Lecture 4: Semiflows, ω-limit Sets, α-limit Sets, Attraction, and Dissipation

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Outline

Let (M, d) be a complete metric space. Typical examples in population dynamics will be

$$M = \mathbb{R}_+^n = [0, \infty)^n.$$

A more general class of subset M are the intervals in $\mathbb{R}^n$. That is a subset of the following form

$$M = [c, \infty) = \{x \in \mathbb{R}^n : c_i \leq x_i\},$$

$$M = [c, d] = \{x \in \mathbb{R}^n : c_i \leq x_i \leq d_i\},$$

or

$$M = (-\infty, d] = \{x \in \mathbb{R}^n : x_i \leq d_i\},$$

where $c, d \in \mathbb{R}^n$ and $c \leq d$.

These subsets will be endowed with usual distance (induced by the norm $\|.\|$ of $\mathbb{R}^n$)

$$d(x,y) = \|x-y\|, \forall x,y \in M.$$

Then (M,d) is a complete metric space (since the subsets M are closed subsets of $\mathbb{R}^n$).

In order to consider both discrete and continuous time dynamical systems, the time will vary either in

$$I_+ = [0,+\infty) \text{ and } I = \mathbb{R} \text{ (**continuous time**)}$$

or in

$$I_+ = \mathbb{N} \text{ and } I = \mathbb{Z} \text{ (**discrete time**)}.$$

Definition 1.1

Let $\{U(t)\}_{t\geq 0}$ be a family of continuous maps from M into itself parameterized by $t \in I_+$. We will say that U is a **continuous semiflow** on M if the following properties are satisfied

(i) $U(0)x = x, \forall x \in M$;

(ii) $U(t)U(s)x = U(t+s)x, \forall t, s \geq 0, \forall x \in M$;

(iii) The map $(t, x) \to U(t)x$ is continuous from $I_+ \times M$ into M.

Moreover we will say that U is a **continuous time semiflow** if $I_+ = [0, +\infty)$ and a **discrete time semiflow** if $I_+ = \mathbb{N}$.

Remark 1.2

Since we assumed that the map $x \to U(t)x$ *continuous (for each* $t \geq 0$*), it follows that a discrete time semiflow is always continuous. That is to say that the property* (iii) *of Definition 1.1 is always satisfied for discrete time semiflow.*

Discrete time semiflow and difference equation

Assume that a discrete semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is given. Define the map $T : M \to M$

$$T(x) = U(1)(x)$$

and the sequence $\{u_n\}_{n\in\mathbb{N}}$

$$u_n := U(n)(x), \forall n \geq 0.$$

Then by using the property (ii), we obtain

$$u_n = U(n)(x) = U(1)(U(n-1)(x)) = T(u_{n-1}), \forall n \in \mathbb{N}, \text{ and } u_0 = x.$$

So we obtain a difference equation

$$u_{n+1} = T\left(u_n\right), \forall n \in \mathbb{N}, \text{ with } u_0 = x \in M.$$

We also observe that

$$\begin{aligned} u_0 &= x \\ u_1 &= T(x) \\ u_2 &= T(T(x)) = T^2(x) \end{aligned}$$

and we obtain

$$u_n = T^n(x), \forall n \in \mathbb{N},$$

where T^n is defined by

$$T^{n+1} = T \circ T^n, \forall n \in \mathbb{N} \text{ and } T^0 = I.$$

Conversely, assume that a map $T : M \to M$ is given. Then the semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is defined by

$$U(n)(x) = T^n(x), \forall n \in \mathbb{N}, \forall x \in M,$$

where $T : M \to M$ is a continuous map.

Semiflow generated by the 1-dimensional logistic equation

Let us consider the family of maps $\{U(t)\}_{t\geq 0}$ defined on $M = \mathbb{R}_+$ as follows

$$U(t)x = \frac{e^{\lambda t}x}{1+\kappa \int_0^t e^{\lambda \sigma} x d\sigma}, \forall t \geq 0, \forall x \geq 0, \tag{1}$$

where $\lambda \in \mathbb{R}$ and $\kappa \geq 0$.

Lemma 1.3

The family $\{U(t)\}_{t\leq 0}$ is a semiflow on $M = [0, +\infty)$.

Remark 1.4

The semiflow $U(t)$ can not be extended (backward in time) to a flow because the solution of the logistic equation $t \to U(t)x$ is blowing up for negative time (whenever $x > \lambda/\kappa$). Because, for $x > \lambda/\kappa$ the solutions blowup for some finite negative time. The map $U(t)$ restricted to $[0, \lambda/\kappa]$ defines a flow (whenever $\lambda > 0$). That is, the map $x \mapsto U(t)x$ redistricted to $[0, \lambda/\kappa]$ is defined for all $t \in \mathbb{R}$.

Proof. Let us verify that $U(t)$ is a continuous semiflow on $\mathbb{R}_+$. Indeed, it is clear that

$$U(0)x = x, \forall x \in \mathbb{R}_+.$$

Let $t, s \geq 0$, we have

$$\begin{aligned} U(t)U(s)x &= \frac{e^{\lambda t}\frac{e^{\lambda s}x}{1+\kappa\int_0^s e^{\lambda r}xdr}}{1+\kappa\int_0^t e^{\lambda\sigma}\frac{e^{\lambda s}x}{1+\kappa\int_0^s e^{\lambda r}xdr}d\sigma} \\ &= \frac{e^{\lambda(t+s)}x}{1+\kappa\int_0^s e^{\lambda r}xdr+\kappa\int_0^t e^{\lambda(\sigma+s)}xd\sigma} \end{aligned}$$

and by using a change of variable, we obtain that

$$\int_0^t e^{\lambda(\sigma+s)}xd\sigma = \int_s^{t+s} e^{\lambda r}xdr,$$

and it follows that

$$U(t)U(s)x = \frac{e^{\lambda(t+s)}x}{1+\kappa\int_0^{t+s} e^{\lambda r} x dr} = U(t+s)x.$$

Therefore U is a semiflow on $\mathbb{R}_+$. $\square$

Remark 1.5

The map $t \to N(t) := U(t)x$ *satisfies the logistic equation*

$$N'(t) = \lambda N(t) - \kappa N^2(t). \tag{2}$$

We refer to Chapter 5 in Volume I for more results.

Semiflow generated by a 2-dimensional Bernoulli-Verhulst equation

Let $\theta > 0$. Consider the family of maps $V(t)$ on $\mathbb{R}^2$ defined by

$$V(t)X = \begin{cases} \left(\frac{U(t)\left(\|X\|_2^\theta\right)}{\|X\|_2^\theta}\right)^{\frac{1}{\theta}} e^{At}X, & \text{if } X = \begin{pmatrix} x \\ y \end{pmatrix} \neq 0, \\ 0, & \text{if } X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{cases} \tag{3}$$

where $U(t)$ is the semiflow defined by (1), and

$$\|X\|_2 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \sqrt{x^2 + y^2}$$

is the Euclidean norm,

and (see Section 3.9.3 in Volume I) we have

$$e^{At}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\omega t)x - \sin(\omega t)y \\ \sin(\omega t)x + \cos(\omega t)y \end{pmatrix},$$

with

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Lemma 1.6

The family $\{V(t)\}_{t\leq 0}$ *is a semiflow on* $\mathbb{R}^2$.

