Hyperbolic wavelet analysis of textures : global regularity and multifractal formalism

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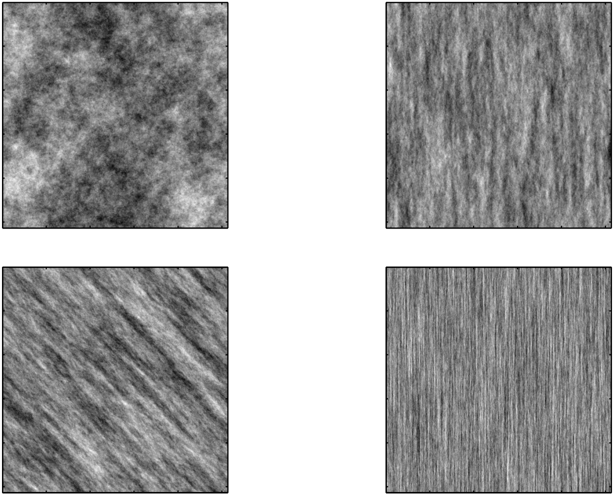
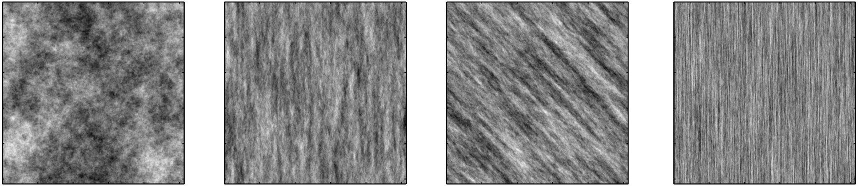
Introduction

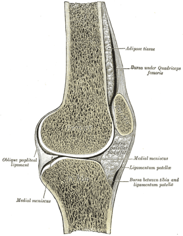


Introduction



- Same maximal directional regularity $s = 0.2$
- More or less roughness in other direction
- A classical wavelet analysis will give $H_1 = 0.2$, $H_2 = \frac{0.2}{1.3} \simeq 0.15$, $H_3 = \frac{0.2}{1.3}$, $H_4 = \frac{0.2}{1.7} \simeq 0.12$
- A multifractal analysis will not distinguish between these textures and a fBm of Hurst exponent H_i.

Introduction



Introduction

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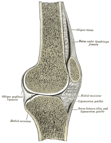
- Numerous tools of harmonic analysis are adapted to isotropic situations (functional spaces, wavelets, multifractal analysis)
- Numerous models to describe rough textures : fractional Brownian motions, multifractal functions, multifractal fields...
- How to characterize phenomena (textures, images) whose regularity depends on the direction ?



Outline

Anisotropy

- Anisotropy is here characterized by
  - The axes
  - A parameter $\alpha \in (0, 2)$

or a basis of $\mathbb{R}^2$ and a matrix $D_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 2-\alpha \end{pmatrix}$

$\alpha = 1 \Rightarrow$ Isotropic case

Anisotropy

- Notion of Operator scaling (anisotropic self-similarity)

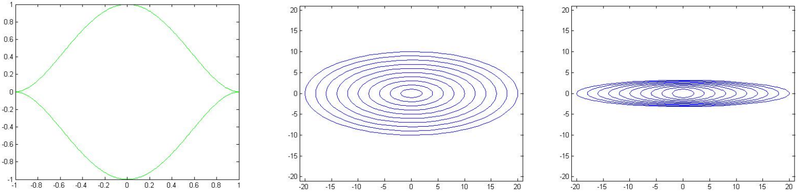
$$\forall\, a > 0 \quad f(a^{D_\alpha} x) = f(a^{\alpha} x_1, a^{2-\alpha} x_2) = a^H f(x_1, x_2)$$

- Quasi (or pseudo) norm : positive continuous function, D_α homogeneous

$$\rho_\alpha(a^{D_\alpha} x) = a\rho_\alpha(x)$$

- Properties :
  - $\rho_\alpha(x+y) \leq C(\rho_\alpha(x) + \rho_\alpha(y))$
  - 2 D_α pseudo-norms define the same topology.
- Example : $\rho_\alpha(x) = \left(|x_1|^{\frac{2}{\alpha}} + |x_2|^{\frac{2}{2-\alpha}}\right)^{1/2}.$

Anisotropy



Anisotropic unit ball - Action of the scaling $x \mapsto \lambda^{D_\alpha} x$ on an ellipse.

Operator scaling Gaussian Random Field

Operator Scaling Gaussian Random Field (OSGRF and extensions to OSSRF) - Biermé et al. (07)

Fix

- α and a pseudo-norm ρ_α
- $0 < H < \min(\alpha, 2-\alpha)$

Define

$$X_{\alpha,H,\rho_\alpha}(x) = \int_{\mathbb{R}^2} \frac{(e^{i\langle x,\xi\rangle} - 1)}{(\rho_\alpha(\xi))^{-(H+1)}} d\widehat{W}(\xi),$$

Some anisotropic models

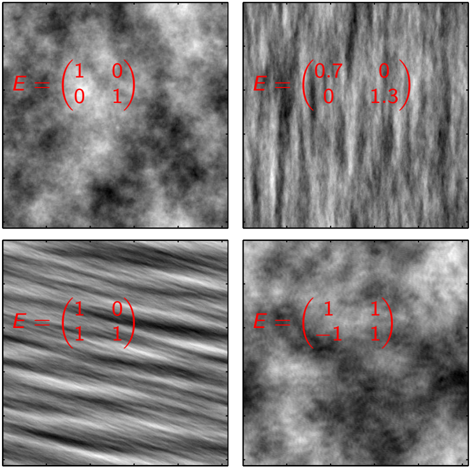
OSGRF (and extensions to OSSRF)

