Affine-invariant harmonic analysis

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June, 2014

Two model problems

Let σ be a probability measure on $\mathbb{R}^d$.

Problem A: Convolution $f \to f * \sigma$

Problem B: Restriction $f \to \widehat{f d\sigma}$

Two model problems: Problem A

Problem A: Convolution $f \to f * \sigma$

- Preserves $L^p(\mathbb{R}^d)$ spaces

Question: For which σ does $f \to f * \sigma$ map L^p to L^q for some $q > p$?

- If $\widehat{\sigma} \in L^r$ for some $r < \infty$, then σ is L^p improving.

Two model problems: Problem A cont'd

$S \subset \mathbb{R}^d$ a k-dimensional submanifold

$$\sigma = \psi dS$$

• σ is L^p improving if and only if $\widehat{\sigma} \in L^r$ for some $r < \infty$ if and only if $|\widehat{\sigma}(\xi)| \leq C|\xi|^{-\delta}$ for some $\delta > 0$ if and only if some curvature of S does not vanish (to infinite order).

Nondegenerate surfaces – convolution case

$$\|f * \sigma\|_q \ \leq \ C\,\|f\|_p \qquad (\dagger)$$

Proposition

Let S be a smooth hypersurface (the case $k = d - 1$) in $\mathbb{R}^d$, σ a compactly supported measure on S, and $b \in S$.

If $(\dagger)$ holds for some σ non-vanishing at $b \in S$, then $(1/p, 1/q)$ lies in the closed triangle with vertices $(0, 0), (1, 1)$, and $(d/(d + 1), 1/(d + 1))$.

There exists a smooth measure σ, non-vanishing at $b \in S$, such that $(\dagger)$ holds for $p = (d + 1)/d$ and $q = 1/(d + 1)$ if and only if the gaussian curvature at $b \in S$ is nonzero.

Two Model Problems: Problem B

Problem B: Fourier restriction $\|\widehat{f}\|_{L^q(d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$.

• Trivial for $p = 1$ since $\|\widehat{f}\|_{L^q(d\sigma)} \leq \|\widehat{f}\|_\infty \leq \|f\|_1$.

Question: For which σ is there a $p > 1$ such that the inequality holds for some q?

• If $\widehat{\sigma} \in L^r$ for some $r < \infty$, then the Fourier restriction phenomenon holds for σ.

Proof: $\|\widehat{f}\|^2_{L^2(d\sigma)} = \int f(x)\overline{f * \widehat{\sigma}(x)}dx \leq \|f\|_p\|f * \widehat{\sigma}\|_{p'} \leq C\|f\|_p^2$ for some $p > 1$ by Hölder and Young's inequalities.

• However $\widehat{\sigma} \in L^r$ for some $r < \infty$ is also necessary!

Dual formulation: $\|\widehat{b\,d\sigma}\|_{L^{p'}(\mathbb{R}^d)} \leq C\|b\|_{L^{q'}(d\sigma)}$. Taking $b \equiv 1$ shows that necessarily $\widehat{\sigma} \in L^{p'}(\mathbb{R}^d)$.

Nondegenerate surfaces – fourier restriction case

$$\|\widehat{f}\|_{L^q(d\sigma)} \;\le\; C\,\|f\|_{L^p(\mathbb{R}^d)} \qquad (\ddagger)$$

Proposition

Let S be a smooth curve (the case $k = 1$) in $\mathbb{R}^d$, σ a compactly supported measure on S, and $b \in S$.

If $(\ddagger)$ holds for some σ non-vanishing at $b \in S$, then $q \le [2/d(d+1)]p'$.

There exists a smooth measure σ, non-vanishing at $b \in S$ such that $(\ddagger)$ holds for some $p > 1$ and $q = [2/d(d+1)]p'$ if and only if all $d-1$ curvatures of S are nonzero at b.

Hausdorff measure

If S is a smooth k-dimensional submanifold with surface measure σ, then $\sigma = c\,H^k\big|_S$ where H^k is k-dim'l Hausdorff measure.

- Recall α - Hausdorff measure $H^\alpha(E) = \lim_{\delta\to 0} H^\alpha_\delta(E)$ where

$$H^\alpha_\delta = \inf\{\sum_j |C_j|^{\alpha/d} : E \subset \cup_j C_j,\ \mathrm{diam}(C_j) < \delta\}$$

where each $C_j = x_j + r_j C$ is a cube.

- Recall $\mathrm{dim}_h(E) = \inf\{\alpha > 0 : H^\alpha(E) < \infty\}$.

Affine measure

We define α - Affine measure $A^\alpha(E) = \lim_{\delta\to 0} A^\alpha_\delta(E)$ where

$$A^\alpha_\delta = \inf\{\sum_j |R_j|^{\alpha/d} : E \subset \cup_j R_j,\ \mathrm{diam}(R_j) < \delta\}$$

where each $R_j = L_j(C)$ is a *rectangle* – affine image of the unit cube.