Proof. We first observe that

$$\begin{aligned}\left\|e^{At}X\right\|_2^2 &= (\cos(\omega t)x)^2 - 2\cos(\omega t)x\sin(\omega t)y + (\sin(\omega t)y)^2 \\ &+ (\sin(\omega t)x)^2 + 2\sin(\omega t)x\cos(\omega t)y + (\cos(\omega t)y)^2 \\ &= x^2 + y^2 \\ &= \|X\|_2^2,\end{aligned}$$

and we deduce that e^{At} preserves the Euclidean norm of X

$$\left\|e^{At}X\right\|_2 = \|X\|_2, \forall t \in \mathbb{R}.$$

It follows that

$$V(t)\left(\mathbb{R}^2 \setminus \{0\}\right) \subset \mathbb{R}^2 \setminus \{0\}, \forall t \geq 0.$$

Let $X \neq 0$. By applying the Euclidean norm on both sides of (3), we deduce that

$$\|V(t)X\|_2^\theta = \frac{U(t)\left(\|X\|_2^\theta\right)}{\|X\|_2^\theta} \left\|e^{At}X\right\|_2^\theta = U(t)\left(\|X\|_2^\theta\right) \forall t \geq 0,$$

and

$$\begin{aligned} V(t)V(s)X &= \left(\frac{U(t)\left(\|V(s)X\|_2^\theta\right)}{\|V(s)X\|_2^\theta}\right)^{\frac{1}{\theta}} e^{At}V(s)X \\ &= \left[U(t)\left(U(s)\left(\|X\|_2^\theta\right)\right)\right]^{\frac{1}{\theta}} e^{At}\frac{V(s)X}{\|V(s)X\|_2} \\ &= \left[U(t+s)\left(\|X\|_2^\theta\right)\right]^{\frac{1}{\theta}} e^{At}\frac{e^{As}X}{\|X\|_2} \\ &= V(t+s)X, \end{aligned}$$

whenever $t \geq 0$ and $s \geq 0$. It follows that $V(t)$ is a continuous semiflow. $\square$

Remark 1.7

The map $t \to X(t) = V(t)X$ satisfies the 2-dimensional Bernoulli-Verhulst equation

$$X'(t) = \left[A + \frac{\lambda}{\theta} I\right] X(t) - \frac{\kappa}{\theta} \|X(t)\|_2^{\theta} X(t), \forall t \geq 0. \quad (4)$$

We refer to chapter 5 in volume I for more results.

Explicit formula for the semiflow of the Poincaré normal form

In the special case $\theta = 2$, we obtain

$$V(t)X = \begin{cases} \sqrt{U(t)\left(\|X\|_2^2\right)}\; e^{At}\left(\frac{X}{\|X\|}\right) & \text{if } X \neq 0, \\ 0 & \text{if } X = 0, \end{cases} \tag{5}$$

and from the above computation we deduce that $t \to V(t)X = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ satisfies the following system of ordinary differential equations

$$\begin{cases} x'(t) = \frac{\lambda}{2}x(t) - \omega y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)x(t), \\ y'(t) = \omega x(t) + \frac{\lambda}{2}y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)y(t). \end{cases} \tag{6}$$

The system (6) is nothing but the Poincaré normal form for Hopf bifurcation. We can say that the Poincaré normal form is a special case of the 2-dimensional Bernoulli-Verhulst equation.

Outline

Definition 2.1

We will say that $\overline{x} \in M$ is an **equilibrium** for U if

$$U(t)\overline{x} = \overline{x}, \forall t \geq 0.$$

Definition 2.2

We will say that an equilibrium $\overline{x} \in M$ is **stable** for U if for each $\varepsilon > 0$, there exists $\eta \in (0, \varepsilon]$ such that

$$U(t)B_M(\overline{x}, \eta) \subset B_M(\overline{x}, \varepsilon), \forall t \geq 0.$$

where $B_M(x, \varepsilon) := \{y \in M : d(x, y) \leq \varepsilon\}$.

We will say that $\overline{x} \in M$ is **unstable** otherwise. That is to say that, there exist $\varepsilon > 0$, and a sequence $x_n \to \overline{x}$ and $t_n \to +\infty$ such that

$$\|U(t_n)x_n - \overline{x}\| > \varepsilon.$$

Definition 2.3

We will say that $\overline{x} \in M$ is **asymptotically stable**, if $\overline{x}$ is stable for U and if there exists $\eta > 0$ such that for each $x \in B_M(\overline{x}, \eta)$

$$\lim_{t \to +\infty} U(t)x = \overline{x}.$$

We will say that $\overline{x} \in M$ is **exponentially asymptotically stable**, if $\overline{x}$ is stable for U and if in addition we can find three constants $\eta > 0$, $\alpha > 0$, $M \geq 1$ such that

$$d(U(t)x, \overline{x}) \leq Me^{-\alpha t} d(x, \overline{x}), \forall t \geq 0, \forall x \in B_M(\overline{x}, \eta).$$

Another equivalent definition of stability is the following. We will say that an equilibrium $\overline{x} \in M$ is stable if for each neighborhood V of $\overline{x}$ in M (that is to say that V contains a ball $B_M(\overline{x}, \varepsilon)$), we can find a neighborhood $W \subset V$ of $\overline{x}$ in M such that

$$U(t)W \subset V, \forall t \geq 0.$$

In this case, by considering

$$\widehat{W} := \bigcup_{t \geq 0} U(t)W,$$

we have $U(0)W = W$ (since $U(0) = I$), so we deduce that

$$W \subset \widehat{W} \subset V.$$

So $\widehat{W}$ is a neighborhood of $\overline{x}$ in M, (since W is a neighborhood of $\overline{x}$ in M), and we have

$$\begin{aligned} U(t)\widehat{W} &= U(t)\bigcup_{s\geq 0} U(s)W = \bigcup_{s\geq 0} U(t)U(s)W \\ &= \bigcup_{s\geq 0} U(s+t)W = \bigcup_{l\geq t} U(l)W. \end{aligned}$$

Thus

$$U(t)\widehat{W} \subset \widehat{W}, \forall t \geq 0.$$

Therefore we obtain the following lemma.

Lemma 2.4

The following properties are equivalent

(i) $\overline{x} \in M$ *is stable equilibrium for* U*;*

(ii) *For each neighborhood* V *of* $\overline{x}$ *in* M*, we can find neighborhood* $W \subset V$ *of* $\overline{x}$ *in* M*, such that*

$$U(t)W \subset V, \forall t \geq 0.$$

(iii) *For each neighborhood* V *of* $\overline{x}$ *in* M*, we can find a neighborhood* $W \subset V$ *of* $\overline{x}$ *in* M*, such that*

$$U(t)W \subset W, \forall t \geq 0.$$

We can observe that an equilibrium $\overline{x}$ is unstable for U if there exists $\varepsilon > 0$ such that for each $\eta > 0$ there exists $t = t(\eta) > 0$ such that

$$U(t)B_M(\overline{x}, \eta) \not\subset B_M(\overline{x}, \varepsilon).$$

Therefore by considering a sequence $\eta_n = 1/(n+1) \to 0$, we obtain the following lemma. For each integer $n \geq 0$ we can find $t_n \geq 0$ such that

$$U(t_n)B_M(\overline{x}, 1/(n+1)) \not\subset B_M(\overline{x}, \varepsilon).$$

Therefore we can find $x_n \in B_M(\overline{x}, 1/(n+1))$ with

$$d(U(t_n)x_n, \overline{x}) \geq \varepsilon.$$

Moreover, we must have $t_n \to +\infty$ because U is a continuous semiflow. Otherwise, we can find a sub-sequence $t_{n_p} \to \widehat{t}$, and by continuity of U, we deduce that

$$\lim_{n \to \infty} d(U(t_n)x_n, \overline{x}) = 0,$$

which is impossible since $\varepsilon > 0$.

Lemma 2.5

An equilibrium $\overline{x} \in M$ *is unstable for* U *if and only if there exists* $\varepsilon > 0$ *and two sequences* $x_n \to \overline{x}$ *and* $t_n \to +\infty$ *such that*

$$d\left(U(t_n)x_n, \overline{x}\right) \geq \varepsilon, \forall n \geq 0.$$

Definition 2.6

Let A be a subset of M. We will say that A is **positively invariant** by U if

$$U(t)A \subset A, \forall t \geq 0.$$

The subset A is positively invariant if and only if for each $x \in A$

$$U(t)x \in A, \forall t \geq 0.$$

We will say that A is **negatively invariant** by U if

$$U(t)A \supset A, \forall t \geq 0.$$

The subset A is negatively invariant if and only if for each $y \in A$ and each $t \geq 0$, there exists $x \in A$ such that

$$U(t)x = y.$$

We will say that A is **invariant** by U if A is both positively and negatively invariant. That is

$$U(t)A = A, \forall t \geq 0.$$

Example 2.7

Consider the map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = 4x(1-x), \forall x \in [0, 1].$$

Then T reaches its maximum on $[0, 1]$ at $x = 1/2$ and

$$T(1/2) = 1.$$

Therefore

$$T([0, 1]) = [0, 1].$$

We deduce that $[0, 1]$ is invariant by the discrete time semiflow

$$U(n) = T^n, \forall n \geq 0.$$

But clearly the map T is not one to one, so the map $x \to U(n)x$ is not invertible.

