Lecture 3: Monotone Semiflows

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Outline

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map. Consider the ordinary differential equation

$$\begin{cases} u'(t) = f(u(t)), \text{ for all } t \geq 0, \\ u(0) = x \in \mathbb{R}. \end{cases}$$

From Chapter 5, we know that there exists a unique maximal semiflow $U(t,x)$ combined with a blow-up time $\tau(x)$ such that $u(t) := U(t,x)$ is the unique solution of the fixed-point problem

$$u(t) = x + \int_0^t f(u(s))\mathrm{d}s \text{ for all } t \in [0, \tau(x)),$$

and

$$\lim_{t \to \tau(x)} |u(t)| = +\infty, \text{ whenever } \tau(x) < +\infty.$$

In this chapter, we will be interested in the comparison property for semi-flows. This type of property can be proved by using very simple arguments for the one-dimensional case.

Lemma 1.1

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1-map. Then the following comparison principle holds

$$x < y \Rightarrow U(t,x) < U(t,y), \text{ for all } 0 \leq t < \min\left(\tau(x), \tau(y)\right).$$

Proof. Let $x < y$ be given. Assume by contradiction that we can find $0 \leq t_0 < \min\left(\tau(x), \tau(y)\right)$ such that

$$U(t_0, x) = U(t_0, y) \text{ and } U(t, x) < U(t, y) \text{ for all } t \in (0, t_0).$$

Set $v_1(t) := U(t_0 - t, x)$ and $v_2(t) := U(t_0 - t, y)$ for each $t \in [0, t_0]$. Then we have

$$v_1(0) = v_2(0)$$

and

$$v_1'(t) = -f(v_1(t)) \text{ and } v_2'(t) = -f(v_2(t)) \text{ for all } t \in [0, t_0].$$

Since the equation $v'(t) = -f(v(t))$ admits at most one solution with initial value $v(0) = v_1(0) = v_2(0)$, we have $v_1(t) = v_2(t)$ for all $t \in [0, t_0]$ and therefore $x = v_1(t_0) = v_2(t_0) = y$. This is a contradiction. □

The following lemma says that if the solutions with positive initial values remain positive for all times, then the solution starting from a larger initial value will blow-up first.

Lemma 1.2

Assume that $f \in C^1(\mathbb{R})$ and $f(0) \geq 0$. Then it holds that

$$0 \leq x < y \Rightarrow \tau(y) \leq \tau(x).$$

Proof. Let $0 \leq x < y$ be given. Assume that $\tau(x) < \tau(y)$. Since $f(0) \geq 0$, we have

$$0 \leq U(t,x) < U(t,y), \text{ for all } t \in [0, \tau(x)).$$

This implies that $\tau(x) < +\infty$, and we must have

$$\lim_{t \to \tau(x)} U(t,x) = +\infty.$$

But $\tau(x) < \tau(y)$ also implies that $t \mapsto U(t,y)$ belongs to $C([0,\tau(x)],\mathbb{R})$. So we obtain

$$+\infty = \lim_{t \to \tau(x)} U(t,x) \leq \limsup_{t \to \tau(x)} U(t,y) < +\infty,$$

which is a contradiction. □

Exercise 1.3

Let $f \in C^1(\mathbb{R})$ be given and assume that (as in the logistic equation) there exists a $\kappa > 0$ such that

$$f(0) = f(\kappa) = 0 \text{ and } f(x) > 0 \text{ for all } x \in (0, \kappa).$$

Consider the ordinary differential equation

$$\begin{cases} u'(t) = f(u(t)) \text{ for all } t \geq 0, \\ u(0) = x \in [0, \kappa]. \end{cases}$$

1. *Prove that, for each $x_0 \in (0, \kappa]$, we have*

$$\lim_{t \to +\infty} u(t) = \kappa.$$

2. *By using a similar argument, show that for each $x_0 \in [0, \kappa)$, we have*

$$\lim_{t \to -\infty} u(t) = 0.$$

Hint: consider reversing the time (i.e. replace $u(t)$ by $v(t) = u(-t)$.

Remark 1.4

Under the conditions of Exercise 1.3, we can show that the points in $(0, \kappa)$ *all belong to a unique heteroclinic orbit joining* 0 *to* κ.

Example 1.5 (Strong Allee effect)

A population may go extinct because the number of individuals is too small. This was the case for the bears of the Pyrenées, where the males and females were unable to meet and therefore could not reproduce. This phenomenon is called the *Allee effect* [2, 3]. The term "Allee's principle" was introduced in the 1950s, a time when the field of ecology was heavily focused on the role of competition among and within species.

In order to describe this mechanism people consider the strong Allee effect

$$u'(t) = \lambda u(t)(u(t) - \varepsilon)(1 - u(t)/\kappa),$$

with a positive initial value

$$u(0) = x \geq 0.$$

We assume that the parameters satisfy

$$\lambda > 0 \text{ and } 0 < \varepsilon < \kappa.$$

Characterization of the Allee effects

Consider a scalar equation of the form

$$N'(t) = (\beta(N) - \mu(N))\, N(t). \tag{1}$$

The birth and the death rate are respectively given by

$$\beta(N) = -aN^2 + bN + c \text{ and } \mu(N) = dN + e,$$

where $a, b, c, d, e \geq 0$ are given parameters.
By summing-up the two terms in (1) we obtain

$$N' = N(-aN^2 + (b-d)N - (e-c)).$$

Therefore, by choosing

$$e - c > 0$$

and $b - d$ large enough, we obtain

$$\Delta = (b-d)^2 - 4a(e-c) > 0,$$

a strong Allee effect.

Example 1.6 (Weak Allee effect)

In order to describe the fact that the speed of growth is sub-linear, one may introduce the so-called *weak Allee effect*. Namely, we consider a population growing as

$$u'(t) = \lambda u(t)^{\alpha}(1 - u(t)/\kappa),$$

with

$$\alpha > 1.$$

It is called a weak Allee effect because the derivative of the right-hand side at zero is null.

It may be consistent to introduce an Allee effect in some epidemic models (see Hilker, Langlais, and Malchow [24]). The resulting systems may undergo a complex dynamic. One may combine the Allee effect with the spatial motion of individuals. This leads to some delicate analysis for the so-called bistable nonlinearity of reaction-diffusion equations. We refer to [18, 6, 46, 19, 1] for more results and references on this topic.

Outline

We continue this chapter with Gronwall's Lemma [20]. Because this result can be very useful when obtaining estimates, and since it can be proved by using a comparison argument, it serves as a good introduction to monotone ordinary differential equations and comparison arguments. There are many variants of Gronwall's Lemma, of which we introduce only the most commonly used in the present section.

Differential form of Gronwall's lemma

Let $u \in C^1([0,\tau],\mathbb{R})$. Assume that u satisfies the following differential inequality

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq \alpha u(t) + \beta \text{ for all } t \in [0,\tau],$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

We remark that this inequality can be rewritten as

$$\begin{aligned}
& \frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq \alpha u(t) + \beta \\
\Leftrightarrow\ & \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \alpha u(t) \leq \beta \\
\Leftrightarrow\ & \mathrm{e}^{\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\alpha t}u(t)) \leq \beta \\
\Leftrightarrow\ & \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\alpha t}u(t)) \leq \beta\mathrm{e}^{-\alpha t},
\end{aligned}$$

and by integrating both sides of this last inequality between 0 and t we obtain

$$\mathrm{e}^{-\alpha t}u(t) - \mathrm{e}^{-\alpha 0}u(0) \leq \int_0^t \mathrm{e}^{-\alpha l}\beta \mathrm{d}l.$$

Thus, we obtain the following lemma, which is the differential form of Gronwall's lemma.

Lemma 2.1 (Gronwall, differential form)

Let $\alpha, \beta \in \mathbb{R}$ be given. Assume that $u \in C^1([0,\tau],\mathbb{R})$ satisfies the following inequality

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq \alpha u(t) + \beta \text{ for all } t \in [0,\tau].$$

Then u satisfies

$$u(t) \leq \mathrm{e}^{\alpha t}u(0) + \int_0^t \mathrm{e}^{\alpha(t-l)}\beta \mathrm{d}l, \text{ for all } t \in [0,\tau].$$

Integral form of Gronwall's lemma

Let $u \in C([0,\tau],\mathbb{R})$ satisfying

$$u(t) \leq \alpha + \beta \int_0^t u(s)\mathrm{d}s, \text{ for all } t \in [0,\tau], \tag{2}$$

for some $\alpha \in \mathbb{R}$ and $\beta \geq 0$.
By setting

$$N(t) = \int_0^t u(s)\mathrm{d}s \text{ for all } t \in [0,\tau],$$

we obtain

$$N'(t) \leq \alpha + \beta N(t) \text{ for all } t \in [0,\tau],$$

and by using the differential form of Gronwall's lemma we deduce that

$$N(t) \leq \int_0^t \mathrm{e}^{\beta(t-l)}\alpha \mathrm{d}l, \text{ for all } t \in [0,\tau].$$

Since $\beta \geq 0$ we obtain

$$u(t) \leq \alpha + \beta N(t) \leq \alpha + \beta \int_0^t \mathrm{e}^{\beta(t-l)} \alpha \mathrm{d}l = \alpha \mathrm{e}^{\beta t},$$

and we obtain the integral form of Gronwall's Lemma.

Lemma 2.2 (Gronwall, integral form)

Let $\alpha \in \mathbb{R}$ and $\beta \geq 0$ be given. Let $u \in C([0,\tau],\mathbb{R})$ such that

$$u(t) \leq \alpha + \beta \int_0^t u(s)\mathrm{d}s \text{ for all } t \in [0,\tau].$$

Then, it holds that

$$u(t) \leq \alpha \mathrm{e}^{\beta t} \text{ for all } t \in [0,\tau].$$

An alternative proof using a comparison argument

Consider the map $\Gamma : C([0,\tau],\mathbb{R}) \to C([0,\tau],\mathbb{R})$ defined by

$$\Gamma(u)(t) = \alpha + \beta \int_0^t u(s)\mathrm{d}s \text{ for all } t \in [0,\tau].$$

Then (2) can be rewritten as

$$u \leq \Gamma(u) \Leftrightarrow u(t) \leq \Gamma(u)(t) \text{ for all } t \in [0,\tau].$$

Since $\beta \geq 0$ the map Γ is monotone non-decreasing, that is,

$$u \leq v \Rightarrow \Gamma(u) \leq \Gamma(v).$$

This implies that

$$u \leq \Gamma(u) \Rightarrow u \leq \Gamma(u) \leq \Gamma^2(u)$$

and by induction we find that for each integer $n \geq 1$,

$$u \leq \Gamma(u) \leq \Gamma^2(u) \leq \cdots \leq \Gamma^n(u).$$

We know that the operator Γ is also the fixed-point operator of the linear scalar ordinary differential equation $u'(t) = \beta u(t)$. Therefore by using the fixed-point argument in Chapter 2, we have

$$\lim_{n \to +\infty} \Gamma^n(u) = v,$$

where $v \in C([0, \tau], \mathbb{R})$ satisfies

$$v = \Gamma(v) \Leftrightarrow v(t) = \alpha + \beta \int_0^t v(s) \mathrm{d}s \text{ for all } t \in [0, \tau],$$

or equivalently,

$$\begin{cases} v'(t) = \beta v(t) \text{ for all } t \in [0, \tau], \\ v(0) = \alpha. \end{cases}$$

We deduce that

$$v(t) = \alpha \mathrm{e}^{\beta t} \text{ for all } t \in [0, \tau],$$

and the result follows.

Exercise 2.3

Let $\alpha, \gamma \in \mathbb{R}$ *and* $\beta \geq 0$ *be given. Let* $u \in C([0,\tau],\mathbb{R})$ *and assume that*

$$u(t) \leq \alpha e^{\gamma t} + \int_0^t \beta e^{\gamma(t-s)} u(s) \mathrm{d}s \text{ for all } t \in [0,\tau].$$

Extend the arguments of the two proofs above to deduce that

$$u(t) \leq \alpha e^{(\gamma+\beta)t} \text{ for all } t \in [0,\tau].$$

Ordinary differential equations with Lipschitz continuous right-hand side

Corollary 2.4

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ *be a locally Lipschitz continuous map (that is, a map that is Lipschitz continuous on bounded sets of* $\mathbb{R}^n$*), and assume that*

$$\|F(x)\| \leq \alpha \|x\| + \beta \text{ for all } x \in \mathbb{R}^n,$$

for some $\alpha \geq 0$ *and* $\beta \geq 0$.
Then there exists a (globally defined) unique continuous semiflow $\{U(t)\}_{t\geq 0}$ *on* $\mathbb{R}^n$ *such that for each* $x \in \mathbb{R}^n$, $u(t) := U(t)x$ *is the unique solution of*

$$\begin{cases} u'(t) = F(u(t)) \text{ for all } t \geq 0, \\ u(0) = x. \end{cases}$$

Moreover we have

$$\|U(t)x\| \leq \mathrm{e}^{\alpha t}\|x\| + \int_0^t \mathrm{e}^{\alpha(t-s)}\beta \mathrm{d}s \text{ for all } t \geq 0.$$

Proof. Let $x \in \mathbb{R}^n$ be given. We know that there exist $\tau(x) \leq +\infty$ and a continuous function $t \mapsto u(t)$ from $[0, \tau(x))$ into $\mathbb{R}^n$ such that

$$u(t) = x + \int_0^t F(u(s))\mathrm{d}s \text{ for all } t \in [0, \tau(x)),$$

and

$$\lim_{t \to \tau(x)^-} \|u(t)\| = +\infty \text{ if } \tau(x) < +\infty.$$

We have

$$\|u(t)\| \leq \|x\| + \int_0^t \|F(u(s))\|\mathrm{d}s \text{ for all } t \in [0, \tau(x)),$$

hence by using the assumption on F we deduce that

$$\|u(t)\| \leq \|x\| + \int_0^t \alpha\|u(s)\| + \beta\mathrm{d}s \text{ for all } t \in [0, \tau(x)).$$

Setting

$$X(t) := \int_0^t \alpha \|u(s)\| + \beta \mathrm{d}s \text{ for all } t \in [0, \tau(x)),$$

we obtain (since $\alpha \geq 0$)

$$X'(t) = \alpha \|u(t)\| + \beta \leq \alpha \|x\| + \alpha X(t) + \beta \text{ for all } t \in [0, \tau(x)).$$

