HERMITE FUNCTIONS AND UNCERTAINTY PRINCIPLES

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Abstract. This notes are outlines of talks given in Vienna at the Gabor analysis conference organized by the NuHaG group in December 2001 and in Cuernavaca, Mexico in january 2002. They do not contain any original work and are not intended for publication. All classical results are either contained in the survey articles by Folland and Sitaram [8], Dembo, Cover and Thomas [7] as well as in the book of Havin and Jöricke [13]. The new results that are given here are issued from my joint work with A. Bonami and B. Demange [5]. I do not intent to give further references for results included in these papers.

1. Introduction and Notations.

Uncertainty principles state that a function and its Fourier transform cannot be simultaneously sharply localized. To be more precise, let $d \geq 1$ be the dimension, and let us denote by $\langle .,. \rangle$ the scalar product and by $\|.\|$ the Euclidean norm on $\mathbb{R}^d$. Then, for $f \in L^2(\mathbb{R}^d)$, define the Fourier transform of f by

$$\widehat{f}(y) = \int_{\mathbb{R}^d} f(t) e^{-2i\pi\langle t,y\rangle} dt.$$

For a set E, denote by $|E|$ the Lebesgues measures of E and by χ_E its characteristic function.

The Hermite function of order $k \in \mathbb{N}$ is defined for $x \in \mathbb{R}$ by

$$h_k(x) = c_k e^{\pi x^2} \left(\frac{d}{dx}\right)^k (e^{2\pi x^2})$$

with $c_k = (-1)^k \frac{2^{1/4}}{\sqrt{k!}(2\sqrt{\pi})^k}$. It is then well known that

(1) $\{h_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$,
(2) h_k is of the form $P(x)e^{-\pi x^2}$ with P a polynomial of degree k,
(3) $\mathcal{F}h_k = i^{-k} h_k$,
(4) $2\sqrt{\pi} x h_k(x) = \sqrt{k+1} h_{k+1}(x) + \sqrt{k} h_{k-1}(x)$.

One can tehn easily prove that

$$\|xf\|_2^2 + \|\xi\hat{f}\|_2^2 = \frac{1}{2\pi}\sum_{k=0}^{+\infty}(2k+1)|\langle f, h_k\rangle|^2 \geq \frac{\|f\|^2}{2\pi}$$

with equality if and only if $f = h_0$. Now, applying this inequality to $f_c(x) = f(cx)$ and minimizing over $c > 0$, we get the celebrated uncertainty principle, due to Heisenberg and Weil :

Heisenberg's inequality. *Let $f \in L^2(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \int_{\mathbb{R}} \xi^2 \left|\widehat{f}(\xi)\right|^2 d\xi \geq \frac{\|f\|_{L^2}^4}{16\pi^2}. \tag{1}$$

1991 *Mathematics Subject Classification.* 42B10;32A15;94A12.
Key words and phrases. Uncertainty principles; windowed Fourier transform.
Research partially financed by : *European Commission* (TMR 1998-2001 Network *Harmonic Analysis*).

Moreover (1) is an equality if and only if f is of the form $f(x) = Ce^{-\alpha x^2}$ where C is a constant and $\alpha > 0$.

The proof given here actually gives more since, if f is orthogonal to $\{h_0, \ldots, h_{n-1}\}$ then

$$\|xf\|_2^2 + \left\|\xi\hat{f}\right\|_2^2 = \frac{1}{2\pi}\sum_{k=0}^{+\infty}(2k+1)|\langle f, h_k\rangle|^2 \geq (2n+1)\frac{\|f\|^2}{2\pi}$$

so that, in (1), the constant $\frac{1}{16\pi^2}$ may be replaced by some bigger constant and equality is then achieved for hermite functions of degree n.

There are also several generalizations to higher dimensions which can take into account directionnal considerations.

One may also give generalizations replacing the L^2 norms in (1) by L^p norms. The following result is due to Cowling and Price [6].

