Quasicrystals are almost periodic patterns

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Quasicrystals are sets of points or pavings of $\mathbb{R}^n$. Following R. Moody a new definition of almost periodicity is used.

A centered patch (or local configuration) of a point set or a paving Λ is defined by

$$\mathcal{P}_0(\lambda, R) = (\Lambda - \lambda) \cap B(0, R)$$

where λ belongs to Λ or is a vertex of Λ.

Let Λ be a quasicrystal (point set or paving). When R is large enough all centered patches $\mathcal{P}_0(\lambda, R),\ \lambda \in \Lambda$, are almost identical. More precisely we have:

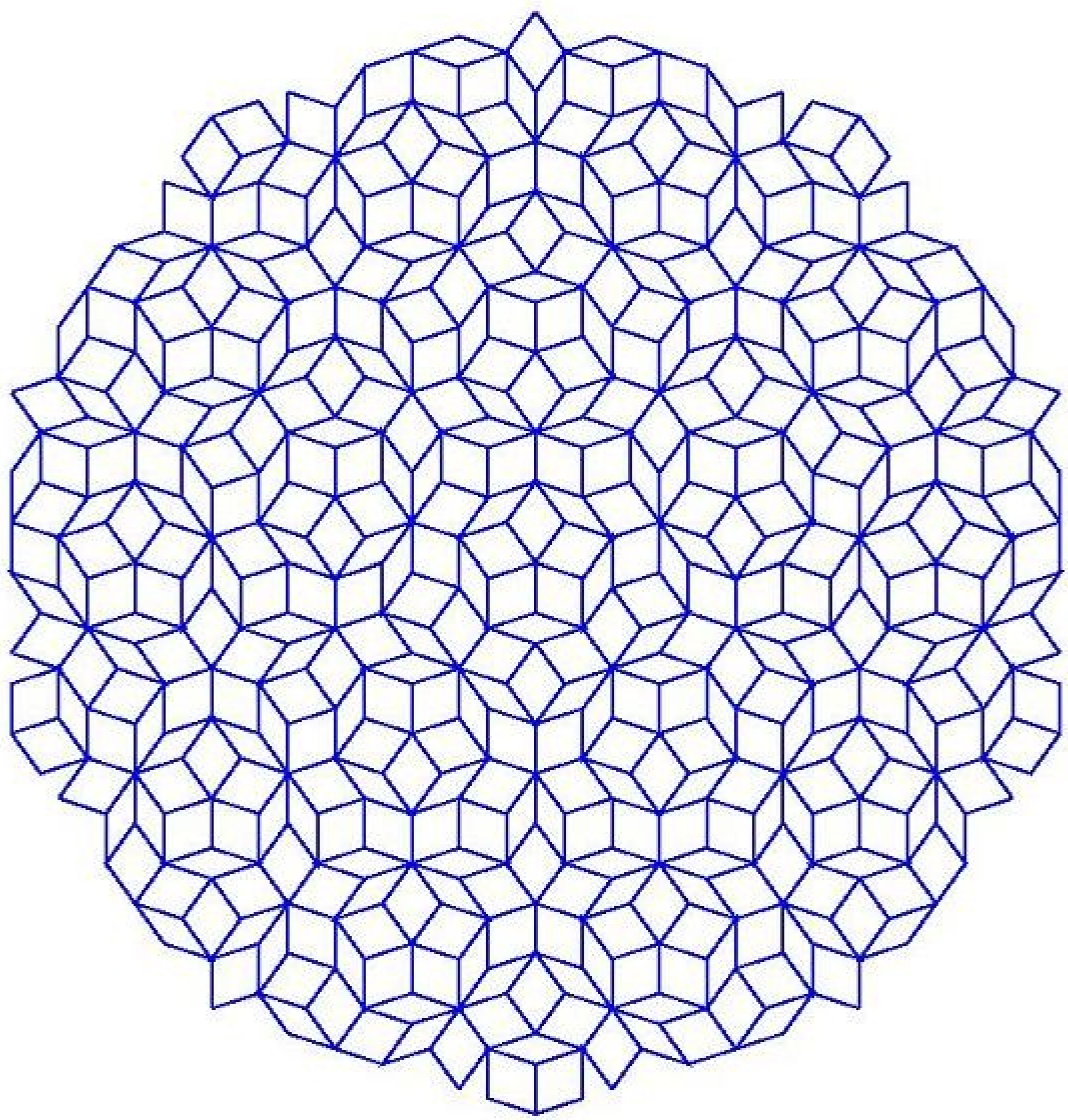
Theorem 1. *For every positive ϵ there exist two positive numbers $R(\epsilon)$ and $C(\epsilon)$ such that the following property holds:*

$\forall x,\ \forall y \in \Lambda$ *there exists a translation* $\tau \in \mathbb{R}^n,\ \tau = \tau(x, y, \epsilon),\ |\tau| \leq C(\epsilon)$ *such that* $\forall R \geq R(\epsilon)$ *we have*

$$\#[\mathcal{P}_0(x, R) \triangle (\mathcal{P}_0(y, R) - \tau)] \leq \epsilon R^n$$

