On Toeplitz products on Bergman space and two-weighted inequalities for the Bergman projection

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Outline

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1. Toeplitz products on the Hardy space
2. Toeplitz products on the Bergman space
3. A dyadic model and a two-weight result for P_B^+

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3. A dyadic model and a two-weight result for P_B^+
4. Sharp Békollé bounds for one weight

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1. Toeplitz products on the Hardy space
2. Toeplitz products on the Bergman space
3. A dyadic model and a two-weight result for P_B^+
4. Sharp Békollé bounds for one weight
5. Characterisation and counterexamples

Outline

The Riesz Projection

Let $\mathbb{D}$ be the unit disk. Recall the Hardy space

$$H^p(\mathbb{T}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \sup_{r<1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(re^{it})|^p dt < \infty \right\}, \qquad (1 \leq p \leq \infty)$$

and the Riesz projection

$$P_R : L^2(\mathbb{D}) \to H^2(\mathbb{D}),$$

$$P_R f(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1 - z\bar{\xi}} d|z|.$$

Toeplitz products on the Hardy space

Let $f \in H^2$. It is well-known that the Toeplitz operator

$$T_f : H^2 \to H^2, \quad h \mapsto P_R f h \quad (h \in H^2)$$

is bounded, if and only if $f \in L^\infty$.

Question, Sarason 1990

For which $f, g \in H^2$ is the Toeplitz product

$$T_g T_f^* : H^2 \to H^2$$

bounded ?

Toeplitz Products and weighted inequalities on the Hardy space

Cruz-Uribe 1992:

$$\begin{array}{ccccc} & H^2(\mathbb{T}) & \xrightarrow{T_g T_f^*} & H^2 & \\ M_{\bar{f}} & \downarrow & & \uparrow & M_g \\ & L^2(\frac{1}{|f|^2}, \mathbb{T}) & \xrightarrow{P_R} & H^2(|g|^2, \mathbb{T}) & \end{array}$$

Toeplitz Products and weighted inequalities on the Hardy space

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Two-weighted Hilbert transform/Riesz projection

When is

$$P_R : L^2(\frac{1}{|f|^2}, \mathbb{T}) \longrightarrow L^2(|g|^2, \mathbb{T})$$

bounded?

Weighted Riesz projection

For which weights w is $P_R : L^p_w(\mathbb{T}) \to H^p_w(\mathbb{T})$ bounded ?

Weighted Riesz projection

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Definition (A_p-condition)

Let $1 < p < \infty$. Then $w \in A_p$, if

$$A_p(w) := \sup_{I \subseteq \mathbb{T}} \left(\frac{1}{|I|} \int_I w(t)dt \right) \left(\frac{1}{|I|} \int_I w^{1-p'}(t)dt \right)^{\frac{1}{p'-1}} < \infty,$$

where I is an interval in $\mathbb{T}$.

For $p = 2$, this is equivalent to

The invariant A_2 condition

$$w \in A_2 \Leftrightarrow \sup_{z \in \mathbb{D}} \mathcal{P}(w)(z)\mathcal{P}(w^{-1})(z) < \infty,$$

where $\mathcal{P}$ denotes the Poisson extension.

Hunt-Muckenhoupt-Wheeden Theorem

Theorem (Hunt, Muckenhoupt, Wheeden)

Let $1 < p < \infty$. *Then*

$$\begin{aligned} & P_R : L^p_w(\mathbb{T}) \to L^p_w(\mathbb{T}) \text{ bounded} \\ \Leftrightarrow \quad & w \in A_p. \end{aligned}$$

Sarason conjecture for Hardy space

Joint A_2 conjecture

$$P_R : L^2_w(\mathbb{T}) \to L^2_v(\mathbb{T}) \text{ bounded} \Leftrightarrow \sup_{z\in\mathbb{D}} \mathcal{P}(v)(z)\mathcal{P}(w^{-1})(z) < \infty?$$