Thus

$$X'(t) \leq [\alpha \|x\| + \beta] + \alpha X(t) \text{ for all } t \in [0, \tau(x)).$$

Now by applying the differential form of Gronwall's lemma, we obtain

$$X(t) \leq \int_0^t \mathrm{e}^{\alpha(t-s)} [\alpha \|x\| + \beta] \mathrm{d}s = \int_0^t \mathrm{e}^{\alpha l} [\alpha \|x\| + \beta] \mathrm{d}l \text{ for all } t \in [0, \tau(x)),$$

and

$$\|u(t)\| \leq \|x\| + \int_0^t \mathrm{e}^{\alpha l} [\alpha \|x\| + \beta] \mathrm{d}l, \text{ for all } t \in [0, \tau(x)).$$

This implies $\tau(x) = +\infty$ and the result follows. $\square$

Theorem 2.5

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous map on $\mathbb{R}^n$. There exists a (global) unique semiflow $\{U(t)\}_{t \geq 0}$ on $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $u(t) := U(t)x$ is the unique solution of the ordinary differential equation

$$\begin{cases} u'(t) = F(u(t)) \text{ for all } t \geq 0 \\ u(0) = x. \end{cases}$$

Moreover, we have

$$\|U(t)x\| \leq \mathrm{e}^{\|F\|_{\mathrm{Lip}} t}\|x\| + \int_0^t \mathrm{e}^{\|F\|_{\mathrm{Lip}}(t-s)}\|F(0)\|\mathrm{d}s, \text{ for all } t \geq 0$$

and

$$\|U(t)x - U(t)y\| \leq \mathrm{e}^{\|F\|_{\mathrm{Lip}} t}\|x - y\| \text{ for all } t \geq 0,$$

where

$$\|F\|_{\mathrm{Lip}} := \sup_{x,y \in \mathbb{R}^n : x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|}.$$

Proof. Observe that

$$\|F(x)\| = \|F(x) - F(0)\| + \|F(0)\| \leq \|F\|_{\mathrm{Lip}} \|x - 0\| + \|F(0)\|.$$

Then by applying Corollary 2.4, the first part of the result follows.
To prove the second part of the theorem, we observe that

$$U(t)x - U(t)y = x - y + \int_0^t F(U(s)x) - F(U(s)y) \mathrm{d}s \text{ for all } t \geq 0.$$

Therefore

$$\|U(t)x - U(t)y\| \leq \|x - y\| + \int_0^t \|F(U(s)x) - F(U(s)y)\| \mathrm{d}s, \text{ for all } t \geq 0,$$

and by using the fact that F is Lipschitz continuous we obtain

$$\|U(t)x - U(t)y\| \leq \|x - y\| + \|F\|_{\mathrm{Lip}} \int_0^t \|U(s)x - U(s)y\| \mathrm{d}s, \text{ for all } t \geq 0.$$

The result follows by using the integral form of Gronwall's lemma. □

Exercise 2.6

Consider the map $F:[0,+\infty)^n \to \mathbb{R}^n$ *defined by* $F(0)=0$ *and*

$$F(x)=\frac{B(x,x)}{\|x\|_1} \text{ if } x>0,$$

where $B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *is a bilinear map satisfying*

$$B(x,x)\geq 0 \text{ for all } x\geq 0$$

and

$$\|x\|_1=\sum_{i=1}^n |x_i|.$$

Consider the ordinary differential equation

$$U'(t)=F(U(t))-U(t) \text{ for all } t\geq 0$$

with initial distribution $U(0)=U_0\geq 0$. *(Observe that it is not clear that* F *can be extended to a Lipschitz continuous map on the whole space* $\mathbb{R}^n$.*)*

Non-autonomous form of Gronwall's Lemma

Recall that

$$W^{1,\infty}(0,\tau) := \{u \in L^\infty(0,\tau) \,:\, u' \in L^\infty(0,\tau)\}.$$

This space is called a *Sobolev space*. The notion of derivative used in the definition of this space is the derivative *in the sense of distributions*, that is,

$$\int_0^\tau \varphi'(s)u(s)\mathrm{d}s = -\int_0^\tau \varphi(s)u'(s)\mathrm{d}s,$$

whenever $\varphi \in C_c^\infty(0,\tau)$ (i.e. φ is of class C^∞ and has a compact support in $(0,\tau)$).

It turns out that this notion of differentiable function is equivalent to saying that

$$u(t) = c + \int_0^t u'(s)\mathrm{d}s \text{ for almost every } t \in [0, \tau],$$

where the last integral is a Lebesgue integral. Therefore by using this representation, u can be regarded as a continuous function. We refer to the book of Brezis [7] for more results on Sobolev spaces. By using the notion of absolutely continuous function, one can also prove that $u(t)$ is differentiable almost everywhere on $[0, \tau]$ with $u' \in L^\infty(0, \tau)$. We refer for instance to the book of Rudin [45] for a proof of this result.

The point of this section is to extend the previous inequalities to the case where α and β depend on the variable t, namely, we consider inequalities of the form

$$u'(t) \leq \alpha(t)u(t) + \beta(t) \text{ for almost every } t \in [0, \tau],$$

where $\alpha, \beta \in L^1(0, \tau)$.
Then for almost every $t \in [0, \tau]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l} u(t)\right] = -\alpha(t)\mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l} u(t) + \mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l} u'(t).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l} u(t)\right] \leq \beta(t)\mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l}$$

and by integrating we obtain

$$\mathrm{e}^{-\int_0^t \alpha(l)\mathrm{d}l} u(t) - u(0) \leq \int_0^t \beta(s)\mathrm{e}^{-\int_0^s \alpha(l)\mathrm{d}l}\mathrm{d}s.$$

Thus

$$u(t) \leq \mathrm{e}^{\int_0^t \alpha(l)\mathrm{d}l} u(0) + \int_0^t \mathrm{e}^{\int_s^t \alpha(l)\mathrm{d}l} \beta(s)\mathrm{d}s, \text{ for all } t \in [0, \tau].$$

Lemma 2.7 (Non-autonomous Gronwall's lemma)

Let $\alpha, \beta \in L^1(0,\tau)$. *Let* $u \in W^{1,\infty}((0,\tau),\mathbb{R})$ *and assume that*

$$u'(t) \leq \alpha(t)u(t) + \beta(t) \textit{ for almost every } t \in [0, \tau].$$

Then

$$u(t) \leq \mathrm{e}^{\int_0^t \alpha(l)\mathrm{d}l} u(0) + \int_0^t \mathrm{e}^{\int_s^t \alpha(l)\mathrm{d}l} \beta(s)\mathrm{d}s, \textit{ for all } t \in [0, \tau].$$

Henry's lemma

In the context of parabolic equations, Henry [23, Lemma 7.1.1, p. 188] proved the following important extension of Gronwall's lemma.

Lemma 2.8 (Henry)

Let $\tau > 0$, $\alpha \geq 0$ and $\beta \geq 0$. Assume that $u \in C([0, \tau], \mathbb{R})$ is a non-negative function satisfying

$$u(t) \leq \alpha t^{\chi - 1} + \beta \int_0^t (t - \sigma)^{\gamma - 1} u(\sigma) \mathrm{d}\sigma \text{ for all } t \in (0, \tau],$$

for some constants

$$\chi > 0 \text{ and } \gamma > 0.$$

Then there exists a constant $\delta > 0$ such that

$$u(t) \leq \delta t^{\chi - 1} \text{ for all } t \in [0, \tau].$$

Proof. For $\eta > 0$ and $\zeta > 0$ we define

$$B(\eta, \zeta) := \int_0^1 (1-r)^{\eta-1} r^{\zeta-1} \mathrm{d}r.$$

Applying twice the inequality satisfied by u, we get

$$\begin{aligned} u(t) &\leq \alpha t^{\chi-1} + \beta \int_0^t (t-\sigma)^{\gamma-1} u(\sigma) \mathrm{d}\sigma \\ &\leq \alpha t^{\chi-1} + \beta \int_0^t (t-\sigma)^{\gamma-1} \alpha \sigma^{\chi-1} \mathrm{d}\sigma \\ &\quad + \beta^2 \int_0^t (t-\sigma_1)^{\gamma-1} \int_0^{\sigma_1} (\sigma_1 - \sigma_2)^{\gamma-1} u(\sigma_2) \mathrm{d}\sigma_2 \mathrm{d}\sigma_1. \end{aligned} \tag{3}$$

By using the change of variable $\sigma = tr$, we rewrite the first integral term of (3) as

$$\int_0^t (t-\sigma)^{\gamma-1} \alpha \sigma^{\chi-1} \mathrm{d}\sigma = \alpha \int_0^1 (t-tr)^{\gamma-1} (tr)^{\chi-1} t \mathrm{d}r = \alpha t^{\gamma+\chi-1} B(\gamma, \chi).$$

Next by using Fubini's theorem in the last term of the inequality (3), we obtain

$$\int_0^t (t-\sigma_1)^{\gamma-1} \int_0^{\sigma_1} (\sigma_1-\sigma_2)^{\gamma-1} u(\sigma_2) \mathrm{d}\sigma_2 \mathrm{d}\sigma_1$$
$$= \int_0^t \int_{\sigma_2}^t (t-\sigma_1)^{\gamma-1} (\sigma_1-\sigma_2)^{\gamma-1} u(\sigma_2) \mathrm{d}\sigma_1 \mathrm{d}\sigma_2$$

and

$$\int_{\sigma_2}^t (t-\sigma_1)^{\gamma-1} (\sigma_1-\sigma_2)^{\gamma-1} \mathrm{d}\sigma_1 = \int_0^{t-\sigma_2} (t-\sigma_2-l)^{\gamma-1} l^{\gamma-1} \mathrm{d}l$$
$$= (t-\sigma_2)^{2(\gamma-1)+1} \int_0^1 (1-r)^{\gamma-1} r^{\gamma-1} \mathrm{d}r.$$

Hence

$$\int_{\sigma_2}^t (t-\sigma_1)^{\gamma-1} (\sigma_1-\sigma_2)^{\gamma-1} \mathrm{d}\sigma_1 = (t-\sigma_2)^{2\gamma-1} B(\gamma,\gamma).$$

Therefore we obtain

$$u(t) \leq [1 + \beta t^{\gamma} B(\gamma, \chi)] \alpha t^{\chi - 1} + \beta^2 B(\gamma, \gamma) \int_0^t (t - \sigma)^{2\gamma - 1} u(\sigma) \mathrm{d}\sigma.$$

By fixing $\alpha_1 = \alpha + \beta \tau^{\gamma} B(\gamma, \chi) > 0$ and $\beta_1 = \beta^2 B(\gamma, \gamma)$ we obtain

$$u(t) \leq \alpha_1 t^{\chi - 1} + \beta_1 \int_0^t (t - \sigma)^{2\gamma - 1} u(\sigma) \mathrm{d}\sigma.$$

By induction, we deduce that for each integer $n \geq 1$ we can find $\alpha_n > 0$ and $\beta_n > 0$ such that

$$u(t) \leq \alpha_n t^{\chi - 1} + \beta_n \int_0^t (t - \sigma)^{2n\gamma - 1} u(\sigma) \mathrm{d}\sigma.$$

Therefore by choosing an integer $n > 0$ such that $2n\gamma > 1$, this yields

$$u(t) \leq \alpha_n t^{\chi - 1} + \hat{\beta}_n \int_0^t u(\sigma) \mathrm{d}\sigma,$$

with $\hat{\beta}_n := \beta_n \tau^{2n\gamma - 1}$.

Setting

$$U(t) := \int_0^t u(\sigma)\mathrm{d}\sigma,$$

we have

$$U'(t) \leq \hat{\beta}_n U(t) + \alpha_n t^{\chi-1}.$$

Thus

$$U(t) \leq \int_0^t \mathrm{e}^{\hat{\beta}_n(t-s)} \alpha_n s^{\chi-1}\mathrm{d}s \leq \alpha_n \mathrm{e}^{\hat{\beta}_n \tau} \int_0^t s^{\chi-1}\mathrm{d}s = \alpha_n \mathrm{e}^{\hat{\beta}_n \tau} \frac{t^{\chi}}{\chi}.$$

The result follows. □

Osgood's lemma

The following lemma can be regarded as a nonlinear version of Gronwall's lemma. This result gives an alternative criterion to show the uniqueness of solutions for an ordinary differential equation when the right-hand side is not Lipschitz continuous. We refer to the book of Hartman [22, Corollary 6.2, p. 33].