Definition 2.8

We will say that $O_+ = \{u(t)\}_{t \geq 0} \subset M$ is a **positive orbit** of U if

$$u(t) = U(t)u(0), \forall t \geq 0,$$

or, equivalently, if

$$u(t+s) = U(s)u(t), \forall t, s \geq 0.$$

We will say that $\{u(t)\}_{t \leq 0} \subset M$ is a **negative orbit** of U if

$$u(-t) = U(s)u(-t-s), \forall t, s \geq 0.$$

Finally we will say that $O = \{u(t)\}_{t \in I} \subset M$ is a **complete orbit** of U if

$$u(t) = U(s)u(t-s), \forall s \geq 0, \forall t \in I.$$

We will say that an orbit (positive, negative or complete) **passes through** $x \in M$ **at time** $t = 0$ if $u(0) = x$.

Remark 2.9

Let $x \in M$ be given. Then there exists at most one positive orbit passing through x at time $t = 0$, which is

$$u(t) := U(t)x, \forall t \geq 0.$$

But in general there is no negative orbit passing through x at time 0. Since the map $U(t)$ is not always onto for $t > 0$. Moreover when there exists a negative orbit passing through x, the negative orbit is not necessarily unique since the map $U(t)$ is not always one to one for $t > 0$. As an example of non-unique negative orbit consider Example 2.7.

Remark 2.10

If $\{u(t)\}_{t\in I} \subset M$ *is a complete orbit passing through* x *then the set*

$$O := \bigcup_{t\in I} \{u(t)\}$$

satisfies

$$U(t)O = O, \forall t \geq 0.$$

This is an example of invariant set.

Outline

Definition 3.1

Let $x \in M$. Let $\{u(t)\}_{t\geq 0} \subset M$ be a positive orbit of U passing through x at time 0. The ω-**limit set** of x is defined as

$$\omega(x) := \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}}.$$

Let $\{u(t)\}_{t\leq 0}$ be a negative orbit of U passing through x at time 0. Then the α-**limit set** of x (with respect to this negative orbit) is

$$\alpha(x) := \bigcap_{t\leq 0} \overline{\bigcup_{s\leq t} \{u(s)\}}.$$

The omega limit set satisfies

$$\begin{aligned}\omega(x) &= \left\{ y \in M : \exists \{t_n\}_{n\in\mathbb{N}} \subset I_+ \to +\infty \text{ such that } \lim_{n\to+\infty} u(t_n) = y \right\} \\ &= \{y \in M : \forall t \geq 0, \forall \varepsilon > 0, \exists s > t \text{ such that } d(u(s), y) \leq \varepsilon\}.\end{aligned}$$

Similarly the alpha limit set satisfies

$$\begin{aligned}\alpha(x) &= \left\{ y \in M : \exists \{t_n\}_{n\in\mathbb{N}} \subset I_+ \to +\infty \text{ such that } \lim_{n\to+\infty} u(-t_n) = y \right\} \\ &= \{y \in M : \forall t \leq 0, \forall \varepsilon > 0, \exists s < t \text{ such that } d(u(s), y) \leq \varepsilon\}.\end{aligned}$$

Example 3.2

We have for example

$$u(t) = \cos(t) \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = [0,1],$$
$$u(t) = t\cos(t) \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = \mathbb{R},$$
$$u(t) = t \Rightarrow \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \{u(s)\}} = \varnothing.$$

So the omega limit sets can be compact, non compact or empty.

Definition 3.3

Let (M, d) be a metric space.

(i) A subset $C \subset M$ is **compact** if and only if any sequence in C has a sub-sequence which converges in C.

(ii) A subset $C \subset M$ is **relatively compact** if and only if $\overline{C}$ (the closure of C in (M, d)) is compact.

In the general case ω-limit set is non empty only whenever the positive orbit

$$O_+ = \{u(t)\}_{t \geq 0}$$

is relatively compact (i.e. its closure is compact).

Theorem 3.4

Let $\{u(t)\}_{t\geq 0} \subset M$ *be a positive orbit passing through* $x \in M$ *at time* $t = 0$. *Assume that the closure of this positive orbit*

$$\overline{\bigcup_{t\geq 0} \{u(t)\}}$$

is compact.

Then the ω*-limit set satisfies the following properties:*

(i) $\omega(x)$ *is a non empty compact subset of* M*;*

(ii) $\omega(x)$ *is invariant by* U*;*

(iii) $\lim_{t\to+\infty} d\left(u(t), \omega(x)\right) = 0$, *where*

$$d\left(x, B\right) := \inf_{y\in B} d(x, y).$$

Remark 3.5

If M is a closed subset of $\mathbb{R}^n$, and the metric d is induced by a norm on $\mathbb{R}^n$ (i.e. $d(x,y) = \|x - y\|$), then $\overline{\bigcup_{t \geq 0} \{u(t)\}}$ is compact if and only if the positive orbit $\bigcup_{t \geq 0} \{u(t)\}$ is a bounded set.

Before proving Theorem 3.4 we need the following lemma.

Lemma 3.6

Let $B \subset M$. Then the map $x \to d(x, B)$ is Lipschitz continuous. More precisely

$$|d(x, B) - d(y, B)| \leq d(x, y), \forall x, y \in M.$$

Proof. Let $x, y \in M$ and $z \in B$. We have

$$d(x, B) \leq d(x, z) \leq d(x, y) + d(y, z).$$

Thus

$$d(x, B) \leq d(x, y) + d(y, B)$$

and the result follows. □

Proof. [of Theorem 3.4] Define

$$A_t := \overline{\bigcup_{s \geq t} \{u(s)\}} \,, \forall\, t \geq 0.$$

By assumption for each $t \geq 0$, the subset A_t is compact. Moreover the family $t \to A_t$ is decreasing, that is to say that

$$t \geq s \Rightarrow A_t \subset A_s.$$

Proof of (i). Let $\{t_n\} \to +\infty$ be an increasing sequence and $x_n \in A_{t_n}, \forall n \geq 0$. Then since A_0 is compact, and the family $t \to A_t$ is decreasing, we have

$$x_n \in A_0, \forall n \geq 0.$$

So, we can find a converging sub-sequence $\{x_n\}_{n\geq 0} \to z \in A_0$ (denoted for notational simplicity by the same index).
Moreover for each $t \geq 0$, we can find an integer $n_0 \geq 0$ such that $t_n \geq t$, $\forall n \geq n_0$, and since the family $t \to A_t$ is decreasing

$$x_n \in A_t, \forall n \geq n_0.$$

But the subset A_t is closed by construction, therefore

$$z \in A_t, \forall t \geq 0.$$

hence $\omega(x)$ is non-empty (since $z \in \omega(x)$).

Next, consider a sequence in the ω-limit set

$$x_n \in \omega(x), \forall n \geq 0.$$

Then

$$x_n \in A_t, \forall n \geq 0, \forall t \geq 0.$$

Since A_0 is compact, we can find a converging sub-sequence, and since this sub-sequence belongs to each subset A_t (which is closed), we deduce that the limit of this converging sub-sequence belongs to $\omega(x)$. Therefore $\omega(x)$ is compact.

Proof of (ii). Observe that for each $t, s \geq 0$

$$U(s)\left(\bigcup_{l \geq t} \{u(l)\}\right) = \bigcup_{l \geq t+s} \{u(l)\}. \tag{7}$$

This equality implies that

$$U(s)\left(\bigcup_{l \geq t} \{u(l)\}\right) \subset A_{t+s}$$

and since the map $x \to U(s)x$ is continuous we obtain

$$U(s)\left(A_t\right) \subset A_{t+s}. \tag{8}$$

From (7), we also have

$$U(s)\left(A_t\right) \supset \bigcup_{l \geq t+s} \{u(l)\}$$

and since by assumption A_t is compact and $x \to U(s)x$ is continuous, it follows that $U(s)\left(A_t\right)$ is compact and we obtain

$$U(s)\left(A_t\right) \supset A_{t+s}. \tag{9}$$

By combining the inclusions (8) and (9) we obtain

$$U(s)\left(A_t\right) = A_{t+s}, \forall t, s \geq 0.$$

The invariance of the ω-limit set follows from the following observation

$$U(s)\left(\omega(x)\right) = U(s)\left(\bigcap_{t\geq 0} A_t\right) = \bigcap_{t\geq 0} U(s)A_t = \bigcap_{t\geq 0} A_{t+s} = \omega(x).$$

Proof of (iii). Assume by contradiction that there exist $\varepsilon > 0$ and a sequence
$t_n \to +\infty$ such that

$$d\left(u(t_n), \omega(x)\right) \geq \varepsilon, \forall n \geq 0.$$

By compactness of A_0, we can find a converging sub-sequence (denoted with the same index) such that

$$\lim_{n \to +\infty} u(t_n) = z \in \omega(x).$$

By Lemma **3.6**, we deduce that

$$\varepsilon/2 \leq \lim_{n \to +\infty} d\left(u(t_n), \omega(x)\right) = d\left(z, \omega(x)\right) = 0,$$

which is a contradiction. □

The proof for alpha limit sets is similar to the proof of Theorem 3.4.