Sarason Conjecture

$$T_g T_f^* : H^2 \to H^2 \text{ bounded} \Leftrightarrow \sup_{z\in\mathbb{D}} \mathcal{P}(|f|^2)(z)\mathcal{P}(|g|^2)(z) < \infty?$$

Saracon conjecture for Hardy space

Joint A_2 conjecture

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$$T_g T_f^* : H^2 \to H^2 \text{ bounded} \Leftrightarrow \sup_{z\in\mathbb{D}} \mathcal{P}(|f|^2)(z)\mathcal{P}(|g|^2)(z) < \infty?$$

Answer: No ! (F.Nazarov, around 2000)
Extensive history of the question (Cotlar, Lacey, Nazarov, Sadosky, Sawyer, Shen, Treil, Volberg, Uriarte-Tuero .....)

Outline

The Bergman Projection

Let $\mathbb{D}$ be the unit disk. Recall the Bergman space

$$A^p(\mathbb{D}) = L^p(\mathbb{D}) \cap \mathrm{Hol}(\mathbb{D}) \qquad (1 \leq p \leq \infty)$$

and the Bergman projection

$$P_B : L^2(\mathbb{D}) \to A^2(\mathbb{D}),$$

$$P_B f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{(1 - z\bar{\xi})^2} dA(\xi).$$

Toeplitz products on the Bergman space

Let $f \in A^2$. It is well-known that the Toeplitz operator

$$T_f : A^2 \to A^2, \quad h \mapsto P_B f h \quad (h \in A^2)$$

is bounded, if and only if $f \in H^\infty$.

Question

For which $f, g \in A^2$ is the Toeplitz product

$$T_g T_f^* : A^2 \to A^2$$

bounded ?

Toeplitz Products and weighted inequalities on the Bergman space

Cruz-Uribe 1992, Zheng/Stroethoff 1999:

$$
\begin{array}{cccccc}
 & A^2(\mathbb{D}) & \xrightarrow{T_g T_f^*} & A^2(\mathbb{D}) & \\
M_{\bar f} & \downarrow & & \uparrow & M_g \\
 & L^2(\frac{1}{|f|^2}, \mathbb{D}) & \xrightarrow{P_B} & A^2(|g|^2, \mathbb{D}) &
\end{array}
$$

Toeplitz Products and weighted inequalities on the Bergman space

Cruz-Uribe 1992, Zheng/Stroethoff 1999:

$$\begin{array}{ccccc} & A^2(\mathbb{D}) & \xrightarrow{T_g T_f^*} & A^2(\mathbb{D}) & \\ M_{\bar f} & \downarrow & & \uparrow & M_g \\ & L^2(\frac{1}{|f|^2}, \mathbb{D}) & \xrightarrow{P_B} & A^2(|g|^2, \mathbb{D}) & \end{array}$$

Two-weighted Bergman projection

When is

$$P_B : L^2(\frac{1}{|f|^2}, \mathbb{D}) \longrightarrow L^2(|g|^2, \mathbb{D})$$

bounded?

The Bergman projection and positive Bergman projection

Theorem

$P_B : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ *bounded for* $1 < p < \infty$.

The Bergman projection and positive Bergman projection

Theorem

$P_B : L^p(\mathbb{D}) \to A^p(\mathbb{D})$ *bounded for* $1 < p < \infty$.

Moreover,

Theorem

$P_B^+ : L^p(\mathbb{D}) \to L^p(\mathbb{D})$ *bounded for* $1 < p < \infty$,

where

$$P_B^+ f(z) = \int_{\mathbb{D}} \frac{f(\xi)}{|1 - z\bar{\xi}|^2} dA(\xi).$$

Weighted Bergman projection

Question

Let $w \geq 0$ be a weight function on $\mathbb{D}$. For which weights w is $P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D})$ bounded ?

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Let $w \geq 0$ be a weight function on $\mathbb{D}$. For which weights w is $P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D})$ bounded ?