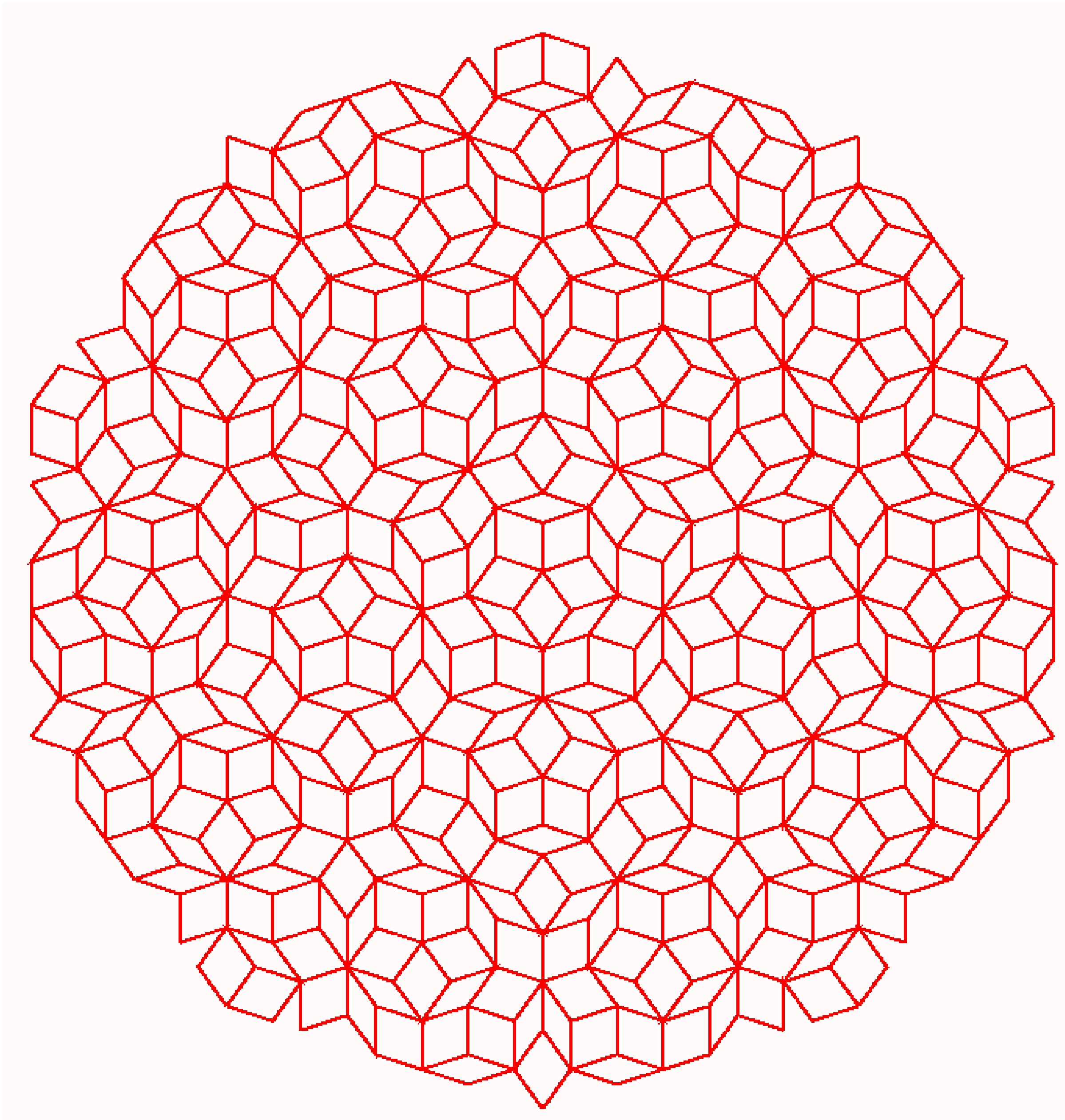
1. Pentagonal symmetry

Nigella.
Beautiful but imperfect pentagonal symmetry.
John Adam

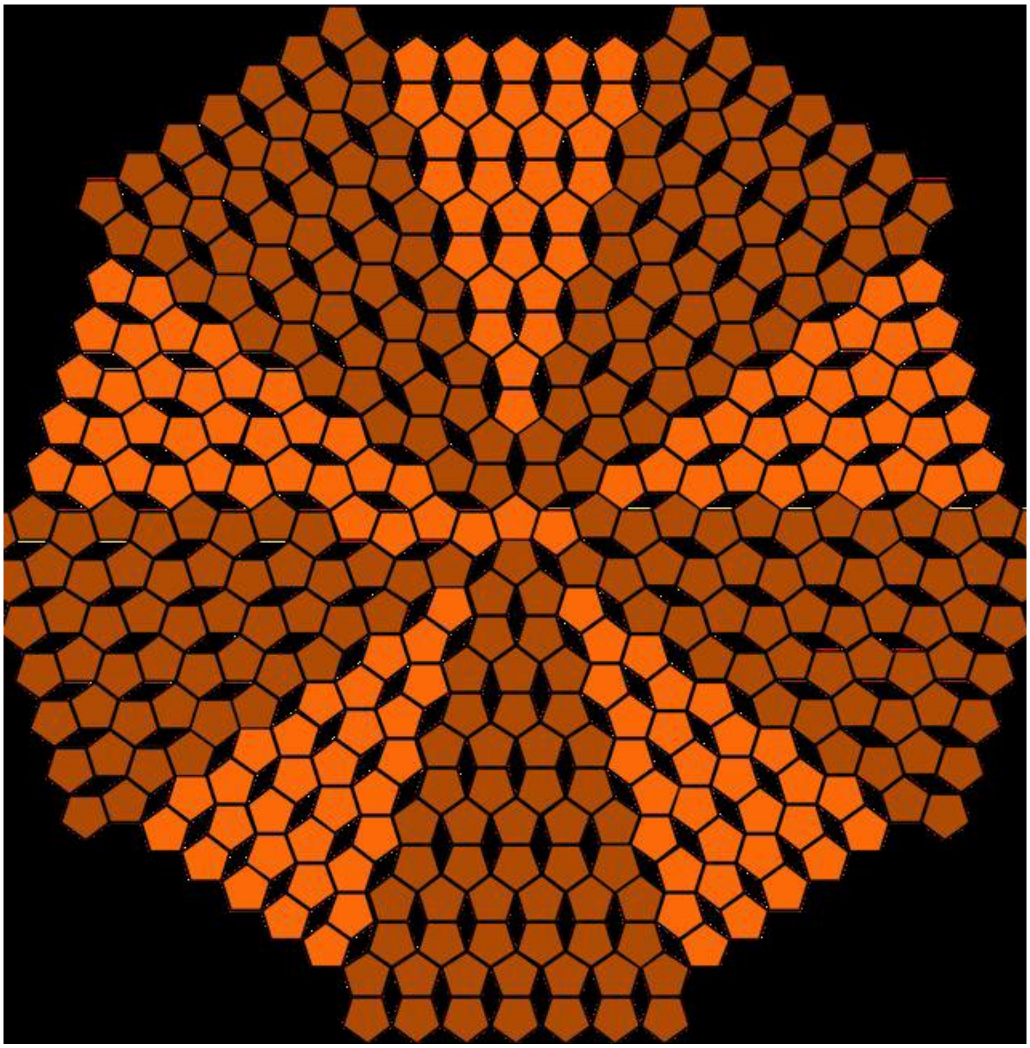


A Penrose tiling with pentagonal symmetry.

