HARMONIC FUNCTIONS ON THE REAL HYPERBOLIC BALL

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1. SETTING

$SO(n,1)$ is the component of the identity of the group of matrices $g=(g_{i,j})_{0\leq i,j\leq n}$ that leaves invariant the quadratic form $-x_0^2+x_1^2+\ldots+x_n^2$.

It acts conformaly on $\mathbb{B}_n$ (unit ball of $\mathbb{R}^n$, $n\geq 3$) by $y=g.x$ with

$$y_p=\frac{\frac{1+|x|^2}{2}g_{p0}+\sum_{l=1}^n g_{pl}x_l}{\frac{1-|x|^2}{2}+\frac{1+|x|^2}{2}g_{00}+\sum_{l=1}^n g_{0l}x_l}$$

for $p=1,\ldots,n$.

Fact *If* $a=g.0$ *then* $g.B(0,\varepsilon)\subset B(a,6(1-|a|^2)\varepsilon)$.

The *invariant Laplacian* for this action is

$$D=(1-|x|^2)^2\Delta+2(n-2)(1-|x|^2)N$$

(Δ the Euclidean Laplacian, $N=\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$ normal derivation).

Functions u on $\mathbb{B}_n$ such that $Du=0$ will be called $\mathcal{H}$*-harmonic*

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The *Hyperbolic Poisson kernel* associated to D is for $x\in\mathbb{B}_n$ and $\xi\in\mathbb{S}^{n-1}$ (unit sphere of $\mathbb{R}^n$)

$$\mathbb{P}_h(x,\xi)=\left(\frac{1-|x|^2}{1+|x|^2-2<x,\xi>}\right)^{n-1}$$

to be compared with the Euclidean Poisson kernel

$$\mathbb{P}_e(x,\xi)=\frac{1-|x|^2}{(1+|x|^2-2<x,\xi>)^{\frac{n}{2}}}.$$

2. MEAN VALUE INEQUALITIES

$\mathcal{H}$-harmonic functions satisfy the following mean value inequalities :

For every $\varepsilon>0$ *small,* $k,d\in\mathbb{N}$, $0<p<\infty$, *there exists a constant* $C=C(\varepsilon,k,d,p)$ *such that for every* $\mathcal{H}$*-harmonic function* u *and every* $a\in\mathbb{B}_n$

$$\left|\nabla^d N^k u(a)\right|\leq C(1-|a|^2)^{-d-\frac{n}{p}}\times\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)}\left|N^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

3. BOUNDARY BEHAVIOR

Definition u has a *boundary distribution* if $\forall\Phi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$,

$$\lim_{r\to 1}\int_{\mathbb{S}^{n-1}}u(r\zeta)\Phi(\zeta)d\sigma(\zeta)\quad\text{exists.}$$

Proposition An $\mathcal{H}$-harmonic function u has a boundary distribution if and only if $u(r\zeta)=O\big((1-r)^{-A}\big)$ for some A.

Theorem *Let* u *be* $\mathcal{H}$*-harmonic with a boundary distribution. Let* $v=N^k u$, *then*

— *if* $k\leq n-2$, v *has a boundary distribution.*

— *if* $k=n-1$, $\int_{\mathbb{S}^{n-1}}v(r\zeta)\Phi(\zeta)d\sigma(\zeta)=O(\log\frac{1}{1-r})$.

Proposition[1] Conversely, for $k = n-1$,

— if n is odd, $\int_{\mathbb{S}^{n-1}} v(r\zeta)\Phi(\zeta)d\sigma(\zeta) = o(\log\frac{1}{1-r}) \Leftrightarrow u$ constant.

— if n is even, $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ then $\mathbb{P}_h[f] \in \mathcal{C}^\infty(\overline{\mathbb{B}_n})$.

4. H^p SPACES

Notation for $u : \mathbb{B}_n \mapsto \mathbb{C}$, $\mathcal{M}[u](\zeta) = \sup_{0<r<1} |u(r\zeta)|,\ \zeta \in \mathbb{S}^{n-1}$.

Definition $\mathcal{H}^p = \{u\ \mathcal{H}\text{−harmonic} \mid \mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})\}$

Proposition — $1 < p < \infty$, $u \in \mathcal{H}^p \Leftrightarrow u = \mathbb{P}_h[f]$, $f \in L^p$.

— $p = 1$, $u \in \mathcal{H}^p \Leftrightarrow u = \mathbb{P}_h[\mu]$, μ measure.

— $p < 1$, $u \in \mathcal{H}^p \Rightarrow u = \mathbb{P}_h[\varphi]$, φ distribution.