Definition

Let $1 < p < \infty$. Then $w \in B_p$, if

$$B_p(w) := \sup_{I \subseteq \mathbb{T}} \left(\frac{1}{|Q_I|} \int_{Q_I} w(z) dA(z) \right) \left(\frac{1}{|Q_I|} \int_{Q_I} w^{1-p'}(z) dA(z) \right)^{\frac{1}{p'-1}} < \infty,$$

where Q_I is the Carleson box

$$Q_I = \{ r\xi : 1 - |I| \leq r < 1, \xi \in I \}.$$

The Theorem of Bonami and Békollé

Theorem (Bonami, Békollé)

Let $1 < p < \infty$. *Then*

$$\begin{aligned} & P_B : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D}) \text{ bounded} \\ \Leftrightarrow \quad & P_B^+ : L^p_w(\mathbb{D}) \to L^p_w(\mathbb{D}) \text{ bounded} \\ \Leftrightarrow \quad & w \in B_p. \end{aligned}$$

"Little" cancellation of the operator!

Joint B_2 condition

Joint B_2 conjecture for weights (w, v)

$$P_B : L^2_w(\mathbb{D}) \to L^2_v(\mathbb{D}) \text{ bounded} \Leftrightarrow$$
$$\sup_{I \subseteq \mathbb{T}} \left(\frac{1}{|Q_I|} \int_{Q_I} v(z)dA(z) \right) \left(\frac{1}{|Q_I|} \int_{Q_I} w^{-1}(z)dA(z) \right) < \infty$$

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Answer: No!

Joint Berezin B_2 condition

Joint Berezin B_2 conjecture for weights (w, v)

$$P_B : L^2_w(\mathbb{D}) \to L^2_v(\mathbb{D}) \text{ bounded } \Leftrightarrow \sup_{z\in\mathbb{D}} \mathcal{B}(v)(z)\mathcal{B}(w^{-1})(z) < \infty?$$

Here,

$$\mathcal{B}(f)(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{f(z)}{|1-\bar{w}z|^4} dA(w)$$

is the Berezin transform.

Joint Berezin B_2 condition

Joint Berezin B_2 conjecture for weights (w, v)

$$P_B : L^2_w(\mathbb{D}) \to L^2_v(\mathbb{D}) \text{ bounded } \Leftrightarrow \sup_{z \in \mathbb{D}} \mathcal{B}(v)(z)\mathcal{B}(w^{-1})(z) < \infty?$$

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is the Berezin transform.
Answer: No!

Saracon conjecture for Bergman space

Sarason Conjecture

$$T_g T_f^* : A^2 \to A^2 \text{ bounded} \Leftrightarrow \sup_{z\in\mathbb{D}} \mathcal{B}(|f|^2)(z)\mathcal{B}(|g|^2)(z) < \infty ?$$

Sarason conjecture for Bergman space

Sarason Conjecture

$$T_g T_f^* : A^2 \to A^2 \text{ bounded} \Leftrightarrow \sup_{z \in \mathbb{D}} \mathcal{B}(|f|^2)(z)\mathcal{B}(|g|^2)(z) < \infty?$$

Answer: No! (Aleman, P., Reguera 2013)

Outline

The dyadic model for P_B^+ on $\mathbb{C}^+$

Now we consider the Bergman projection P_B and P_B^+ on $\mathbb{C}^+$,

$$P_B f(z) = \int_{\mathbb{C}^+} \frac{f(\xi)}{(z-\bar{\xi})^2} dA(\xi).$$

Let

$$Q = \sum_{K \in \mathcal{D}} \frac{1}{|K|^2} \chi_{Q_K} \otimes \chi_{Q_K}.$$

For each $\beta \in \{0,1\}^{\mathbb{Z}}$, consider the dyadic grid $\mathcal{D}_\beta$ of intervals of the form $I + \sum_{i>j} 2^{-i}\beta_i$, where I is an dyadic interval of length 2^{-j} in $\mathcal{D}_0$.