X_{α,H,ρ_α}

- is a gaussian random field
- has stationary increments
- is Operator Scaling, that is

$$X_{\alpha,H,\rho_\alpha}(a^{\alpha}x_1, a^{2-\alpha}x_2) \overset{d}{=} a^H X_{\alpha,H,\rho_\alpha}(x_1, x_2).$$

Typical OSGRF (H=0.6)



Anisotropic functional spaces

- Basic example 1 : Anisotropic Sobolev space ($\alpha = 1/2$)

$$H^{3,\alpha}(\mathbb{R}^2) = \{f \in L^2;\ \frac{\partial^6}{\partial x_1} f \in L^2 \text{ and } \frac{\partial^2}{\partial x_2} f \in L^2 \text{ and } \ldots\}$$

$(s_1 = 6 = \frac{3}{\alpha},\ s_2 = 2 = \frac{3}{2-\alpha})$.

Anisotropic functional spaces

- Basic example 2 : Anisotropic Hölder spaces :

$$\begin{aligned} \exists C>0,\ \forall (x,y)\in\mathbb{R}^2,\ |f(y)-f(x)| &\leq C\left(\rho_{\alpha}(y-x)\right)^s \\ &\leq C\left(|y_1-x_1|^{\frac{2}{\alpha}}+|y_2-x_2|^{\frac{2}{2-\alpha}}\right) \\ (\alpha=1/2) \qquad &\leq C\left(|y_1-x_1|^{4}+|y_2-x_2|^{\frac{4}{3}}\right)^{s/2} \end{aligned}$$

Anisotropic functional spaces

Anisotropic Littlewood-Paley analysis : let $\varphi_0^\alpha \geq 0, \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\varphi_0^\alpha(x) = 1 \quad \text{if} \quad \|\xi\|_\infty \leq 1 \ ,$$
$$\text{and} \ \ \varphi_0^\alpha(x) = 0 \quad \text{if} \quad \|2^{-D_\alpha}\xi\|_\infty \geq 1 \ .$$

For $j \in \mathbb{N}$, we define

$$\varphi_j^\alpha(x) = \varphi_0^\alpha(2^{-jD_\alpha}\xi) - \varphi_0^\alpha(2^{-(j-1)D_\alpha}\xi) \ .$$

$$\text{Then} \quad \sum_{j=0}^{+\infty} \varphi_j^\alpha \equiv 1 \ ,$$

and $(\varphi_j^\alpha)_{j\geq 0}$ is called an *anisotropic resolution of the unity*.

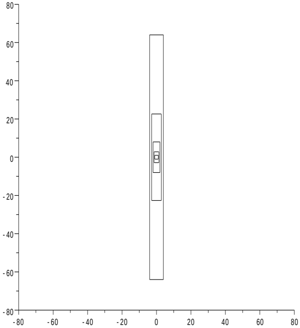
$$\mathrm{supp}\,(\varphi_0^\alpha) \subset R_1^\alpha, \quad \mathrm{supp}\,\left(\varphi_j^\alpha\right) \subset R_{j+1}^\alpha \setminus R_{j-1}^\alpha \ ,$$

where $R_j^\alpha = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2;\ |\xi_1| \leq 2^{j\alpha}, \text{ and } |\xi_2| \leq 2^{j(2-\alpha)}\}$.

Anisotropic functional spaces

Anisotropic Littlewood analysis :

For $f \in \mathcal{S}'(\mathbb{R}^2)$ let $\Delta_j^{\alpha} f = \mathcal{F}^{-1}\left(\varphi_j^{\alpha}\widehat{f}\right)$.



Anisotropic functional spaces

Anisotropic Besov spaces :

$$\Delta_j^{\alpha} f = \mathcal{F}^{-1}\left(\varphi_j^{\alpha}\widehat{f}\right) .$$

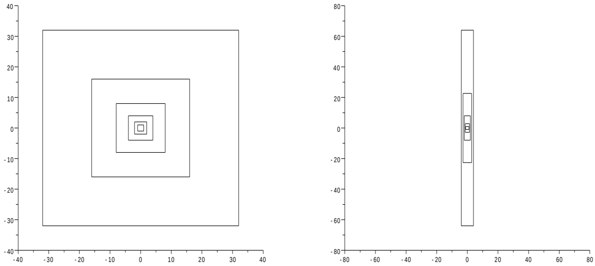
Definition

The Besov space $B_{p,q}^{s,\alpha}(\mathbb{R}^2)$, for $0 < p \leq +\infty$, $0 < q \leq +\infty$, $s \in \mathbb{R}$, is defined by

$$B_{p,q}^{s,\alpha}(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2);\ \left(\sum_{j\geq 0} 2^{jsq}\|\Delta_j^{\alpha} f\|_p^q\right)^{1/q} < +\infty \right\} .$$

Anisotropic wavelets

Classical wavelets are clearly not adapted



It does not fit with the anisotropic Littlewood Paley analysis

Anisotropic wavelets : Triebel basis

Hochmuth, Triebel : construction of a orthonormal anisotropic wavelet basis adapted to an anisotropy α :
Looks like :

$$\psi(2^{j\alpha}x_1 - k_1)\psi(2^{j(2-\alpha)}x_2 - k_2), \quad j \geq 0,\ k_1, k_2 \in \mathbb{Z}$$

$$\varphi(2^{j\alpha}x_1 - k_1)\psi(2^{j(2-\alpha)}x_2 - k_2), \quad j \geq 0,\ k_1, k_2 \in \mathbb{Z}$$

$$\psi(2^{j\alpha}x_1 - k_1)\varphi(2^{j(2-\alpha)}x_2 - k_2), \quad j \geq 0,\ k_1, k_2 \in \mathbb{Z}$$