Theorem 3.7

Let $\{u(t)\}_{t \leq 0} \subset M$ *be a negative orbit passing through* $x \in M$ *at time* $t = 0$*. Assume that* $\overline{\bigcup_{t \leq 0} \{u(t)\}}$ *is compact. Then the* α*-limit set satisfies the following properties*

(i) $\alpha(x)$ *is a non empty compact subset of* M*;*

(ii) $\alpha(x)$ *is invariant by* U*.*

(iii) $\lim_{t \to -\infty} d\left(u(t), \alpha(x)\right) = 0.$

Outline

Proposition 4.1

Let $\{u(t)\}_{t\geq 0} \subset M$ *be a positive orbit passing through* $x \in M$ *at time* $t=0$. *Then*

$$\omega(x) = \{\overline{x}_+\} \Leftrightarrow \lim_{t\to+\infty} u(t) = \overline{x}_+.$$

Moreover in that case, $\overline{x}_+$ *must be an equilibrium of* U.

Let $\{u(t)\}_{t\leq 0} \subset M$ *be a negative orbit passing through* $x \in M$ *at time* $t=0$. *Then*

$$\alpha(x) = \{\overline{x}_-\} \Leftrightarrow \lim_{t\to-\infty} u(t) = \overline{x}_-.$$

Moreover in that case, $\overline{x}_-$ *must be an equilibrium of* U.

Proof. By using the definition of $\omega(x)$, we deduce that $\omega(x) = \{\overline{x}_+\}$ is equivalent to

$$\lim_{t \to +\infty} u(t) = \overline{x}_+.$$

Let $s \geq 0$. Since $t \to u(t)$ is a positive orbit, we have

$$u(t+s) = U(s)u(t), \forall t \geq 0,$$

and by taking the limit when $t \to +\infty$ on both side, we obtain

$$\overline{x}_+ = U(s)\overline{x}_+.$$

By using the definition of $\alpha(x)$, we deduce that $\alpha(x) = \{\overline{x}_-\}$ is equivalent to

$$\lim_{t \to -\infty} u(t) = \overline{x}_-.$$

Let $s \geq 0$. Since $t \to u(t)$ is a negative orbit, we have

$$u(t+s) = U(s)u(t), \forall t \leq -s,$$

and by taking the limit when $t \to -\infty$ on both side, we obtain

$$\overline{x}_- = U(s)\overline{x}_-.$$

□

Definition 4.2

A complete orbit $\{u(t)\}_{t\in I}$ is called a **heteroclinic orbit** if there exist $\overline{x}_{-\infty} \in M$ and $\overline{x}_{+\infty} \in M$ (with $\overline{x}_{-\infty} \neq \overline{x}_{+\infty}$) such that

$$\lim_{t\to-\infty} u(t) = \overline{x}_{-\infty} \text{ and } \lim_{t\to+\infty} u(t) = \overline{x}_{+\infty}.$$

That is equivalent to say that both the omega limit set and alpha limit set are reduced to a single point. That is

$$\alpha(x) = \{\overline{x}_{-\infty}\} \text{ and } \omega(x) = \{\overline{x}_{+\infty}\}.$$

Definition 4.3

A complete orbit $\{u(t)\}_{t\in I}$ is called a **homoclinic orbit** if this orbit is not constant and there exists $\overline{x} \in M$ such that

$$\lim_{t\to+\infty} u(t) = \overline{x} \text{ and } \lim_{t\to+\infty} u(t) = \overline{x}.$$

That is equivalent to say that this orbit is not constant and the omega limit set and the alpha limit set are reduced to the same single point. That is

$$\alpha(x) = \omega(x) = \{\overline{x}\}.$$

Outline

Let (M,d) be a complete metric space. For any subsets A, $B \subset M$, we define Hausdorff's semi-distance of B to A as

$$\delta(B,A) := \sup_{x \in B} d(x,A),$$

where

$$d(x,A) := \inf_{z \in A} d(x,z).$$

For each $\varepsilon > 0$, we define an **open** ε**-neighborhood of** A (see also Section 2 in Chapter 2 for more results) as

$$N(A,\varepsilon) := \{x \in M : d(x,A) < \varepsilon\},$$

and a **closed** ε**-neighborhood of** A as

$$\overline{N}(A,\varepsilon) := \{x \in M : d(x,A) \leq \varepsilon\}.$$

Now by using the fact that $x \to d(x, A)$ is a continuous map, we deduce that $N(A, \varepsilon)$ is an open neighborhood of A and $\overline{N}(A, \varepsilon)$ is a closed neighborhood of A. From these observations, it becomes clear that $\delta(B, A)$ is measuring the distance of B to A (and not the converse). Therefore $\delta(B, A)$ is only a semi-distance (since $\delta(B, A) = 0$ does not imply $A = B$).

Remark 5.1

The open ball $B_M(x,\varepsilon)$ *(respectively the closed ball* $\overline{B}_M(x,\varepsilon)$*) centered at* x *with radius* $\varepsilon > 0$ *satisfies*

$$B_M(x,\varepsilon) = \{y \in M : d(y,x) < \varepsilon\} = N(\{x\},\varepsilon),$$

and

$$\overline{B}_M(x,\varepsilon) = \{y \in M : d(y,x) \leq \varepsilon\} = \overline{N}(\{x\},\varepsilon).$$

Definition 5.2

The distance between two subsets $A, B \subset M$ is measured by using the so called **Hausdorff's distance** which is defined by

$$d_H(A, B) = \max(\delta(B, A), \delta(A, B)).$$

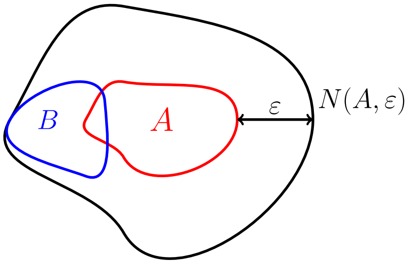


Figure: *The figure illustrates the notion of Hausdorff's semi-distance of B to A. In the figure $\varepsilon = \delta(B, A)$. The black curve corresponds to the boundary of $N(A, \varepsilon)$ a ε-neighborhood of A.*

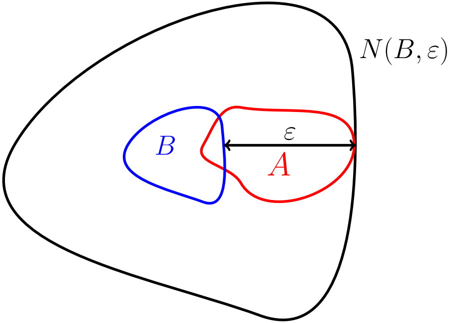


Figure: *The figure illustrates the notion of Hausdorff's semi-distance of A to B. In the figure $\varepsilon = \delta(A, B)$. The black curve corresponds to the boundary of $N(B, \varepsilon)$ a ε-neighborhood of B.*

Definition 5.3

We say that a subset $A \subset M$ **attracts** a subset $B \subset M$ for a semiflow U on M if

$$\lim_{t \to +\infty} \delta(U(t)B, A) = 0.$$

Remark 5.4

This means that for each $\varepsilon > 0$, *we can find* $t_0 = t_0(\varepsilon) > 0$ *(large enough) such that for each* $t \geq t_0$, *the subset* $U(t)B = \{U(t)x : x \in B\}$ *is included in* $N(A, \varepsilon)$.

To illustrate the notion of attraction, we first prove that the positive orbit is attracted by the omega limit sets (whenever it is exists).