Theorem 1.1 (Cowling-Price)**.** *Let $1 \leq p, q \leq \infty$ and $a, b > 0$ be such that*

$$a + \frac{1}{p} = b + \frac{1}{q} \quad , \quad a > \frac{1}{2} - \frac{1}{p} \quad , \quad b > \frac{1}{2} - \frac{1}{q}$$

then, there exists a constant $K > 0$ such that, for every $f \in L^2(\mathbb{R})$,

$$\||x|^a f\|_p \||\xi|^b \hat{f}\|_q \geq K\|f\|_2.$$

2. Support conditions

It is well known that, if a funtion $f \in L^2(\mathbb{R})$ is compactly supported, then its Fourier transform $\hat{f}$ is an entire function. In particular, $\hat{f}$ can not be compacty supported, unless $\hat{f}$ thus f is zero.

The following theorem, independently du to Benedicks [4] and Amrein-Berthier [1] shows that the previous fact can be generalized when compact support is replaced by support of finite measure.

Theorem 2.1 (Benedicks/Amrein-Berthier)**.** *Let $f \in L^2(\mathbb{R}^d)$. Assume that f has support of finite measure and is such that its Fourier transform has support of finite measure, then $f = 0$.*

Proof. Let us outline the proof for dimension $d = 1$, the generalization to higher dimension is obvious.

First, f having support of finite measure, $f \in L^1$. Let Σ be the support of f and $\hat{\Sigma}$ be the support of $\hat{f}$. Up to rescaling f, we may assume that $|\Sigma| < 1$.

Now,

$$\int_0^1 \sum_{k\in\mathbb{Z}} \chi_{\hat{\Sigma}}(\xi + k)d\xi = \left|\hat{\Sigma}\right| < +\infty.$$

In particular, for (almost) every $\xi \in [0, 1]$, $\sum_{k\in\mathbb{Z}} \chi_{\hat{\Sigma}}(\xi + k) < +\infty$, that is $\hat{f}(\xi + k) \neq 0$ for finitely many k's.

For such a ξ, define

$$L^1(\mathbb{T}) \ni Z_\xi(x) = \sum_{k\in\mathbb{Z}} f(x + k)e^{2i\pi\xi(x+k)} = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + k)e^{2i\pi kx}.$$

Then, Z_ξ is a trigonometric polynomial, so it can not be zero on a set of > 0 measure, unless it is zero everywhere, in which case $\hat{f} = 0$ and this would conclude the proof. But,

$$\int_0^1 \sum_{k\in\mathbb{Z}} \chi_\Sigma(x + k) = |\Sigma| < 1$$

so that $\sum_{k\in\mathbb{Z}} \chi_\Sigma(x+k)=0$ on a set E with $|E|>0$, that is $f(x+k)=0$ for all $x\in E$ and all $k\in\mathbb{Z}$. It follows that for $x\in E$,

$$|Z_\xi(x)|\leq\sum_{k\in\mathbb{Z}}|f(x+k)|=0$$

so that Z_ξ is zero on a set of positive measure, as desired. □

Using a compacity argument (see [13]), one may deduce from this result the existence of a constant C depending only on $\Sigma,\hat\Sigma$ such that for all $f\in L^2(\mathbb{R}^d)$,

$$\|f\|_2\leq C(\|f\chi_{\mathbb{R}^d\setminus\Sigma}\|_2+\|\hat f\chi_{\mathbb{R}^d\setminus\hat\Sigma}\|_2).$$

An estimate of this constant is not known, excepted in dimension $d=1$, where Nazarov [18] has proved :

Theorem 2.2 (Nazarov). *Let $\Sigma,\hat\Sigma$ be two subsets of $\mathbb{R}$ of finite measure. Then, for every $f\in L^2(\mathbb{R}^d)$,*

$$\|f\|_2\leq 133e^{133|\Sigma||\hat\Sigma|}\|f\chi_{\mathbb{R}\setminus\Sigma}\|_2+\|\hat f\chi_{\mathbb{R}^d\setminus\hat\Sigma}\|_2).$$

