w} := \langle v_1, w_1 \rangle - \langle v_2, w_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^n$.

This conjugate linear form is degenerate; in fact, $\underline{v} \odot \underline{v} = 0$, if $\underline{v} \in \mathbb{T}^n \times \mathbb{T}^n$.

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- A subset $S \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is called a set of *mutually null-vectors* if $\forall\ \underline{v}, \underline{w} \in S$, we have $\underline{v} \odot \underline{w} = 0$.
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Let $\Omega = \cup_{j=1}^{n} [a_j, a_j + r_j)$ be a union of n disjoint intervals with $a_1 = 0$, $r_j > 0$, and $|\Omega| = \sum_1^n r_j = 1$. In what follows Ω will always denote a set like this. Observe that

$$\widehat{\chi_\Omega}(\lambda) = \frac{1}{e^{2\pi i\lambda}} \sum_1^n (e^{2\pi i\lambda(a_j+r_j)} - e^{2\pi i\lambda a_j})$$

We define a map φ_Ω from $\mathbb{R}$ to $\mathbb{T}^n \times \mathbb{T}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n$ by

$$x \to \varphi_\Omega(x) = (\varphi_1(x); \varphi_2(x)),$$

where

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- For a set $\Lambda \subset \mathbb{R}$, the mutual orthogonality of the set of exponentials $E_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ is equivalent to saying that the set $\varphi_\Omega(\Lambda) = \{\varphi_\Omega(\lambda); \lambda \in \Lambda\}$ is a set of mutually null vectors.

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If (Ω, Λ) is a spectral pair, then the spectrum Λ can be characterized:

Lemma

Let (Ω, Λ) be a spectral pair and let $\mathcal{B} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B}) := \{\varphi_\Omega(y) : y \in \mathcal{B}\}$ forms a basis of $V_\Omega(\Lambda)$. Then $x \in \Lambda$ if and only if $\varphi_\Omega(x) \odot \varphi_\Omega(y) = 0, \ \forall\, y \in \mathcal{B}$.

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Next, we give two criteria for the periodicity of the spectrum.

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Lemma

Let $dim(V_\Omega(\Lambda)) = m \leq n$ and $\mathcal{B} = \{y_1, \dots, y_m\} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B})$ is a basis for $V_\Omega(\Lambda)$. If for some $d \in \mathbb{R}$, we have $\mathcal{B} + d = \{y_1 + d, \dots, y_m + d\} \subseteq \Lambda$ then Λ is d-periodic, i.e., $\Lambda = \{\lambda_1, \dots, \lambda_d\} + d\mathbb{Z}$.

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Density

- In order to show repeated patterns and conclude periodicity, we now use Landau's density theorem.

- Define $n^+(R)$, $n^-(R)$ respectively, as the largest and smallest number of elements of Λ contained in any interval of length R, i.e.,

$$n^+(R) = \max_{x \in \mathbb{R}} \#\{\Lambda \cap [x - R, x + R]\}$$

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- A uniformly discrete set Λ is called a *set of sampling* for $L^2(\Omega)$, if there exists a constant K such that $\forall f \in L^2(\Omega)$ we have $\|f\|_2^2 \leq K \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2$.
- Λ is called a *set of interpolation* for $L^2(\Omega)$, if for every square summable sequence $\{a_\lambda\}_{\lambda \in \Lambda}$, there exists an $f \in L^2(\Omega)$ with $\hat{f}(\lambda) = a_\lambda$, $\lambda \in \Lambda$.
- Clearly if (Ω, Λ) is a spectral pair, then Λ is both a set of sampling and a set of interpolation for $L^2(\Omega)$.

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- Clearly if (Ω, Λ) is a spectral pair, then Λ is both a set of sampling and a set of interpolation for $L^2(\Omega)$.

Theorem(Landau)

Let Ω be a union of a finite number of intervals with total measure 1, and Λ a uniformly discrete set. Then

1. If Λ is a set of sampling for $L^2(\Omega)$,

$$n^-(R) \geq R - A\log^+ R - B$$

2. If Λ is a set of interpolation for $L^2(\Omega)$,

$$n^+(R) \leq R - A\log^+ R - B$$

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It follows that if (Ω, Λ) is a spectral pair then Λ has uniform asymptotic density 1.

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Step 1. We first prove that the spectrum Λ can be modified to a set Λ_d which is d-periodic and is such that (Ω, Λ_d) is a spectral pair. For this we use Landau's density result to extract a "patch" from Λ which has some periodic structure and a large enough density. Then Λ_d will be a suitable periodization of this patch.

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A Structure Theorem

For a spectral pair (Ω, Λ), the periodicity of Λ implies that

$$\Lambda = \cup_{j=0}^{d-1} (\lambda_j + d\mathbb{Z})$$

Then

Theorem

A d-periodic spectrum Λ as above is also a Spectrum for a set Ω_1 which is a union of d equal intervals (each of length $1/d$), with endpoints lying in $\mathbb{Z}/d$.