Lemma 5.5

Let $\{u(t)\}_{t\geq 0} \subset M$ *be a positive orbit passing through* $x \in M$ *at time* $t = 0$. *Assume that*

$$O_+(x) := \bigcup_{t\geq 0} \{u(t)\}$$

is relatively compact. Then $\omega(x)$ *attracts* $O_+(x)$ *for* U.

Proof. We first observe that

$$U(t)O_+(x) = \bigcup_{s \geq t} \{u(s)\}, \forall t \geq 0.$$

Assume by contradiction that $\omega(x)$ does not attract $O_+(x)$ for U. Then we can find $\varepsilon > 0$ and a sequence $t_n \to +\infty$ such that

$$\delta(u(t_n), \omega(x)) \geq \varepsilon,$$

and we obtain a contradiction with Theorem **3.4**-(iii). □

Remark 5.6

If we consider the family of subsets $A_t := \bigcup_{s \geq t} \{u(s)\}$ *, then* $\omega(x)$ *attracts* $O_+(x)$ *for* U *if, and only if,*

$$\lim_{t \to +\infty} \delta\left(A_t, \omega(x)\right) = 0.$$

Let $\{u(t)\}_{t \leq 0} \subset M$ *be a negative orbit passing through* $x \in M$ *at time* $t = 0$*. We can adapt this last notion of attractivity for the alpha limit sets, by saying that if*

$$O_-(x) = \bigcup_{t \leq 0} \{u(t)\}$$

is relatively compact, then

$$\lim_{t \to -\infty} \delta\left(B_t, \alpha(x)\right) = 0,$$

where

$$B_t = \bigcup_{s \leq t} \{u(s)\}.$$

The Hausdorff semi-distance measure the distance of B to A. Therefore if $B \subset A$ then $\delta(B, A) = 0$. Moreover, if $\varepsilon > 0$ then

$$\delta(B, A) = \varepsilon \Rightarrow B \subset \overline{N}(A, \varepsilon).$$

This means that we can find a sequence $x \in B$ such that

$$d(x, A) \leq \varepsilon,$$

and there exists a sequence $x_n \in B$ such that

$$\lim_{n \to \infty} d(x_n, A) = \varepsilon.$$

The Hausdorff distance can also be defined as follows

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon)\}.$$

Actually the Hausdorff distance is not a real distance, because we only have

$$d_H(A,B) = 0 \Leftrightarrow \overline{A} = \overline{B}.$$

But the Hausdorff is a real distance if we restrict to the closed subsets.

Proposition 5.7

Let (M, d) be a metric space. Then the set of closed subsets of M is a metric space endowed with the Hausdorff distance.

Proof. It remains to prove the triangle inequality for the Hausdorff distance. Assume that

$$d_H(A,B) < \varepsilon \text{ and } d_H(B,C) < \varepsilon'.$$

From the proof of Lemma 3.6, we know that for each $x \in C$, $z \in B$,

$$d(x,A) \leq d(x,z) + d(z,A).$$

Let $x \in C$ be fixed. Since

$$d_H(B,C) < \varepsilon' \Rightarrow C \subset N(B,\varepsilon'),$$

we can choose $z \in B$ such that $d(z,x) \leq \varepsilon'$, and

$$d_H(A,B) < \varepsilon \Rightarrow B \subset N(A,\varepsilon).$$

We deduce that

$$d(x,A) \leq \varepsilon' + \varepsilon, \forall x \in C.$$

By taking the supremum in x, we obtain

$$\delta(C, A) \leq d_H\left(A, B\right) + d_H\left(B, C\right),$$

and by symmetry of the problem, we obtain

$$\delta(A, C) \leq d_H\left(A, B\right) + d_H\left(B, C\right).$$

We conclude that

$$d_H(C, A) \leq d_H\left(A, B\right) + d_H\left(B, C\right).$$

□

Outline

Internally chain transitive ω-limit sets and α-limit sets

The fundamental property of omega limit sets is the fact that each couple of points in a omega limit set can be almost connected by an orbit staying in a small neighborhood of the omega limit set.

The following result is making this statement more precise.

Lemma 6.1

Assume that the positive orbit $O_+(x)$ (respectively the negative orbit $O_-(x)$) is relatively compact. For each $a, b \in \omega(x)$ (respectively $a, b \in \alpha(x)$) and each $\varepsilon > 0$ we can find $a_\varepsilon, b_\varepsilon \in M$ and $t_\varepsilon > 0$ such that

$$d(a, a_\varepsilon) \leq \varepsilon,\; d(b, b_\varepsilon) \leq \varepsilon,$$

$$U(t_\varepsilon)a_\varepsilon = b_\varepsilon,$$

and

$$U(t)a_\varepsilon \in N(\omega(x), \varepsilon), \forall t \in [0, t_\varepsilon],$$

which is equivalent to

$$d\left(U(t)a_\varepsilon, \omega\left(x\right)\right) \leq \varepsilon, \forall t \in [0, t_\varepsilon].$$

Proof. We have $\lim_{t\to+\infty}\delta(U(t)x,\omega(x))=0$. So we can find $t_0>0$ such that

$$\delta(U(t)x,\omega(x))\leq\varepsilon,\ \forall t\geq t_0.$$

Since a belongs to $\omega(x)$ we can find $t_1>t_0$ such that $d(U(t_1)x,a)\leq\varepsilon$. Set $a_\varepsilon=U(t_1)x$. Then since b belongs to $\omega(x)$ we can find $t_2>t_1>t_0$ such that $d(U(t_2)x,b)\leq\varepsilon$. Set $b_\varepsilon=U(t_2)x$ and the result follows. □

Definition 6.2

Let A be a subset of M. We say that $a \in A$ **is chained to** $b \in A$ **in** A, if for each $t^\star > 0$, for each $\varepsilon > 0$, and each $\eta > 0$, there exist $\tau \in [t^\star, t^\star + \eta] \cap I$, and $x_1, x_2, ..., x_m \in A$ (with $m \geq 2$) such that

$$x_1 = a, \; x_m = b, \; \text{and } d\left(U(\tau)x_i, x_{i+1}\right) \leq \varepsilon, \forall i = 1, ..., m-1.$$

We will say that A is **internally chain transitive**, if for each $a, b \in A$, a is chained to b in A.

Theorem 6.3

Let $\{U(t)\}_{t\in I}$ be a continuous semiflow on (M, d). Then the omega limit set (respectively alpha limit set) of a relatively compact positive orbit (respectively negative orbit) is internally chain transitive.

Lemma 6.4

Let $\{U(t)\}_{t\in I}$ be a continuous semiflow on (M,d). Let C be a compact subset of M. Then for each $\varepsilon>0$, and each $t^\star\in I$, there exists $\delta>0$, such that $s\in[t^\star,t^\star+\delta]$, $u,v\in M$,

$$d(u,C)\leq\delta, d(v,C)\leq\delta, \text{ and } d(u,v)\leq\delta\Rightarrow d(U(s)u,U(s)v)\leq\varepsilon.$$

Proof. Assume by contradiction that there exists $\varepsilon > 0$, and two sequences $u_n \in M$, and $v_n \in M$, and $s_n \in [t^\star, t^\star + 1/n]$, such that for each integer $n > 0$,

$$d(u_n, C) \leq 1/n, d(v_n, C) \leq 1/n, \text{ and } d(u_n, v_n) \leq 1/n,$$

and

$$d(U(s_n)u_n, U(s_n)v_n) \geq \varepsilon. \tag{10}$$

By definition of distance $d(x, C)$, we can find two sequences $u_n^C \in C$ and $v_n^C \in C$ such that

$$d\left(u_n, u_n^C\right) \leq 2/n \text{ and } d\left(v_n, v_n^C\right) \leq 2/n.$$

By using the triangle inequality we obtain

$$d\left(u_n^C, v_n^C\right) \leq d\left(u_n^C, u_n\right) + d(u_n, v_n) + d\left(v_n, v_n^C\right) \leq 5/n.$$