A slightly different result has been obtained in [20] using real varaible methods. Let $\rho(x)=\max(1,1/\|x\|)$, a set $E\subset\mathbb{R}^d$ is ε-thin if for every $x\in\mathbb{R}^d$, $\big|E\cap B\big(x,\rho(x)\big)\big|<\varepsilon\big|B\big(x,\rho(x)\big)\big|$ where $B\big(x,\rho\big)$ is the ball of center x and radius ρ. Then, one can prove :

Theorem 2.3 (Shubin, Vakilian, Wolff). *There is $\varepsilon_0>$ such that, for every $0<\varepsilon<\varepsilon_0$, there exists $C<1$ such that if Σ and $\hat\Sigma$ are ε-thin sets in R^d then for any $f\in L^2(\mathbb{R}^d)$,*

$$\|f\|_2\leq C(\|f\chi_{\mathbb{R}^d\setminus\Sigma}\|_2+\|\hat f\chi_{\mathbb{R}^d\setminus\hat\Sigma}\|_2).$$

3. Decreasing conditions

An other famous uncertainty principle is du to Hardy [12] and gives conditions under which a function and its Fourier transform can be both fastly decreasing. More precisely

Theorem 3.1 (Hardy). *Let $f\in L^2(\mathbb{R})$ be such that, for some $C,N>0$,*

i/ $f(x)\leq C(1+|x|)^Ne^{-a\pi x^2}$,
ii/ $\hat f(\xi)\leq C(1+|\xi|)^Ne^{-b\pi\xi^2}$.

Then

(1) *If $ab>1$, $f=0$.*
(2) *If $ab=1$, $f(x)=P(x)e^{-ax^2}$ for some polynomial of degree $\leq N$.*
(3) *If $ab<1$, there is a dense subspace of functions $f\in L^2$ satisfying conditions i and ii.*

There are many other results of this type, both in dimension $d\geq1$ and involving L^p norms (Cowling-Price [6]) or bounds of the form e^{-ax^p} with $1<p<\infty$ (Morgan [17], Gelfand-Shilov [9]). This results are all partially implied in dimension $d=1$ by the following theorem du to Beurling-Hörmander [14].

Theorem 3.2 (Beurling-Hörmander). *Let $f\in L^2(\mathbb{R})$. Then*

$$\iint_{\mathbb{R}\times\mathbb{R}}|f(x)||\widehat f(y)|e^{2\pi|xy|}dxdy<+\infty \tag{2}$$

if and only if $f=0$.

Despite its strength, this theorem still has drawbacks as it is only in dimension $d=1$ and further as it does not contain any equality cases. In particular, it does not imply the equality case in Hardy's Theorem.

It is not fully obvious of what $|xy|$ should be replaced with in dimension $d\geq 1$. Three main options are given to us, first $|x||y|$ or $\displaystyle\sum_{k=1}^{d} x_i y_i$, which has been achieved by Bagchi and Ray [3], or by $|\langle x,y\rangle|$, as proved by Ray and Naranayan [19]. Both proofs use the one dimensional theorem and then either an occillary function or a Radon transform.

However, this author do not get equality cases. We have been able to prove the following :

Theorem 3.3 (Beurling-Hörmander type, [5]). *Let $f\in L^2(\mathbb{R}^d)$ and $N\geq 0$. Then*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(x)||\widehat{f}(y)|}{(1+\|x\|+\|y\|)^N}e^{2\pi|\langle x,y\rangle|}dxdy<+\infty \tag{3}$$

if and only if f may be written as

$$f(x)=P(x)e^{-\pi\langle Ax,x\rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial of degree $<\frac{N-d}{2}$.

In particular, for $N\leq d$, the function f is identically 0.

Remarks.

— If one replaces the term $e^{2\pi|\langle x,y\rangle|}$ in (3) by $e^{2\pi\sum_{k=1}^d|x_iy_i|}$, then one further restricts the matrix A to be diagonal. If one further replaces $e^{2\pi\sum_{k=1}^d|x_iy_i|}$ by $e^{2\pi\|x\|\|y\|}$, then A has to be $A=-\beta I$ with $\beta>0$.