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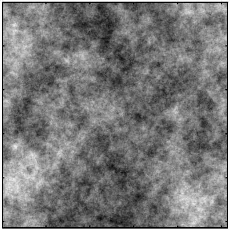
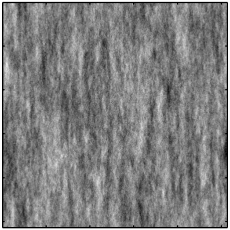
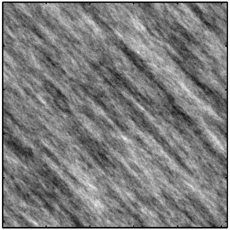
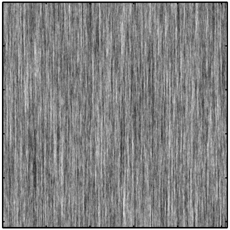
$\Longrightarrow$ provides characterizations of Hölder, Sobolev, Besov spaces with anisotropy α...
$\Longrightarrow B^{s,\alpha}_{p,q}(\mathbb{R}^2) \simeq B^{s}_{p,q}(\mathbb{R}^2)$
$\Longrightarrow$ Transference method...
(Triebel, Theory of functions III)

What to do when α is not known ?









Anisotropic wavelets

- Directional wavelets (J.P. Antoine et al, 93)
- Hyperbolic wavelet analysis (De Vore et al, 98)
- Curvelets (Candès - Donoho, 02)
- Bandlets (Le Pennec - Mallat, 03)
- Anisotropic wavelets (Triebel, 10)
- Shearlets (Kutyniok et al.,10)
- ...

Hyperbolic wavelet analysis

- Hyperbolic wavelet basis :

$$\psi_{j_1,j_2,k_1,k_2}(x_1,x_2) = \psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2), \quad j_1, j_2 \geq 0,\ k_1, k_2 \in \mathbb{Z}$$

(DeVore, Konyagin, Temlyakov, 98)
'Rectangular supports' of size $2^{-j_1} \times 2^{-j_2}$

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- Hyperbolic wavelet coefficients of f

$$c_{j_1,j_2,k_1,k_2} = 2^{j_1+j_2}\langle f,\ \psi_{j_1,j_2,k_1,k_2}\rangle$$

Hyperbolic wavelet analysis

Theorem (P. Abry, M. Clausel, S. Jaffard, S. Roux, B.V.)

- If $f \in \mathcal{C}^{s,\alpha}(\mathbb{R}^2)$ then

$$\forall (j_1, j_2) \quad \sup_{k_1,k_2} |c_{j_1,j_2,k_1,k_2}(f)| \leq C 2^{-\max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha})s} \quad (\star)$$

(starting with a 1D Meyer wavelet).

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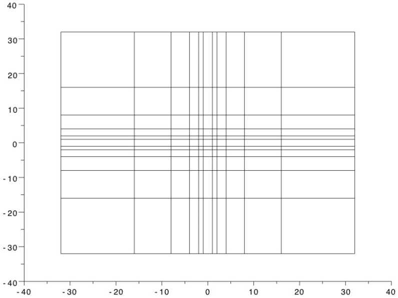
- Conversely, if $(\star)$ holds, $f \in \mathcal{C}^{s,\alpha}_{\log}(\mathbb{R}^2)$.
- General version of this result for anisotropic Besov spaces :

$$f \in B^{s,\alpha}_{p,q} \overset{\log}{\simeq} \sum_{(j_1,j_2)\in\mathbb{N}_0^2} 2^{\max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha})sq} 2^{-\frac{(j_1+j_2)q}{p}} \, \| c_{j_1,j_2,\cdot,\cdot} \|^q_{\ell^p} < +\infty \, .$$

(starting with a 1D Meyer wavelet).

Sketch of the proof

• 1st step : To obtain a characterization of $B^{s,\alpha}_{p,q}(\mathbb{R}^2)$ in term of Hyperbolic wavelet characterization :



$\theta_{j_1,j_2} = \tau_{j_1} \times \tau_{j_2}$ with τ a 1D resolution of unity.
$\Delta_{j_1,j_2}(f) = \mathcal{F}^{-1}(\Delta_{j_1,j_2}(\widehat{f}))$.

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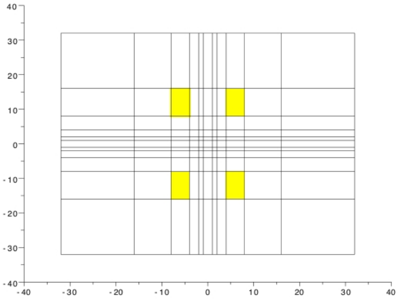

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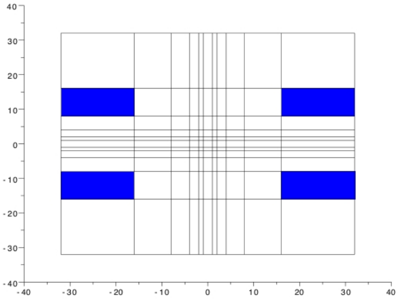