Now, by using the fact that C is compact, we can find some converging sub-sequences (denoted with the same index), such that

$$u_n^C \to w, \text{ and } v_n^C \to w, \text{ as } n \to +\infty.$$

By using the continuity of $(t, x) \to U(t)x$, we deduce from (10) that

$$0 = d\left(U(t^\star)w, U(t^\star)w\right) \geq \varepsilon > 0.$$

A contradiction. The proof is completed. □

Proof. [of Theorem 6.3] Let us prove the result for omega limit set (the proof for alpha limit set is similar). Let $x \in M$, and assume that $\overline{\gamma^+(x)}$ is compact. Then $\omega(x)$ is nonempty, compact, invariant and

$$\lim_{t \to +\infty} d\left(U(t)x, \omega(x)\right) = 0.$$

Let $t^\star \in I$ (with $t^\star > 0$), $\varepsilon > 0$, and $\eta > 0$ be fixed. By continuity of U, and compactness of $\omega(x)$, we can find $\delta \in (0, \varepsilon/3) \cap (0, \eta)$, with the following property: If $s \in [t^\star, t^\star + \delta]$, $u, v \in M$,

$$d\left(u, \omega(x)\right) \leq \delta, d\left(v, \omega(x)\right) \leq \delta, \text{ and } d\left(u, v\right) \leq \delta \Rightarrow d\left(U(s)u, U(s)v\right) \leq \varepsilon/3.$$

Since

$$\lim_{t \to +\infty} d\left(U(t)x, \omega(x)\right) = 0,$$

we can find $t_1 \in I$, such that

$$d\left(U(t)x, \omega(x)\right) < \delta, \forall t \geq t_1.$$

Let $a, b \in \omega(x)$. Then we can find $t_a \geq t_1$, such that

$$d\left(U(t_a)x, a\right) < \delta.$$

Let $k \in \mathbb{N}$, $k \geq 2$, such that $t^\star/k \leq \delta$. Then we can find $t_b \geq t_a + kt^\star$, such that

$$d\left(U(t_b)x, b\right) < \delta.$$

However there exists $m \geq k+1$, such that $(m-1)\,t^\star \leq t_b - t_a < mt^\star$. We set $\tau = \dfrac{t_b - t_a}{m-1}$. Then by construction of $t^\star$, we have $\tau \in I$, and

$$\tau \in \left[t^\star, \left(1 + \frac{1}{m-1}\right) t^\star\right] \subset \left[t^\star, \left(1 + \frac{1}{k}\right) t^\star\right] \subset \left[t^\star, t^\star + \delta\right].$$

We set

$$y_1 = a, y_2 = U(\tau) U(t_a)x, \ldots, y_{m-1} = U((m-2)\tau) U(t_a)x, y_m = b.$$

Then

$$d(U(\tau) y_1, y_2) = d(U(\tau) a, U(\tau) U(t_a)x)$$

and since

$$a \in \omega(x), d(U(t_a)x, \omega(x)) < \delta, \text{ and } d(U(t_a)x, a) < \delta,$$

and $\tau \in [t, t+\delta]$, we deduce that

$$d(U(\tau) y_1, y_2) = d(U(\tau) a, U(\tau) U(t_a)x) \leq \varepsilon/3,$$

$$U(\tau) y_j = y_{j+1}, \forall j = 2, ..., m-2,$$

and

$$d(U(\tau) y_{m-1}, y_m) = d(U(t_b)x, b) \leq \delta \leq \varepsilon/3.$$

Now for each $j = 2, ..., m-1$, there exists $x_j \in \omega(x)$, such that $d(x_j, y_j) \leq \delta$. By setting $x_1 = a$ and $x_m = b$, we obtain for $j = 1, ..., m-1$,

$$\begin{array}{rcl} d\left(U\left(\tau\right)x_j, x_{j+1}\right) & \leq & d\left(U\left(\tau\right)x_j, U\left(\tau\right)y_j\right) \\ & & +d\left(U\left(\tau\right)y_j, y_{j+1}\right) \\ & & +d\left(y_{j+1}, x_{j+1}\right) \\ & \leq & \varepsilon/3 + \varepsilon/3 + \delta \leq \varepsilon. \end{array}$$

The proof is completed. $\square$

Invariantly connected ω-limit and α-limit Sets

Definition 6.5

A compact invariant set A is said to be **invariantly connected** if it is not the union of two nonempty disjoint compact invariant subsets. That is to say that if $A_1 \neq \emptyset$, and $A_2 \neq \emptyset$ are non empty and compact subsets satisfying

$$A = A_1 \cup A_2, \text{ with } A_1 \cap A_2 = \emptyset.$$

Then either A_1 or A_2 are not invariant by U. That is,

$$\text{either } U(t^\star) A_1 \neq A_1, \text{ or } U(t^\star) A_2 \neq A_2.$$

for some $t^\star > 0$.

Theorem 6.6

Let $\{U(t)\}_{t\in I}$ be a continuous semiflow on (M, d). Then the omega limit set (respectively alpha limit set) of a relatively compact positive orbit (respectively negative orbit) is invariantly connected.

Theorem 6.6 follows from Theorem 6.3 and the following lemma.

Lemma 6.7

Let A be a compact subset of M which is invariant by U. If A is internally chain transitive then A is also invariantly connected.

Proof. We can prove the result by contradiction. Assume that A is the union of two disjoint closed invariant sets $A = A_1 \cup A_2$. We get a contradiction because the subsets A_1 and A_2 are invariant. So if we fix $\tau > 0$ then when $d(x, A_1) \leq \varepsilon$, then $d(U(\tau)x, A_2) > 2\varepsilon$ whenever $\varepsilon > 0$ is small enough. So a point of A_1 can not be chained to a point of A_2. □

Connected ω-limit and α-limit sets

Definition 6.8

Let (M, d) be a metric space. Let $C \subset M$ be a subset of M. We will say that a pair of of non empty subsets $A \subset C$ and $B \subset C$ is a **partition of** C if

$$A \cup B = C \text{ and } A \cap B = \emptyset.$$

Definition 6.9

Let (M, d) be a complete metric space. A subset C of M is said to be **connected** if there exists no partition of C in two subsets A and B which are both open subsets for the topology of (C, d). We will say that C is **disconnected** whenever C is not connected.

Remark 6.10

Recall that a subset is closed in (C, d) if its complementary set in C is open in (C, d). Therefore, in the above definition, it is equivalent to say that both subsets A and B are also closed in (C, d).

Assume that C is not connected. Then we can there exists a partition of C in two subsets A and B which are both open subsets for the topology of (C, d). Recall that a subset A is open in (C, d), if and only if for each $x \in A$, there exists $\varepsilon_x > 0$ such that

$$B_C(x, \varepsilon_x) = \{y \in C : d(x, y) \leq \varepsilon_x)\} \subset A,$$

where $B_C(x, \varepsilon_x)$ is the ball of center x and radius ε_x (in C).
Remark that

$$B_C(x, \varepsilon_x) = B_M(x, \varepsilon_x) \cap C,$$

where

$$B_M(x, \varepsilon_x) = \{y \in M : d(x, y) \leq \varepsilon_x\}.$$

So we must have

$$B_M(x, \varepsilon_x) \cap B = \emptyset.$$

Now we can define

$$U = \bigcup_{x \in A} B_M(x, \varepsilon_x),$$

and

$$V = \bigcup_{x \in B} B_M(x, \varepsilon_x).$$

where for each $x \in B$, ε_x is chosen small enough to guaranty

$$B_M(x, \varepsilon_x) \cap A = \emptyset.$$

We observe that U and V satisfy

$$U \cap C = A \text{ and } V \cap C = B$$

and

$$U \cap V = \emptyset.$$

Therefore we obtain the following proposition.

Proposition 6.11

Let (M, d) be a complete metric space. A subset C is disconnected if there exist two open subsets $U \subset M$ and $V \subset M$, such that

(i) $C \subset U \cup V$;

(ii) $C \cap U \neq \emptyset$

(iii) $C \cap V \neq \emptyset$

(iv) $U \cap V = \emptyset$.

Definition 6.12

An **interval** in $\mathbb{R}$ is a subset I such that

$$a < c < b \text{ and } a, b \in I \Rightarrow c \in I.$$

Remark 6.13

This is definition can be extended to $\mathbb{R}^n$ endowed with some partial order $\leq$.

Theorem 6.14

A connected set of real numbers is an interval.

Remark 6.15

The converse is also true. An interval of real numbers is a connected set.

