L^p-Estimates of the Bergman Projection in some homogeneous domains of $\mathbb{C}^n$.

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- We present here more recent results about continuity of the Bergman projector on convex homogeneous domains of $\mathbb{C}^n$.
- We shall start by recalling some fundamental definitions and examples to fix our idea.
- Later we state the problem, the tools used to investigate solutions and present the results.
- We shall state and prove at the end of this presentation the Korányi's Lemma.

Siegel Domains of $\mathbb{C}^n$.

Definition

- *A subset Ω of $\mathbb{R}^n$ is a cone if $\forall x \in C$, $\lambda > 0$, $\lambda x \in \Omega$.*
- *A cone Ω is said to be convex if $\forall x, y \in \Omega, \lambda x + (1-\lambda)y \in \Omega,\ 0 < \lambda < 1$.*
- *A cone Ω is said to be homogeneous if the group $G(\Omega)$ of all linear transformations of $\mathbb{R}^n$ leaving invariant the cone acts transitively on Ω.*
- *A cone C is said to be symmetric if it is homogeneous and identical to its dual cone*

$$C^* = \{\xi \in \mathbb{R}^n : (\xi|x) > 0,\ \forall x \in \bar{C} \setminus \{0\}\}.$$

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- *Let Ω be an open convex cone in $\mathbb{R}^n$. The point set $T_\Omega = \mathbb{R}^n + i\Omega = \{z \in \mathbb{C}^n : \Im m\, z \in \Omega\}$ in $\mathbb{C}^n$ is called a Siegel domain of type I or a tube domain over the cone Ω in $\mathbb{C}^n$.*

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Definition

Let $m, n \in \mathbb{N}$.

- *Let Ω be a convex homogenous cone. The linear functional $F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ satisfying*
  1. *$F(u, u) = 0$ if and only if $u = 0$;*
  2. *$F(u, u) \in \overline{\Omega}$, for all $u \in \mathbb{C}^m$*

  where $\overline{\Omega}$ is the closure of Ω is called a Ω-Hermitian form.
- *The point set $D(\Omega, F) = \{(z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \Im m\, z - F(u, u) \in \Omega\}$ in $\mathbb{C}^{m+n}$ is called a Siegel domain of type II over Ω.*

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Examples of Siegel Domains

Example

- As an example of a Siegel domain of type I, we have the upper half plane of $\mathbb{C}$, $\Pi^+ = \mathbb{R} + i(0, +\infty)$.
- Let $\Omega = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 > 0,\ y_2 - \frac{y_3^2}{y_1} > 0\}$ be the spherical cone of $\mathbb{R}^3$. Define $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^3$ by

$$F(u, v) = (u\bar{v}, 0, 0).$$

  Then the point set $D = \{(z, u) \in \mathbb{C}^3 \times \mathbb{C} : \Im m\, z - F(u, u) \in \Omega\}$ is a Siegel domain of type II in $\mathbb{C}^4$ called the Piatetski-Shapiro domain.

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Isomorphic domains of $\mathbb{C}^n$

Let D and D' be two domains of $\mathbb{C}^n$. We denote by $dv(z) = dxdy$, $(z = x + iy \in \mathbb{C}^n)$ the Lebesgue measure of $\mathbb{C}^n$.

Definition

- *We say that $f : D \to D'$ is a biholomorphism if F is a holomorphic bijection with its inverse holomorphic.*
- *The domains D and D' of $\mathbb{C}^n$ are said to be isomorphic if there is a biholomorphism that carries D onto D'.*
- *A biholomorphism $f : D \to D$ is called an automorphism of D.*

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- *We say that $f : D \to D'$ is a **biholomorphism** if F is a holomorphic bijection with its inverse holomorphic.*
- *The domains D and D' of $\mathbb{C}^n$ are said to be **isomorphic** if there is a biholomorphism that carries D onto D'.*
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We denote by $Aut(D)$ the group of all automorphisms of D.

Definition

- *The domain D is* homogeneous *if the group $Aut(D)$ acts transitively on D; that is, for all $z, w \in D$, there is $\Phi \in Aut(D)$ such that $z = \Phi(w)$.*
- *The domain D is* symmetric *if it is homogeneous and it exists $z_0 \in D$ and an involution $s \in Aut(D)$ such that z_0 is an isolated fixed point.*

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Isomorphic domains of $\mathbb{C}^n$

In 1963, E. B. Vinberg, S. G. Gindikin and I. I. Piatetski-Shapiro proved that homogenous Siegel domains of $\mathbb{C}^n$ are isomorphic to bounded homogeneous domains of $\mathbb{C}^n$.