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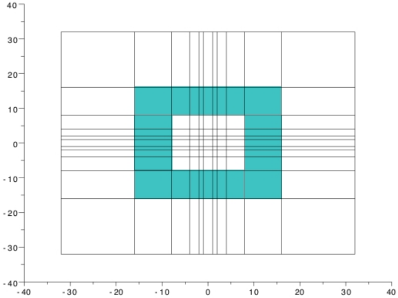
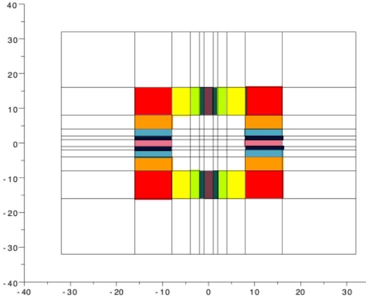
Sketch of the proof

Isotropic case :

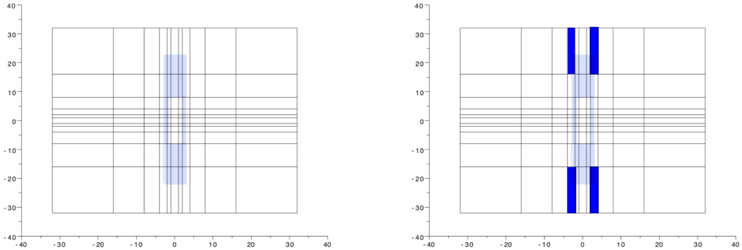




we can replace $\Delta_j(f)$ with $\sum_{j_1,j_2} \Delta_{j_1,j_2}(f)$ with $\max(j_1, j_2) = j$. $\Rightarrow$
About $3 \times j$ terms

Sketch of the proof

Anisotropic case



Sketch of the proof

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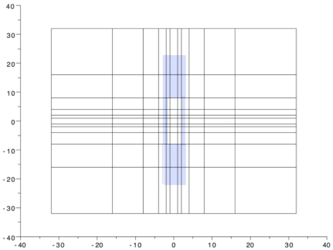

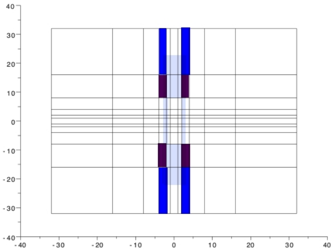

Sketch of the proof

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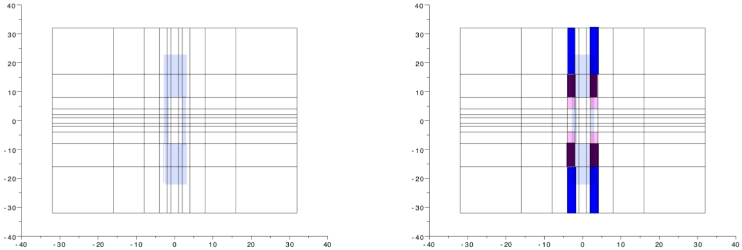

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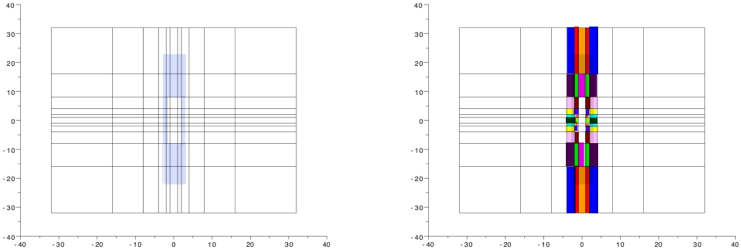

$\Rightarrow$ we can replace $\Delta_j^{\alpha}(f)$ with $\sum_{j_1,j_2} \Delta_{j_1,j_2}(f)$ with $\max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha}) \simeq j$.

$\Rightarrow$ About $C_\alpha \times j$ terms

Sketch of the proof

$$f \in B^{s,\alpha}_{p,q}(\mathbb{R}^2) \quad \Leftrightarrow \quad \left(\sum_{j \geq 0} 2^{jsq} \| \Delta^{\alpha}_j f \|^q_p \right)^{1/q} < +\infty$$

$$\Leftrightarrow \quad \left(\sum_{j \geq 0} 2^{jsq} \| \sum_{j_1, j_2;\ \max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha}) \simeq j} \Delta_{j_1, j_2}(f) \|^q_p \right)^{1/q} < +\infty$$

$$\overset{?}{\Leftrightarrow} \quad \left(\sum_{j \geq 0} 2^{jsq} \sum_{j_1, j_2;\ \max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha}) \simeq j} \| \Delta_{j_1, j_2}(f) \|^q_p \right)^{1/q} < +\infty$$

Sketch of the proof

$$f \in B^{s,\alpha}_{p,q}(\mathbb{R}^2) \Leftrightarrow \left(\sum_{j \geq 0} 2^{jsq} \|\Delta^{\alpha}_j f\|^q_p \right)^{1/q} < +\infty$$

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OK if $p = q = 2$

Sketch of the proof

$$f \in B^{s,\alpha}_{p,q}(\mathbb{R}^2) \quad \Leftrightarrow \quad \left(\sum_{j \geq 0} 2^{jsq} \| \Delta^{\alpha}_j f \|^q_p \right)^{1/q} < +\infty$$

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$$\overset{?}{\Leftrightarrow} \quad \left(\sum_{j \geq 0} 2^{jsq} \sum_{j_1, j_2;\, \max(\frac{j_1}{\alpha}, \frac{j_2}{2-\alpha}) \simeq j} \| \Delta_{j_1, j_2}(f) \|^q_p \right)^{1/q} < +\infty$$

OK but with a logarithmic correction in the other cases.

Sketch of the proof

• 2nd step : The wavelet coefficients are obtained by the Fourier analysis of $\widehat{\Delta_{j_1,j_2}(f)}$.