5. ATOMIC DECOMPOSITION $(0 < p \leq 1)$

Definition A p-atom a on $\mathbb{S}^{n-1}$ is a function on $\mathbb{S}^{n-1}$ with support in a ball $B(\xi_0, r_0)$ such that

(1) $|a(\xi)| \leq |B(\xi_0, r_0)|^{-1/p}$
(2) for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$

$$\left|\int_{\mathbb{S}^{n-1}} a(\xi)\Phi(\xi)d\sigma(\xi)\right| \leq \left\|\nabla^{k(p)}\Phi\right\|_{L^\infty\left(B(\xi_0,r_0)\right)} \times r_0^{k(p)}|B(\xi_0,r_0)|^{1-1/p} \quad (1)$$

with $k(p)$ an integer $> (n-1)(1-1/p)$ (*cancellation property*).

Definition $\mathcal{H}^p_{at} = \{u = \sum \lambda_i \mathbb{P}_h[a_i] \mid (\lambda_i) \in \ell^p,\ a_i\ p-\text{atom}\}$

Proposition $\exists C, \forall a$ p-atom, $\|\mathbb{P}_h[a]\|_{\mathcal{H}^p} \leq C$, in particular $\mathcal{H}^p_{at} \subset \mathcal{H}^p$.

Conversely if $u \in \mathcal{H}^p$ then $u = \mathbb{P}_h[\varphi]$, φ distribution.

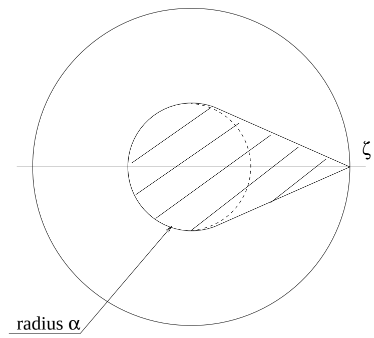
$$\begin{array}{ccc} & & u = \mathbb{P}_h[\varphi] \\ & \nearrow & \\ \varphi & & \\ & \searrow & \\ & & v = \mathbb{P}_e[\varphi] \end{array}$$

Lemma $\exists \eta : [0,1] \times [0,1] \mapsto \mathbb{R}_+$ s.t. $\int_0^1 \eta(r,s)ds \leq C$ and $v(r\zeta) = \int_0^1 \eta(r,s)u(rs\zeta)ds \Rightarrow v \in H^p \Rightarrow (Garnett - Latter)\ f = \sum \lambda_i a_i \Rightarrow u \in \mathcal{H}^p_{at}$.

Theorem *For* $0 < p \leq 1$, $\mathcal{H}^p = \mathcal{H}^p_{at}$.

6. CHARACTERIZATIONS

For $0 < \alpha < 1$ and $\zeta \in \mathbb{S}^{n-1}$, let $\mathcal{A}_\alpha(\zeta)$ be the interior of the convex hull of $B(0,\alpha)$ and ζ ; $\mathcal{A}_\alpha(\zeta)$ will be called *non-tangential approach region.*



non-tangential approach region $\mathcal{A}_\alpha(\zeta)$

Definition For $u : \mathbb{B}_n \mapsto \mathbb{C}$ define the following functionals :

$\mathcal{M}_\alpha[u](\xi) = \sup_{x \in \mathcal{A}_\alpha(\xi)} |u(x)|$.

$S_\alpha[u](\xi) = \left(\int_{\mathcal{A}_\alpha(\xi)} |\nabla u(x)|^2 \frac{dx}{(1-|x|^2)^{n-2}}\right)^{\frac{1}{2}}$.

$g[u](\xi) = \left(\int_0^1 |\nabla u(r\xi)|^2(1-r^2)dr\right)^{\frac{1}{2}}$.

S^N_α and g^N same definition with ∇ replaced by N.

Theorem *Let* $0 < p < \infty$. *For* u $\mathcal{H}$-*harmonic, the following are equivalent to* $u \in \mathcal{H}^p$ *:*

(1) $\mathcal{M}_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some (all) $0 < \alpha < 1$;
(2) $S_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some (all) $0 < \alpha < 1$;
(3) $S^N_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some (all) $0 < \alpha < 1$;
(4) $g[u] \in L^p(\mathbb{S}^{n-1})$;
(5) $g^N[u] \in L^p(\mathbb{S}^{n-1})$.

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[1]The complex case has been proved in **A. Bonami, J. Bruna and S. Grellier** *On Hardy, BMO and Lipschitz spaces of invariant harmonic functions in the unit ball* Proc. London Math. Soc. (*to appear*)