Penrose, Roger (1974), "Role of aesthetics in pure and applied research", Bulletin of the Institute of Mathematics and its Applications 10.



L’étonnant pavage d’Albrecht Dürer



Dürer, *Underweysung der Messung*. Nürnberg, 1525.

Version française : *Géométrie*. Trad. et présentation de Jeanne Peiffer, Seuil, 1995.

Ce pavage se trouve à la page 218 de la version française.

Il est invariant par rotation d’angle $2\pi/5$ et préfigure donc les pavages de Penrose.

Underweysung der Messung, premier ouvrage en allemand et donc accessible aux artisans et aux artistes, consacré à la géométrie euclidienne et à la science de la perspective telle que pratiquée à la Renaissance.

Peu de temps après sa publication en 1525, le traité de Dürer fut réuni, dans une reliure unique, à une édition du *Geometricorum elementorum* (1516) d’Euclide et à la première édition illustrée du *De architectura* (1511) de Vitruve.

Des annotations manuscrites dans les marges du Vitruve renvoient à l’œuvre d’Érasme et à d’autres humanistes d’Europe du Nord.

Ce volume montre combien l’érudition et la pratique s’influencent l’une l’autre et révèle la place centrale qu’occupe l’architecture dans l’expansion du savoir de la Renaissance

The Dürer paving looks like a Penrose paving since it is invariant by a $2\pi/5$ rotation. However it is not a quasicrystal. Here comes the recipe.

> *Voici une cinquième manière d'assembler des pentagones. Construis d'abord un pentagone et accole à chacun de ses côtés un pentagone régulier. Rajoute à ces cinq pentagones d'autres pentagones, un respectivement sur deux côtés de chacun des pentagones. Entre les pentagones se formeront cinq losanges étroits. Engage ensuite les pentagones dans les angles qui se forment tout autour et fais en sorte que leurs sommets touchent ceux des losanges étroits. Continue ensuite tant que tu voudras.*
>
> Albrecht Dürer.

[Communicated by Serge Boucheron and Denis Gratias.]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2. What is a quasicrystal ?

Is it (1) a paving of the plane with tiles $T_j,\ j \in J$, which are isometric copies of finitely many polygonal prototiles or (2) an almost periodic discrete set $\Lambda \subset \mathbb{R}^2$? This set Λ could be the set of vertices of the tiles $T_j,\ j \in J$, and conversely the paving could be the Voronoï tessellation generated by Λ. In this essay quasicrystals will be defined as almost periodic patterns (Definition 2).

A subset Λ of $\mathbb{R}^n$ is a Delone set if there exist two radii $R_2 > R_1 > 0$ such that

(a) each ball with radius R_1, whatever be its location, shall contain at most one point in Λ

(b) each ball with radius R_2, whatever be its location, shall contain at least one point in Λ.

A Delone set Λ is of finite type if and only if $\Lambda - \Lambda$ is a discrete closed set. Then Λ has only a finite number of different local neighborhoods of radius $2R_2$ around its points $\lambda \in \Lambda$, up to translations (J. Lagarias). Moreover the Voronoï tessellation generated by Λ is a paving of the plane with tiles which are translated copies of finitely many prototiles. The converse implication holds.

A centered patch (or local configuration) is defined by

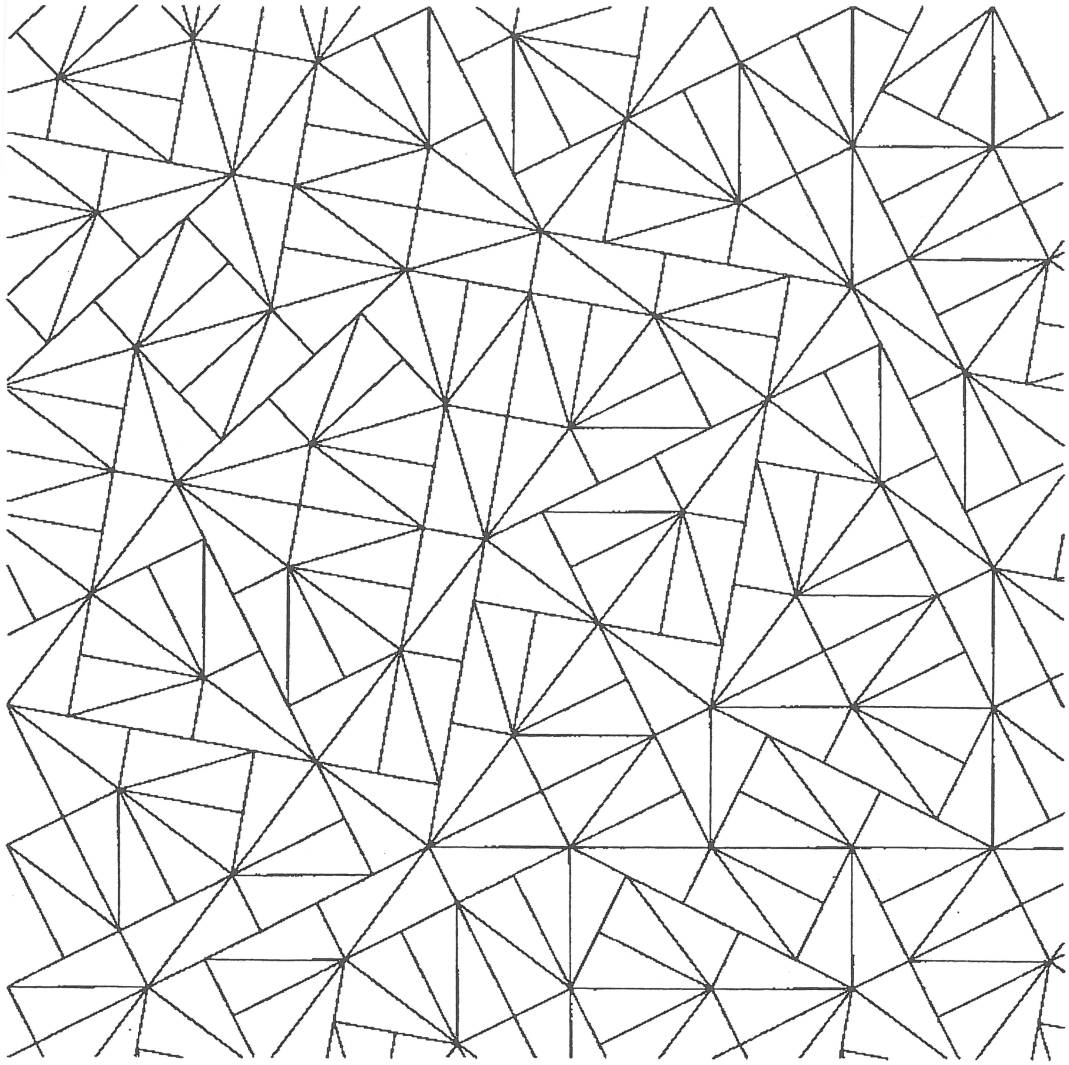
$$\mathcal{P}_0(\lambda, R) = (\Lambda - \lambda) \cap B(0, R),\ \ \lambda \in \Lambda.$$