Random grids

Lemma

For $\beta \in \{0,1\}^{\mathbb{Z}}$, let $Q^\beta = \sum_{K \in \mathcal{D}_\beta} \frac{1}{|K|^2} \chi_{Q_K} \otimes \chi_{Q_K}$. Let μ denote the canonical probability measure on $\{0,1\}^{\mathbb{Z}}$ such that the coordinates β_j are independent with $\mu(\beta_j = 0) = \mu(\beta_j = 1) = 1/2$. Let $K_\beta(z,w)$ denote the kernel of Q_β and $K_B^+(z,w)$ denote the kernel of P_B^+. Then for each $\beta \in \{0,1\}^{\mathbb{Z}}$,

$$K_\beta(z,w) \lesssim K_B^+(z,w) \lesssim \int_{\{0,1\}^{\mathbb{Z}}} K_{\beta'}(z,w) d\mu(\beta') \qquad (z,w \in \mathbb{C}^+)$$

The test function result for Q

Theorem

Let u, v be weights on $\mathbb{D}$. Then the following are equivalent:

1. $Q : L^2_u(\mathbb{D}) \to L^2_v(\mathbb{D})$ *bounded*
2. *There exist* K_1, K_2, K_3 *such that*
   1. *joint* B_2 *condition:*

$$\sup_{I \subset \mathbb{T} \text{ interval}} \left(\frac{1}{|Q_I|} \int_{Q_I} u^{-1}(z)dA(z) \right) \left(\frac{1}{|Q_I|} \int_{Q_I} v(z)dA(z) \right) \leq K_1$$

   2. *test condition* $\|Q_{in,I} u^{-1} \chi_{Q_I}\|_{L^2_v} \leq K_2 \|\chi_{Q_I}\|_{L^2_{u^{-1}}}$
   3. *adjoint test condition* $\|Q_{in,I} v \chi_{Q_I}\|_{L^2_{u^{-1}}} \leq K_3 \|\chi_{Q_I}\|_{L^2_v}$

Moreover, in this case $\|Q\|_{L^2_u \to L^2_v} \lesssim K_1 + K_2 + K_3$.

Proof: Adaptation of Eric Sawyer's test function result for positive kernels.

A two-weight result for P_B^+

Theorem

Let u, v be weights on $\mathbb{D}$. Then the following are equivalent:

1. $P_B^+ : L^2_u(\mathbb{D}) \to L^2_v(\mathbb{D})$ *bounded*
2. *There exist* K_1, K_2, K_3 *such that*
   1. *joint* B_2 *condition:*
   $$\sup_{I \subset \mathbb{T} \text{ interval}} \left(\frac{1}{|Q_I|} \int_{Q_I} u^{-1}(z)dA(z) \right) \left(\frac{1}{|Q_I|} \int_{Q_I} v(z)dA(z) \right) \leq K_1$$
   2. *test condition* $\|\chi_{Q_I} P_B^+ u^{-1} \chi_{Q_I}\|_{L^2_v} \leq K_2 \|\chi_{Q_I}\|_{L^2_{u^{-1}}}$
   3. *adjoint test condition* $\|\chi_{Q_I} P_B^+ v \chi_{Q_I}\|_{L^2_{u^{-1}}} \leq K_3 \|\chi_{Q_I}\|_{L^2_v}$

Moreover, in this case $\|P_B^+\|_{L^2_u \to L^2_v} \lesssim K_1 + K_2 + K_3$.

Proof of the two-weight result

Proof:

$$\text{test conditions for } P_B^+ \Rightarrow \text{test conditions for all } Q^\beta$$
$$\Rightarrow \text{ all } Q^\beta \text{ are uniformly bounded} \Rightarrow P_B^+ \text{ is bounded}$$

Outline

Sharp A_p estimates

Theorem (Hytönen)

Let T be a Calderón-Zygmund operator, and let w be an A_p weight. We have the inequality

$$\|T\|_{L^p(w)\mapsto L^p(w)} \lesssim \|w\|_{A_p}^{\max(1,\frac{1}{p-1})}.$$

and the estimate is sharp.