— Let $f(x)=P(x)e^{-\pi\langle(A+iB)x,x\rangle}$ with P polynomial, A,B positive definite matrixes. Then, if F satisfies (3), some calculus shows that A is positive definite, $B=0$ and the degree of P is $<\frac{N-d}{2}$. So, our aim will be to show that f is of that form.

— In condition (3), we may replace $\frac{1}{(1+\|x\|+\|y\|)^N}$ by $\frac{1}{(1+\|x\|)^{N/2}(1+\|y\|)^{N/2}}$.

This theorem admits several corollaries. The first one is du to Cowling and Price in dimension $d=1$ and with $N=0$, [6].

Theorem 3.4 (Cowling-Price type, [5]). *Let $N\geq 0$. Assume that $f\in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d}|f(x)|\frac{e^{\pi a\|x\|^2}}{(1+\|x\|)^N}dx<+\infty\quad\text{and}\quad\int_{\mathbb{R}^d}|\widehat{f}(y)|\frac{e^{\pi b\|y\|^2}}{(1+\|y\|)^N}dy<+\infty$$

for some positive constants a and b with $ab=1$. Then $f(x)=P(x)e^{-a\|x\|^2}$ for some polynomial P.

This results follows from Theorem 3.3 and Cauchy Schwarz : $2\|x\|\|y\|\leq a\|x\|^2+\frac{1}{a}\|y\|^2$.

If one further uses H"older instead of Cauchy-Schwarz, one would obtain a result in terms of $e^{\pi a|x|^p}$. However, this result is not optimal since we have:

Theorem 3.5 (Morgan type, [5]). *Let $1<p<2$, and let q be the conjugate exponent. Assume that $f\in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d}|f(x)|e^{2\pi\frac{a^p}{p}\|x\|^p}dx<+\infty\quad\text{and}\quad\int_{\mathbb{R}^d}|\widehat{f}(y)|e^{2\pi\frac{b^q}{q}\|y\|^q}dy<+\infty$$

for some $j=1,\cdots,d$ and for some positive constants a and b. Then $f=0$ if $ab>\left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$. If $ab<\left|\cos(\frac{p\pi}{2})\right|^{\frac{1}{p}}$, one may find a dense subset of functions which satisfy the above conditions for all j.

Finally, note that in theorem 3.4, on may replace $|f(x)|$ by $|f(x)|^p$ using the following

$$\int_{\mathbb{R}^d}|f(x)|\frac{e^{\pi a\|x\|^2}}{(1+\|x\|)^{N/p+K}}dx\leq\left(\int_{\mathbb{R}^d}|f(x)|^p\frac{e^{\pi a\|x\|^2}}{(1+\|x\|)^N}dx\right)^p\left(\int_{\mathbb{R}^d}\frac{1}{(1+\|x\|)^{Kq}}dx\right)^q<+\infty$$

provided K is big enough and $\int_{\mathbb{R}^d}|f(x)|^p\frac{e^{\pi a\|x\|^2}}{(1+\|x\|)^N}dx<+\infty$. So, one gets

Theorem 3.6 (Cowling-Price type, [5]). *Let $N \geq 0$, $1 \leq p < +\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies*

$$\int_{\mathbb{R}^d} |f(x)|^p \frac{e^{\pi a\|x\|^2}}{(1+\|x\|)^N}dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\widehat{f}(y)|^q \frac{e^{\pi b\|y\|^2}}{(1+\|y\|)^N}dy < +\infty$$

for some positive constants a and b. Then, if $\frac{ab}{pq} > 1$, $f = 0$ whereas if $\frac{ab}{pq} = 1$. Then $f(x) = P(x)e^{-\pi\frac{a}{p}\|x\|^2}$ for some polynomial P.

4. Proof of Theorem 3.3

We will here only give the full proof in the case $d = 1$ and $N = 0$, that is, the proof of Beurling-Hörmander's original Theorem. This proof simplifies Hörmander's argument and contains already the main ideas for the general case. We will then sketch the main steps of our theorem.