Lemma 2.9 (Osgood)

Let $\tau > 0$, $\alpha \geq 0$ *and* $\beta \in C([0,\tau],[0,+\infty))$ *be given. Let* $\chi \in C([0,+\infty),[0,+\infty))$ *be a non-decreasing function satisfying*

$$\chi(0) = 0 \text{ and } \chi(x) > 0 \text{ for all } x > 0.$$

Assume that $u \in C([0,\tau],[0,+\infty))$ *satisfies the following inequality*

$$u(t) \leq \alpha + \int_0^t \beta(s)\chi(u(s))\mathrm{d}s \text{ for all } t \in [0,\tau].$$

If $\alpha > 0$ *then the following inequality holds*

$$\int_\alpha^{u(t)} \frac{1}{\chi(\sigma)}\mathrm{d}\sigma \leq \int_0^t \beta(\sigma)\mathrm{d}\sigma \text{ for all } t \in [0,\tau].$$

If $\alpha = 0$ *then we have*

$$\int_0^\varepsilon \frac{1}{\chi(\sigma)}\mathrm{d}\sigma = \infty \text{ for all } \varepsilon > 0 \Rightarrow u(t) = 0 \text{ for all } t \in [0,\tau].$$

Proof. Assume first that $\alpha > 0$. Define

$$U(t) = \int_0^t \beta(s)\chi(u(s))\mathrm{d}s.$$

Then we have

$$U'(t) = \beta(t)\chi(u(t)) \leq \beta(t)\chi(\alpha + U(t))$$

and since $\alpha > 0$ we obtain

$$\frac{U'(t)}{\chi(\alpha + U(t))} \leq \beta(t).$$

By integrating this inequality, we obtain

$$\int_0^t \frac{U'(\sigma)}{\chi(\alpha + U(\sigma))}\mathrm{d}\sigma \leq \int_0^t \beta(\sigma)\mathrm{d}\sigma.$$

We use the change of variable $r = \alpha + U(\sigma)$ to obtain

$$\int_{\alpha}^{\alpha+U(t)} \frac{1}{\chi(r)} \mathrm{d}r \leq \int_0^t \beta(\sigma)\mathrm{d}\sigma, \text{ for all } t \in [0,\tau],$$

and since $u(t) \leq \alpha + U(t)$ we deduce the first statement in the lemma. Next we consider the case where $\alpha = 0$ and assume by contradiction that there exists a $t^* \in (0,\tau]$ such that $u(t^*) > 0$. Then

$$u(t) \leq \int_0^t \beta(s)\chi(u(s))\mathrm{d}s \text{ for all } t \in [0,\tau],$$

and this implies that

$$u(t) \leq \alpha + \int_0^t \beta(s)\chi(u(s))\mathrm{d}s \text{ for all } t \in [0,\tau],$$

for each $\alpha > 0$. Proceeding as above we obtain for each $\alpha < u(t^*)$

$$\int_{\alpha}^{u(t^*)} \frac{1}{\chi(\sigma)} \mathrm{d}\sigma \leq \int_0^t \beta(\sigma)\mathrm{d}\sigma,$$

and we get a contradiction by choosing α sufficiently small. $\square$

Example.[Lifetime of a blow-up solution] Osgood's lemma can be used to estimate the lifetime of solutions in the presence of a blow-up. Consider for instance a nonlinearity $f \in C^1([0,+\infty),[0,+\infty))$ such that

$$f(u) \leq \beta u^2.$$

Then it is known that there exists a function $\tau(x) \in (0,+\infty]$ such that, for all $x \geq 0$, there is a unique solution $u(t)$ of the equation

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = x, \end{cases}$$

which is defined on $[0,\tau(x))$, and

$$\lim_{t \to \tau(x)^-} u(t) = +\infty \text{ if } \tau(x) < +\infty.$$

Integrating the equation satisfied by u we find that

$$u(t) \leq u(0) + \int_0^t f(u(s))\mathrm{d}s \leq u(0) + \int_0^t \beta u(s)^2 \mathrm{d}s \text{ for all } t \in [0, \tau(x)).$$

Therefore by applying Osgood's Lemma, we have

$$\int_{u(0)}^{u(t)} \frac{1}{\sigma^2}\mathrm{d}\sigma \leq \int_0^t \beta \mathrm{d}s = \beta t,$$
$$\frac{1}{2}\left(\frac{-1}{u(t)} + \frac{1}{x}\right) \leq \beta t,$$

so that finally

$$u(t) \leq \frac{x}{1 - 2x\beta t}.$$

Therefore, we end-up with a lower estimate of the lifetime of u

$$\tau(x) \geq \frac{1}{2\beta x}.$$

End example.

Exercise 2.10

Apply the Osgood criterion to prove the uniqueness of the solution in the following example

$$U'(t) = \begin{cases} 0, & \textit{if } U(0) = 0, \\ U(t)\ln(|U(t)|), & \textit{if } U(0) \neq 0. \end{cases}$$

Solution. If $U(0) \neq 0$ then the uniqueness of the solution follows from the standard theory of locally Lipschitz continuous equations. What we need to show is that there cannot exist a solution $U(t)$ not identically equal to 0 and such that $U(t) \to 0$ when $t \to 0$. Suppose by contradiction that there exists such a solution $U(t)$ defined for all $t \in [0, \tau)$. Integrating the equation satisfied by U and $-U$, we get

$$U(t) = 0 + \int_0^t U(s) \ln(|U(s)|) \mathrm{d}s \leq \int_0^t |U(s)| \, |\ln(|U(s)|)| \mathrm{d}s, \quad \forall t \in [0, \tau),$$

$$-U(t) = 0 + \int_0^t -U(s) \ln(|U(s)|) \mathrm{d}s \leq \int_0^t |U(s)| \, |\ln(|U(s)|)| \mathrm{d}s, \quad \forall t \in [0, \tau),$$

so that

$$|U(t)| \leq \int_0^t |U(s)| \, |\ln(|U(s)|)| \mathrm{d}s, \text{ for all } t \in [0, \tau).$$

We remark that $\int_0^{\varepsilon} \frac{1}{\sigma|\ln(\sigma)|} d\sigma$ is a Bertrand integral which is divergent, that is,

$$\int_0^{\varepsilon} \frac{1}{\sigma|\ln(\sigma)|} d\sigma = +\infty \text{ for all } \varepsilon > 0, \tag{4}$$

so that Osgood's lemma implies that $|U(t)| = 0$ for all $t \in [0, \tau]$. This is a contradiction.

To prove (4) we change the variable $\ln(\sigma) = s$ in $\int_{\varepsilon_1}^{\varepsilon} \frac{1}{\sigma|\ln(\sigma)|} d\sigma$,

$$\int_{\varepsilon_1}^{\varepsilon} \frac{1}{\sigma|\ln(\sigma)|} d\sigma = \int_{\ln(\varepsilon_1)}^{\ln(\varepsilon)} \frac{1}{|s|} ds$$

and let $\varepsilon_1 \to 0$. **End solution.**

Remark 2.11 (A counterexample for the uniqueness of the solution)

If $f(u) = \sqrt{u}$ *then one can check that*

$$u(t) := \left(\sqrt{u_0} + \frac{t}{2}\right)^2$$

is a solution of the ordinary differential equation

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0, \end{cases}$$

for all $u_0 \geq 0$*. In particular, there are two solutions corresponding to the initial condition* $u_0 = 0$*:* $u(t) = \frac{t^2}{4}$ *and* $u(t) = 0$*.*

Outline

In this section we will assume that the semiflow U is positive (see Chapter 7 for more results), that is,

$$x \geq 0 \Rightarrow U(t)x \geq 0 \text{ for all } t \in [0, \tau(x)),$$

and we will focus on the following property.

Definition 3.1

We will say that a maximal semiflow $U : D_\tau \to \mathbb{R}^n$ is *monotone* on $\mathbb{R}^n_+$ if

$$0 \leq x \leq y \Rightarrow 0 \leq U(t)x \leq U(t)y \text{ for all } t \in \left[0, \min\left(\tau(x), \tau(y)\right)\right).$$

From now on we assume that the positive cone $\mathbb{R}^n_+$ is *normal*, that is,

$$0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|,$$

or (equivalently) that the norm on $\mathbb{R}^n$ is monotone. One can choose for example

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Lemma 3.2

If U is a positive monotone semiflow on $\mathbb{R}^n_+$, then for each $x, y \in \mathbb{R}^n$,

$$0 \leq x \leq y \Rightarrow \tau(y) \leq \tau(x).$$

Proof. We have

$$0 \leq U(t)x \leq U(t)y \text{ for all } t \in \left[0, \min\left(\tau(x), \tau(y)\right)\right),$$

and since the norm is monotone, we have

$$\|U(t)x\| \leq \|U(t)y\| \text{ for all } t \in \left[0, \min\left(\tau(x), \tau(y)\right)\right).$$

Assume for instance that $\tau(x) < \tau(y)$. Then we have

$$+\infty = \lim_{\substack{t \to \tau(x) \\ t < \tau(x)}} \|U(t)x\| \leq \lim_{\substack{t \to \tau(x) \\ t < \tau(x)}} \|U(t)y\| < +\infty,$$

which is a contradiction. □

Lemma 3.3

Let $A \in C([0,\tau], M_n(\mathbb{R}))$ *be given for some* $\tau > 0$. *For each* $s \in [0,\tau]$ *and each* $x \in \mathbb{R}^n$ *consider the unique solution* $t \mapsto U(t,s)x$ *of the non-autonomous ordinary differential equation*

$$\begin{cases} \dfrac{\partial U(t,s)x}{\partial t} = A(t)U(t,s)x \text{ for all } t \in [s,\tau], \\ U(s,s)x = x \in \mathbb{R}^n. \end{cases}$$

Then the two following properties are equivalent.

(i) *For each* $x \in \mathbb{R}^n$,

$$x \geq 0 \Rightarrow U(t,s)x \geq 0, \text{ for all } t,s \in [0,\tau] \text{ with } t \geq s.$$

(ii) *For each* $t \in [0,\tau]$, *the off-diagonal elements of* $A(t)$ *are non-negative.*

Remark 3.4

Since $t \to A(t)$*, assertion (ii) is equivalent to the existence of a* $\lambda > 0$ *such that*

$$A(t) + \lambda I \geq 0 \text{ for all } t \in [0, \tau].$$

Proof. Proof of (i)$\Leftarrow$(ii). Assume that (ii) is satisfied, then there exists a $\lambda_0 > 0$ such that $A(t) + \lambda_0 I \geq 0$ for all $t \in [0, \tau]$. We have

$$\frac{\partial U}{\partial t}(t,s)x = -\lambda_0 U(t,s)x + [A(t) + \lambda_0 I]U(t,s)x,$$

and by using the non-autonomous variation of constant formula we obtain

$$U(t,s)x = \mathrm{e}^{-\lambda_0(t-s)}x + \int_s^t \mathrm{e}^{-\lambda_0(t-l)}[A(l) + \lambda_0 I]U(l,s)\mathrm{d}l.$$

Now by using the fixed-point method, (i) follows.
Proof of (i)$\Rightarrow$(ii). Let $s \in [0, \tau)$. Assume that

$$U(\varepsilon + s, s)x \geq 0 \text{ for all } \varepsilon \in [0, \tau - s] \text{ and } x \geq 0.$$

Let $x^* \geq 0$ be given. Then we have

$$\langle x^*, U(\varepsilon + s, s)x\rangle = \langle x^*, x\rangle + \int_s^{\varepsilon+s} \langle x^*, A(l)U(l,s)x\rangle \mathrm{d}l.$$

If $\langle x^*, x\rangle = 0$ ($\Leftrightarrow x^* \perp x$) then by dividing by ε we obtain

$$0 \leq \frac{\langle x^*, U(\varepsilon + s, s)x\rangle}{\varepsilon} = \frac{1}{\varepsilon}\int_s^{\varepsilon+s} \langle x^*, A(l)U(l,s)x\rangle \mathrm{d}l,$$

and by taking the limit as ε approaches 0, we obtain

$$0 \leq \langle x^*, A(s)x\rangle.$$

We deduce that for each $x^* \geq 0$ and $x \geq 0$, one has

$$\langle x^*, x\rangle = 0 \Rightarrow \langle x^*, A(s)x\rangle \geq 0 \text{ for all } s \in [0, \tau).$$

Choosing $x^* = e_i$ and $x = e_j$ with $i \neq j$ (where e_i is the i-th element of the canonical basis), since

$$\langle e_i, A(s)e_j \rangle = A_{ij}(s) \geq 0 \text{ for all } s \in [0, \tau),$$

we deduce that

$$i \neq j \Rightarrow A_{ij}(s) \geq 0 \text{ for all } s \in [0, \tau).$$

Therefore all the off-diagonal elements of $A(s)$ are non-negative. □

Theorem 3.5 (Positive and monotone semiflow)

Let $F \in C^1(\mathbb{R}^n)$ be given. Let $U : D_\tau \to \mathbb{R}^n$ be the maximal semiflow generated by

$$\begin{cases} U'(t)x = F(U(t)x), \\ U(0)x = x \in \mathbb{R}^n. \end{cases}$$

Then the following properties are equivalent.

(i) U *is positive and monotone on* $\mathbb{R}^n_+$.

(ii) $F(0) \geq 0$ *and* $\dfrac{\partial F_i}{\partial x_j}(x) \geq 0$ *whenever* $x \geq 0$ *and* $i \neq j$.

(iii) *For each* $M > 0$, *there exists a* $\lambda = \lambda(M) > 0$ *such that for each* $x, y \in B_{\mathbb{R}^n}(0, M)$,

$$0 \leq x \leq y \Rightarrow (\lambda I + F)(x) \leq (\lambda I + F)(y).$$

That is, $x \mapsto (\lambda I + F)(x)$ *is non-negative and monotone non-decreasing on* $B_{\mathbb{R}^n}(0, M) \cap \mathbb{R}^n_+$.