Hyperbolic wavelet analysis

- Benefits :

  • No a priori knowledge of α (Universal basis... but axis are known)

  • Orthonormal basis (in particular, no redundancy $\longrightarrow$ useful also for synthesis)

Hyperbolic wavelet analysis

- Benefits :
  • No a priori knowledge of α (Universal basis... but axis are known)
  • Orthonormal basis (in particular, no redundancy $\longrightarrow$ useful also for synthesis)
- Applications :
  Texture analysis : OSGRF (cf. last part)

Beyond global regularity

- Analysis of complex textures taking into account local regularity and anisotropy ?

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- Definition of an extended multifractal analysis

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- In the isotropic case, a well-known tool : Multifractal analysis
- Definition of an extended multifractal analysis
- Done for a fixed anisotropy using Triebel bases (Ben Braiek, Ben Slimane, 2011)

Beyond global regularity

Pointwise regularity : isotropic case

f locally bounded on $\mathbb{R}^2$, $s > 0$. $f \in \mathcal{C}^s(x_0)$ if $\exists C > 0, \delta$ and P_{x_0} with $deg(P_{x_0}) \leq s$ such that

$$|x - x_0| \leq \delta \Longrightarrow |f(x) - P_{x_0}(x)| \leq C|x - x_0|^s$$

where $|\cdot| = \rho$ is the euclidian norm

isotropic pointwise exponent :

$$h_f(x_0) = \sup\{s,\ f \in \mathcal{C}^s(x_0)\}$$

Characterization of $h_f(x_0)$ via the classical wavelet basis.

Beyond global regularity

Pointwise regularity : Anisotropic case

f locally bounded on $\mathbb{R}^2$, $s > 0$. $f \in \mathcal{C}^{s,\alpha}(x_0)$ if $\exists C > 0, \delta$ and P_{x_0} with $deg_\alpha(P_{x_0}) \leq s$ such that

$$|x - x_0| \leq \delta \Longrightarrow |f(x) - P_{x_0}(x)| \leq C|x - x_0|_\alpha^s$$

where $|\cdot|_\alpha = \rho_\alpha$ is an anisotropic pseudo-norm.

Anisotropic pointwise exponent :

$$h_{f,\alpha}(x_0) = \sup\{s,\ f \in \mathcal{C}^{s,\alpha}(x_0)\}$$

Characterization of $h_{f,\alpha}(x_0)$ via the Triebel wavelet basis.

Wavelet leaders

- dyadic rectangle :

$$\lambda = \lambda(j_1, j_2, k_1, k_2) = [\frac{k_1}{2^{j_1}}, \frac{k_1+1}{2^{j_1}}[\times[\frac{k_2}{2^{j_2}}, \frac{k_2+1}{2^{j_2}}[\,,$$

- Hyperbolic wavelet leaders

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|\,.$$

For any $x_0 = (a, b) \in \mathbb{R}^2$,

$$d_{j_1,j_2}(x_0) = \sup_{\lambda' \subset 3\lambda_{j_1,j_2}(x_0)} |c_{\lambda'}|\,.$$

where

$$3\lambda_{j_1,j_2}(x_0) = [\frac{[2^{j_1}a]-1}{2^{j_1}}, \frac{[2^{j_1}a]+2}{2^{j_1}}[\times[\frac{[2^{j_2}b]-1}{2^{j_2}}, \frac{[2^{j_2}b]+2}{2^{j_2}}[\,,$$

Local regularity

- Hyperbolic wavelet characterizations of $\mathcal{C}^{s,\alpha}(x_0)$
  - $f \in \mathcal{C}^{s,\alpha}(x_0) \Longrightarrow \exists C > 0$ such that for any j_1, j_2 one has
$$|d_{j_1,j_2}(x_0)| \leq C 2^{-\max(\frac{j_1}{\alpha},\frac{j_2}{2-\alpha})s} \,. \tag{1}$$
  - $f \in \mathcal{C}^{\varepsilon_0^*}(\mathbb{R}^2) + (1) \Longrightarrow f \in \mathcal{C}^{s,\alpha}_{|\log|^2}(x_0)$.

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- Again, same tool for all anisotropies.

Beyond global regularity

- Iso–anisotropic–Hölder sets :

$$E_f(H, \alpha) = \{x \in \mathbb{R}^2, h_{f,\alpha}(x) = H\}$$

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- Hyperbolic spectrum of singularities of f on $(\mathbb{R}^+ \cup \{\infty\}) \times (0,2)$

$$d(H,\alpha) = \dim_H(E_f(H,\alpha)) \,.$$

Beyond global regularity

For a locally bounded function

- Hyperbolic partition functions

$$S(j,p,\alpha) = 2^{-2j} \sum_{(j_1,j_2);\ \max(\frac{j_1}{\alpha},\frac{j_2}{\alpha})=j} \ \sum_{(k_1,k_2)\in\mathbb{Z}^2} d^{p}_{j_1,j_2,k_1,k_2} \ ,$$

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- Anisotropic scaling function

$$\omega_f(p,\alpha) = \liminf_{j\to\infty} \frac{\log S(j,p,\alpha)}{\log 2^{-j}},$$

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- Legendre Hyperbolic Spectrum :

$$\mathcal{L}_f(H,\alpha) = \inf_{p\in\mathbb{R}^*} \{Hp - \omega_f(p,(\alpha,2-\alpha)) + 2\} \ .$$

Beyond global regularity

Extended multifractal formalism

For $f \in \mathcal{C}^{\varepsilon_0^*}(\mathbb{R}^2)$, one has

$$\forall (H, \alpha) \in (\mathbb{R}_+^*) \times (0, 2), \quad d_f(H, \alpha) \leq \mathcal{L}_f(H, \alpha).$$