A Delone set is of finite type iff for $R \geq 1$ there are finitely many patches $\mathcal{P}_0(\lambda, R),\ \lambda \in \Lambda$.

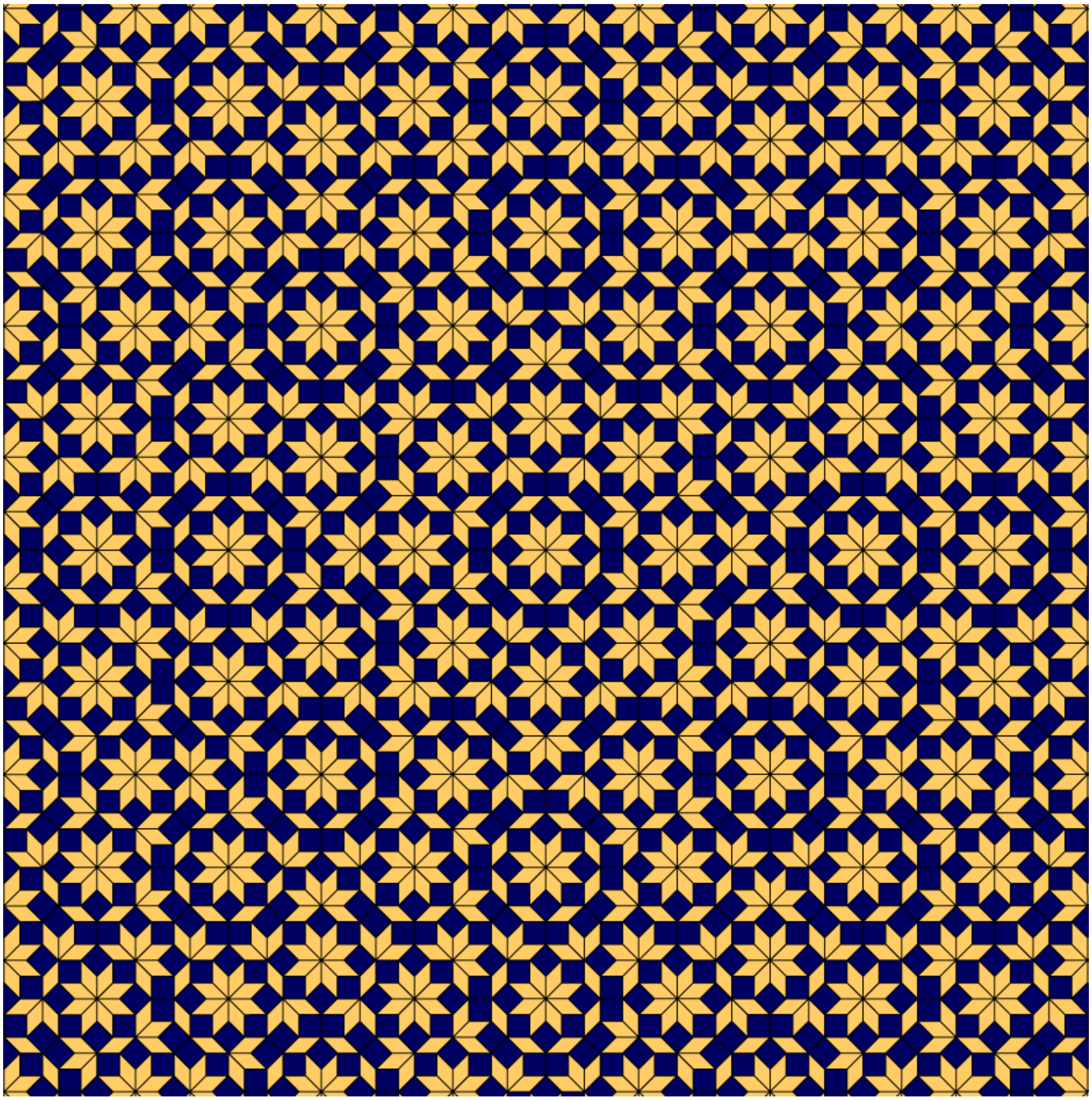
3. The pinwheel tiling

The beautiful paving which will appear now is the *pinwheel tiling*. It has been constructed by John Conway and Charles Radin (1994). It is not of finite type. Therefore it is not a quasicrystal.