Some Remarks on Rationality

1. All known spectra of sets in $\mathbb{R}$ are rational; however it is not known whether this must always be so.

2. From the structure theorem stated above, we see that to resolve the problem of rationality of a d-periodic spectrum, it is sufficient to assume that the set Ω is a union of d equal intervals, with end points lying in $\mathbb{Z}/d$. Such sets are called clusters.

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3. For clusters, the only result on rationality of the spectrum known to us is due to Izabella Laba.

Theorem (Laba)

Suppose that $\Omega = A + [0, 1)$, $A \subset \mathbb{N}$ where $|A| = n$ is a spectral set. If $A \subset [0, M]$, with $M < \frac{5n}{2}$, then any spectrum for Ω is rational.

The proof of this theorem uses Galois theory.

4. It is easy to see that spectrum is either rational or it has elements which are transcendental.

Infact, we have assumed that

$$0 \in \Lambda \subset \Lambda - \Lambda \subset \mathbb{Z}(\widehat{\chi_{\Omega_1}}) \cup \{0\}$$

where $\mathbb{Z}(\widehat{\chi_{\Omega_1}}) = \{\xi \in \mathbb{R} : \widehat{\chi_{\Omega_1}} = 0\}$. Hence, every $\lambda_j, j = 1, 2, ..., d-1$ satisfies

$$1 + e^{2\pi i a_1 \lambda_j / d} + ... + e^{2\pi i a_{d-1} \lambda_j / d} = 0$$

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We can now use the following theorem from Number Theory:

Theorem (Gelfond-Schneider)

If α and β are algebraic numbers with $\alpha \neq 0, 1$, and if β is not a rational number, then any value of $\alpha^\beta = exp(\beta \log \alpha)$ is a transcendental number.

Take $\alpha = e^{\pi i} = -1$, and $\beta = 2\lambda_j/d$ and apply G-S theorem above. Since $\alpha^\beta = e^{2\pi i \lambda_j/d}$ is an algebraic integer, $2\lambda_j$ is either rational or is not an algebraic number.

5. Further investigation into the problem of rationality of the spectrum leads us to the study of integer zeros of exponential polynomials.

Let $\Lambda = \Gamma + d\mathbb{Z}$ be a periodic spectrum for a set Ω. Then using Poisson Summation Formula, we see that, (Ω, Λ) is a spectral pair

$$
\begin{aligned}
&\iff && |\widehat{\chi_\Omega}|^2 * \delta_\Lambda \equiv 1 \\
&\iff && \widehat{\chi_\Omega}|^2 * \delta_\Gamma * \delta_{d\mathbb{Z}} \equiv 1 \\
&\iff && \frac{1}{d}(\chi_\Omega * \chi_{\tilde{\Omega}})\widehat{\delta_\Gamma}\delta_{\mathbb{Z}/d} \equiv \delta_0 \\
&\iff && (\Omega - \Omega|_{\mathbb{Z}/d} \subset \mathbb{Z}_{1/d}(\widehat{\delta_\Gamma})
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- The Skolem-Mahler-Lech Theorem says that the integer zero set of an exponential polynomial is of the form $E \cup F$, where E is a finite union of complete arithmetic progressions, and F is a finite set.
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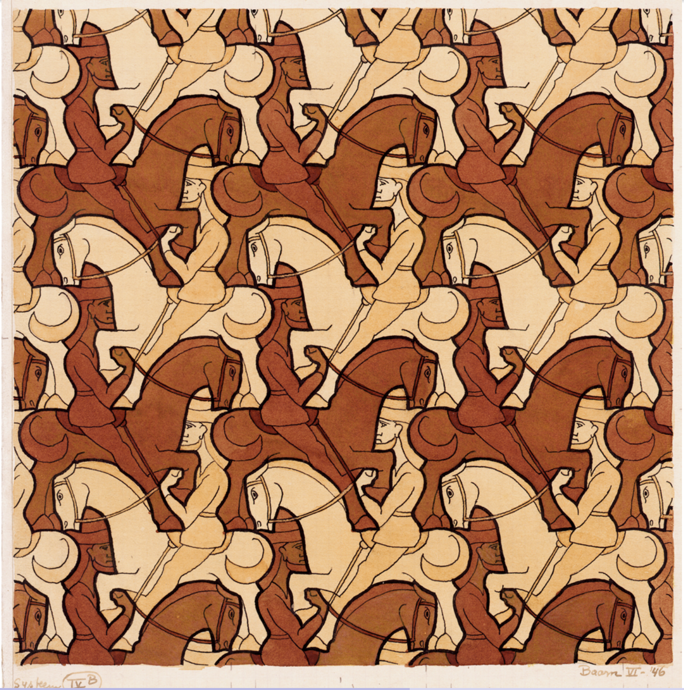
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