Examples of Isomorphic domains of $\mathbb{C}$.

Example

- The unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{C}$ is a bounded symmetric domain of $\mathbb{C}$. The automorphisms of $\mathbb{C}$ are the Möbius transforms

$$\phi_a : \mathbb{D} \to \mathbb{D}, \;\; \phi_a(z) = \lambda \frac{z - a}{1 - \bar{a}z}, \; a \in \mathbb{D}, \; \lambda \in \mathbb{C}, \; |\lambda| = 1.$$

- The unit disc $\mathbb{D}$ of $\mathbb{C}$ is isomorphic to the upper half-plane $\Pi^+ = \{z \in \mathbb{C} : \Im m\, z > 0\}$ under the biholomorphic transformation $\Phi : U \to \Pi^+$ given by the Cayley transform

$$\Phi(z) = i\frac{1 + z}{1 - z}.$$

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Minimal domains of $\mathbb{C}^n$.

Proposition (Maschler, 1956)

- *A domain D is minimal (with centre z_0) if any other domain isomorphic to D has a volume greater than the volume of D.*
- *The volume of a minimal domain of centre z_0 is given by $\frac{1}{B(z_0, z_0)}$.*
- *A necessary and sufficient condition for a domain D to be a minimal domain with centre at z_0 is that $B(z, z_0) = \frac{1}{Vol(D)}$ for all $z \in D$.*

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Example

- The unit disc of $\mathbb{C}$ is a minimal domain centered at 0 since $B_{\mathbb{D}}(z,0) = \frac{1}{\pi} = \frac{1}{\text{Vol}(\mathbb{D})}$.
- The upper half plane of $\mathbb{C}$ is not minimal since it is isomorphic to the unit disc with a bigger volume.
- Every circular domain of $\mathbb{C}^n$, is a minimal domain.
- It is not true that every minimal domain is circular.
- Actually the Cayley transform image of a homogeneous non symmetric Siegel domain of $\mathbb{C}^n$ is a minimal domain of $\mathbb{C}^n$.

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The Lebesgue spaces L^p-The Bergman space

Let Ω be a region in $\mathbb{C}^n$ and $p \in [1, +\infty[$. We denote $dv(z) = dxdy$ the Lebesgue measure of $\mathbb{C}^n$.

- We denote $L^p(\Omega, dv)$ the class of measurable functions f satisfying the estimate

$$\int_\Omega |f(z)|^p dv(z) < +\infty.$$

- The space $L^p(\Omega, dv)$ is equipped with the norm

$$\|f\|_p = \left(\int_\Omega |f(z)|^p dv(z)\right)^{\frac{1}{p}}$$

since this norm is complete, $L^p(\Omega, dv)$ a Banach space. (Fischer-Riesz).

- The Bergman space $A^p(\Omega, dv)$ is the closed subspace of $L^p(\Omega, dv)$ consisting of holomorphic functions. That is

$$A^p(\Omega, dv) = L^p(\Omega, dv) \cap H(\Omega).$$

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The case $p = 2$.

- The space $L^2(\Omega, dv)$ is equipped with the inner product

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g(z)}dv(z).$$

- Hence $L^2(\Omega, dv)$ is a Hilbert space. Since $A^2(\Omega, dv)$ is closed in the Hilbert space $L^2(\Omega, dv)$, it is also a Hilbert space.
- Let $z \in \Omega$. The map $\phi_z : A^2(\Omega, dv) \to \mathbb{C}$ defined by

$$\phi_z(f) = f(z)$$

  if a continuous linear functional.
- It follows from the Riesz representation theorem that there is a unique function $b_z \in A^2(\Omega, dv)$ such that

$$f(z) = \langle f, b_z \rangle = \int_\Omega f(w)\overline{b_z(w)}dv(w), \ \forall f \in A^2(\Omega, dv).$$

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- The space $L^2(\Omega, dv)$ is equipped with the inner product

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Bergman kernel-Bergman Projection