Federation Square, a building complex in Melbourne, Australia features the pinwheel tiling. In the project, the tiling pattern is used to create the structural sub-framing for the facades, allowing for the facades to be fabricated off-site, in a factory and later erected to form the facades. The pinwheel tiling system was based on the single triangular element, composed of zinc, perforated zinc, sandstone or glass (known as a tile), which was joined to 4 other similar tiles on an aluminum frame, to form a "panel". Five panels were affixed to a galvanized steel frame, forming a "mega-panel", which were then hoisted onto support frames for the facade. The rotational positioning of the tiles gives the facades a more random, uncertain compositional quality, even though the process of its construction is based on pre-fabrication and repetition. The same pinwheel tiling system is used in the development of the structural frame and glazing for the "Atrium" at Federation Square, although in this instance, the pinwheel grid has been made "3-dimensional" to form a portal frame structure.



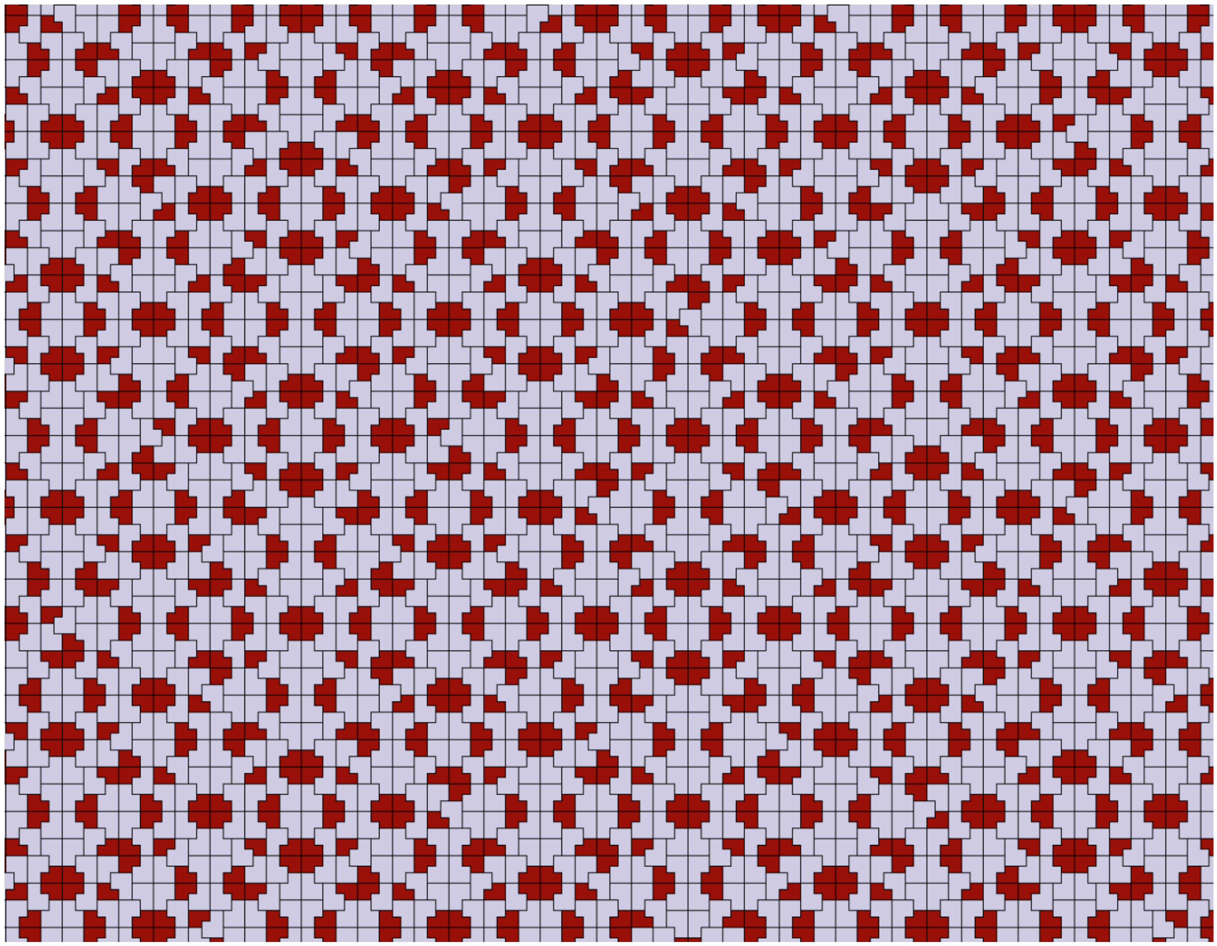
LAB architecture studio + Bates Smart: federation square, along flinders st- photography trevor mein

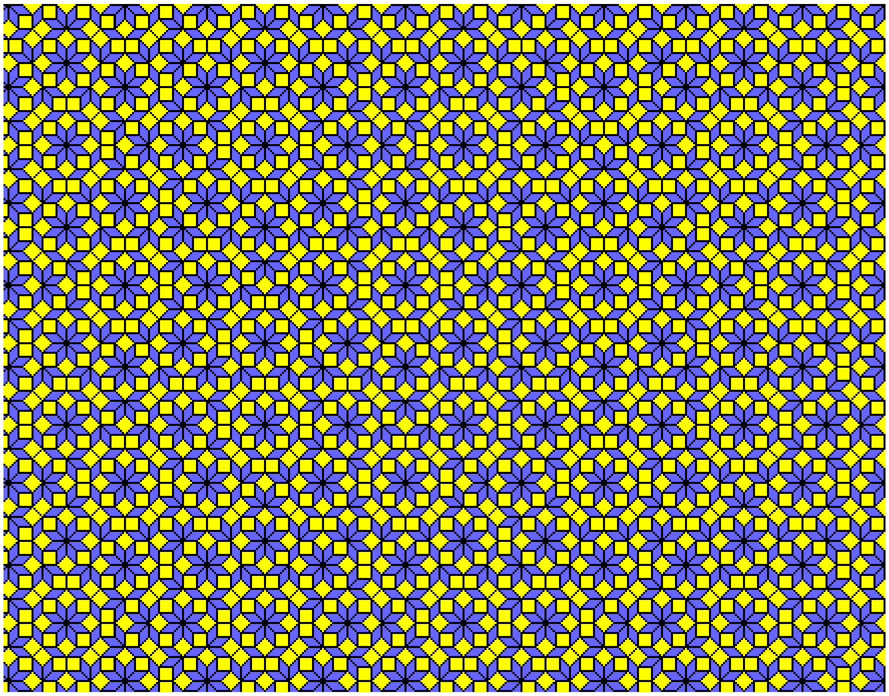


The set of vertices

of the Ammann-Beenker tiling

is a model set.