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Previous work

Petermichl-Volberg, Petermichl, Dragičević-Volberg, Wittwer, Beznosova, Lacey-Petermichl-Reguera, Cruz-Uribe-Martell-Pérez, Pérez-Treil-Volberg, Nazarov-Reznikov-Treil-Volberg, Lerner...

A_2-A_∞ estimates

Theorem (Hytönen, Pérez)

Let T be a Calderón-Zygmund operator, and let w be an A_p weight. We have the inequality

$$\|T\|_{L^p(w)\mapsto L^p(w)} \lesssim (\|w\|_{A_\infty} + \|w^{-1}\|_{A_\infty})^{1/2}\|w\|_{A_2}^{1/2}.$$

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A_2-A_∞ estimates

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$$\|T\|_{L^p(w)\mapsto L^p(w)} \lesssim (\|w\|_{A_\infty} + \|w^{-1}\|_{A_\infty})^{1/2}\|w\|_{A_2}^{1/2}.$$

and the estimate is sharp.

A B_∞-condition

Definition

w belongs to the class B_∞, if and only if

$$B_\infty(w) := \sup_{\substack{I \text{ interval} \\ I \subset \mathbb{T}}} \frac{1}{w(Q_I)} \int_{Q_I} M(w 1_{Q_I}) < \infty, \qquad (1)$$

where M stands for the Hardy-Littlewood maximal function over Carleson cubes.

A consequence of the two-weight theorem

Theorem (Aleman, P., Reguera)

Let $w \in B_2$ and let P_B^+ be the positive Bergman projection. Then

$$\|P_B^+\|_{L^2_w(\mathbb{D}) \to L^2_w(\mathbb{D})} \lesssim (B_\infty(w) + B_\infty(w^{-1}))^{1/2} B_2(w)^{1/2}$$

A consequence of the two-weight theorem

Theorem (Aleman, P., Reguera)

Let $w \in B_2$ and let P_B^+ be the positive Bergman projection. Then

$$\|P_B^+\|_{L^2_w(\mathbb{D})\to L^2_w(\mathbb{D})} \lesssim (B_\infty(w) + B_\infty(w^{-1}))^{1/2} B_2(w)^{1/2}$$

Corollary

$$\begin{aligned}\|P_B\|_{L^2_w(\mathbb{D})\to L^2_w(\mathbb{D})} &\lesssim (B_\infty(w) + B_\infty(w^{-1}))^{1/2} B_2(w)^{1/2} \\ &\lesssim B_2(w)\end{aligned}$$

A consequence of the two-weight theorem

Theorem (Aleman, P., Reguera)

Let $w \in B_2$ and let P_B^+ be the positive Bergman projection. Then

$$\|P_B^+\|_{L^2_w(\mathbb{D})\to L^2_w(\mathbb{D})} \lesssim (B_\infty(w) + B_\infty(w^{-1}))^{1/2} B_2(w)^{1/2}$$

Corollary

$$\|P_B\|_{L^2_w(\mathbb{D})\to L^2_w(\mathbb{D})} \lesssim (B_\infty(w) + B_\infty(w^{-1}))^{1/2} B_2(w)^{1/2} \lesssim B_2(w)$$

The result is sharp .

Proof

Set $u = w$ and check that the constants K_2, K_3 in the testing of Q^β satisfy the required bounds.

One can find sharp lower bounds using $w(z) = \left(1 - |z|^2\right)^{(1-\delta)}$.

Verifying the testing conditions

To finish the proof of the Corollary, we just need to prove

$$\|Q^{\beta}_{I,in}(\sigma 1_{Q_I})\|^2_{L^2(w)} \lesssim (B_\infty(w) + B_\infty(w^{-1}))B_2(w)\sigma(Q_I),$$

Verifying the testing conditions

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$$\|Q^{\beta}_{I,in}(\sigma 1_{Q_I})\|^2_{L^2(w)} \lesssim (B_\infty(w) + B_\infty(w^{-1}))B_2(w)\sigma(Q_I),$$