4.1. **The case $d = 1$, $N = 0$.** We want to prove that a function $f \in L^1(\mathbb{R})$ which satisfies the inequality

$$\iint_{\mathbb{R}\times\mathbb{R}} |f(x)||\widehat{f}(\xi)|e^{2\pi|x||\xi|}dxd\xi < +\infty \tag{4}$$

is identically 0. It is sufficient to show that the function $g = e^{-\pi x^2} * f$ is identically 0. Indeed, the Fourier transform of g is equal to $e^{-\pi\xi^2}\widehat{f}$. If it is 0, then f vanishes also. Now g extends to an entire function of order 2 in the complex plane. We note also g its extension. We claim that, moreover,

$$\iint_{\mathbb{R}\times\mathbb{R}} |g(x)||\widehat{g}(\xi)|e^{2\pi|x||\xi|}dxd\xi < +\infty \tag{5}$$

Indeed, replacing g and $\widehat{g}$ by their values in terms of f and $\widehat{f}$ and using Fubini's theorem, we are led to prove that the quantity

$$\int_{\mathbb{R}} e^{-\pi[(x-y)^2-2|x||\xi|+2|y||\xi|+\xi^2]}dx$$

is bounded independently of y and ξ. Taking $x-y$ as the variable, it is sufficient to prove that

$$\int_{\mathbb{R}} e^{-\pi[x^2-2|x||\xi|+\xi^2]}dx$$

is bounded by 2, which follows from the fact that $x^2-2|x||\xi|+\xi^2$ is either $(x-\xi)^2$ or $(x+\xi)^2$.

Now, for all $z \in \mathbb{C}$, we have the elementary inequality

$$|g(z)| \leq \int_{\mathbb{R}} |\widehat{g}(\xi)|e^{2\pi|z||\xi|}d\xi\,,$$

so that there exists some constant C such that

$$\int_{-\infty}^{+\infty} |g(x)| \sup_{|z|=|x|} |g(z)|dx \leq C\,. \tag{6}$$

We claim that the holomorphic function

$$G(z) = \int_0^z g(u)g(iu)du$$

is bounded by C. Once we know this, the end of the proof is immediate: G is constant by Liouville's Theorem; so $g(u)g(iu)$ is identically 0, which implies that g is identically 0.

We will need the following version of Phragmèn-Lindelhöf's principle which may be found in [10]: *let ϕ be an entire function of order 2 in the complex plane and let $\alpha \in]0,\pi/2[$; assume that $|\phi(z)|$ is* *bounded by $C(1+|z|)^N$ on the boundary of some angular sector $\{re^{i\beta}\ :r\geq 0,\beta_0\leq\beta\leq\beta_0+\alpha\}$. Then the same bound is valid inside the angular sector (when replacing C by 2^NC).*

It is clear from (6) that G is bounded by C on the axes. Let us prove that it is bounded by C for $z=re^{i\theta}$ in the first quadrant. Assume that θ is in the interval $(0,\pi/2)$. By continuity, it is sufficient to prove that

$$G_\alpha(z)=\int_0^z g(e^{-i\alpha}u)g(iu)du$$

is bounded by C for all $\alpha\in(0,\theta)$. But the function G_α is an entire function of order 2, which is bounded by C on the y-axis and on the half-line $\rho e^{i\alpha}$. By Phagmèn–Lindelhöf principle, it is bounded by C inside the angular sector, which gives the required bound for $|G_\alpha(z)|$. A similar proof gives the same bound in the other quadrants.

4.2. **Sketch of the general case.** Our aim is now to prove that if $f\in L^2(\mathbb{R}^d)$ is such that, for some positive integer N,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(x)||\widehat{f}(y)|}{(1+\|x\|+\|y\|)^N}e^{2\pi|\langle x,y\rangle|}dxdy<+\infty\,, \tag{7}$$

then f may be written as

$$f(x)=P(x)e^{-\pi\langle(A+iB)x,x\rangle},$$

where A and B are two symmetric matrices and P is a polynomial.

We may assume that $f\neq 0$.

First step. Both f and $\widehat{f}$ are in $L^1(\mathbb{R}^d)$.