The Ammann-Beenker tiling is a model set.

Ammann tilings are quasicrystals. Robert Ammann (1946-1994) was an amateur mathematician who made several signicant and groundbreaking contributions to the theory of quasicrystals and aperiodic tilings. Ammann attended Brandeis University, but generally did not go to classes, and left after three years. He worked as a programmer for Honeywell. After ten years, his position was eliminated as part of a routine cutback, and Ammann ended up working as a mail sorter for a post office. In 1975, Ammann read an announcement by Martin Gardner of new work by Roger Penrose. Penrose had discovered two simple sets of aperiodic tiles, each consisting of just two quadrilaterals. Since Penrose was taking out a patent, he wasn't ready to publish them, and Gardner's description was rather vague. Ammann wrote a letter to Gardner, describing his own work, which duplicated one of Penrose's sets, plus a foursome of golden rhombohedra that formed aperiodic tilings in space. More letters followed, and Ammann became a correspondent with many of the professional researchers. 20 He discovered several new aperiodic tilings, each among the simplest known examples of aperiodic sets of tiles. He also showed how to generate tilings using lines in the plane as guides for lines marked on the tiles, now called Ammann bars. The discovery of quasicrystals in 1982 by Dan Shechtman changed the status of aperiodic tilings and Ammann's work from mere recreational mathematics to respectable academic research. After more than ten years of coaxing, he agreed to meet various professionals in person, and eventually even went to two conferences and delivered a lecture at each. Afterwards, Ammann dropped out of sight, and died of a heart attack a few years later. News of his death did not reach the research community for a few more years

5. Almost periodic patterns

Definition 1. *If* Λ *is a discrete set,* $R > 0$*, and if*

$$D_R^+(\Lambda) = \sup_{x \in \mathbb{R}^n} |B(x,R)|^{-1} \#[(B(x,R)) \cap \Lambda] \tag{5.1}$$

The uniform upper density of Λ *is defined by*

$$D^+(\Lambda) = \limsup_{R \to \infty} D_R^+(\Lambda) \tag{5.2}$$

The symmetric difference between $A, B \subset \mathbb{R}^n$ is denoted by $A \triangle B$. Almost periodic patterns are defined as follows.

Definition 2. *A Delone set* Λ *is an almost periodic pattern if and only if for every positive* ϵ *there exists a* $R(\epsilon) > 0$ *and a relatively dense set* $M(\epsilon)$ *such that*

$$R \geq R(\epsilon),\ \tau \in M(\epsilon) \Rightarrow D_R^+[(\Lambda + \tau) \triangle \Lambda] \leq \epsilon \tag{5.3}$$

This implies the weaker property

$$D^+[(\Lambda + \tau) \triangle \Lambda] \leq \epsilon \tag{5.4}$$

Robert Moody introduced an even weaker property in [3] where the uniform upper density is replaced by

$$d(\Lambda) = \limsup_{R \to \infty} |B(0,R)|^{-1} \#[(B(0,R)) \cap \Lambda] \tag{5.5}$$

and (5.4) by

$$d[(\Lambda + \tau) \bigtriangleup \Lambda] \leq \epsilon \tag{5.6}$$

Here is an example of a set of integers which is almost periodic if the definition given by Robert Moody is used but is not an almost periodic pattern. Let $E \subset \mathbb{Z}$ be the union of the intervals $[2^j, 2^j + j]$, $j \geq 1$, and let $\Lambda = \mathbb{Z} \setminus E$. For every τ we have $d[(\Lambda + \tau) \bigtriangleup \Lambda] = 0$. However Λ is not an almost periodic pattern. We argue by contradiction and assume that for every $\epsilon > 0$ there exists a relatively dense set $M(\epsilon)$ of almost periods of Λ. Therefore for every j there exists a $\tau \in M(\epsilon)$ such that $|\tau - j| \leq C(\epsilon)$. Then $[2^j, 2^j + j]$ is disjoint from $[2^j, 2^j + j] + \tau$ up to an interval of length less than $2C(\epsilon)$. It implies $D^+[(\Lambda+\tau) \bigtriangleup \Lambda] \geq 1 - 2C(\epsilon)/j$ and the expected contradiction will be reached when $1 - 2C(\epsilon)/j > \epsilon$.

6. Regular model sets are almost periodic patterns

A regular model set Λ is defined by the cut and projection method. Let us assume that $\Gamma \subset \mathbb{R}^{n+m}$ is a lattice. Then $p_1 : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ is defined by $p_1(x, y) = x$ when $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and similarly $p_2(x, y) = y$. We are assuming that $p_1 : \Gamma \mapsto \mathbb{R}^n$ is injective and that $p_2 : \Gamma \mapsto \mathbb{R}^m$ has a dense range. Let $W \subset \mathbb{R}^m$ be a Riemann integrable compact set.

Definition 3. *With these notations the regular model set Λ is defined by*

$$\Lambda = \{\lambda = p_1(\gamma); \;\; \gamma \in \Gamma, \;\; p_2(\gamma) \in W\} \tag{6.1}$$

We know that Λ has a uniform density given by dens $\Lambda = D^+(\Lambda) = c|W|$ where $c = c(\Gamma)$ and $|W|$ is the Lebesgue measure of W. Keeping in mind Definition 2 we have

Theorem 2. *Regular model sets are almost periodic patterns in the sense given by Definition* 2.

In other terms for every positive ϵ there exists a Delone set $M(\epsilon)$ such that (5.3) holds for every $\tau \in M(\epsilon)$ and for every $R \geq R(\epsilon)$. Most of the points in the model set Λ also belong to $\tau + \Lambda$ (this set of "good points" in Λ depends on τ).