For all $z, w \in \Omega$, define $B(z, w) = \overline{b_z(w)}$. Then the previous identity becomes

$$f(z) = \langle f, b_z \rangle = \int_\Omega f(w)B(z, w)dv(w), \ \forall f \in A^2(\Omega, dv). \quad (1)$$

- The function $B : \Omega \times \Omega \to \mathbb{C}$ is called the Bergman kernel for Ω. Thanks to (1), the Bergman kernel is the reproducing kernel of the Bergman space $A^2(\Omega, dv)$.
- The orthogonal projection P of the Hilbert space $L^2(\Omega, dv)$ onto its closed subspace $A^2(\Omega, dv)$ is called the Bergman projection. It is an integral operator defined by

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Main problem

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Let D be a homogeneous Siegel domain. Determine the best possible values of $p \in [1, +\infty]$ such that the Bergman projector P extends to a bounded operator from $L^p(\mathrm{D})$ onto the Bergman space $A^p(\mathrm{D})$.

Scientific Motivations

Corollaries

When the Bergman projection P is continuous,

- we have a good knowledge of the dual of the Bergman space $A^p(D, dv)$;
  - this dual is identified with the Bergman space $A^{p'}(D, dv)$ where p' is the conjugate index of p;
- we have a molecular or atomic decomposition of functions in the Bergman space $A^p(D, dv)$; (Coifman and Rochberg).

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  - every function $f \in A^p(\mathrm{D}, dv)$ has this type of decomposition

$$f = \sum_j \lambda_j B(z_j, z_j)^{\alpha} B(\cdot, z_j)^{1-\alpha}$$

where the sequence $\{z_j\}_j$ is a δ-lattice in D, the sequence $\{\lambda_j\}_j \in l^p$ and α is a real number depending of D and the dimension.

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Tools used

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  - The converse is not true.
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Tube domain over the Lorentz cone

The following results have been obtained these last few years : Tube domain over the Lorentz cone $T_{\Lambda_n} = \mathbb{R}^n + i\Lambda_n$ where the Lorentz cone is defined by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1 > 0,\ y_1^2 - y_2^2 - \cdots - y_n^2 > 0\},\ n \geq 3.$$

Theorem (D. Békollé and A. Bonami, 1995 ; D. Békollé, A. Bonami, M. M. Peloso and F. Ricci, 2001)

- *The operator with positive Bergman kernel $P^+ : L^p(T_{\Lambda_n}) \to L^p(T_{\Lambda_n})$ is bounded if and only if $\frac{2n-2}{n} < p < \frac{2n-2}{n-2}$. Therefore the Bergman projector $P : L^p(T_{\Lambda_n}) \to A^p(T_{\Lambda_n})$ is bounded if $\frac{2n-2}{n} < p < \frac{2n-2}{n-2}$.*
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Tube domain over convex symmetric cone

Consider the tube domains over general symmetric cones $T_\Omega = \mathbb{R}^n + i\Omega$ where Ω is a convex symmetric cone of rank r.

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The operator with positive Bergman kernel $P^+ : L^p(T_\Omega) \to L^p(T_\Omega)$ *is bounded if and only if* $(\frac{2n-r}{n-r})' < p < \frac{2n-r}{n-r}$.
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Homogeneous Siegel domains of type I

Tube domain over an example of homogeneous non-symmetric cone : the Vinberg cone (1963) defined by

$$V = \left\{ x = \begin{pmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & 0 \\ x_5 & 0 & x_3 \end{pmatrix} : Q_j(x) > 0,\ j = 1, 2, 3 \right\}$$

with $Q_1(x) = x_1,\ Q_2(x) = x_2 - \frac{x_4^2}{x_1},\ Q_3(x) = x_3 - \frac{x_5^2}{x_1}$. Its dual is given by

$$V^* = \left\{ \xi = \begin{pmatrix} \xi_1 & \xi_4 & \xi_5 \\ \xi_4 & \xi_2 & 0 \\ \xi_5 & 0 & \xi_3 \end{pmatrix} : Q_j^*(\xi) > 0,\ j = 1, 2, 3 \right\}$$

with $Q_1^*(\xi) = \xi_1 - \frac{\xi_4^2}{\xi_2} - \frac{\xi_5^2}{\xi_3},\ Q_2^*(\xi) = \xi_2,\ Q_3^*(\xi) = \xi_3$.

