Affine-invariant harmonic analysis

James Wright

University of Edinburgh

June, 2014
Two model problems

Let $\sigma$ be a probability measure on $\mathbb{R}^d$.

**Problem A:** Convolution $f \rightarrow f \ast \sigma$

**Problem B:** Restriction $f \rightarrow \hat{f} d\sigma$
Problem A: Convolution $f \rightarrow f \ast \sigma$

- Preserves $L^p(\mathbb{R}^d)$ spaces

Question: For which $\sigma$ does $f \rightarrow f \ast \sigma$ map $L^p$ to $L^q$ for some $q > p$?

- If $\hat{\sigma} \in L^r$ for some $r < \infty$, then $\sigma$ is $L^p$ improving.
Two model problems: Problem A cont’d

$S \subset \mathbb{R}^d$ a $k$-dimensional submanifold

$$\sigma = \psi dS$$

- $\sigma$ is $L^p$ improving if and only if $\hat{\sigma} \in L^r$ for some $r < \infty$ if and only if $|\hat{\sigma}(\xi)| \leq C|\xi|^{-\delta}$ for some $\delta > 0$ if and only if some curvature of $S$ does not vanish (to infinite order).
∥f ∗ σ∥q ≤ C ∥f∥p \quad (†)

Proposition

Let S be a smooth hypersurface (the case k = d − 1) in \( \mathbb{R}^d \), \( \sigma \) a compactly supported measure on S, and \( b \in S \).

If (†) holds for some \( \sigma \) non-vanishing at \( b \in S \), then \((1/p, 1/q)\) lies in the closed triangle with vertices \((0, 0), (1, 1)\), and \((d/(d + 1), 1/(d + 1))\).

There exists a smooth measure \( \sigma \), non-vanishing at \( b \in S \), such that (†) holds for \( p = (d + 1)/d \) and \( q = 1/(d + 1) \) if and only if the gaussian curvature at \( b \in S \) is nonzero.
Two Model Problems: Problem B

**Problem B**: Fourier restriction $\|\hat{f}\|_{L^q(d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$.

- Trivial for $p = 1$ since $\|\hat{f}\|_{L^q(d\sigma)} \leq \|\hat{f}\|_{\infty} \leq \|f\|_1$.

**Question**: For which $\sigma$ is there a $p > 1$ such that the inequality holds for some $q$?

- If $\hat{\sigma} \in L^r$ for some $r < \infty$, then the Fourier restriction phenomenon holds for $\sigma$.

**Proof**: $\|\hat{f}\|_{L^2(d\sigma)}^2 = \int f(x) \overline{\hat{f} \ast \hat{\sigma}(x)} dx \leq \|f\|_p \|f \ast \hat{\sigma}\|_{p'} \leq C \|f\|_p^2$ for some $p > 1$ by Hölder and Young's inequalities.

- However $\hat{\sigma} \in L^r$ for some $r < \infty$ is also necessary!

**Dual formulation**: $\|b d\sigma\|_{L^{p'}(\mathbb{R}^d)} \leq C\|b\|_{L^{q'}(d\sigma)}$. Taking $b \equiv 1$ shows that necessarily $\hat{\sigma} \in L^{p'}(\mathbb{R}^d)$. 
\[ \| \hat{f} \|_{L^q(d\sigma)} \leq C \| f \|_{L^p(\mathbb{R}^d)} \]  \hspace{1cm} (\dagger)

**Proposition**

Let \( S \) be a smooth curve (the case \( k = 1 \)) in \( \mathbb{R}^d \), \( \sigma \) a compactly supported measure on \( S \), and \( b \in S \).

If (\dagger) holds for some \( \sigma \) non-vanishing at \( b \in S \), then \( q \leq [2/d(d+1)]p' \).

There exists a smooth measure \( \sigma \), non-vanishing at \( b \in S \) such that (\dagger) holds for some \( p > 1 \) and \( q = [2/d(d+1)]p' \) if and only if all \( d - 1 \) curvatures of \( S \) are nonzero at \( b \).
If \( S \) is a smooth \( k \)-dimensional submanifold with surface measure \( \sigma \), then 
\[
\sigma = c \left. H^k \right|_S \quad \text{where} \quad H^k \quad \text{is} \quad k \text{-dim’l Hausdorff measure.}
\]

- Recall \( \alpha \)-Hausdorff measure \( H^\alpha(E) = \lim_{\delta \to 0} H^\alpha_\delta(E) \) where 
\[
H^\alpha_\delta = \inf \left\{ \sum_j |C_j|^\alpha/d : E \subset \bigcup_j C_j, \ \text{diam}(C_j) < \delta \right\}
\]
where each \( C_j = x_j + r_jC \) is a cube.

- Recall \( \dim_h(E) = \inf\{\alpha > 0 : H^\alpha(E) < \infty\} \).
Affine measure

We define $\alpha$ - Affine measure $A^\alpha(E) = \lim_{\delta \to 0} A^\alpha_\delta(E)$ where

$$A^\alpha_\delta = \inf \left\{ \sum_j |R_j|^\alpha/d : E \subset \bigcup jR_j, \text{ diam}(R_j) < \delta \right\}$$

where each $R_j = L_j(C)$ is a rectangle – affine image of the unit cube.