The proof is not difficult. Let $N(\eta)$ the model set defined by

$$(6.2) \qquad N(\eta) = \{x = p_1(\gamma);\ \ \gamma \in \Gamma,\ \ |p_2(\gamma)| \leq \eta\}$$

Then $N(\eta)$ is the $M(\epsilon)$ we are looking for if η is small enough. More precisely we have

Lemma 1. *There exists a regular model set $Q(\epsilon)$ such that $D^+(Q(\epsilon)) \leq \epsilon$ and if η is small enough*

$$(6.3) \qquad \tau \in N(\eta) \Rightarrow (\Lambda + \tau) \triangle \Lambda \subset Q(\epsilon)$$

Lemma 1 obviously implies Theorem 2.

Let us treat the set $\Lambda + \tau \setminus \Lambda$ when $\tau \in N(\eta)$. The treatment of $\Lambda \setminus \Lambda + \tau$ will be similar. If $x \in \Lambda + \tau$ we have $x = p_1(\gamma) + p_1(\gamma_0) = p_1(\gamma + \gamma_0)$ where γ, $\gamma_0 \in \Gamma$, $p_2(\gamma) \in W$ and $|p_2(\gamma_0| \leq \eta$. If $x \notin \Lambda$ we have $p_2(\gamma + \gamma_0) \notin W$. It implies that $p_2(\gamma+\gamma_0) \in W_\eta$ where $W_\eta \subset W$ is defined as the set of points $y \notin W$ such that the distance from y to the boundary of W does not exceed η. We have proved the following

$$(6.4) \qquad \Lambda + \tau \setminus \Lambda \subset Q_\eta = \{\lambda = p_1(y);\ \ y \in \Gamma,\ \ p_2(y) \in W_\eta\}$$

Let us stress that the model set Q_η does not depend on τ. We now observe that $|W_\eta|$ tends to 0 as η tends to 0. The uniform density of the model set Q_η defined by the window W_η does not exceed ϵ if η is small enough and we then set $Q(\epsilon) = Q_\eta$. As it was said the treatment of $\Lambda \setminus \Lambda + \tau$ is similar, W_η being replaced by the set W^η

of points $y \in W$ such that the distance from y to the boundary of W does not exceed η. This ends the proof.

Is the converse implication true? Let us assume that a Delone set Λ is an almost periodic pattern. Is $\Lambda - \Lambda$ a Delone set? Here is a one dimensional counter example.

Lemma 2. *Let $\Lambda = \cup_0^\infty \Lambda_j$ where $\Lambda_j = 2^j + r_j + 2^{j+1}\mathbb{Z}$. If $r_j \in (0, 1/3]$ then Λ is an almost periodic pattern. If $r_j \in (0, 1/3]$ tends to 0 as j tends to infinity then $\Lambda - \Lambda$ cannot be a Delone set.*

Let us observe that $2^j + 2^{j+1}\mathbb{Z} = 2^j\mathbb{Z} \setminus 2^{j+1}\mathbb{Z}$ which implies that Λ is a Delone set. Moreover Λ is an almost periodic pattern since $\Lambda_0 \cup \ldots \cup \Lambda_{j-1}$ is 2^j-periodic and the uniform upper density of $\Lambda_j \cup \ldots$ is 2^{-j}. Let us directly check that $\Lambda - \Lambda$ is not a Delone set. We have $2^j \in 2\mathbb{Z} = \Lambda_0 - \Lambda_0$. Moreover $2^j + r_j + 2^k \in \Lambda_j$ if $k \geq j+1$. Finally $r_k + 2^k \in \Lambda_k$ which implies $2^j + r_j - r_k \in \Lambda - \Lambda$. But 2^j also belongs to $\Lambda - \Lambda$. Therefore $\Lambda - \Lambda$ cannot be a Delone set since $r_j - r_k$, $k \geq j+1$, tends to 0 as j tends to infinity. Let us observe that Λ is also an almost periodic pattern if the definition given in [2] is adopted.

7. Large patches of model sets

Let $B(x,R) \subset \mathbb{R}^n$ be the ball centered at x with radius R. Patches of Λ are defined as $\mathcal{P}(x,R) = \Lambda \cap B(x,R),\ x \in \Lambda,\ R > 0$ and the corresponding centered patch is $\mathcal{P}_0(x,R) = \mathcal{P}(x,R) - x$. Finally $\#E$ denotes the number of elements of E. Then we have

Theorem 3. *Let Λ be an almost periodic pattern in the sense given by Definition 2. Then for every positive ϵ there exist two positive numbers $R(\epsilon)$ and $C(\epsilon)$ such that the following property holds:*

$\forall x, \forall y \in \Lambda$ *there exists a translation* $\tau \in \mathbb{R}^n,\ \tau = \tau(x,y,\epsilon),\ |\tau| \le C(\epsilon)$ *such that:*

(7.1)
$$\forall R \ge R(\epsilon), \quad \#[\mathcal{P}_0(x,R) \bigtriangleup (\mathcal{P}_0(y,R) - \tau)] \le \epsilon R^n$$

This was discovered by Pierre-Antoine Guihéneuf. We do not know if (5.1) characterizes almost periodic patterns. The proof of Theorem 3 gives more. Indeed there exists a Delone set $M(\epsilon)$ such that one can impose $y - x - \tau \in M(\epsilon)$ in (7.1). This improved statement is then a characterization of almost periodic patterns.

Property (7.1) is obvious if $|x-y| \le R_0$ and $R \ge R_0(C_n\epsilon)^{-1}$. Then (7.1) holds with $\tau = 0$. Indeed we then have $|B(y,R) \bigtriangleup B(x,R)| \le C_n|x-y|R^{n-1} \le \epsilon R^n$. It yields (7.1) since Λ is a Delone set. It is the trivial case. Property (7.1) is only relevant if the distance between x and y is extremely large.