Lemma

$$\sum_{K:\, K\subset I} w(Q_K) \leq 2B_\infty(w)w(Q_I),$$

Verifying the testing conditions

$$\|Q_{I,in}^{\beta}(\sigma 1_{Q_I})\|_{L^2(w)}^2 = \int_{Q_I} \left| \sum_{K:\, K\subset I} \langle \sigma 1_{Q_I}, \frac{1_{Q_K}}{|K|} \rangle \frac{1_{Q_K}}{|K|} \right|^2 wdA$$

$$:= D + 2OD,$$

Verifying the testing conditions

$$\|Q^{\beta}_{I,in}(\sigma 1_{Q_I})\|^2_{L^2(w)} = \int_{Q_I} \left| \sum_{K:\,K\subset I} \langle \sigma 1_{Q_I}, \frac{1_{Q_K}}{|K|} \rangle \frac{1_{Q_K}}{|K|} \right|^2 wdA$$
$$:= D + 2OD,$$

Let us consider the diagonal part D,

$$\begin{aligned} D &= \int_{Q_I} \sum_{K:\,K\subset I} \langle \sigma 1_{Q_I}, \frac{1_{Q_K}}{|K|} \rangle^2 \frac{1_{Q_K}}{|K|^2} wdA \\ &= \sum_{K:\,K\subset I} \frac{\sigma(Q_K)^2}{|K|^2} \frac{w(Q_K)}{|K|^2} \\ &\leq B_2(w) \sum_{K:\,K\subset I} \sigma(Q_K) \leq 2B_\infty(\sigma) B_2(w) \sigma(Q_I). \end{aligned}$$

Verifying the testing conditions

For the off-diagonal part *OD*, consider

$$\begin{aligned} OD &= \int_{Q_I} \sum_{K,L:\, L\subset K\subset I} \langle \sigma 1_{Q_I}, \frac{1_{Q_K}}{|K|}\rangle \frac{1_{Q_L}}{|K||L|}\langle \sigma 1_{Q_I}, \frac{1_{Q_L}}{|L|}\rangle w dA \\ &= \sum_{K:\, K\subset I} \frac{\sigma(Q_K)}{|K|^2} \sum_{L:\, L\subset K} \frac{\sigma(Q_L)w(Q_L)}{|L|^2} \\ &\leq \sum_{K:\, K\subset I} \frac{\sigma(Q_K)}{|K|^2} B_2(w) \sum_{L:\, L\subset K} |L|^2 \lesssim B_\infty(\sigma)B_2(w)\sigma(Q_I). \end{aligned}$$

□

Outline

A characterisation

Theorem

Let $f, g \in A^2$. Then the Toeplitz product $T_g T_f^* : A^2 \to A^2$ is bounded, if and only if $M_g P_B^+ M_{\bar{f}} : A^2 \to L^2(\mathbb{D})$ is bounded.

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Let $f, g \in A^2$. Then the Toeplitz product $T_g T_f^* : A^2 \to A^2$ is bounded, if and only if $M_g P_B^+ M_{\bar{f}} : A^2 \to L^2(\mathbb{D})$ is bounded.

This gives a full characterisation of bounded Toeplitz products, though not in terms of the Sarason conjecture.

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This gives a full characterisation of bounded Toeplitz products, though not in terms of the Sarason conjecture.
Notice that the weights given by Bergman space functions on the disk are far from general, in contrast to weights given by Hardy space functions on the circle. They have to be at least subharmonic.

Counterexamples

A simple counterexample

There exist weights (u, w) satisfying the joint B_2 condition, such that $P_B : L^2_u(\mathbb{C}^+) \to L^2_w(\mathbb{C}^+)$ is not bounded.

Proof of the counterexample

Let Q_1 be a Carleson cube of side length 2^{-N} inside the unit cube Q_0, $\sigma = u^{-1} = \chi_{Q_1} 2^{2N}$ and $w = \sum_{j=0}^{N} 2^{-2N+2j} \chi_{T_{Q_1^{(j)}}}$.

It is easy to see that u and w satisfy joint the B_2 condition.