• $\mathrm{dim}_a(E) = \inf\{\alpha > 0 : A^\alpha(E) < \infty\}$ is the affine dimension of E.

• E smooth curve in $\mathbb{R}^2$: $\mathrm{dim}_a(E) = 0$ if E is a line segment and $\mathrm{dim}_a(E) = 2/3$ otherwise.

Examples of affine measures

If S a C^2 hypersurface in $\mathbb{R}^d$, then

$$A^{d(d-1)/(d+1)}(E) \sim \sigma(E), \quad E \subset S$$

where $d\sigma = |K_S|^{1/(d+1)} dS$ and K_S is the Gaussian curvature of S.

If Γ is a $C^{(d)}$ curve in $\mathbb{R}^d$, then

$$A^{2/(d+1)}(E) \sim \sigma(E), \quad E \subset \Gamma$$

where $d\sigma = |L_\Gamma(t)|^{2/d(d+1)} dt$ and $L_\Gamma(t) = \det\big(\Gamma'(t), \dots, \Gamma^{(d)}(t)\big)$.

Affine measure cont'd

Proposition

Let σ be a positive Borel measure on $\mathbb{R}^d$.

(a) If $\|f * \sigma\|_q \leq c\|f\|_p$ holds, then $\sigma \leq C(c)A^{d(1/p-1/q)}$.

(b) If $\|\widehat{f}\|_{L^q(d\sigma)} \leq c\|f\|_{L^p(\mathbb{R}^d)}$, then $\sigma \leq C(c)A^{dq/p'}$.

Proof.

Note that $\sigma \leq cA^\alpha$ if and only if $\sigma(R) \leq c|R|^{\alpha/d}$ for all rectangles R.

Assume $\|f * \sigma\|_q \leq c\|f\|_p$ holds. Let R be a rectangle in $\mathbb{R}^d$ and set $R' = R - R$. Then

$$|R|\sigma(R) \leq \int \chi_R * \chi_{R'}(x) d\sigma(x) = \langle \chi_{R'} * \sigma, \chi_R \rangle \leq c|R'|^{1/p}|R|^{1/q'}.$$

□

Affine-invariant questions

Let S by *any* smooth hypersurface and let $d\sigma = |K|^{1/(d+1)} dS$ be affine surface measure.

(I) For $p_d = (d+1)/d$ and $q_d = d+1$, does

$$\|f * \sigma\|_{L^{q_d}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_d}(\mathbb{R}^d)} \qquad \text{hold?}$$

(II) For $q = [(d-1)/(d+1)]p'$, does

$$\|\widehat{f}\|_{L^q(S,d\sigma)} \leq \|f\|_{L^p(\mathbb{R}^d)}$$

hold for some $1 < p$?

Affine-invariant questions

Let Γ by *any* smooth curve and let $d\sigma = |L(t)|^{1/(d+1)}dt$ be affine arclength measure.

(III) For $p_d = (d+1)/2$ and $q_d = d(d+1)/2(d-1)$, does

$$\|f * \sigma\|_{L^{q_d}(\mathbb{R}^d)} \leq C\|f\|_{L^{p_d}(\mathbb{R}^d)} \qquad \text{hold?}$$

(IV) For $q = [2/d(d+1)]p'$, does

$$\|\widehat{f}\|_{L^q(S,d\sigma)} \leq C\,\|f\|_{L^p(\mathbb{R}^d)}$$

hold for some $1 < p$?

Some results – fourier restriction

- (Sjölin, 1972) Let Γ be a smooth convex curve in $\mathbb{R}^2$. Let $d\sigma = |K(t)|^{1/3}dt$. Then $\|\widehat{f}\|_{L^q(\Gamma,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$ for $q = p'/3$ and $1 \leq p < 4/3$.

- (Dendrinos, W – Stovall) Let Γ be a polynomial curve in $\mathbb{R}^d$ and let σ be affine arclength measure. Then $\|\widehat{f}\|_{L^q(\Gamma,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$ for $q = [2/d(d+1)]p'$ and $1 \leq p < (d^2+d+2)/(d^2+d)$.

Some results – convolution

- (D. Oberlin, P. Gressman) Let Γ be a smooth convex curve in $\mathbb{R}^2$. Let $d\sigma = |K(t)|^{1/3}dt$. Then $\|f * \sigma\|_{L^3(\mathbb{R}^2)} \le C\|f\|_{L^{3/2}(\mathbb{R}^2)}$.

- (Dendrinos, Laghi, W – Stovall) Let Γ be a polynomial curve in $\mathbb{R}^d$ and let σ be affine arclength measure. Then

$$\|f * \sigma\|_{L^{q_d}(\mathbb{R}^d)} \le C\|f\|_{L^{p_d}(\mathbb{R}^d)}$$

where $p_d = (d+1)/2$ and $q_d = d(d+1)/2(d-1)$.