Nazarov's uncertainty principle in higher dimension

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Outline of talk

Definitions

Definition

Let S, Σ subsets of $\mathbb{R}^d$.

- (S, Σ) *is an annihilating pair if*

$$\operatorname{supp} f \subset S \quad \& \quad \operatorname{supp} \widehat{f} \subset \Sigma \quad \Rightarrow \quad f = 0;$$

- (S, Σ) *is a strong annihilating pair if* $\exists C = C(S, \Sigma)$ *s.t.* $\forall f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq C \left(\int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi \right).$$

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$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq D \int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, dx$$

Motivation

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Let $S, \Sigma \subset \mathbb{R}^d$ have finite measure. Then

- *(Benedicks 1974-1985) (S, Σ) is weakly annihilating.*
- *(Amrein-Berthier 1977) (S, Σ) is strongly annihilating.*
- *(Nazarov $d = 1$ 1993) $C(S, \Sigma) \leq ce^{c|S||\Sigma|}$*
- *(J. $d \geq 2$ 2007) $C(S, \Sigma) \leq ce^{c \min\left(|S||\Sigma|, |S|^{1/d}\omega(\Sigma), \omega(S)|\Sigma|^{1/d}\right)}$*
  $\omega(S)$ = mean width of S.

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- $f = e^{-\pi|x|^2} = \widehat{f}$, $S = \Sigma = B(0, R)$

$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq c e^{(2\pi+\varepsilon)R^2} \left(\int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|x|^2} \, dx + \int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|\xi|^2} \, d\xi \right).$$

- Optimal: $C(S, \Sigma) \leq c e^{(2\pi+\varepsilon)(|S||\Sigma|)^{1/d}}$.
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Proof 1/2

$|S|, |\Sigma| < +\infty$, $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset S$ & $\operatorname{supp} \widehat{f} \subset \Sigma$.

1. WLOG $|S| < 1$
2. $\displaystyle\int_{[0,1]} \sum_k \chi_\Sigma(\xi + k)\, d\xi = |\Sigma| < +\infty \Rightarrow$ for a.a. $\xi \in \mathbb{R}$, $\mathrm{Card}\,\{k \in \mathbb{Z} : \xi + k \in \Sigma\}$ *finite*
3. $\displaystyle\int_{[0,1]} \underbrace{\sum_k \chi_S(\xi + k)}_{=0 \text{ or } \geq 1}\, d\xi = |S| < 1 \Rightarrow \exists F \subset [0,1], |F| > 0$ s.t. $\forall x \in F, k \in \mathbb{Z}, f(x+k) = 0$.

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4. by Poisson Summation
$$\sum_{k\in\mathbb{Z}} f(x+k)e^{2i\pi\xi(x+k)} = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi+k)e^{2i\pi kx}.$$
By 2, the RHS is a trigonometric polynomial $Z(f)(x)$ in x (for a.a. ξ)
By 3, the LHS is supported in $[0,1]\setminus F$
5. $Z(f)=0 \Rightarrow \widehat{f}=0 \Rightarrow f=0.$

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Lemma (Nazarov, $d = 1$)

$\varphi \in L^1(\mathbb{R})$, $\varphi \geq 0$,

$$\int_1^2 \sum_{k \in \mathbb{Z} \quad \setminus \{0\}} \varphi(v \quad k \;) \, \mathrm{d}v \qquad \simeq \int_{\|x\| \geq 1} \varphi(x) \, \mathrm{d}x$$

Lemma (Nazarov, $d = 1$)

$\varphi \in L^1(\mathbb{R}^d)$, $\varphi \geq 0$,

$$\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\,\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho) \simeq \int_{\|x\| \geq 1} \varphi(x)\,\mathrm{d}x$$

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$$\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\,\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho)$$

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Turan type Lemma

Lemma (Nazarov, $d = 1$, Fontes-Merz $d \geq 2$)

- $p(\theta_1, \dots, \theta_d) = \sum_{k_1=0}^{m_1} \cdots \sum_{k_d=0}^{m_d} c_{k_1,\dots,k_d} e^{2i\pi(r_{1,k_1}\theta_1 + \cdots r_{r_d,k_d}\theta_d)}$ *a trigonometric polynomial in d variables.*
- $E \subset \mathbb{T}^d$,
- *Then*

$$\sup_{(\theta_1,\dots,\theta_d)\in\mathbb{T}^d} |p(\theta_1, \dots, \theta_d)| \leq \left(\frac{14d}{|E|}\right)^{m_1+\cdots+m_d} \sup_{(\theta_1,\dots,\theta_d)\in E} |p(\theta_1, \dots, \theta_d)|.$$

— ord $p := m_1 + \cdots + m_d$ is called *the order of* p.