Theorem 3 in its full generality follows easily from the trivial case and from Lemma 1. We obviously have

(7.2)
$$B(y,R)\cap[(\Lambda+\tau)\triangle\Lambda]=\mathcal{P}(y,R)\triangle(\mathcal{P}(y-\tau,R)+\tau)$$

Therefore Λ is an almost periodic pattern if and only if there exists a relatively dense Delone set $M(\epsilon)$ such that $y\in\mathbb{R}^n,\ R\geq R(\epsilon),\ \tau\in M(\epsilon)$ implies

$$\frac{\#[\mathcal{P}(y,R)\triangle(\mathcal{P}(y-\tau,R)+\tau)]}{|B(y,R)|}\leq\epsilon \tag{7.3}$$

This is a restatement of Lemma 1 and it settles the case of the two patches $\mathcal{P}(y,R)$ and $\mathcal{P}(y-\tau,R)$. To compare $\mathcal{P}(y,R)$ to $\mathcal{P}(x,R)$ it suffices to observe that there exists a $\tau\in M(\epsilon)$ such that $|y-x-\tau|\leq C(\epsilon)$. We are finally led to compare $\mathcal{P}(x,R)$ to $\mathcal{P}(y-\tau,R)$ which is the trivial case.

8. Almost periodic measures

Let us begin with the definition by Hermann Weyl of almost periodic functions.

Definition 4. *The Weyl norm $\|f\|_{w,1}$ of a function $f\in L^1_{loc}(\mathbb{R}^n)$ is defined by*

(8.1)
$$\|f\|_{w,p}=\limsup_{R\to\infty}\Big[\sup_{x\in\mathbb{R}^n}\frac{1}{|B(x,R)|}\int_{B(x,R)}|f(y)|\,dy\Big]$$

whenever the right hand side of (8.1) *is finite.*

The space $\mathcal{W}_1$ of almost periodic functions in the sense of H. Weyl is defined as follows.

Definition 5. *A function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to $\mathcal{W}_1$ if it is the limit for the norm $\|\cdot\|_{w,1}$ of a sequence of generalized trigonometric polynomials.*

Besicovitch used a less demanding norm defined as

$$\|f\|_{b,1} = \limsup_{R\to\infty} \frac{1}{|B(0,R)|} \int_{B(0,R)} |f(y)|\, dy \tag{8.2}$$

Definition 6. *A function $f \in L^1_{loc}(\mathbb{R}^n)$ is almost periodic in the sense of Besicovitch if it is the limit for the norm $\|\cdot\|_{b,1}$ of a sequence of generalized trigonometric polynomials. We then write $f \in \mathcal{B}_1$.*

The Besicovitch space $\mathcal{B}_1$ is larger than the Weyl space $\mathcal{W}_1$. For example we have

Proposition 1. *The function $f(x)$ of the real variable x defined by*

$$f(x) = \sum_1^\infty \frac{1}{n} \sin(2^{-n}x) \tag{8.3}$$

belongs to $\mathcal{B}_1$ but not to $\mathcal{W}_1$.

Definition 7. *The Weyl norm of a Borel measure* μ *on* $\mathbb{R}^n$ *is defined by*

$$\|\mu\|_w = \limsup_{R\to\infty} \sup_{x\in\mathbb{R}^n} |B(x,R)|^{-1}|\mu|(B(x,R)) \tag{8.4}$$

whenever the right hand side of (8.4) *is finite.*

If $\mu = f\,dx$ where f is locally integrable then $\|\mu\|_w = \|f\|_{w,1}$.

For $\tau \in \mathbb{R}^n$ we denote by μ_τ the measure μ translated by τ. Almost periodic measures are defined as follows.

Definition 8. *A Borel measure* μ *is almost periodic in the sense of Weyl if for every positive* ϵ *there exists a relatively dense set* $M(\epsilon) \subset \mathbb{R}^n$ *such that*

$$\tau \in M(\epsilon) \Rightarrow \|\mu_\tau - \mu\|_w \le \epsilon \tag{8.5}$$

Proposition 2. *Let r_k, $k \in \mathbb{Z}$, be a sequence of real numbers tending to* 0*. Then the perturbed Dirac comb $\sigma = \sum_{k\in\mathbb{Z}} \delta_{k+r_k}$ is an almost periodic measure if and only if the upper density of the set $E = \{k;\, r_k \neq 0\}$ is* 0.

If $\tau \notin \mathbb{Z}$ then $k+r_k \neq j+r_j+\tau$ for $|k|, |j| \geq k_0$ which implies $\|\sigma_\tau - \sigma\|_w \geq 2$. Then $\|\sigma_\tau - \sigma\|_w \leq 1$ implies $\tau \in \mathbb{Z}$. Finally the conclusion follows immediately from Definition 8.

Definition 9. *A Borel measure μ is uniformly almost periodic if for every positive ϵ there exists a relatively dense set $M(\epsilon)$ and a positive number $R(\epsilon)$ such that $\tau \in M(\epsilon),\ x \in \mathbb{R}^n,\ R \geq R(\epsilon)$ implies*

$$\frac{1}{|B(x,R)|}\int_{B(x,R)} d\,|\mu_\tau - \mu| \leq \epsilon \tag{8.6}$$

The sum $\sigma = \sum_{k=1}^{\infty} \delta_k$ is the right half of the Dirac comb. It is an example of an almost periodic measure which is not uniformly almost periodic. Here $M(\epsilon) = \mathbb{Z}$ and $\sigma_\tau - \sigma = -\sum_{k=1}^{\tau} \delta_k$ if $\tau \geq 1$. When $\tau \leq -1$ we have $\sigma_\tau - \sigma = \sum_{k=\tau+1}^{0} \delta_k$. Therefore

$$\|\sigma_\tau - \sigma\|_w = 0 \quad (\forall \tau \in \mathbb{Z}) \tag{8.7}$$

Therefore σ is almost periodic in the sense of Weyl. It is not uniformly almost periodic. Indeed (8.6) implies $R \geq |\tau|/\epsilon$. Uniformity with respect to τ makes the difference between (8.5) and (8.6). An equivalent definition of an almost periodic pattern is given now.

Proposition 3. *A Delone set Λ is an almost periodic pattern if the sum of Dirac masses $\mu = \sum_{\lambda \in \Lambda} \delta_\lambda$ is a uniformly almost periodic measure (in the sense given by Definition* 9*).*

References

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[3] Robert V. Moody *Mathematical Quasicrystals: a tale of two topologies.* Proceedings of the International Congress of Mathematical Physics, Lisbon (2003).