Remark 3.6

In practice we will use the property (ii) *to show the monotony of a semiflow. A map* $F \in C^1(\mathbb{R}^n)$ *is quasi-monotone on* $\mathbb{R}^n_+$ *if*

$$\frac{\partial F_i}{\partial x_j}(x) \geq 0 \text{ whenever } x \geq 0 \text{ and } i \neq j.$$

Proof. Proof of (i)⇒(ii). Since the semiflow is positive, we know from Chapter 7 that we must have $F(0) \geq 0$. Moreover, for each $x, y \geq 0$ and each $t \in [0, \tau(x))$, since the semiflow is monotone we must have

$$\lim_{\varepsilon \to 0^+} \frac{U(t)(x + \varepsilon y) - U(t)x}{\varepsilon} \geq 0,$$

and we obtain

$$V(t,0)y := \partial_x U(t)(x)(y) \geq 0, \text{ for all } x, y \geq 0 \text{ and } t \in [s, \tau(x)).$$

Similarly by replacing x by $U(s)x$ (for some $s \in [0, \tau(x))$) we must have

$$V(t,s)y = \partial_x U(t)(U(s)x)(y) \geq 0 \text{ for all } x, y \geq 0 \text{ and } t \in [s, \tau(x)).$$

From Chapter 6 we know that for each $s \in [0, \tau(x))$, the map $t \mapsto V(t,s)y$ is the unique solution of the non-autonomous ordinary differential equation

$$\partial_t V(t,s)y = \mathrm{D}F(U(t+s)x)V(t,s)y \text{ for all } t \in [s, \tau(x)), V(s,s)y = y.$$

By using Lemma 3.3 (with $A(t) = \mathrm{D}F(U(t+s)x))$ we deduce that for each $x \in \mathbb{R}^n_+$ there exists a $\lambda > 0$ such that

$$\mathrm{D}F(x) + \lambda I \geq 0,$$

and (ii) follows.
Proof of (ii)⇒(iii). Since the map $x \mapsto \mathrm{D}F(x)$ is continuous, we deduce that for each $M > 0$ there exists a $\lambda > 0$ such that

$$\mathrm{D}F(x) + \lambda I \geq 0,$$

whenever $x \geq 0$ and $\|x\| \leq M$.
Let $y \geq x \geq 0$. We have (see Lemma **??**)

$$F(y) - F(x) = \int_0^1 \mathrm{D}F(sy + (1-s)x)(y-x)\mathrm{d}s.$$

Therefore

$$(\lambda I + F)(y) - (\lambda I + F)(x) = \lambda(y-x) + F(y) - F(x)$$
$$= \int_0^1 (\lambda I + \mathrm{D}F)\left[sy + (1-s)x\right](y-x)\mathrm{d}s \geq 0,$$

and (iii) follows.
Proof of (iii)⇒(i). Let $M > 0$ be given and $x \in \mathbb{R}^n$ satisfy $\|x\| \leq M$. We choose $\lambda_0 := \lambda(2M)$ so that, for all $x, y \in B(0, 2M)$,

$$0 \leq x \leq y \Rightarrow (\lambda_0 I + F)(x) \leq (\lambda_0 I + F)(y).$$

Define

$$E := \{u \in C([0,\tau], \mathbb{R}^n) : u(t) \geq 0 \text{ and } \|u(t)\| \leq 2M \text{ for all } t \in [0,\tau]\}.$$

Recall that $t \mapsto u(t)$ is a solution of

$$\begin{cases} u'(t) = F(u(t)) \text{ for all } t \in [0, \tau], \\ u(0) = x \end{cases}$$

if and only if

$$u(t) = \mathrm{e}^{-\lambda t} x + \int_0^t \mathrm{e}^{-\lambda(t-s)} (\lambda I + F)(u(s)) \mathrm{d}s \text{ for all } t \in [0, \tau].$$

Therefore, it is natural to consider the fixed-point operator

$$\Psi_x(u)(t) = \mathrm{e}^{-\lambda t} x + \int_0^t \mathrm{e}^{-\lambda(t-s)} (\lambda I + F)(u(s)) \mathrm{d}s,$$

whenever $x \in B_{\mathbb{R}^n}(0, M) \cap \mathbb{R}^n_+$, $u \in E$ and $\lambda \geq \lambda_0$.
We observe that $(u, x) \mapsto \Psi_x(u)$ is a monotone non-decreasing map from $E \times B(0, M) \cap \mathbb{R}^n_+$ into $C([0, \tau], \mathbb{R}^n)$ and

$$\Psi_x(0) \geq 0$$

whenever $x \geq 0$.

Let $x \in B_{\mathbb{R}^n}(0,M) \cap \mathbb{R}^n_+$, $u \in E$ and $\lambda \geq \lambda_0$. For each $t \in [0,\tau]$ we have

$$\begin{aligned}\|\Psi_x(u)(t)\| &\leq \mathrm{e}^{-\lambda t}M + \int_0^t \mathrm{e}^{-\lambda(t-s)}\|(\lambda I + F)(u(s)) - (\lambda I + F)(0)\|\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-\lambda(t-s)}\|(\lambda I + F)(0)\|\mathrm{d}s \\ &\leq M + \tau\big[(\lambda + k(2M))2M + \|F(0)\|\big],\end{aligned}$$

where $k(2M)$ denotes the Lipschitz constant of F on the ball $B_{\mathbb{R}^n}(0,2M)$, and when $\tau > 0$ is sufficiently small

$$\tau\big[((\lambda + k(2M))2M + \|F(0)\|\big] < M, \tag{5}$$

and then

$$\Psi_x(E) \subset E.$$

Moreover, under the condition (5) the map Ψ_x is K-Lipschitz with $K := \tau(\lambda + k(2M)) < \frac{1}{2}$.

Let $x, y \in B_{\mathbb{R}^n}(0, M)$ with $y \geq x$. Set

$$u_0(t) = x \text{ and } v_0(t) = y \text{ for all } t \in [0, \tau].$$

By using the monotone properties of Ψ we obtain

$$\Psi_x(u_0)(t) \leq \Psi_y(v_0)(t) \text{ for all } t \in [0, \tau],$$

and

$$\Psi_x^2(u_0)(t) = \Psi_x(\Psi_x(u_0))(t) \leq \Psi_x(\Psi_y(v_0))(t) \leq \Psi_y(\Psi_y(v_0))(t)$$

so that

$$\Psi_x^2(u_0)(t) \leq \Psi_y^2(v_0)(t) \text{ for all } t \in [0, \tau].$$

By induction we obtain for each $n \geq 1$

$$\Psi_x^m(u_0)(t) \leq \Psi_y^m(v_0)(t) \text{ for all } t \in [0, \tau],$$

and by taking the limit when m goes to $+\infty$ on both sides, we obtain

$$U(t)x = \lim_{m \to +\infty} \Psi_x^m(u_0)(t) \leq \lim_{m \to +\infty} \Psi_y^m(v_0)(t) = U(t)y \text{ for all } t \in [0, \tau].$$

The proof is complete. $\square$

Corollary 3.7

Let $F \in C^1(\mathbb{R}^n)$. *Let* $U : D_\tau \to \mathbb{R}^n$ *be the maximal semiflow generated by*

$$\begin{cases} U'(t)x = F(U(t)x), \\ U(0)x = x \in \mathbb{R}^n. \end{cases}$$

Assume that U *is positive and monotone on* $\mathbb{R}^n_+$.

Then we have the following properties.

(i) *If* $F(x) \geq 0$ *then the map* $t \to U(t)x$ *is non-decreasing on* $[0, \tau(x))$ *(i.e.* $t \leq s \Rightarrow U(t)x \leq U(s)x$*).*

(ii) *If* $F(x) \leq 0$ *then the map* $t \to U(t)x$ *is non-increasing on* $[0, \tau(x))$ *(i.e.* $t \leq s \Rightarrow U(t)x \geq U(s)x$*).*

Proof. We observe that $U(t+h)x = U(t)U(h)x$. Then differentiating with respect to h and evaluating at $h = 0$ we have

$$\partial_t U(t)x = \partial_x U(t)(x)\left(\partial_t U(0)x\right) = \partial_x U(t)(x)\left(F(x)\right).$$

Moreover since the semiflow is monotone we have

$$\partial_x U(t)(x) \geq 0_{M_n(\mathbb{R})} \text{ for all } t \geq 0 \text{ and } x \geq 0.$$

The result follows. □

Corollary 3.8

Let $F \in C^1(\mathbb{R}^n)$. *Let* $U : D_\tau \to \mathbb{R}^n$ *be the maximal semiflow generated by*

$$\begin{cases} U'(t)x = F(U(t)x) \\ U(0)x = x \in \mathbb{R}^n. \end{cases}$$

Assume that U *is positive and monotone on* $\mathbb{R}^n_+$.

Then we have the following properties.

(i) *If* $F(x) \leq 0$ *then there is an equilibrium* $\bar{y} \geq 0$ *(i.e.* $F(\bar{y}) = 0$*) such that*

$$\lim_{t \to +\infty} U(t)x = \bar{y}.$$

(ii) *If* $F(x) \geq 0$ *then the map* $t \mapsto U(t)x$ *is bounded if and only if there exists an equilibrium* $\bar{x} \geq x$. *Moreover, in that case there is an equilibrium* $\bar{y} \leq \bar{x}$ *such that*

$$\lim_{t \to +\infty} U(t)x = \bar{y}.$$

Proof. If $F(x) \leq 0$, the map $t \mapsto U(t)x$ is non-increasing and non-negative and we can find $\bar{x} \geq 0$ such that

$$\bar{x} = \lim_{t \to +\infty} U(t)x.$$

We have

$$U(s+t)x = U(s)U(t)x.$$

Thus

$$\bar{x} = \lim_{t \to +\infty} U(t+s)x = U(s)\left(\lim_{t \to +\infty} U(t)x\right) = U(s)\bar{x}.$$

Hence

$$U(s)\bar{x} = \bar{x}, \text{ for all } s \geq 0.$$

When $F(0) \geq 0$ and $t \mapsto U(t)x$ is bounded, the proof is similar. □

Outline

Theorem 4.1 (Comparison principle, integral form)

Let $F \in C^1(\mathbb{R}^n)$. Let $U : D_\tau \to \mathbb{R}^n$ be the maximal semiflow (where $\tau(x)$ is the maximal time of existence of the solution starting from x) generated by

$$\begin{cases} U'(t)x = F(U(t)x), \forall t \in [0, \tau(x)), \\ U(0)x = x. \end{cases}$$

Assume that U is positive and monotone on $\mathbb{R}^n_+$. Then we have the following properties.

(i) ***(Sub-solution)*** *Let $x \geq 0$ and $v \in C([0, \sigma], \mathbb{R}^n)$ (with $\sigma > 0$), and $v \geq 0$. Assume that*

$$v(t) \leq \mathrm{e}^{-\lambda t} x + \int_0^t \mathrm{e}^{-\lambda(t-s)} (\lambda I + F)(v(s)) \mathrm{d}s$$

for each $\lambda > 0$ sufficiently large and each $t \in [0, \sigma]$.
Then

$$v(t) \leq U(t)x \text{ for all } t \in [0, \sigma] \cap [0, \tau(x)).$$

Theorem 4.2 (Comparison principle, integral form)

(ii) ***(Super-solution)*** *Let* $x \geq 0$ *and* $w \in C([0,\sigma],\mathbb{R}^n)$ *(with* $\sigma > 0$*), and* $w \geq 0$*. Assume that*

$$w(t) \geq \mathrm{e}^{-\lambda t}x + \int_0^t \mathrm{e}^{-\lambda(t-s)}(\lambda I + F)(w(s))\mathrm{d}s$$

for each $\lambda > 0$ *sufficiently large and for each* $t \in [0,\sigma]$*.*
Then

$$w(t) \geq U(t)x \text{ for all } t \in [0,\sigma] \cap [0,\tau(x)).$$

Proof. For each $\lambda > 0$ sufficiently large we have

$$v(t) \leq \mathrm{e}^{-\lambda t} x + \int_0^t \mathrm{e}^{-\lambda(t-s)} (\lambda I + F)(v(s)) \mathrm{d}s =: \Psi_x(v(t)).$$

Therefore by using similar arguments as in the proof of Theorem 3.5 we have for all $t \in [0, \tau]$ with τ sufficiently small and λ sufficiently large,

$$v \leq \Psi_x(v) \leq \cdots \leq \Psi_x^m(v) \xrightarrow[m \to +\infty]{} U(t)x.$$

(i) follows. The proof of (ii) is similar. □

Theorem 4.3 (Comparison principle, differential form)

Let $F \in C^1(\mathbb{R}^n)$. Let $U : D_\tau \to \mathbb{R}^n$ be the maximal (where $\tau(x)$ is the maximal time of existence of the solution starting from x) semiflow generated by

$$\begin{cases} U'(t)x = F(U(t)x), \forall t \in [0, \tau(x)), \\ U(0)x = x \in \mathbb{R}^n. \end{cases}$$

Assume that U is positive and monotone on $\mathbb{R}^n_+$. Then the following properties hold

(i) ***(Sub-solution)*** *Let $x \geq 0$ and $v \in C^1([0, \sigma], \mathbb{R}^n)$ (with $\sigma > 0$), and $v \geq 0$. Assume that*

$$\begin{cases} v'(t) \leq F(v(t)) \text{ for all } t \in [0, \sigma], \\ v(0) \leq x. \end{cases}$$

Then

$$v(t) \leq U(t)x \text{ for all } t \in [0, \sigma] \cap [0, \tau(x)).$$

Theorem 4.4 (Comparison principle, differential form)

(ii) ***(Super-solution)*** *Let* $x \geq 0$ *and* $w \in C^1([0,\sigma],\mathbb{R}^n)$ *(with* $\sigma > 0$*), and* $w \geq 0$*. Assume that*

$$\begin{cases} w'(t) \geq F(w(t)) \text{ for all } t \in [0,\sigma], \\ w(0) \geq x. \end{cases}$$

Then

$$w(t) \geq U(t)x \text{ for all } t \in [0,\sigma] \cap [0,\tau(x)).$$

Proof. We only proof (ii), the proof for (i) being similar. Let $\lambda \in \mathbb{R}$. Assume (ii), then for each $t \in [0, \sigma]$,

$$\begin{aligned} w'(t) \geq F(w(t)) \quad &\Leftrightarrow w'(t) \geq -\lambda w(t) + (F + \lambda I)(w(t)) \\ &\Leftrightarrow e^{\lambda t}(w'(t) + \lambda w(t)) \geq e^{\lambda t}(F + \lambda I)(w(t)) \end{aligned}$$

hence

$$\left(e^{\lambda t} w(t)\right)' \geq e^{\lambda t}(F + \lambda I)(w(t)).$$

By integrating both sides of the above inequality between 0 and t, we obtain

$$e^{\lambda t} w(t) - w(0) \geq \int_0^t e^{\lambda s}(F + \lambda I)(w(s))\mathrm{d}s,$$

therefore

$$w(t) \geq e^{-\lambda t} w(0) + \int_0^t e^{-\lambda(t-s)}(F + \lambda I)(w(s))\mathrm{d}s$$

and since $w(0) \geq x$, we obtain for $t \in [0, \sigma]$,

$$w(t) \geq e^{-\lambda t} x + \int_0^t e^{-\lambda(t-s)}(F + \lambda I)(w(\mathrm{s}))\mathrm{d}s.$$

The result follows from Theorem 4.2. □

Exercise 4.5

Let $f \in C^1(\mathbb{R})$ be given with $f(0) \geq 0$. Consider the scalar ordinary differential equation

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = x \geq 0. \end{cases} \tag{6}$$

1. *Assume that there exist three constants $x^* > 0$, $\beta > 0$ and $\lambda \in \mathbb{R}$ such that*

   $$f(x) \leq \beta x + \lambda \text{ for all } x > x^*.$$

   By using the comparison principle, prove that for each $x \geq 0$ the solution of equation (6) *exists for all positive time (i.e. $\tau(x) = +\infty$).*

2. *Assume that there are four constants $x^* > 0$, $\beta > 0$, $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that*

   $$f(x) \geq \beta x^2 + \lambda x + \gamma, \text{ for all } x > x^*.$$

   By using the comparison principle, show that for each $x \geq 0$ sufficiently large the solution of (6) *blows-up in finite time (i.e. $\tau(x) < +\infty$).*

Outline

The Lotka–Volterra model with two species

To illustrate the notion of the competitive and cooperative Lotka–Volterra model we refer to [34, 35, 36, 59, 60]. The model is written as

$$\begin{cases} u_1'(t) = u_1(t)\left(\lambda_1 + \alpha_{11}u_1(t) + \alpha_{12}u_2(t)\right), \\ u_2'(t) = u_2(t)\left(\lambda_2 + \alpha_{21}u_1(t) + \alpha_{22}u_2(t)\right), \end{cases} \tag{7}$$

where $u_i(t)$ $(i = 1, 2)$ denotes the number of individuals in the species i at time t. The parameter $\lambda_i \in \mathbb{R}$ $(i = 1, 2)$ is the growth rate of the species i, and $\alpha_{ii} \leq 0$ describes the intra-specific growth limitations for the species i. The coefficient α_{ij} (with $i \neq j$) corresponds either to a term of inter-specific competition when $\alpha_{ij} \leq 0$, or to a term of inter-specific cooperation when $\alpha_{ij} \geq 0$.