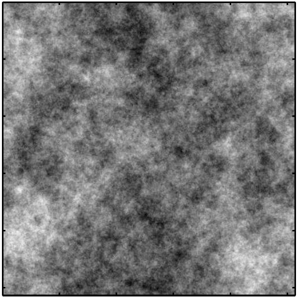
If

$$\forall (H, \alpha) \in (\mathbb{R}_+^*) \times (0, 2), \quad d(H, \alpha) = \mathcal{L}_f(H, \alpha),$$

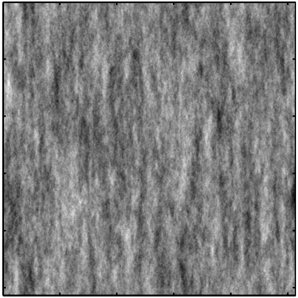
then f is said to satisfy the *hyperbolic multifractal formalism.*

Analysis of OSGRF

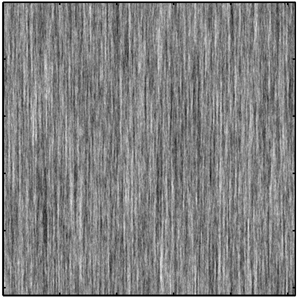
$(\theta_0 = 0,\ \alpha_0 = 1,\ H_0 = 0.2)$



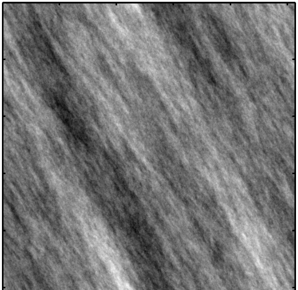
$(\theta_0 = 0,\ \alpha_0 = 0.7, H_0 = 0.2)$



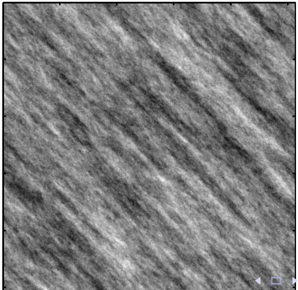
$(\theta_0 = 0,\ \alpha_0 = 0.3,\ H_0 = 0.2)$



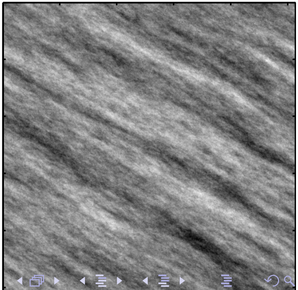
$(\theta_0 = \pi/6,\ \alpha_0 = 0.7,\ H_0 = 0.2)$



$(\theta_0 = \pi/4,\ \alpha_0 = 0.7,\ H_0 = 0.2)$



$(\theta_0 = \pi/3,\ \alpha_0 = 0.7,\ H_0 = 0.2)$



Directional Regularity : Anisotropic spaces

Theorem

Let $D_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 2-\alpha \end{pmatrix}$ and $0 < \alpha_0 < 2$. One has

1. (Biermé, Lacaux, 09)

$$\sup\{s,\ X_{D_{\alpha_0},H_0,\rho} \in C^{s,\alpha_0}_{loc}\} = H_0$$

2. (Clausel, V.)

$$\sup\{s,\ X_{D_{\alpha_0},H_0,\rho} \in C^{s,\alpha}\} < H_0$$

Proof : expansion on anisotropic wavelet bases.

Estimation procedure

Wavelet coefficients d_{j_1,j_2,k_1,k_2}
Structure function

$$Z(q,j_1,j_2) = \frac{1}{N_{j_1,j_2}} \sum_{j_1,j_2} |d_{j_1,j_2,k_1,k_2}|^q$$

N_{j_1,j_2} : number of coefficients at scale (j_1,j_2).

$$\zeta(q,\alpha) = \liminf_{j\to\infty} \frac{-\alpha \log_2(Z(q,[\alpha j],[(2-\alpha)j]))}{j}$$

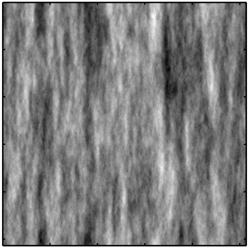
Estimators

$$\widehat{\alpha_q} = argmax_\alpha \zeta(q,\alpha)$$

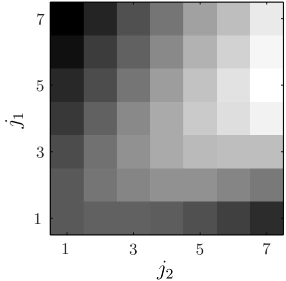
$$\widehat{H_q} = \zeta(q,\alpha)/q.$$

Estimation procedure

Image

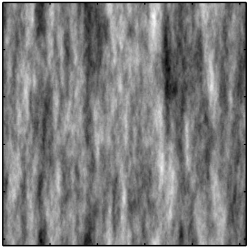


$log_2(Z(q=2,j_1,j_2))$



Estimation procedure

Image



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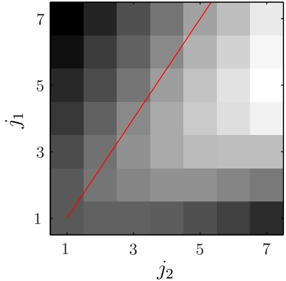
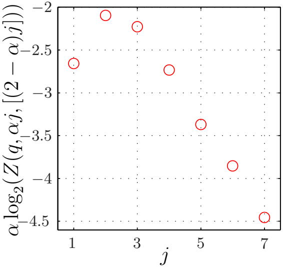