Proof. Let C be a connected set in $\mathbb{R}$. Assume by contradiction that C is not an interval. Then we can find $a, b \in C$ and $c \notin C$ with $a < c < b$. The subsets $U = (-\infty, c)$ and $V = (c, \infty)$ are both open in $\mathbb{R}$, $C \subset U \cap V$ and

$$a \in C \cap U \neq \emptyset, \text{ and } b \in C \cap V \neq \emptyset.$$

Therefore $A = C \cap U$ and $B = C \cap V$ are both open in C. We deduce that C is disconnected. □

Theorem 6.16

Let $T : M \to \widetilde{M}$ be a continuous map from a metric space (M, d) to a metric space $(\widetilde{M}, \widetilde{d})$. Then M is connected implies that $T(M)$ is connected.

Proof. Assume by contradiction that $T(M)$ is not connected. Then there exists $\widetilde{A}$ and $\widetilde{B}$ two open subset of $(T(M), \widetilde{d})$

$$\widetilde{A} \cup \widetilde{B} = T(M) \text{ and } \widetilde{A} \cap \widetilde{B} = \emptyset.$$

By continuity of T, we deduce that $A = T^{-1}\left(\widetilde{A}\right)$ and $B = T^{-1}\left(\widetilde{B}\right)$ are open in (M, d) and

$$A \cup B = M \text{ and } A \cap B = \emptyset.$$

This contradicts the fact that M is connected. The proof is completed. □

As a consequence of the Proposition 6.11, Theorem 6.14, and Theorem 6.16, we obtain the following results that will be useful in the applications.

Theorem 6.17

Let $M \subset X$ be a subset of Banach space $(X, \|.\|)$. Then we have the following properties:

(i) *M is convex $\Rightarrow$ M is connected.*

(ii) *If M is connected and $x^* : X \to \mathbb{R}$ is a bounded linear map then*

$$I = \{x^*(x) : x \in M\} \subset \mathbb{R}$$

is an interval in $\mathbb{R}$.

As a consequence of the above theorem we have for example, a connected set in $\mathbb{R}^n$ becomes an interval when it is projected onto the axes.

Example of not connected ω-limit and α-limit sets: The ω-limit set (respectively the α-limit set) of a relatively compact positive orbit (respectively negative orbit) generated by a discrete time semiflow is not connected in general. Indeed, assume that $T : M \to M$ has a 2-periodic orbit

$$T(a) = b \text{ and } T(b) = a,$$

with

$$a \neq b.$$

If we define the complete orbit

$$u(n) = \begin{cases} a, & \text{if } n = 2k, \text{ for some integer } k \in \mathbb{Z}, \\ b, & \text{if } n = 2k+1, \text{ for some integer } k \in \mathbb{Z}, \end{cases}$$

Then the ω-limit set of the solution starting from a or b is

$$\omega(a) = \omega(b) = \{a, b\}\,,$$

and

$$\alpha(a) = \alpha(b) = \{a, b\}\,.$$

This provides an example of disconnected ω-limit and α-limit sets.

The case of continuous time semiflow is different.

Theorem 6.18

Let (M, d) be a complete metric space. The ω-limit set of a relatively compact orbit generated by a continuous time semiflow $\{U(t)\}_{t \in \mathbb{R}_+}$ is connected.

Proof. [of Theorem 6.18] Assume that $\omega(x)$ is disconnected. Then there would be disjoint open subsets U and V of M such that $U \cap \omega(x)$ and $V \cap \omega(x)$ are nonempty and $\omega(x) \subset U \cup V$. Let $a \in U \cap \omega(x)$ and $b \in V \cap \omega(x)$. Then we can have a sequence $t_1, t_2, \ldots t_k \to \infty$ and a sequence $s_1, s_2, \ldots s_k \to \infty$ (with $t_k \leq s_k$) such that $U(t_k)x \in U \to a$, and $U(s_k)x \in V \to b$. But $\{U(t)x : t \in [t_k, s_k]\}$ is a connected curve going from a point in U to a point in V. Therefore, there must be able to find $\tau_k \in (t_k, s_k)$ such that $U(\tau_k)x \in M \setminus (U \cup V)$. But the sequence $k \to U(\tau_k)x$ is relatively compact, so up to a sub-sequence (denoted with the same index) we can assume that

$$U(\tau_k)x \to c.$$

But by construction $M \setminus (U \cup V)$ is a closed subset, and it follows that

$$c \in M \setminus (U \cup V) \text{ and } c \in \omega(x).$$

We obtain a contradiction since $\omega(x) \subset (U \cup V)$. □

Outline

Definition 7.1

A continuous semiflow $\{U(t)\}_{t \in I}$ on a metric space (M, d) is said to be **point dissipative (respectively compact dissipative, bounded dissipative)** if there exists a bounded set $B_0 \subset M$ attracting the points (respectively the compact subsets, the bounded subsets).

Definition 7.2

The notion of dissipative semiflow can be expressed by using one of the two following equivalent properties:

(i) There exists $B_0 \subset M$ a bounded subset such that

$$\lim_{t \to +\infty} \delta(U(t)B, B_0) = 0,$$

whenever B is a point (respectively a compact subset, a bounded subset).

(ii) For each $\varepsilon > 0$, and each subset $B \subset M$ that is a point (respectively a compact subset, a bounded subset), there exists $t_0 = t_0(\varepsilon, B) > 0$ such that

$$U(t)B \subset \overline{N}(B_0, \varepsilon), \forall t \geq t_0,$$

where $\overline{N}(B_0, \varepsilon)$ is a closed ε-neighborhood of B_0 defined by

$$\overline{N}(B_0, \varepsilon) := \{x \in M : d(x, B_0) \leq \varepsilon\}.$$

Definition 7.3

A subset $B_0 \subset M$ is called **point absorbing, compact absorbing, bounded absorbing** if for each subset $B \subset M$ which is respectively a single point, a compact subset, a bounded subset, there exists $t_0 = t_0(B) \geq 0$ such that

$$U(t)B \subset B_0, \forall t \geq t_0.$$

A bounded absorbing subset is called **absorbing subset**.

Outline

Logistic equations: heteroclinic orbit

Consider the scalar logistic equation

$$N'(t) = N(t) - N(t)^2, \forall t \geq 0 \text{ and } N(0) = x. \tag{11}$$

The solution is explicitly given by

$$N(t) = \frac{e^t x}{1 + \int_0^t e^l x dl}, \forall t \geq 0.$$

Define the maximal backward time of existence

$$\tau^-(x) = \inf \left\{ t < 0 : 1 - \int_s^0 e^l x dl > 0, \, \forall s \in [t, 0] \right\}.$$

Then

$$\int_{-\infty}^{0} e^{l} dl = 1,$$

therefore

$$\tau^{-}(x) = -\infty, \forall x \in [0, 1]$$

and the solution is global:

$$N(t) = \frac{e^{t} x}{1 + \int_0^t e^{\sigma} x d\sigma}, \forall t \in \mathbb{R}. \quad (12)$$

It is clear that 0 and 1 are equilibrium solutions. Moreover for each $x \in (0, 1)$, the solution (12) is a heteroclinic orbit and

$$\lim_{t \to -\infty} N(t) = 0 \text{ and } \lim_{t \to +\infty} N(t) = 1.$$

That is equivalent to say that

$$\alpha(x) = \{0\} \text{ and } \omega(x) = \{1\}.$$

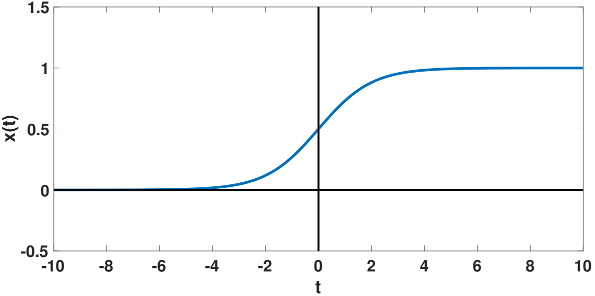


Figure: *In this figure the blue curve represents the heteroclinic orbit* (6) *whenever* $x = 0.5$.