Definition 5.1

System (7) is said to be *cooperative* if and only if

$$\alpha_{12} \geq 0 \text{ and } \alpha_{21} \geq 0.$$

This means that all the inter-specific interactions are cooperative.

System (7) is said to be *competitive* if and only if

$$\alpha_{12} \leq 0 \text{ and } \alpha_{21} \leq 0.$$

This means that all the inter-specific interactions are competitive.

The Lotka–Volterra model with multiple species

The model is written as

$$\begin{cases} u_1' = u_1\left(\lambda_1 + \alpha_{11}u_1 + \cdots + \alpha_{1n}u_n\right), \\ u_2' = u_2\left(\lambda_2 + \alpha_{21}u_1 + \cdots + \alpha_{2n}u_n\right), \\ \vdots \\ u_n' = u_n\left(\lambda_n + \alpha_{n1}u_1 + \cdots + \alpha_{nn}u_n\right), \end{cases} \tag{8}$$

where $u_i(t)$ $(i = 1, \ldots, n)$ denotes the number of individuals in the species i at time t. The parameter $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ is the growth rate of the species i, and $\alpha_{ii} \leq 0$ describes the intra-specific growth limitations for the species i. The coefficient α_{ij} (with $i \neq j$) corresponds either to a term of inter-specific competition whenever $\alpha_{ij} \leq 0$, or to a term of inter-specific cooperation whenever $\alpha_{ij} \geq 0$.

Definition 5.2

The system (8) is said to be *cooperative* if and only if

$$\alpha_{ij} \geq 0, \forall i \neq j.$$

The system (8) is said to be *competitive* if and only if

$$\alpha_{ij} \leq 0, \forall i \neq j.$$

Definition 5.3

Consider the system

$$\begin{cases} u_1' = F_1(u_1, u_2, \ldots, u_n), \\ u_2' = F_2(u_1, u_2, \ldots, u_n), \\ \qquad \vdots \\ u_n' = F_n(u_1, u_2, \ldots, u_n). \end{cases} \tag{9}$$

The system (9) is *cooperative* if and only if

$$\partial_{x_i} F_j(x) \geq 0, \quad \forall i \neq j, \forall x \geq 0. \tag{10}$$

The system (9) is *competitive* if and only if

$$\partial_{x_i} F_j(x) \leq 0, \quad \forall i \neq j, \forall x \geq 0. \tag{11}$$

Remark 5.4

A system $u' = F\left(u\left(t\right)\right)$ *is competitive if and only if the system* $v'(t) = -F\left(v\left(t\right)\right)$ *is cooperative. This means that a competitive system is nothing but a cooperative system when we reverse the time (i.e.* $v(t) = u(-t)$*).*

Outline

Logistic equations and diffusion between two cities with local limitations in space

We investigate the system

$$\begin{cases} u_1'(t) = \gamma\left[u_2(t) - u_1(t)\right] + \lambda_1 u_1(t) - \kappa_1^{-1} u_1(t)^2, \\ u_2'(t) = \gamma\left[u_1(t) - u_2(t)\right] + \lambda_2 u_2(t) - \kappa_2^{-1} u_2(t)^2, \end{cases} \tag{12}$$

where $\lambda_1 \in \mathbb{R}$ is the growth rate of group 1 and $\lambda_2 \in \mathbb{R}$ is the growth rate of group 2. The parameter $\kappa_1 > 0$ (respectively $\kappa_2 > 0$) describes the growth limitation in group 1 (respectively 2). The parameter $\gamma > 0$ is the rate at which individuals from group 1 (respectively 2) are leaving to the other group 2 (respectively 1). For the movement of individuals, we consider a two-patch model with symmetric transfer rates.

By setting $U(t) = (u_1(t), u_2(t))^T$, the system (12) can be written in the more condensed form

$$U'(t) = F(U(t)), \forall t \geq 0, \text{ and } U(0)x = x \geq 0,$$

where

Positivity

The positivity follows by using the criterion proved in Chapter 7. Namely for each $M > 0$ we can find a $\lambda = \lambda(M) > 0$ such that

$$F(u) + \lambda u \geq 0, \forall u \in [0, M\mathbb{1}] := \left\{ U \in \mathbb{R}^2 : 0 \leq U \leq M\mathbb{1} \right\},$$

where

$$\mathbb{1} = (1,1)^T.$$

Remark 6.1 (An alternative proof of the positivity)

We have

$$u_i' \geq (\lambda_i - \gamma) u_i - \kappa_i^{-1} u_i^2.$$

Therefore by using the comparison principle, we deduce that each component $u_i(t)$ is bounded from below by the solution of the logistic equation

$$v_i' = (\lambda_i - \gamma) v_i - \kappa_i^{-1} v_i^2,$$

and the positivity follows. We can also deduce that the species i will

Monotone semiflow

We have $F(0) = 0_{\mathbb{R}^2}$ and the Jacobian matrix of F is

$$\partial F(U) = \begin{pmatrix} \lambda_1 - \gamma - 2\kappa_1^{-1} u_1 & \gamma \\ \gamma & \lambda_2 - \gamma - 2\kappa_2^{-1} u_2 \end{pmatrix}.$$

Since the off-diagonal entries of $\partial F(U)$ are non-negative for all $U \in \mathbb{R}^2$ with $U \geq 0$, a direct application of Theorem 3.5 shows that the differential system (12) generates a unique maximal semiflow which is positive and monotone.

Remark 6.2

The off-diagonal terms in the Jacobian matrix correspond to the fluxes from group 1 (respectively 2) to group 2 (respectively 1) and are equal to $\gamma > 0$.

The linearized equation at 0

The behavior of the system close to 0 will be characterized by the sign of the dominant eigenvalue of the Jacobian matrix of F at $u = 0$. In particular, the lower solutions will start from a multiple of the associated positive eigenvector. We have

$$\partial F(0) = \begin{pmatrix} \lambda_1 - \gamma & \gamma \\ \gamma & \lambda_2 - \gamma \end{pmatrix}.$$

The characteristic equation is

$$\begin{aligned} & (\lambda_1 - \gamma - \Lambda)(\lambda_2 - \gamma - \Lambda) - \gamma^2 = 0 \\ \Leftrightarrow\ & \Lambda^2 - [\lambda_1 + \lambda_2 - 2\gamma]\Lambda + (\lambda_1 - \gamma)(\lambda_2 - \gamma) - \gamma^2 = 0. \end{aligned} \tag{13}$$

The discriminant is

$$\Delta = [\lambda_1 + \lambda_2 - 2\gamma]^2 - 4\left[\lambda_1\lambda_2 - \gamma\left(\lambda_1 + \lambda_2\right)\right].$$

Hence by developing this formula we obtain

$$\Delta = (\lambda_1 + \lambda_2)^2 - 4\gamma\left(\lambda_1 + \lambda_2\right) + 4\gamma^2 - 4\lambda_1\lambda_2 + 4\gamma\left(\lambda_1 + \lambda_2\right).$$

After simplification and since $\gamma > 0$, we obtain

$$\Delta = (\lambda_1 - \lambda_2)^2 + 4\gamma^2 > 0. \tag{14}$$

Let Λ be the maximal eigenvalue (or dominant eigenvalue) of $\partial F(0)$, that is,

$$\Lambda := \frac{1}{2}\left([\lambda_1 + \lambda_2 - 2\gamma] + \sqrt{\Delta}\right). \tag{15}$$

Right eigenvector: The corresponding eigenvector $\mathbb{V}$ satisfies

$$\begin{pmatrix} \lambda_1 - \gamma & \gamma \\ \gamma & \lambda_2 - \gamma \end{pmatrix} \begin{pmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \end{pmatrix} = \Lambda \begin{pmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \end{pmatrix} \Leftrightarrow \begin{cases} [\Lambda - (\lambda_1 - \gamma)]\, \mathbb{V}_1 = \gamma \mathbb{V}_2, \\ [\Lambda - (\lambda_2 - \gamma)]\, \mathbb{V}_2 = \gamma \mathbb{V}_1. \end{cases}$$

We observe that (independently of the sign of Λ) we always have

$$\begin{aligned} \Lambda - (\lambda_1 - \gamma) &= \frac{1}{2}\left([\lambda_2 - \lambda_1] + \sqrt{\Delta}\right) > 0 \text{ and} \\ \Lambda - (\lambda_2 - \gamma) &= \frac{1}{2}\left([\lambda_1 - \lambda_2] + \sqrt{\Delta}\right) > 0. \end{aligned} \tag{16}$$

Therefore, the sign of $\mathbb{V}_1$ is the same as the sign of $\mathbb{V}_2$. In other words, the eigenspace associated with Λ is spanned by a positive vector.

Global asymptotic stability of the trivial equilibrium when $\Lambda \leq 0$

Assume that $\Lambda \leq 0$. Then we can find a left eigenvector $\mathbb{W} \gg 0$ (i.e. $\mathbb{W}_1 > 0$ and $\mathbb{W}_2 > 0$) of $\partial F(0)$ associated to Λ. That is,

$$\mathbb{W}^T \partial F(0) = \Lambda \mathbb{W}^T.$$

Therefore

$$\mathbb{W}^T u(t)' = \mathbb{W}^T \, \partial F(0) u(t) - \mathbb{W}_1 \, \kappa_1^{-1} u_1(t)^2 - \mathbb{W}_2 \, \kappa_2^{-1} u_2(t)^2.$$

It follows that the quantity $V(u(t)) = \mathbb{W}^T u(t)$ (i.e. in the sense that $V(u(t))$ is a functional of $u(t)$) is strictly decreasing as long as $u(t) \neq 0$. Since $\mathbb{W}_1 > 0$ and $\mathbb{W}_2 > 0$ we deduce the following proposition.

Proposition 6.3 (Global stability of 0)

Assume that $\Lambda \leq 0$. Then the trivial equilibrium 0 is globally asymptotically stable.

Proof. This proof is left as an exercise. *Hint:* Set $x(t) = \mathbb{W}^T u(t)$ and observe that $x(t)' \leq \Lambda\, x(t)$. □

Dissipativity

To prove the dissipativity of (12) we use a family of upper solutions starting from

$$\overline{U}_\alpha = \alpha \mathbb{1}.$$

We choose such an initial value in order to get rid of the γ terms in (12) at $t = 0$.
Indeed, we have

$$F(\alpha \mathbb{1}) = \alpha \begin{pmatrix} \lambda_1 - \kappa_1^{-1}\alpha \\ \lambda_2 - \kappa_2^{-1}\alpha \end{pmatrix}.$$

By choosing $\alpha^+ := \max(\lambda_1\kappa_1, \lambda_2\kappa_2) > 0$ we deduce that

$$F(\alpha \mathbb{1}) \ll 0, \forall \alpha > \alpha^+. \quad (17)$$

Therefore by applying Corollary 3.7 and using the fact that 0 is an equilibrium we deduce the following.

Lemma 6.4 (Upper solutions)

For each $\alpha > \max(\alpha^+, 0)$, the interval $[0, \alpha\mathbb{1}] = \{x \in \mathbb{R}^2 : 0 \leq x \leq \alpha\mathbb{1}\}$ is positively invariant for the semiflow generated by (12). That is,

$$U(t)\,[0, \alpha\mathbb{1}] \subset [0, \alpha\mathbb{1}]\,, \forall t \geq 0.$$

Moreover the solution $t \to u(t) = U(t)\,(\alpha\mathbb{1})$ starting from $\alpha\mathbb{1}$ converges to the largest equilibrium of (12) in the interval $[0, \alpha\mathbb{1}]$ (this equilibrium can be 0).

Lower solutions

Assume that $\Lambda > 0$. We can also compute a right eigenvector $\mathbb{V} \gg 0$ (i.e. $\mathbb{V}_1 > 0$ and $\mathbb{V}_2 > 0$) of $\partial F(0)$ associated with Λ. We can choose for example

$$\mathbb{V} := \begin{pmatrix} \gamma \\ \Lambda - (\lambda_1 - \gamma) \end{pmatrix} \tag{18}$$

or

$$\mathbb{V} := \begin{pmatrix} \Lambda - (\lambda_2 - \gamma) \\ \gamma \end{pmatrix}. \tag{19}$$

We then have

$$F(\alpha \mathbb{V}) = \alpha \begin{pmatrix} \left(\Lambda - \kappa_1^{-1} \alpha \mathbb{V}_1\right) \mathbb{V}_1 \\ \left(\Lambda - \kappa_2^{-1} \alpha \mathbb{V}_2\right) \mathbb{V}_2 \end{pmatrix}.$$

Therefore if $\Lambda > 0$ we obtain

$$F(\alpha \mathbb{V}) \gg 0, \quad \forall \alpha \in (0, \alpha^-), \tag{20}$$

with

$$\alpha^- := \min \left(\frac{\Lambda \kappa_1}{\mathbb{V}_1}, \frac{\Lambda \kappa_2}{\mathbb{V}_2} \right) > 0. \tag{21}$$

By a direct application of Corollary 3.7 we obtain the following result.