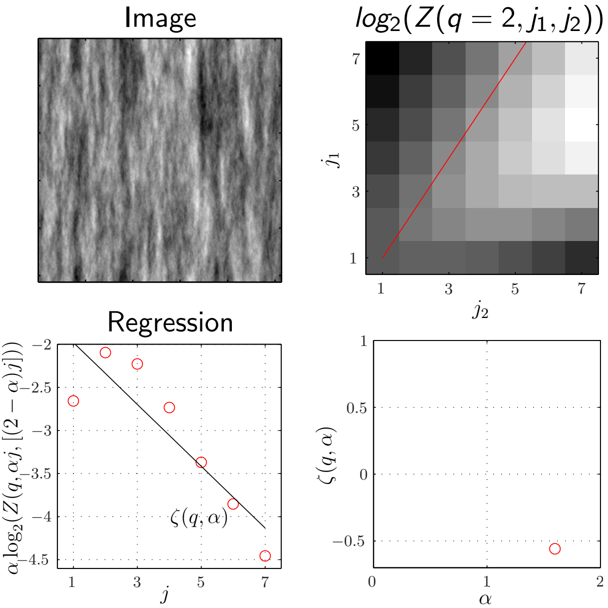
$\alpha = 1.5\,;$

$\alpha j_1 = (2 - \alpha) j_2\,;$

Interpolation



Estimation procedure



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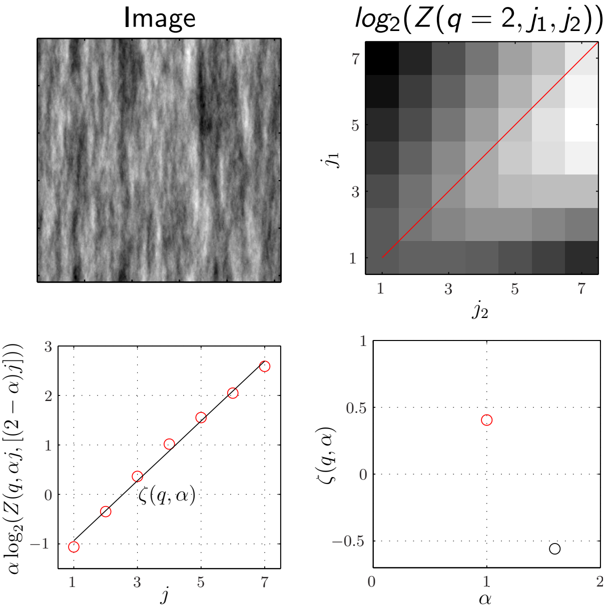
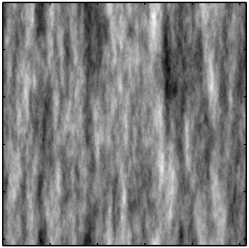


$\alpha = 1\,;$

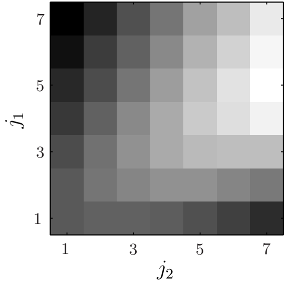
$\alpha j_1 = (2-\alpha) j_2\,;$

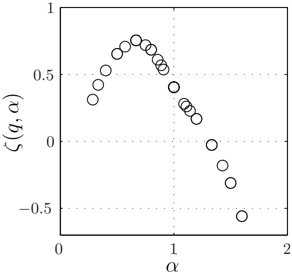
Estimation procedure

Image

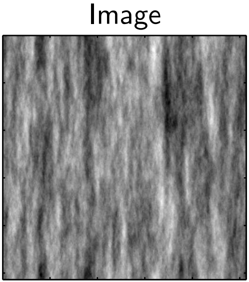


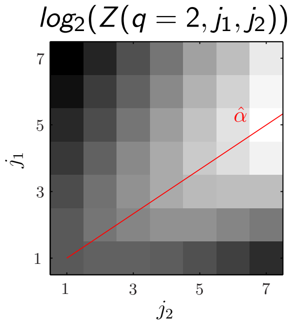
$log_2(Z(q = 2, j_1, j_2))$



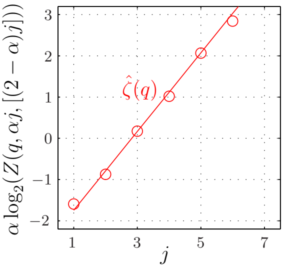


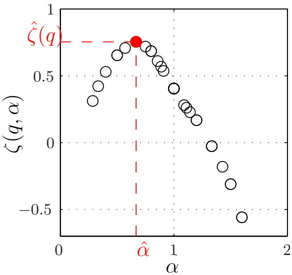
Estimation procedure





$$\hat{\alpha} j_1 = (2 - \hat{\alpha}) j_2 \,;$$



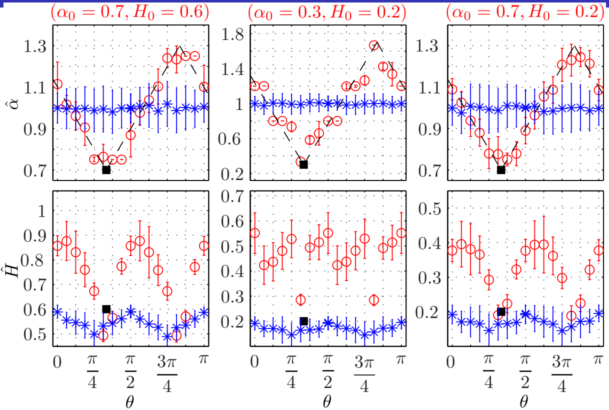


$$\hat{\alpha}_{\zeta(q)} = \mathrm{argmax}_{\alpha} \zeta(q, \alpha) \,;$$

$$\hat{\zeta}(q) = \zeta(q, \hat{\alpha}_{\zeta(q)}) \,;$$

Estimation : θ_0 unknown

Isotropic
Anisotropic



- Efficient and fast estimation procedures.
- Joint estimation of parameters $(\alpha_0, H_0, \theta_0)$
- Bootstrap : test for isotropy (and multifractality).