- $\dim_a(E) = \inf \{ \alpha > 0 : A^\alpha(E) < \infty \}$ is the affine dimension of $E$.
- $E$ smooth curve in $\mathbb{R}^2$: $\dim_a(E) = 0$ if $E$ is a line segment and $\dim_a(E) = 2/3$ otherwise.
Examples of affine measures

If $S$ a $C^2$ hypersurface in $\mathbb{R}^d$, then

$$A^{d(d-1)/(d+1)}(E) \sim \sigma(E), \quad E \subset S$$

where $d\sigma = |K_S|^{1/(d+1)}dS$ and $K_S$ is the Gaussian curvature of $S$.

If $\Gamma$ is a $C^{(d)}$ curve in $\mathbb{R}^d$, then

$$A^{2/(d+1)}(E) \sim \sigma(E), \quad E \subset \Gamma$$

where $d\sigma = |L_{\Gamma}(t)|^{2/d(d+1)}dt$ and $L_{\Gamma}(t) = \det(\Gamma'(t), \ldots, \Gamma^{(d)}(t))$. 
Proposition

Let $\sigma$ be a positive Borel measure on $\mathbb{R}^d$.

(a) If $\|f * \sigma\|_q \leq c \|f\|_p$ holds, then $\sigma \leq C(c)A^{d(1/p-1/q)}$.

(b) If $\|\hat{f}\|_{L^q(d\sigma)} \leq c \|f\|_{L^p(\mathbb{R}^d)}$, then $\sigma \leq C(c)A^{dq/p'}$.

Proof.

Note that $\sigma \leq cA^\alpha$ if and only if $\sigma(R) \leq c|R|^{\alpha/d}$ for all rectangles $R$. Assume $\|f * \sigma\|_q \leq c\|f\|_p$ holds. Let $R$ be a rectangle in $\mathbb{R}^d$ and set $R' = R - R$. Then

$$|R|\sigma(R) \leq \int \chi_R * \chi_{R'}(x)d\sigma(x) = \langle \chi_{R'} * \sigma, \chi_R \rangle \leq c|R'|^{1/p} |R|^{1/q'}.$$
Let $S$ be any smooth hypersurface and let $d\sigma = |K|^{1/(d+1)}dS$ be affine surface measure.

(I) For $p_d = (d + 1)/d$ and $q_d = d + 1$, does

$$\| f * \sigma \|_{L^{q_d}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_d}(\mathbb{R}^d)}$$

hold?

(II) For $q = [(d - 1)/(d + 1)]p'$, does

$$\| \hat{f} \|_{L^q(S,d\sigma)} \leq \|f\|_{L^p(\mathbb{R}^d)}$$

hold for some $1 < p$?
Affine-invariant questions

Let $\Gamma$ be any smooth curve and let $d\sigma = |L(t)|^{1/(d+1)}dt$ be affine arclength measure.

(III) For $p_d = (d + 1)/2$ and $q_d = d(d + 1)/2(d - 1)$, does $\|f * \sigma\|_{L^{q_d}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_d}(\mathbb{R}^d)}$ hold?

(IV) For $q = [2/d(d + 1)]p'$, does $\|\hat{f}\|_{L^q(S,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$ hold for some $1 < p$?
Some results – fourier restriction

• (Sjölin, 1972) Let $\Gamma$ be a smooth convex curve in $\mathbb{R}^2$. Let $d\sigma = |K(t)|^{1/3}dt$. Then $\|\hat{f}\|_{L^q(\Gamma,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$ for $q = p'/3$ and $1 \leq p < 4/3$.

• (Dendrinos, W – Stovall) Let $\Gamma$ be a polynomial curve in $\mathbb{R}^d$ and let $\sigma$ be affine arclength measure. Then $\|\hat{f}\|_{L^q(\Gamma,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$ for $q = [2/d(d+1)]p'$ and $1 \leq p < (d^2 + d + 2)/(d^2 + d)$. 
(D. Oberlin, P. Gressman) Let \( \Gamma \) be a smooth convex curve in \( \mathbb{R}^2 \). Let \( d\sigma = |K(t)|^{1/3} \, dt \). Then \( \| f * \sigma \|_{L^3(\mathbb{R}^2)} \leq C \| f \|_{L^{3/2}(\mathbb{R}^2)} \).

(Dendrinos, Laghi, W – Stovall) Let \( \Gamma \) be a polynomial curve in \( \mathbb{R}^d \) and let \( \sigma \) be affine arclength measure. Then

\[
\| f * \sigma \|_{L^{q_d}(\mathbb{R}^d)} \leq C \| f \|_{L^{p_d}(\mathbb{R}^d)}
\]

where \( p_d = (d + 1)/2 \) and \( q_d = d(d + 1)/2(d - 1) \).