However

$$\|Q_{in}(\sigma \chi_{Q_0})\|^2_{L^2_w} = \| \sum_{j=0}^{N} \chi_{Q_1^{(j)}} \langle \sigma \rangle_{Q_1^{(j)}} \|^2_{L^2_w}$$

$$\gtrsim \sum_{j=0}^{N} |Q_1^{(j)}| \langle w \rangle_{T_{Q_1^{(j)}}} \langle \sigma \rangle^2_{Q_1^{(j)}} \gtrsim \sum_{j=0}^{N} |Q_1^{(j)}| \langle \sigma \rangle_{Q_1^{(j)}} = N.$$

$\square$

Note that one of the weights is ”far from subharmonic”.

A counterexample with one analytic weight

Let $g(z) = 1 - |z|^2,\ z \in \mathbb{D}$. Then

$$B(|g|^2)(z) \sim (1 - |z|^2)^2 \log \frac{2}{1 - |z|^2}\,, \tag{2}$$

when $z \in \mathbb{D}$ and $|z| \mapsto 1$. Given $f \in L^2(\mathbb{D})$, the Berezin B_2 condition is equivalent to

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^2 \log \frac{2}{1 - |z|^2} B(|f|^2)(z) < \infty\,. \tag{3}$$

Proposition

A function $f \in L^2(\mathbb{D})$ satisfies (3) and only if

$$\gamma(f)^2 := \sup_{I\subset\mathbb{T}\ subarc} \log \frac{2}{|I|} \int_{Q_I} |f|^2 dA < \infty. \tag{4}$$

Carleson measure on Dirichlet space

Recall that the Dirichlet D space consists of analytic functions h on $\mathbb{D}$ with

$$\|h\|^2 = |h(0)|^2 + \int_{\mathbb{D}} |h'|^2 dA < \infty .$$

A positive measure μ on $\mathbb{D}$ is called a Carleson measure for the Dirichlet space D, if D is continuously embedded in $L^2(\mu)$.

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A positive measure μ on $\mathbb{D}$ is called a Carleson measure for the Dirichlet space D, if D is continuously embedded in $L^2(\mu)$.
It is well known that the condition

$$\mu(Q_I) \lesssim \frac{1}{\log \frac{2}{|I|}}, \qquad (I \subset \mathbb{T} \text{ subarc})$$

is necessary but not sufficient for μ to be a Carleson measure for D.

Finishing the counterexample

Proposition

If $g(z) = 1 - |z|^2,\ z \in \mathbb{D}$, and $f \in L^2(\mathbb{D})$ then $M_g T_f^*$ is bounded on A^2 if and only if $|f|^2 dA$ is a Carleson measure for the Dirichlet space.

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However, it is known (Stegenga) that there exists measures $\mu = |f|^2 dA$ for which $\gamma(f) < \infty$, but which are not Carleson measures for the Dirichlet space.

□

A counterexample with two analytic weights

Given $f \in A^2$ let

$$\gamma^2(f) = \sup_I \log \frac{2\pi}{|I|} \int_{Q_I} |f|^2 dA\,,$$

and

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Then

$$\gamma(f) \lesssim \delta(f)$$

but the reverse is not true (Stegenga).

Constructing g

Lemma

Let $f \in L^2_a$, and let g be a Lipschitz analytic function in $\mathbb{D}$ with

$$|g(z)| \geq c(1 - |z|)\,, \tag{5}$$

for some constant $c > 0$ and all $z \in \mathbb{D}$.
(i) If $fg \in H^\infty$ and $\gamma(f) < \infty$ then $b_{f,g} < \infty$.
(ii) If $T_f T_g^*$ is bounded then $\delta(f) < \infty$.

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(i) $f_n g \in H^\infty$ uniformly bounded, $\gamma(f_n)$ bounded
(ii) $\delta(f_n) \to \infty$
(f_n taken from Stegenga counterexample, g constructed by means of a result of Dynkin.)
□

THANK YOU

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and
Happy Birthday, Aline!