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- *The operator with positive Bergman kernel* $P^+ : L^p(T_V) \to L^p(T_V)$ *is bounded if* $\frac{3}{2} < p < 3$. *Therefore the Bergman projector* $P : L^p(T_V) \to A^p(T_V)$ *is bounded if* $\frac{3}{2} < p < 3$.
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Remarks and Perspectives

There are still many open questions concerning the L^p-boundedness of the Bergman projection in homogeneous Siegel domains. Actually,

- For the tube domain over the Lorentz cone T_{Λ_3}, the conjecture is that $\frac{7}{6} < p < 7$ and so far we have $\frac{5}{4} < p < 5$.
- It is not yet done the fact that the indices obtained for the operator P^+ are always necessary and sufficient. Nevertheless, we have not yet been able to exhibit a cone for which the necessary condition and the sufficient do not coincide.
- Investigations continue with the team D. Békollé (Cameroon), A. Bonami (France), G. Garrigós (Spain), C. Nana (Cameroon), M. Peloso (Italy), F. Ricci (Italy), B. F. Sehba (Cameroon), B. Trojan (Poland) and any other interested person !

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Korányi's Lemma

- This lemma consists of showing that the modulus of the Bergman kernel $B(z, \zeta)$, is "almost constant" uniformly with respect to z when ζ varies inside a Bergman ball.
- The control is expressed in terms of the Bergman distance.
- This result was proved by A. Korányi for symmetric Siegel domains of type II.
- R. R. Coifman and R. Rochberg used this result to establish an atomic decomposition theorem and an interpolation theorem by functions in Bergman spaces A^p on these domains.
- D. Békollé and A. Temgoua proved later Korányi's Lemma for two homogeneous non symmetric domains of $\mathbb{C}^4$ and $\mathbb{C}^5$ respectively and extended to these two domains results by R. R. Coifman and R. Rochberg.

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- All these results were recently generalized to general homogeneous Siegel domain of type II.
- Let $\mathcal{D} \subset \mathbb{C}^n$ be a homogeneous Siegel domain of type II and let $B : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ denote the Bergman kernel of $\mathcal{D}$.

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For every $\rho > 0$, there exists a constant $M_\rho > 0$ such that

$$\left| \frac{B(z,\zeta_2)}{B(z,\zeta_1)} - 1 \right| \leq M_\rho d(\zeta_1,\zeta_2) \qquad (2)$$

for all $z, \zeta_1, \zeta_2 \in \mathcal{D}$ with $d(\zeta_1,\zeta_2) < \rho$.

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The proof is based on the uniform boundedness of Cayley transform images of the homogeneous Siegel domain $\mathcal{D}$ of type II. We shall give some elements of the proof here.

- Let $\sigma : \mathcal{D} \to \sigma(\mathcal{D})$ be a biholomorphic mapping such that $\sigma((ie, 0)) = 0$. The domain $D := \sigma(\mathcal{D})$ is a homogeneous bounded domain.
- Precisely, D is a bounded minimal domain of $\mathbb{C}^n$ with center 0 (Ishi-Kai, 2010),
- so that the Bergman kernel B_D of the domain D has the following property :

$$B_D(w, 0) = \frac{1}{\text{Vol}(D)} \qquad (w \in D). \tag{3}$$

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where B denotes again the Bergman kernel of $\mathcal{D}$.

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The key points of the proof are the following :

- this lemma already used by Békollé and Temgoua

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There exists a positive constant $C > 0$ such that

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for all $w \in \mathcal{K}$ and $z \in \mathcal{D}$.

- the following well known change of variable formula for the Bergman kernel

$$B_D(z, w) = B_{\mathcal{D}}(\sigma(z), \sigma(w)) J_{\mathbb{C}}\sigma(z)\overline{J_{\mathbb{C}}\sigma(w)}.$$

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Proposition (Ishi-Yamaji, 2011)

Let D be a minimal domain of $\mathbb{C}^n$. *For any* $\rho > 0$, *there exists a constant* $C_\rho > 0$ *such that*

$$C_\rho^{-1} \leq \left| \frac{B(z, a)}{B(a, a)} \right| \leq C_\rho$$

for all $z, a \in D$ *for which* $d(z, a) < \rho$.

THANK YOU FOR YOUR KIND ATTENTION