Lemma 6.5 (Lower solutions)

Assume that $\Lambda > 0$. *For each* $\alpha \in (0, \alpha^-)$, *the interval* $[\alpha \mathbb{V}, +\infty) = \{x \in \mathbb{R}^2 : x \geq \alpha \mathbb{V}\}$ *is positively invariant for the semiflow generated by* (12). *Moreover, the solution* $t \to u(t) = U(t)(\alpha \mathbb{V})$ *starting from* $\alpha \mathbb{V}$ *converges to the smallest equilibrium of* (12) *in the interval* $[\alpha^- \mathbb{V}, \alpha^+ \mathbb{1}]$.

Equilibrium

It follows from Proposition 6.3 that there cannot exist a positive equilibrium when $\Lambda \leq 0$. When $\Lambda > 0$, Lemma 6.4 and 6.5 show that there exists a positive equilibrium and any solution starting from a non-negative and nontrivial initial value $U(0)$ converges to such an equilibrium. Here we show that there exists a *unique* positive equilibrium which attracts any non-negative and non-trivial initial value.

Proposition 6.6 (Uniqueness of the positive equilibrium)

Assume $\Lambda > 0$. *There exists a unique positive equilibrium for* (12), *which attracts any non-negative and non-zero initial value.*

Proof. Assume by contradiction that there exist two non-negative equilibria $U := (u_1, u_2)^T \neq 0_{\mathbb{R}^2}$ and $V := (v_1, v_2)^T \neq 0_{\mathbb{R}^2}$, that is,

$$F(U) = F(V) = 0_{\mathbb{R}^2}.$$

First we show that U and V are both positive, that is, $U \gg 0$ and $V \gg 0$. Indeed by considering the first equation of the system

$$\begin{cases} 0 = \gamma\left[u_2 - u_1\right] + \lambda_1 u_1 - \kappa_1^{-1} u_1^2 \\ 0 = \gamma\left[u_1 - u_2\right] + \lambda_2 u_2 - \kappa_2^{-1} u_2^2, \end{cases} \tag{22}$$

we deduce that $u_1 = 0$ implies $u_2 = 0$. Similarly the second equation of (22) shows that $u_2 = 0$ implies $u_1 = 0$. Therefore, since U is not zero, both components of U are positive (i.e. $U \gg 0$). A similar argument shows that $V \gg 0$.

Next we show that U and V are ordered, that is, $U \leq V$. To do so we use the fact that F is strictly sub-homogeneous, that is,

$$\eta F(U) \ll F(\eta U) \text{ for all } \eta \in (0,1).$$

Define

$$\eta^* := \sup\{\eta > 0 \,:\, \eta U \leq V\}$$

and assume by contradiction that $\eta^* < 1$. Then

$$0_{\mathbb{R}^2} = \eta^* F(U) \ll F(\eta^* U). \tag{23}$$

On the other hand, there must be an $i \in \{1,2\}$ such that $\eta^* u_i = v_i$. Otherwise we would get a contradiction with the definition of η^*, namely, we could find $\tilde{\eta} > \eta$ such that $\tilde{\eta} u_i \leq v_i$. Then, letting $j \neq i$ and recalling that $\eta^* u_i \leq v_j$ by definition of η^*, we have

$$\begin{aligned} F_i(\eta^* U) &= \gamma(\eta^* u_j - \eta^* u_i) + \lambda_i(\eta^* u_i) - \kappa_i^{-1}(\eta^* u_i)^2 \\ &= \gamma(\eta^* u_j) - \gamma v_j + \gamma(v_j - v_i) + \lambda_i v_i - \kappa_i^{-1} v_i^2 \\ &\leq F_i(V) = 0, \end{aligned}$$

which contradicts (23).

We conclude that $\eta \geq 1$. Therefore, U and V can be compared with

$$U \leq V.$$

By exchanging the roles of U and V, we also have $V \leq U$, which shows that $U = V$. Therefore there cannot exist more than one positive equilibrium. This completes the proof of Proposition 6.6. □

Remark 6.7

The argument used to prove the uniqueness of the equilibrium in the previous proposition is borrowed from Krasnoselskii [29]. It is an important tool in the study of monotone systems.

Invariant subregion in $\mathbb{R}^2_+$

We have seen in the previous sections that there exist positive subregions of $\mathbb{R}^2_+$, starting with $\mathbb{R}^2_+$ itself. Thanks to the upper solution of Lemma 6.4, the interval $[0, \alpha\mathbb{1}]$ is an invariant subregion of $\mathbb{R}^2_+$ for α sufficiently large (more precisely, for $\alpha \geq \alpha^+$). Then, in Lemma 6.5 we have shown that $[\alpha\mathbb{V}, +\infty)$ is invariant for $\alpha \in (0, \alpha^-]$ and, therefore, so is $[\alpha^-\mathbb{V}, \alpha^+\mathbb{1}]$.
Let us consider the case where $\lambda_1 = \lambda_2 =: \lambda$ and $\kappa_1 = \kappa_2 =: \kappa$. Then (12) becomes

$$\begin{cases} u_1'(t) = \gamma\left[u_2(t) - u_1(t)\right] + \lambda u_1(t) - \kappa^{-1} u_1(t)^2 \\ u_2'(t) = \gamma\left[u_1(t) - u_2(t)\right] + \lambda u_2(t) - \kappa^{-1} u_2(t)^2. \end{cases} \tag{24}$$

In this case one can check that the solution starting from an initial condition $u_0\mathbb{1}$ stays aligned with the vector $\mathbb{1}$. Indeed, if the function $u(t)$ is a solution to the scalar ODE

$$\begin{cases} u'(t) = \lambda u(t)\big(1 - \kappa^{-1}u(t)\big), \\ u(0) = u_0, \end{cases}$$

then the vector function $U(t) := u(t)\mathbb{1}$ solves (24) with initial condition $U(0) := u_0\mathbb{1}$. Thus the vector space spanned by $\mathbb{1}$ is an invariant subregion of $\mathbb{R}^2_+$.

Exercise 6.8

Extend the above results to the following two-patch model

$$\begin{cases} u_1'(t) = \gamma_2 u_2(t) - \gamma_1 u_1(t) + \lambda_1 u_1(t) - \kappa_1^{-1} u_1(t)^2 \\ u_2'(t) = \gamma_1 u_1(t) - \gamma_2 u_2(t) + \lambda_2 u_2(t) - \kappa_2^{-1} u_2(t)^2 \end{cases} \tag{25}$$

with $\gamma_1 > 0$ *is the leaving rate from patch* 1 *and* $\gamma_2 > 0$ *is leaving rate from patch* 2.

N-dimensional logistic equations with diffusion: local limitations in space

Consider the system

$$\begin{cases} U'(t) = \gamma D U(t) + \lambda U(t) - \kappa^{-1} U(t)^2 \\ U(0) = U_0 \geq 0, \end{cases} \tag{26}$$

where $U_i(t)$ is the number of individuals in the i-th compartment, $\lambda := (\beta - \mu)$ is the growth rate of individuals and the matrix describes the spatial motion of individuals between the $N \geq 3$ compartments

$$D = \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \ddots & & \vdots \\ 0 & 1 & -2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}$$

$U(t)^2$ is the column vector formed by the square of the components of $U(t)$ and

$$F(U) := \gamma DU + \lambda U - \kappa^{-1} U^2 = (F_1(U), F_2(U), \ldots, F_N(U))^T.$$

The system can be rewritten component by component as follows

$$
\begin{aligned}
U_1'(t) &= \gamma\big[U_2(t) - U_1(t)\big] + \lambda U_1(t) - \kappa^{-1} U_1(t)^2 = F_1(U), \\
U_2'(t) &= \gamma\big[U_3 + U_1\big] - 2\gamma U_2 + \lambda U_2(t) - \kappa^{-1} U_2(t)^2 = F_2(U), \\
&\vdots \\
U_i'(t) &= \gamma\big[U_{i+1} + U_{i-1}\big] - 2\gamma U_i + \lambda U_i(t) - \kappa^{-1} U_i(t)^2 = F_i(U), \\
&\vdots \\
U_{N-1}'(t) &= \gamma\big[U_N + U_{N-2}\big] - 2\gamma U_{N-1} + \lambda U_{N-1}(t) - \kappa^{-1} U_{N-1}(t)^2 = F_{N-1}(U) \\
U_N'(t) &= \gamma\big[U_{N-1} - U_N(t)\big] + \lambda U_N(t) - \kappa^{-1} U_N(t)^2 = F_N(U),
\end{aligned}
\tag{27}
$$

where $i \in \{2, \dots, N-1\}$ and γ^{-1} is the average time spent in the i-th compartment. Most of the arguments from the previous section can be adapted to the case of an N-component system.

Positivity

We can use a criterion from Chapter 7. More precisely, for each $M > 0$ we can find a $\Lambda = \Lambda(M) > 0$ such that

$$F(u) + \Lambda u \geq 0 \text{ for all } u \in [0, M\mathbb{1}],$$

where

$$\mathbb{1} = (1, 1, \ldots, 1)^T$$

is the N-component vector with all entries equal to 1. Actually we can take

$$\Lambda(M) = \kappa^{-1} M - \min(\lambda, 0) + 2\gamma.$$

Monotone semiflow

We have $F(0) = 0_{\mathbb{R}^N}$ and the Jacobian of F is

$$\partial F(U) = \gamma D + \lambda I - 2\kappa^{-1} \mathrm{diag}\,(U_1, U_2, \ldots, U_N)\,. \tag{28}$$

Since the off-diagonal entries of $\partial F(U)$ are non-negative for all $U \in \mathbb{R}^N_+$, a direct application of Theorem 3.5 shows that the differential system (26) generates a unique semiflow which is positive and monotone.

The linearized equation at 0

The behavior of the system close to 0 can be characterized by the sign of the dominant eigenvalue of the Jacobian matrix of F at $u = 0$. In the case of equation (26), we admit that this eigenvalue is equal to λ and corresponds to the right eigenvector $\mathbb{1}$. The fact that $(\lambda, \mathbb{1})$ is an eigenpair for $\partial F(0)$ can be checked from (28). The fact that λ is the dominant eigenvalue of $\partial F(0)$ is a consequence of the Perron–Frobenius Theorem in Chapter 4.

Global asymptotic stability of $0_{\mathbb{R}^N}$ when $\lambda \leq 0$

Assume $\lambda \leq 0$. Then since $\partial F(0) = \gamma D + \lambda I$ is symmetric, $\mathbb{1}$ is a left eigenvector of $\partial F(0)$ associated with λ,

$$\mathbb{1}^T \partial F(0) = \lambda \mathbb{1}^T.$$

Therefore

$$\mathbb{1}^T U'(t) = \lambda \mathbb{1}^T U(t) - \kappa^{-1} \mathbb{1}^T U(t)^2.$$

It follows that the quantity $V(u(t)) := \mathbb{1}^T U(t) = \sum_{i=1}^N U_i$ is strictly decreasing as long as $U(t) \neq 0$. Since $\mathbb{1} \gg 0$ we deduce the following proposition.

Proposition 6.9 (Global stability of $0_{\mathbb{R}^N}$)

Assume that $\lambda \leq 0$. Then the trivial equilibrium 0 is globally asymptotically stable.

Proof. The result has been proved already, and we now provide an alternative proof. Let us assume first that $\lambda < 0$. Define $\overline{U}(t) := M\mathrm{e}^{\lambda t}\mathbb{1}$. Then $\overline{U}(t)$ is a super-solution in the sense that it satisfies the differential inequality

$$\begin{aligned}\frac{\mathrm{d}\overline{U}(t)}{\mathrm{d}t} &= \lambda M\mathrm{e}^{\lambda t}\mathbb{1} = \lambda\overline{U}(t) = (\gamma D + \lambda I)\,\overline{U}(t) \\ &\geq (\gamma D + \lambda I)\,\overline{U}(t) - \kappa^{-1}\overline{U}(t)^2 = F\big(\overline{U}(t)\big).\end{aligned}$$

Therefore by Theorem 4.4 (ii) we have

$$x \leq M\mathbb{1} \Rightarrow U(t)x \leq \overline{U}(t) = M\mathrm{e}^{\lambda t}\mathbb{1}, \forall t \geq 0.$$

Hence $\overline{U}(t)$ is a super-solution in the sense of Theorem 4.4 (ii). In particular, for each initial condition $U_0 \in \mathbb{R}_+^N$, we can choose $M = \|U_0\|_\infty$ (so that $\overline{U}(0) = \|U_0\|_\infty \mathbb{1} \geq x$) and by a direct application of Theorem 4.4 we find that

$$U(t) \leq \|U_0\|_\infty \mathrm{e}^{\lambda t} \mathbb{1}, \text{ therefore } \lim_{t \to +\infty} U(t) = 0_{\mathbb{R}^N}.$$

If $\lambda = 0$, we observe that the function

$$\overline{U}(t) := \frac{U_0}{1 + \kappa^{-1} U_0 t} \mathbb{1}$$

solves the differential system

$$\begin{aligned} \frac{\mathrm{d}\overline{U}(t)}{\mathrm{d}t} = -\kappa^{-1}\overline{U}(t)^2 &= \frac{\gamma}{2} D\overline{U}(t) - \kappa^{-1}\overline{U}(t)^2 \\ &\geq F\left(\overline{U}(t)\right). \end{aligned}$$

Therefore we obtain from Theorem 4.4 that for all $U_0 \geq 0$

$$U(t) \leq \frac{\|U_0\|_\infty}{1 + \kappa^{-1}\|U_0\|_\infty t} \mathbb{1},$$

so that $\lim_{t \to +\infty} U(t) = 0_{\mathbb{R}^N}$. □

Dissipativity

In order to prove the dissipativity of (26) we use a family of upper solutions starting from

$$\overline{U}_\alpha := \alpha \mathbb{1}.$$

Indeed, since $\alpha D \mathbb{1} = 0$, we have

$$F(\alpha \mathbb{1}) = \alpha D \mathbb{1} + \lambda \alpha \mathbb{1} - \alpha^2 \kappa^{-1} \mathbb{1} = \alpha(\lambda - \kappa^{-1}\alpha)\mathbb{1},$$

so that by choosing $\overline{\alpha} := \lambda\kappa$, we deduce that

$$F(\alpha \mathbb{1}) \ll 0 \text{ for all } \alpha > \overline{\alpha}.$$

Therefore by applying Corollary 3.7 and using the fact that $0_{\mathbb{R}^N}$ is an equilibrium we deduce the following.