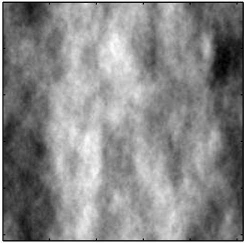
Extended Fractional Brownian Fields (EFBF)

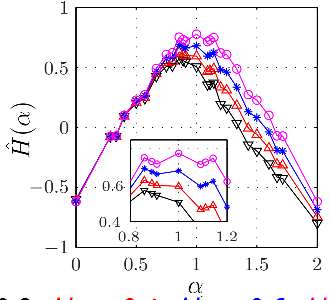
Bonami, Estrade (03)

$$X_{(a,\beta)} = \int_{\mathbb{R}^2} (e^{i\langle \underline{x}, \underline{\xi} \rangle} - 1) f(\underline{\xi}) d\widehat{W}(\underline{\xi}),$$

$$f(\underline{\xi}) = |\underline{\xi}|^{-2h(arg(\underline{\xi}))-2},$$

$H_1 = \max h(arg(\underline{\xi}))$, $H_2 = \min h(arg(\underline{\xi}))$. $H_2 = H_1 \Rightarrow$ isotropic.



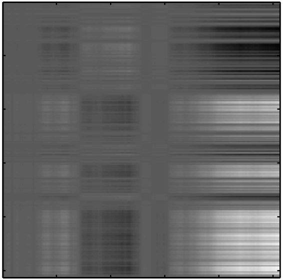


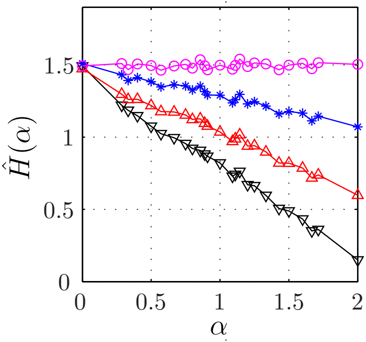
100 real. $(2^{10}, 2^{10})$

$H_1 = 0.8$ et $H_2 = 0.2$, $H_2 = 0.4$, $H_2 = 0.6$, $H_2 = 0.8$.

Fractional Brownian Sheet (FBS)

$$B_{H_1,H_2}(x) = \int_{\mathbb{R}^2} \frac{(e^{i<x_1,\xi_1>} - 1)(e^{i<x_2,\xi_2>} - 1)}{|\xi_1|^{H_1+\frac{1}{2}} |\xi_2|^{H_2+\frac{1}{2}}} d\widehat{W}_{\xi_1,\xi_2},$$





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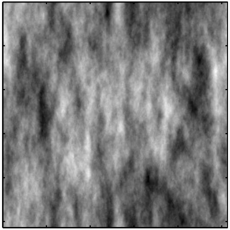
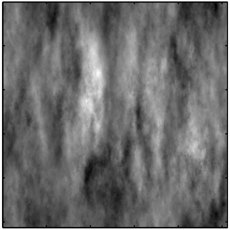
Multifractal case :

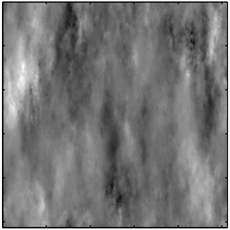
$(\alpha_0 = 0.8,\ H_0 = 0.7, \lambda_0 = 0)$



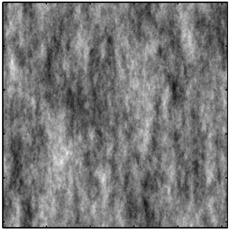
$(\alpha_0 = 0.8, H_0 = 0.7, \lambda_0 = 0.1)$



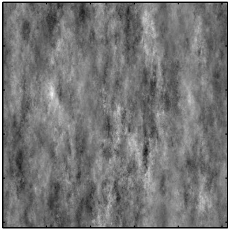
$(\alpha_0 = 0.8,\ H_0 = 0.7, \lambda_0 = 0.2)$



$(\alpha_0 = 0.8,\ H_0 = 0.3, \lambda_0 = 0)$



$(\alpha_0 = 0.8,\ H_0 = 0.3, \lambda_0 = 0.1)$



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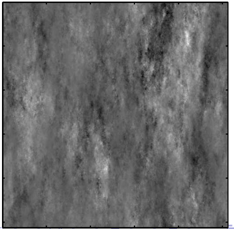
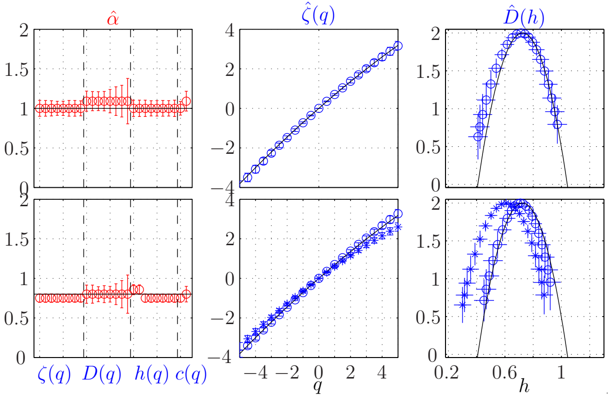


Estimation :

Isotropic

Anisotropic



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- What about axes (global, local) ?
- Anisotropic multifractal models

Thank you for your attention !