For almost every y,

$$|\widehat{f}(y)|\int_{\mathbb{R}^d}\frac{|f(x)|}{(1+\|x\|)^N}e^{2\pi|\langle x,y\rangle|}dx<+\infty.$$

As $f\neq 0$, the set of all y's such that $\widehat{f}(y)\neq 0$ has positive measure. In particular, there is a basis $y^1\dots y^d$ of $\mathbb{R}^d$ such that, for $i=1,\dots,d$, $\widehat{f}(y^{(i)})\neq 0$ and

$$\int_{\mathbb{R}^d}\frac{|f(x)|}{(1+\|x\|)^N}e^{2\pi|\langle x,y^{(i)}\rangle|}dx<+\infty.$$

Since, clearly, there exists a constant C such that

$$(1+\|x\|)^N\leq C\sum_{i=1}^d\exp\left[2\pi\left|\left\langle x,y^{(i)}\right\rangle\right|\right],$$

we get $f\in L^1(\mathbb{R}^d)$. Exchanging the roles of f and $\widehat{f}$, we get $\widehat{f}\in L^1(\mathbb{R}^d)$. ⋄

Second step. The function g defined by $\widehat{g}(y)=\widehat{f}(y)e^{-\pi\|y\|^2}$ satisfies the following properties (with C depending only on f)

$$\int_{\mathbb{R}^d}|\widehat{g}(y)|e^{\pi\|y\|^2}dy<\infty\,; \tag{8}$$

$$|\widehat{g}(y)|\leq Ce^{-\pi\|y\|^2}\,; \tag{9}$$

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|g(x)||\widehat{g}(y)|}{(1+\|x\|+\|y\|)^N}e^{2\pi|\langle x,y\rangle|}dxdy<+\infty\,; \tag{10}$$

$$\int_{\|x\|\leq R}\int_{\mathbb{R}^d}|g(x)||\widehat{g}(y)|e^{2\pi|\langle x,y\rangle|}dxdy<C(1+R)^N\,. \tag{11}$$

Property (8) is obvious from the definition of g and the fact that $\widehat{f}$ is in $L^1(\mathbb{R}^d)$. As $f\in L^1(\mathbb{R}^d)$, $\widehat{f}$ is bounded thus (9) is also obvious. To prove (10), we have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|g(x)||\widehat{g}(y)|}{(1+\|x\|+\|y\|)^N}e^{2\pi|\langle x,y\rangle|}dxdy\le\iint|f(t)||\widehat{f}(y)|A(t,y)e^{2\pi|\langle t,y\rangle|}dtdy$$

with

$$A(t,y)=\int\frac{e^{-\pi\|x\|^2-\pi\|y\|^2+2\pi|\langle x,y\rangle|}}{(1+\|t-x\|+\|y\|)^N}dx.$$

It is thus enough to prove that

$$A(t,y)\le C(1+\|t\|+\|y\|)^{-N}\,,\tag{12}$$

which is done by seperating the cases of $\langle x,y\rangle$ being positive or negative, and then suitably cutting the resulting integrals.

To prove (11), fix $c>2$, and cut the integral over y into two parts:

$$\begin{aligned}\int_{\|x\|\le R}|g(x)|&\left(\int_{\|y\|>cR}|\widehat{g}(y)|e^{2\pi|\langle x,y\rangle|}dy+\int_{\|y\|<cR}|\widehat{g}(y)|e^{2\pi|\langle x,y\rangle|}dy\right)dx\\
&\le\int_{\|x\|\le R}|g(x)|\left(\int_{\|y\|>cR}Ce^{-(\pi-2\frac{\pi}{c})\|y\|^2}dy+\int_{\|y\|<cR}|\widehat{g}(y)|e^{2\pi|\langle x,y\rangle|}dy\right)dx\\
&\le K\|g\|_{L^1}+\int_{\|x\|\le R}\int_{\|y\|<cR}|g(x)||\widehat{g}(y)|e^{2\pi|\langle x,y\rangle|}dxdy.\end{aligned}$$

Then, if we multiply and divide by $(1+\|x\|+\|y\|)^N$ in the integral of right side, we get the required inequality (11). This completes the proof of the claim. $\diamond$

Third step. The function g admits an holomorphic extension to $\mathbb{C}^d$ that is of order 2. Moreover, there exists a polynomial R such that for all $z\in\mathbb{C}^d$, $g(z)g(iz)=R(z)$.