Lemma 6.10 (Upper solutions)

For each $\alpha > \max(\overline{\alpha}, 0)$, the interval $[0, \alpha \mathbb{1}] = \left\{ U \in \mathbb{R}^N : 0 \leq U \leq \alpha \mathbb{1} \right\}$ is positively invariant for the semiflow generated by (26). Moreover, the solution $t \to U(t)(\alpha \mathbb{1})$ starting from $\alpha \mathbb{1}$ converges to the largest equilibrium of (26) in the interval $[0, \alpha \mathbb{1}]$ (this equilibrium can be 0).

Lower solutions

Assume that $\lambda > 0$. Since $\mathbb{1}$ is an eigenvector of D, we have that for all $\alpha \in (0, \overline{\alpha})$

$$F(\alpha \mathbb{1}) = \alpha(\lambda - \kappa^{-1}\alpha)\mathbb{1} \gg 0 \text{ for all } \alpha \in (0, \overline{\alpha}),$$

where $\overline{\alpha} := \lambda\kappa$. Therefore applying Corollary 3.7 we have the following result.

Lemma 6.11 (Lower solutions)

Assume that $\lambda > 0$. For each $\alpha \in (0, \overline{\alpha})$, the interval $[\alpha\mathbb{1}, +\infty)$ is positively invariant for the semiflow generated by (26). Moreover, the solution $t \mapsto U(t)$ starting from $U(0) = \alpha\mathbb{1}$ converges to the smallest equilibrium of (26) in the interval $[\alpha\mathbb{1}, +\infty)$.

Equilibrium

It follows from Proposition 6.9 that there cannot exist a positive equilibrium when $\lambda \leq 0$. When $\lambda > 0$, Lemma 6.10 and Lemma 6.11 show that there exists a positive equilibrium such that any solution starting from a non-negative and non-trivial initial value $U(0)$ converges to such an equilibrium. Here we show the uniqueness of the positive equilibrium, which consequently attracts any non-negative and non-trivial initial value.

Proposition 6.12 (Uniqueness of the positive equilibrium)

Assume $\lambda > 0$. Then $\overline{\alpha}\mathbb{1}$ is the unique positive equilibrium of (26)*, which attracts any non-negative and non-zero initial value.*

Proof. Let us first remark that the solution of (26) starting from an initial condition $U(0) = U_0 \mathbb{1}$ for $U_0 > 0$ always stays in the vector space spanned by $\mathbb{1}$, that is, $\mathrm{Vect}\,\mathbb{1}$ is a positively invariant subregion of $\mathbb{R}_+^N$. Actually we have an explicit solution for the solutions of (26) when $U(0) = U_0 \mathbb{1}$, which is

$$U^*(t, U_0) := \frac{\lambda \kappa U_0 \mathrm{e}^{\lambda t}}{\lambda \kappa + U_0(\mathrm{e}^{\lambda t} - 1)} \mathbb{1} \text{ if } U_0 \neq \lambda\kappa \text{ and } U^*(t, \lambda\kappa) := \lambda\kappa.$$

Next, we show that a solution that starts from the boundary of the positive cone will immediately enter the interior of the cone. Let

$$U_0 = (U_1(0), \ldots, U_N(0))^T \in \mathbb{R}^N$$

be a non-negative non-trivial vector.

Assume that there exists a $j \in \{1, \ldots, N\}$ such that $U_j = 0$. Then select $j_0 \in \{1, \ldots, N\}$ such that $U_{j_0} = 0$ and $U_{j_0+1} + U_{j_0-1} > 0$, and according to (26) we have

$$U'_{j_0}(0) = \frac{\gamma}{2}\left(U_{j_0+1} + U_{j_0-1}\right) > 0$$

and therefore U_{j_0} is positive on $(0, \tau)$ for some $\tau > 0$.

By induction we can prove that $U_j(t) > 0$ on an interval $(0, \tau)$ for all $j \in \{1, \ldots, N\}$ and some $\tau > 0$.

Therefore, we can restrict the analysis to initial conditions that start from a

$$U_0 = (U_1(0), \ldots, U_N(0))^T$$

with $U_i(0) > 0$ for all $i \in \{1, \ldots, N\}$.

Finally, we construct a pair of upper and lower solutions which allow us to prove the convergence of $U(t)$ to the positive equilibrium. Let $U_0 := (U_1(0), \dots, U_N(0))^T$ with $U_i(0) > 0$ for all $i \in \{1, \dots, N\}$ be given. Define

$$\underline{U}_0 := \min(U_1(0), \dots, U_N(0)) > 0 \text{ and } \overline{U}_0 := \max(U_1(0), \dots, U_N(0)).$$

Then we have

$$\underline{U}_0 \mathbb{1} \leq U_0 = U(0) \leq \overline{U}_0 \mathbb{1},$$

and therefore by the comparison principle (Theorem 4.4) we have

$$U^*(t, \underline{U}_0) \leq U(t) \leq U^*(t, \overline{U}_0) \text{ for all } t \geq 0.$$

Since

$$\lim_{t \to +\infty} U^*(t, \underline{U}_0) = \lim_{t \to +\infty} U^*(t, \overline{U}_0) = \lambda \kappa \mathbb{1},$$

we have shown that $U(t) \to \lambda \kappa \mathbb{1}$ when $t \to +\infty$. This completes the proof of Proposition 6.12.

□

Invariant subregions of $\mathbb{R}_+^N$

We have seen in the previous sections that there exist positive subregions of $\mathbb{R}_+^N$, starting with $\mathbb{R}_+^N$ itself. Thanks to the upper solution of Lemma 6.10, the interval $[0, \alpha\mathbb{1}]$ is an invariant subregion of $\mathbb{R}_+^N$ for α sufficiently large (more precisely, for $\alpha \geq \overline{\alpha}$). Then, in Lemma 6.11 we have shown that $[\overline{\alpha}\mathbb{1}, +\infty)$ is invariant for $\alpha \in (0, \overline{\alpha}]$. Finally, the vector space spanned by $\mathbb{1}$ is an invariant subregion of $\mathbb{R}_+^N$.

N-dimensional systems with arbitrary growth rates

The analysis of the previous subsection can be adapted in a much more general framework. More precisely, let $M = (m_{ij})_{1 \le i,j \le N}$ be a square matrix of size N with non-negative off-diagonal entries, which is irreducible in the sense that for all partitions I, J of $\{1, \ldots, N\}$ (i.e. $I \cup J = \{1, \ldots, N\}$ and $I \cap J = \varnothing$) there are $i \in I$ and $j \in J$ such that $m_{ij} > 0$.
Let $f_1, \ldots, f_N : \mathbb{R}_+ \to \mathbb{R}_+$ be positive, locally Lipschitz continuous and sub-1-homogeneous functions, that is,

$$f_i(\lambda x) \le \lambda f_i(x) \text{ for all } \lambda \in (0,1) \text{ and } x \ge 0 \text{ and } i \in \{1, \ldots, N\}.$$

We investigate the problem

$$U'(t) = MU(t) - F(U(t))U(t), \tag{29}$$

where $F(U)U$ is defined as

$$F(U)U := (f_1(U_1)U_1, \ldots, f_N(U_N)U_N)^T.$$

We define the *spectral bound* of M as

$$\Lambda := \sup\{\lambda \, : \, \lambda \in \sigma(M)\}.$$

Theorem 6.13

If $\Lambda \leq 0$ *then* $0_{\mathbb{R}^N}$ *is the only non-negative equilibrium of* (29)*. If* $\Lambda > 0$ *then there exists a positive equilibrium which attracts every non-negative non-trivial initial condition of* (29)*.*

Exercise 6.14 (For advanced readers)

Prove Theorem 6.13. The case $\Lambda \leq 0$ can be done by using an explicit super-solution. Here we detail some of the important steps of the proof when $\Lambda > 0$.

1. *Use the Perron–Frobenius theorem to show that there is a unique (up to multiplication by a positive scalar) $\Phi \in \mathbb{R}^N$, $\Phi \geq 0$, satisfying*

$$M\Phi = \Lambda\Phi.$$

   Notice that $\eta\Phi$ is a sub-solution to (29) *for sufficiently small $\eta > 0$.*

2. *Show that every orbit is bounded. Deduce that for all $U_0 \in \mathbb{R}^N$ with $U_0 \geq 0$ the corresponding solution to* (29) *satisfies*

$$\lim_{t \to +\infty} U(t) = \bar{U}.$$

3. *Show that there exists a unique equilibrium to* (29)*. One can use an argument similar to the one found in the proof of Proposition 6.6. The interested reader may have a look at [29, Theorem 6.3, p. 188].*

Outline

Monotone dynamics consist of both monotone semiflows (continuous-time systems) and monotone mappings (discrete-time systems). Autonomous (or non-autonomous) monotone systems occur in ordinary differential equations, delay differential equations, parabolic equations, abstract non-densely defined Cauchy problems, random dynamical systems, control systems, etc. Schneider and Vidyasagar [47] introduced the quasi-monotone condition for autonomous finite-dimensional linear systems. This has been extended by Volkmann [58] to nonlinear infinite-dimensional systems and we refer to the works of Hirsch and Smith [25], Walter [61], Uhl [57] for more results and references. Monotonicity methods and comparison arguments are largely developed in ordinary differential equations, delay differential equations, and partial differential equations. We refer for instance to Smoller [54], Smith [52], Zhao [71, 72], Hirsch and Smith [25], and Chueshov [11] for more results and references on this subject. Here we have only selected a few topics and a few references.

Monotone dynamical systems

Monotone ordinary differential equations We refer to Smoller [54], Smith [52], Zhao [71], Hirsch and Smith [25] for more results on monotone ordinary differential equations. In Section 6, we have analyzed logistic equations with diffusion in detail, which are examples of autonomous monotone ordinary differential equations.

Monotone theory for delay differential equations As already mentioned, the theory developed in this chapter can also be extended and applied to delay differential equations. We refer to Hirsch and Smith [25] for results on monotone theory for differential equations with bounded delays. For systems of delay differential equations with unbounded and infinite delay, we refer for instance to Wu [66] and to Krisztin and Wu [31] for the neutral delay differential equations.

Monotone abstract non-densely defined Cauchy problems A recent extension of the monotone theory to non-densely defined Cauchy problems has been obtained in Magal, Seydi and Wang [41]. Several examples of differential equations, such as delay differential equations [33, 17], parabolic equations with nonlinear and non-local boundary conditions [8, 16, 15] can be put into the abstract non-densely defined Cauchy problems. More results and examples of abstract non-densely defined Cauchy problems can be found in Magal and Ruan [39, 40]. Thus, the theory of monotone semiflows, the comparison principle, and invariance of solutions for abstract non-densely defined Cauchy problems obtained in [41] will have a wide range of applicability. In particular, they provide an application to age-structured population dynamics models (see the book of Webb [64] and Magal and Ruan [40] for more results on this topic), where a monotone semiflow theory and some comparison principles for age-structured models are consequently obtained. As a special case, the Kermack and McKendrick model with age of infection can be handled, as will be seen below in this section.

Monotone mappings A continuous map $T : X \to X$ on an ordered metric space X is monotone if

$$x \leq y \Rightarrow Tx \leq Ty.$$

Monotone maps play an important role in the study of periodic solutions to periodic quasi monotone systems of ordinary differential equations (see, e.g., the monographs of Krasnoselskii [29, 30] and the papers [4, 55, 42, 21, 49, 50, 63]). Furthermore, monotone maps frequently arise as mathematical models (see, e.g., [55, 65], and the references therein). For instance, the following system is a discrete Lotka–Volterra competition model:

$$\begin{aligned}(u_{n+1}, v_{n+1}) &= T(u_n, v_n) \\ &= (u_n \exp[r(1 - u_n - bv_n)], v_n \exp[s(1 - cu_n - v_n)]) .\end{aligned}$$

We refer to Smith [51] for more results on this Lotka–Volterra competition model.

Monotone random dynamical systems

Monotonicity methods as well as comparison arguments have mainly been used to study one-dimensional random or stochastic differential equations (see, e.g., [32]). The book of Chueshov [12] proposes a systematic treatment of the basic ideas and methods for monotone random dynamical systems with infinite-dimensional phase space. This book focuses on the qualitative behavior of monotone random dynamical systems and its applications on finite cooperative random and stochastic ordinary differential equations, that occur in the field of ecology, epidemiology, economics, and biochemistry (see Smith [52]). For results on monotone methods and comparison arguments to random and stochastic parabolic partial differential equations and non-autonomous parabolic equations, we refer for instance to the papers by Chueshov (see, e.g.,[9, 10]) and the paper by Shen and Yi [48].

Various examples Monotone systems occur in many fields, including ecological systems, chemistry, economic models, epidemiology, and biological models. Population dynamics is an important subject in mathematical biology, and a central aim is to study the long-term behavior of the associated models. Monotonicity methods and comparison principles are the main tools in the investigation of the global dynamics of such systems. The theory of monotone dynamical systems has been widely used in population dynamics. There is a long history of the application of monotone methods and comparison arguments (see, e.g., [52, 12] and the literature quoted there).

The chemostat system The chemostat is a laboratory device that is a dynamic system with continuous material inputs and outputs. The continuous turnover follows the input and removal of nutrients. The specific death, predation, or emigration which always occurs in nature is equivalent to the washout of organisms. The apparatus consists of three connected vessels: the feed bottle, the culture vessel, and the collection vessel. The culture vessel, where the "action" takes place, contains a mixture of nutrients and organisms. The following model is a classical chemostat model

$$\begin{cases} S'(t) = D(S^0 - S) - \dfrac{1}{\gamma}\dfrac{rSN}{a+S} \\ N'(t) = \dfrac{rSN}{a+S} - DN. \end{cases}$$

Here $S(t)$ denotes the concentration of the nutrient in the culture vessel at time t, $N(t)$ denotes the concentration of the organism at time t.