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Lemma

- Σ *be a relatively compact open set with* $0 \in \Sigma$
- $\Lambda = \Lambda(\rho, v) := \{v\, {}^t\rho(j) \ : \ j \in \mathbb{Z}^d\}$ *a random lattice*
- $\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z}^d \ : \ v\, {}^t\rho(k) \in \Sigma\} = \Lambda \cap \Sigma$
- *then*

$$\mathbb{E}_{\rho,v}\big(\mathrm{ord}\,\mathcal{M}_{\rho,v} - d\big) \leq C\omega(\Sigma).$$

remark

If order $\to$ size of support, $\mathbb{E}_{\rho,v}\big(\mathrm{Card}\,\mathcal{M}_{\rho,v} - d\big) \leq C|\Sigma|$.

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End of Proof 1/4

Scale to have $|S| = 2^{-d-1}$ and take $f \in L^2$ with $\operatorname{supp} f \subset S$

Set $\Gamma_{\rho,v}(t) = \dfrac{1}{v^{d/2}} \displaystyle\sum_{k \in \mathbb{Z}^d} f\left(\frac{\rho(k+t)}{v}\right)$

Set $E_{\rho,v} = \{t \in [0,1] : \Gamma_{\rho,v}(t) = 0\}$

$$\Gamma_{\rho,v}(t) = v^{d/2} \sum_{m \in \mathbb{Z}^d} \widehat{f}(v^t \rho(m)) e^{2i\pi mt} \text{ (Poisson summation)}$$

$$= \sum_{m \in \mathcal{M}_{\rho,v}} + \sum_{m \notin \mathcal{M}_{\rho,v}} := P_{\rho,v} + R_{\rho,v}$$

with $\mathcal{M}_{\rho,v} = \{m \in \mathbb{Z}^d : v^t \rho(m) \in \Sigma\}$

End of Proof 1/4

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End of Proof 2/4

From the Lattice averaging lemma, one can choose ρ, v s.t.

— $\|R_{\rho,v}\|_2^2 \leq C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, d\xi$ (w.h.p)

— ord $P_{\rho,v} \leq C(\omega(\Sigma) + d)$ (w.h.p)

— $|E_{\rho,v}| \geq 1/2$ (certain)

— $\widehat{f}(0) \leq |P_{\rho,v}(0)|$ (certain).

ρ, v s.t. all 4 properties hold.

End of Proof 2/4

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— $\operatorname{ord} P_{\rho,v} \leq C(\omega(\Sigma) + d)$ (w.h.p)

— $|E_{\rho,v}| \geq 1/2$ (certain)

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End of Proof 2/4

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ρ, v s.t. all 4 properties hold.

End of Proof 3/4

On $E_{\rho,v}$, we have $\Gamma_{\rho,v}=0$, thus $P_{\rho,v}=-R_{\rho,v}$ so

$$\int_{E_{\rho,v}}|P_{\rho,v}(t)|^2\,\mathrm{d}t=\int_{E_{\rho,v}}|R_{\rho,v}(t)|^2\,\mathrm{d}t\leq C\int_{\mathbb{R}^d\setminus\Sigma}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi$$

So $E:=\{t\in E_{\rho,v}\ :\ |P_{\rho,v}(t)|^2\leq 16C^2\int_{\mathbb{R}^d\setminus\Sigma}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\}$ has $|E|\geq 1/4$.

End of Proof 4/4

$$
\begin{aligned}
|\widehat{f}(0)|^2 &\leq |\widehat{P_{\rho,v}}(0)|^2 \leq \left(\sum_{k\in\mathbb{Z}^d} |\widehat{P_{\rho,v}}(k)|\right)^2 \leq \left(\sup_{x\in\mathbb{T}^d} |P_{\rho,v}(x)|\right)^2 \\
&\leq \left[\left(\frac{14d}{|E|}\right)^{\mathrm{ord}P_{\rho,v}-1} \sup_{x\in E} |P_{\rho,v}(x)|\right]^2 \\
&\leq \left[\left(\frac{14d}{1/4}\right)^{\mathrm{ord}P_{\rho,v}-1} 4\left(C\int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\right]^2 \\
&\leq Ce^{C\omega(\Sigma)}\int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi.
\end{aligned}
$$

Apply to $f \to f_y(x) = f(x)e^{-2i\pi xy}$, $\Sigma \to \Sigma_y = \Sigma - y$ and integrate over $y \in \Sigma$ QED

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