It follows from (9) and Fourier inversion that g admits an holomorphic extension to $\mathbb{C}^d$ which we again denote by g. Moreover,

$$|g(z)|\le Ce^{\pi\|z\|^2}\,,$$

with C the L^1 norm of $\widehat{g}$. It follows that g is of order 2. On the other hand, for all $x\in\mathbb{R}^d$ and $e^{i\theta}$ of modulus 1,

$$|g(e^{i\theta}x)|\le\int_{\mathbb{R}^d}|\widehat{g}|(y)e^{2\pi|\langle x,y\rangle|}dy.\tag{13}$$

The occilary function that we consider is essentially the same, up to an adaptation to dimension $d\ge 1$:

$$G\,:\,z\to\int_0^{z_1}\dots\int_0^{z_d}g(u)g(iu)du.$$

As g is entire of order 2, so is G. By differentiation of G, the proof of this step is complete once we show that G is a polynomial.

We first reduce the problem to a one complex dimensional problem by considering $G_\xi(z)=G(z\xi)$ with $z\in\mathbb{C},\xi\in\mathbb{C}^d$. The proof is then essentially the same as in the previous section : we prove that G_ξ has at most polynomial growth inside each quadrant by using property 11 and then apply Phragmèn-Lindelhöf's principle. (There is again a small adaptation to stay inside the quadrants).

Step 4. A lemma about entire functions of several variables.

In the previous section, the polynomial R was 0 and the proof was complete after step 3. Here, we can conclude with the help of the following Lemma :

Lemma 4.1. *Let φ be an entire function of order 2 on $\mathbb{C}^d$ such that, on every complex line, either φ is identically 0 or it has at most N zeros. Then, there exists a polynomial P with degree at most N and a polynomial Q with degree at most 2 such that $\varphi(z) = P(z)e^{Q(z)}$.*

This is essentially du to the fact that, in one variable, entire functions of finite type are characterized by there zeros.

5. The short time Fourier transform

One way one might hope to overcome uncertainty principles is by the use of the short-time Fourier transform, also known as the windowed Fourier transform.

To define it, fix a function $v \in L^2(\mathbb{R}^d)$, called the windowed. The windowed Fourier transform of $u \in L^2(\mathbb{R}^d)$ is then the Fourier of u seen through the slides $v(\cdot - x)$ of v. More precisely,

$$\mathcal{F}_v u(x,y) = \mathcal{F}[u(\cdot)\overline{v(\cdot - x)}](y) = \int_{\mathbb{R}^d} u(t)\overline{v(t-x)}e^{-2i\pi ty}dt.$$

A particular case is when v is a Gaussian, in which case $\mathcal{F}_v$ is known as the Bargmann transform.

We prefer working with the radar ambiguity function, defined by

$$A(u,v)(x,y) = \int_{\mathbb{R}^d} u\left(t+\frac{x}{2}\right)\overline{v\left(t+\frac{x}{2}\right)}e^{-2i\pi ty}dt.$$

Note that $A(u,v)$ and $\mathcal{F}_v u$ have same modulus, so there is no change in using $A(u,v)$ instead of $\mathcal{F}_v u$ bellow. Basic properties of $A(u,v)$ can be found in [2].

Another closely linked transform is the Wigner transform, that is the $\mathbb{R}^{2d}$ Fourier transform of $A(u,v)$, and is given by

$$W(u,v)(x,y) = \int_{\mathbb{R}^d} u\left(x+\frac{t}{2}\right)\overline{v\left(x+\frac{t}{2}\right)}e^{-2i\pi ty}dt.$$

This, up to a change of variable, may also be expressed as an ambiguity function (exercise) so that all results of this section can be stated in terms of W.

Let us recall here a few properties of the ambiguity function that we may use in the sequel.