The constant S^0 represents the concentration of the input nutrient while D is the dilution (or washout) rate. It is defined as $D = F/V$, where V denotes the volume of the culture vessel and F denotes the volumetric flow rate. Finally, $\frac{rSN}{a+S}$ corresponds to the consumption term, where r is the maximal growth rate, a is the Michaelis–Menten (or half-saturation) constant, while γ is a "yield" constant reflecting the conversion of nutrients to organisms. Note that D and S^0 are environmental parameters, and r, a and γ are biological parameters.

The book of Smith and Waltman [53] is devoted to the theoretical description of ecological models based on the chemostat. The theory of the chemostat (and/or dynamics of microbial competition) is a very good example of the application of monotone systems. We refer to the books of Waltman [62] and Hsu and Waltman [27], and references therein for more references and results on this subject. We also refer to the book of Perthame [43] for more results about chemostats.

Lotka–Volterra equations

Competitive and cooperative Lotka–Volterra systems are also a very important class of monotone systems. This covers many different formulations, including autonomous ordinary differential equations, non-autonomous ordinary differential equations, partial differential equations and so on. We refer for instance to [13, 26, 37, 70, 67, 68, 69, 73] and the references cited therein.

Application to the Kermack and McKendrick model

We now present an important monotone property of the Kermack–McKendrick model that will be used in Chapter 12 to construct various algorithms. The Kermack–McKendrick model takes the following form

$$\begin{cases} S'(t) = -\nu S(t)I(t), \\ I'(t) = \nu S(t)I(t) - \gamma I(t), \\ R'(t) = \gamma I(t), \end{cases} \tag{30}$$

supplemented with the initial values

$$S(0) = S_0 \geq 0 \text{ and } I(0) = I_0 \text{ and } R(0) = R_0.$$

The Kermack and McKendrick system is not monotone in itself. Nevertheless, it becomes monotone by considering the cumulative number of I, that is, the quantity $CI(t)$ given by

$$CI(t) = \int_0^t I(s)\mathrm{d}s, \; t \geq 0. \tag{31}$$

Indeed, integrating the S-equation yields

$$S(t) = S_0 \exp\left(-\nu C I(t)\right), \forall t \geq 0.$$

So that the I-equation can be rewritten as

$$I'(t) = S_0 \exp\left(-\nu C I(t)\right) \nu C I'(t) - \gamma I(t). \tag{32}$$

By integrating the I-equation we obtain

$$C I'(t) = I_0 + S_0 \left[1 - \exp\left(-\nu C I(t)\right)\right] - \gamma C I(t). \tag{33}$$

Now since the map

$$G(x) = I_0 + S_0 \left[1 - \exp\left(-\nu x\right)\right]$$

is monotone increasing and

$$G(0) = I_0 > 0,$$

we obtain the following result.

Theorem 7.1

Assume that $S_0 > 0$ *and* $I_0 > 0$ *are given. Let* $t > 0$ *be given and fixed. The quantity* $CI(t)$ *is increasing with respect to* S_0, I_0, ν *and* $-\gamma$. *Moreover,* $t \to CI(t)$ *is strictly increasing and*

$$\lim_{t \to \infty} CI(t) = CI_\infty,$$

where $CI_\infty > 0$ *is called the final size of the epidemic and* $CI_\infty > 0$ *is the unique positive solution of*

$$I_0 + S_0 \left[1 - \exp\left(-\nu CI_\infty\right)\right] = \gamma CI_\infty.$$

Application to the Kendall model The model introduced by Kendall in 1957 [28] is the following

$$\begin{cases} \partial_t s(t,x) = -\nu s(t,x) b(t,x), \\ \partial_t i(t,x) = \quad \nu s(t,x) b(t,x) - \eta i(t,x), \\ \partial_t r(t,x) = \eta i(t,x), \end{cases} \tag{34}$$

and

$$\varepsilon^2 b(t,x) - \partial_x^2 b(t,x) = \chi i(t,x). \tag{35}$$

The above system is supplemented with the initial distributions

$$s(0,x) = s_0(x) \in BC_+(\mathbb{R}),\ i(0,x) = i_0(x) \in BC_+(\mathbb{R}),$$
$$\text{and } r(0,x) = r_0(x) \in BC_+(\mathbb{R}),$$

where $BC_+(\mathbb{R})$ is the space of positive bounded continuous functions on $\mathbb{R}$.

In this model $s(t,x)$ is the distribution of susceptible at time t, $i(t,x)$ is the distribution of infectious at time t, and $r(t,x)$ is the distribution of recovered at time t. In this model, the individuals are not moving, because by summing the three equations of (34) we obtain

$$\partial_t \left(s(t,x) + i(t,x) + r(t,x) \right) = 0.$$

Here the spatial movement of the pathogen is described by equation (35). Indeed, the equation (35) expresses that the distribution of the pathogen is locally around the location of infectious (which released the pathogen). Indeed, the equation (35) is equivalent to

$$b(t,x) = \frac{\chi}{2\varepsilon} \int_{\mathbb{R}} \mathrm{e}^{-\varepsilon |x-\sigma|} i(t,\sigma) \mathrm{d}\sigma. \tag{36}$$

Remark 7.2

Kendall's original model does not specify the kernel $k(x) = \dfrac{\chi}{2\varepsilon} \mathrm{e}^{-\varepsilon|x|}$ *as we do here. The advantage of this special form is that it can be used on a bounded domain with suitable boundary conditions. Such models have been introduced for the influenza epidemic on the island of Puerto Rico by Magal et al. [38].*

Define

$$I(t,x)=\int_0^t i(\sigma,x)\mathrm{d}\sigma \text{ and } B(t,x)=\int_0^t b(\sigma,x)\mathrm{d}\sigma.$$

Without loss of generality, we can assume that

$$\chi=\varepsilon^2.$$

The S-equation of (2) gives

$$s(t,x)=s_0(x)\exp\left(-\nu\int_0^t b(\sigma,x)\mathrm{d}\sigma\right).$$

Therefore by integrating the I-equation in time, we obtain the following monotone equation

$$\partial_t I(t,x) = i_0(x) + s_0(x)\left[1 - \exp\left(-\nu\, B(t,x)\right)\right] - \eta I(t,x) \tag{37}$$

and the equation (35) becomes

$$\varepsilon^2 B(t,x) - \partial_x^2 B(t,x) = \varepsilon^2 B(t,x), \tag{38}$$

which is equivalent to

$$B(t,x) = \frac{\varepsilon}{2}\int_{\mathbb{R}} \mathrm{e}^{-\varepsilon|x-\sigma|} I(t,\sigma)\mathrm{d}\sigma. \tag{39}$$

The above procedure to transform the Kendall epidemic model into a single monotone system is originally due to Aronson [5]. We refer to Aronson [5], Diekmann [14] and Thieme [56] for more results about this class of equations. We also refer to the survey paper by Ruan [44] for more results about this subject.

Application to the Kermack and McKendrick model with age of infection

In this subsection, we derive a similar monotone property for the Kermack and McKendrick model with age of infection, denoted by $a > 0$. Let $\eta > 0$, $\beta \in L^\infty_+(0,\infty)$ and $\nu \in L^\infty_+(0,\infty)$ be given. Let us first consider the equation for the susceptible individuals that reads as

$$\begin{cases} S'(t) = -\eta S(t) \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \\ S(0) = S_0 \geq 0. \end{cases} \tag{40}$$

By integrating the above S-equation, we obtain

$$S(t) = S_0 \exp\left(-\eta \int_0^t \int_0^{+\infty} \beta(a) i(s,a) \mathrm{d}a \mathrm{d}s \right), \quad \forall t \geq 0.$$

Next the equation for the density of the infected is the following

$$\begin{cases} \partial_t i(t,a) + \partial_a i(t,a) = -\nu(a) i(t,a), \text{ for } a \in (0,\infty), \\ i(t,0) = \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \\ i(0,\cdot) = i_0 \in L^1_+((0,+\infty),\mathbb{R}). \end{cases} \tag{41}$$

Now consider the function

$$CI(t,a) = \int_0^t i(s,a)\mathrm{d}s.$$

Integrating the i-equation, we obtain (at least formally)

$$\begin{cases} \partial_t CI(t,a) + \partial_a CI(t,a) = i_0(a) - \nu(a) CI(t,a), \text{ for } a \in (0,\infty), \\ CI(t,0) = \int_0^t S_0 \exp\left(-\eta \int_0^s \int_0^{+\infty} \beta(a) i(\sigma,a) \mathrm{d}a \mathrm{d}\sigma\right) \eta \int_0^{+\infty} \beta(a) i(s,a) \mathrm{d}a \,\mathrm{d} \\ CI(0,\cdot) = 0 \in L^1_+((0,+\infty),\mathbb{R}). \end{cases}$$

By integrating the boundary condition and by applying Fubini's theorem, we obtain

$$\begin{cases} \partial_t CI(t,a) + \partial_a CI(t,a) = i_0(a) - \nu(a)CI(t,a), \text{ for } a \in (0,\infty), \\ CI(t,0) = S_0 \left[1 - \exp\left(-\eta \int_0^{+\infty} \beta(a)CI(\sigma,a)\mathrm{d}a\mathrm{d}\sigma\right)\right], \\ CI(0,\cdot) = 0 \in L^1_+((0,+\infty),\mathbb{R}). \end{cases} \tag{42}$$

Now since the functional $G : L^1_+(0,\infty) \to [0,\infty)$ given by

$$G(CI) = S_0 \left[1 - \exp\left(-\eta \int_0^{+\infty} \beta(a)CI(a)\mathrm{d}a\mathrm{d}\sigma\right)\right]$$

is monotone increasing, and

$$G(0_{L^1}) = 0_{\mathbb{R}},$$

we can apply the result in Magal, Seydi and Wang [41], and this yields the following (new) result on the Kermack and McKendrick model with age of infection.

Theorem 7.3

Assume that $S_0 > 0$ *and* $i_0 = i_0(a) \in L^1_+(0,\infty) \setminus \{0\}$. *Let* $t > 0$ *be given and fixed. Then the distribution* $a \to CI(t,a)$ *is monotone increasing with respect to* S_0, $a \to i_0(a)$ *and* η *and* $a \to -\nu(a)$.

Moreover, the map $t \to CI(t,a)$ *is increasing in time and*

$$\lim_{t\to\infty} CI(t,a) = CI_\infty(a), \text{ in } L^1(0,\infty),$$

where $a \to CI_\infty(a) > 0$ *is the so-called final size age distribution of the epidemic and* $a \to CI_\infty(a) > 0$ *is the unique positive solution of the system*

$$\begin{cases} \partial_a CI_\infty(a) = i_0(a) - \nu(a) CI_\infty(a), \text{ for } a \in (0,\infty), \\ CI_\infty(0) = S_0 \left[1 - \exp\left(-\eta \int_0^{+\infty} \beta(a) CI_\infty(a) \mathrm{d}a\right)\right]. \end{cases}$$

Remark 7.4

We can calculate the distribution $a \to CI_\infty(a)$ *a little further. Indeed the first equation of* (42) *gives*

$$CI_\infty(a) = \Pi(a) CI_\infty(0) + \int_0^a \frac{\Pi(a)}{\Pi(s)} i_0(s)\mathrm{d}s \text{ with } \Pi(a) = \exp\left(-\int_0^a \nu(s)\mathrm{d}s\right).$$

By substituting this formula into the second equation of (42) *we obtain the following scalar equation for the unknown* $CI_\infty(0) > 0$

$$CI_\infty(0) = S_0 \left[1 - \exp\left(-A \times CI_\infty(0) - B\right)\right],$$

wherein A *and* B *are given by*

$$A = \eta \int_0^{+\infty} \beta(a)\Pi(a)\mathrm{d}a \text{ and } B = \eta \int_0^{+\infty} \beta(a) \int_0^a \frac{\Pi(a)}{\Pi(s)} i_0(s)\mathrm{d}s\mathrm{d}a.$$

Volterra's formulation of the Kermack and McKendrick model with age of infection To explain the partial differential equation formulation of the Kermack–McKendrick model with age of infection, we can use Volterra's integral formulation. Let $a > 0$ denote the time since an individual has become infected by the pathogen. Then the model of Kermack and McKendrick with age of infection can be rewritten as follows. The number of susceptible individuals $S(t)$ satisfies the following equation

$$S'(t) = -\nu\, S(t) \int_0^\infty \beta(a) i(t,a) \mathrm{d}a, \text{ for } t \geq 0, \text{ with } S(0) = S_0 \geq 0.$$

Now recall that the function $a \to \beta(a) \in L^\infty_+(0,\infty)$ is the fraction of infectious individuals (capable of transmitting the pathogen to the susceptible) with infection age a, and let $a \to \Pi(a)$ be the probability for individuals to remain infected after infection age a,

$$\Pi(a) = \exp\left(-\int_0^a \nu(\sigma)\mathrm{d}\sigma\right).$$

Next define

$$B(t) = \int_0^\infty \beta(a) i(t,a) \mathrm{d}a,$$

so that the distribution of the infected population $a \to i(t,a)$ at time t satisfies

$$i(t,a) = \begin{cases} \dfrac{\Pi(a)}{\Pi(a-t)} i_0(a-t), & \text{if } a > t, \\ \Pi(a)\, \nu\, S(t-a)\, B(t-a), & \text{if } t > a, \end{cases} \tag{43}$$

where $a \to i_0(a) \in L^1_+(0,\infty)$ denotes the initial distribution of the population of the infected.

Using (43), we deduce that $t \to B(t) \in C([0,\infty),\mathbb{R})$ becomes the unique non-negative solution of the following Volterra integral equation for $t \geq 0$,

$$B(t) = \left[\int_t^{\infty} \beta(a) \frac{\Pi(a)}{\Pi(a-t)} i_0(a-t)\mathrm{d}a + \int_0^t \beta(a)\Pi(a)\,\nu\, S(t-a)\, B(t-a)\mathrm{d}a\right.$$

Moreover, by applying the variation of constant formula to the S-equation, the map $t \to S(t)$ is given by

$$S(t) = \mathrm{e}^{-\int_0^t \nu\, B(\sigma)\mathrm{d}\sigma} S_0.$$

To conclude, one may alternatively use Volterra's integral formulation to prove some results about monotonicity for age-structured models, as in Magal, Seydi and Wang [41].

Thank you for listening

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