For u, v in $L^2(\mathbb{R}^d)$, $A(u,v)$ is continuous on $\mathbb{R}^{2d}$ and $A(u,v) \in L^2(\mathbb{R}^{2d})$. Further,

$$\|A(u,v)\|_{L^2(\mathbb{R}^{2d})} = \|u\|_{L^2(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}$$

In particular, $A(u,v) = 0$ if and only if $u = 0$ or $v = 0$.

Finally, we will also need the following lemma from [15], [16]:

Lemma 5.1. *Let $u, v, w \in L^2(\mathbb{R}^d)$. Then, for every $x, y \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^{2d}} A(u,v)(s,t)\overline{A(v,w)(s,t)}e^{2i\pi(\langle s,x\rangle + \langle t,y\rangle)}dsdt = A(u,v)(-y,x)\overline{A(v,w)(-y,x)}.$$

This lemma is the key fact in proving uncertainty principles for the ambiguity function as it states that $A(u,v)\overline{A(v,w)}$ is essentially its own Fourier transform. One can then for instance prove

Theorem 5.2 (Jaming [15], Janssen [16], Wilczok [21]). *If the support of $A(u,v)$ is of finite Lebesgue measure, then $u = 0$ or $v = 0$.*

Proof. Assume $v \neq 0$. For every $w \in L^2$, $A(u,v)\overline{A(v,w)}$ has support of finite measure. As this function is essentially its own Fourier transform (up to a rotation of the variables), its Fourier transform also has a support of finite measure. By Benedicks/Amrien-Berthier, this implies that $A(u,v)(x,y)\overline{A(v,w)}(x,y) = 0$, and this, whatever w. But, for x, y fixed, it is not hard, looking at $A(v,w)$ as a scalar product, to find a w such that $A(v,w)(x,y) \neq 0$ so that for every x, y, $A(u,v)(x,y) = 0$ and then $u = 0$. $\square$

One might also prove uncertainty principles for the ambiguity function. There are some results in this direction in [11]. Other results can be found in [5], for instance we proved :

Theorem 5.3. *For $u, v \in L^2(\mathbb{R})$, one has the following inequality:*

$$\iint_{\mathbb{R}^2} |x|^2 |A(u,v)(x,y)|^2 dxdy \iint_{\mathbb{R}^2} |y|^2 |A(u,v)(x,y)|^2 dxdy \geq \frac{\|u\|^4_{L^2(\mathbb{R}^d)} \|v\|^4_{L^2(\mathbb{R}^d)}}{4\pi^2}. \tag{14}$$

Moreover equality holds in (14), with u and v non identically 0, if and only if there exists $\mu, \nu \in L^2(\mathbb{R}^{d-1})$, $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$ such that

$$u(t) = e^{2i\pi\beta t} e^{-\alpha/2|t-\gamma|^2},$$
$$v(t) = e^{2i\pi\beta t} e^{-\alpha/2|t-\gamma|^2}.$$

Finally, we have extended Theorem 3.3 to the ambiguity function:

Theorem 5.4. *Let $u, v \in L^2(\mathbb{R}^d)$ be non identically vanishing. If*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|A(u,v)(x,y)|^2}{(1 + \|x\| + \|y\|)^N} e^{\pi(\|x\|^2 + \|y\|^2)} dxdy < +\infty\,, \tag{15}$$

then there exists $a, w \in \mathbb{R}^d$ such that both u and v are of the form $P(x)e^{2i\pi\langle w,x\rangle}e^{-\pi\|x-a\|^2}$ with P a polynomial.

Note that we have not been able to replace $e^{\pi(\|x\|^2+\|y\|^2)}$ by $e^{2\pi|\langle x,y\rangle|}$ in this theorem.

As a conclusion, we would like to stress out that most of the theorems stated here admit slightly storonger statements in [5], in particular, they admit directionnal versions.

Note also that Gröchenig and Zimmermann [11] have slightly simpler arguments to deduce uncertainty principles for the short-time Fourier transform from there counterparts for the usual Fourier transform and Lemma 5.1.

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