Dissipation: Consider the map

$$V(N) = (N-1)^2.$$

We have

$$V'(N) = 2(N-1)N' = 2(N-1)N(1-N) = -2N(1-N)^2 \leq 0.$$

Theorem 8.1

The semiflow

$$U(t)x = \frac{e^t x}{1 + \int_0^t e^\sigma x d\sigma}, \forall t \in \mathbb{R}$$

of scalar logistic equation (11) *is bounded dissipative on* $\mathbb{R}_+$. *More precisely, each such that* $B_0 = [0, N_1]$ *(with* $N_1 > 2$*) is bounded absorbing set. That is to say that for each bounded set* $B \subset [0, +\infty)$, *there exists* $t_0 = t_0(B)$, *such that*

$$U(t)B \subset B_0, \forall t \geq t_0.$$

Proof. We first observe that

$$\sup_{x\in[0,N_1]} V(x) = (N_1-1)^2, \forall N_1 > 2,$$

therefore

$$\left\{x \geq 0 : V(x) \leq (N_1-1)^2\right\} = [0,N_1], \forall N_1 > 2.$$

We choose $B_0 = \{x \geq 0 : V(x) \leq (N_1-1)^2\}$ for some $N_1 > 2$. Next, we observe that B_0 is positively invariant by U. Indeed, we have for each $x \in B_0$

$$V(U(t)x) \leq V(x) \leq \sup_{x\in B_0} V(x) = (N_1-1)^2.$$

Therefore

$$\sup_{x\in B_0} V(U(t)x) \leq (N_1-1)^2 \Rightarrow U(t)B_0 \subset B_0, \forall t \geq 0.$$

Assume by contradiction, that there exists a bounded set $B \subset [0, \infty)$, such that

$$\sup_{x \in B} V(U(t)x) \geq (N_1 - 1)^2, \forall t \geq 0.$$

Then we can construct a sequence $x_n \in B$ such that for each integer $n \geq 0$,

$$V(U(n)x_n) \geq (N_1 - 1)^2, \forall n \in \mathbb{N}.$$

Since the sequence x_n is bounded, we can find a sub-sequence (denoted with the same index) $x_n \to x_\infty$ and by continuity of U we deduce that

$$V(U(t)x_\infty) \geq (N_1 - 1)^2, \forall t \geq 0. \tag{13}$$

But since $N_1 \geq 2$, we must have

$$U(t)x_\infty > N_1, \forall t \geq 0. \tag{14}$$

Finally observe that

$$V'(U(t)x_\infty) = -U(t)x_\infty \times (1 - U(t)x_\infty)^2 = -U(t)x_\infty \times V\left(U(t)x_\infty\right),$$

so by integrating this formula and by using (14), we obtain

$$V\left(U(t)x_\infty\right) = e^{-\int_0^t U(\sigma)x_\infty d\sigma} V\left(U(t)x_\infty\right) \leq e^{-tN_1} V(x) \to 0, \text{ as } t \to +\infty.$$

We obtain a contradiction with (13). $\square$

Theorem 8.2

Consider the semiflow $U(t)$ *of scalar logistic equation* (11) *restricted to* $(0, \infty)$*. Then the subset* $B_0 = [N_0, N_1]$ *(with* $0 < N_0 < 1 < N_1$*) is a compact absorbing set, but* B_0 *is not a bounded absorbing set. Moreover precisely*

(i) *The subset* $(0, 1]$ *attracts all the bounded subsets in* $(0, +\infty)$*;*

(ii) *The subset* $(0, 1]$ *is invariant by* U*. That is*

$$U(t)(0, 1] = (0, 1], \forall t \geq 0;$$

(iii) *The subset* $(0, 1]$ *is not compact in* $(0, +\infty)$*;*

(iv) *The subset* $(0, N_1]$ *is a bounded absorbing set in* $(0, +\infty)$*.*

Poincaré normal form: periodic orbit

Consider the Poincaré normal form

$$\begin{cases} x'(t) = \frac{\lambda}{2}x(t) - \omega y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)x(t) \\ y'(t) = \omega x(t) + \frac{\lambda}{2}y(t) - \frac{\kappa}{2}\left(x(t)^2 + y(t)^2\right)y(t) \end{cases} \tag{15}$$

From Section 1, we know that the semiflow generated by (15) is defined by

$$\begin{aligned} V(t)\begin{pmatrix} x \\ y \end{pmatrix} &= \sqrt{\frac{e^{\lambda t}\left(x^2+y^2\right)}{1+\kappa\int_0^t e^{\lambda s}\left(x^2+y^2\right)ds}} \times \frac{1}{\sqrt{x^2+y^2}} \\ &\times \begin{pmatrix} \cos(\omega t)x - \sin(\omega t)y \\ \sin(\omega t)x + \cos(\omega t)y \end{pmatrix}, \end{aligned} \tag{16}$$

whenever $(x, y) \neq (0, 0)$ and

$$V(t)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 8.3

Derive the above formula by using the following changes of variables. Consider

$$r(t)^2 = x^2(t) + y(t)^2$$

and prove that

$$(r(t)^2)' = \lambda r(t)^2 - \kappa r(t)^4.$$

Consider

$$X(t) = \frac{x(t)}{\sqrt{x^2(t) + y^2(t)}}$$

and

$$Y(t) = \frac{y(t)}{\sqrt{x^2(t) + y^2(t)}}$$

whenever $(x, y) \neq (0, 0)$. *Prove that*

$$X' = -\omega Y, \text{ and } Y' = \omega X.$$

Figures 4, 5 and 6 illustrate the behavior of the solutions of the Poincaré normal form.

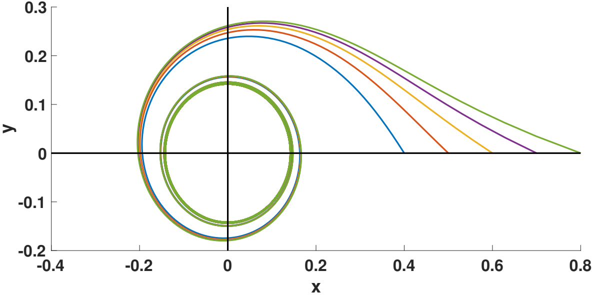


Figure: *We plot some solutions of* (15) *in the phase plane* $(x(t), y(t))$ *whenever* $\lambda = 0.02$, $\omega = 0.1$ *and* $\kappa = 1$. *We choose several initial values where* $x = 0.2, 0.3, \ldots, 0.8$ *and* $y = 0$. *One may observe that the omega limit set of these solutions is the central circle, while the alpha limit set is empty since the norm of the solutions eventually blowup when the time goes to* $-\infty$.

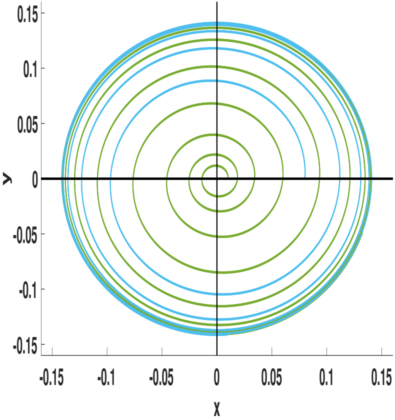
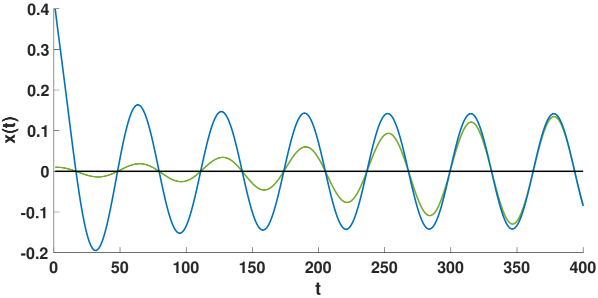
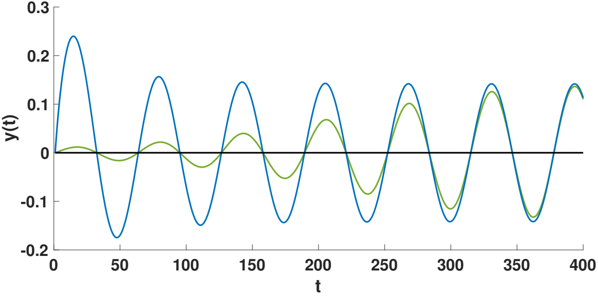


Figure: *We plot some solutions of* (15) *in the phase plane* $(x(t), y(t))$ *whenever* $\lambda = 0.02$, $\omega = 0.1$ *and* $\kappa = 1$. *We choose several initial values, where* $x = 0.01, 0.08$ *and* $y = 0$. *The solutions are part of a complete orbits jointing the trivial equilibrium* 0 *to the circular periodic orbit.*



0.4
0.3
0.2
0.1
0
-0.1
-0.2
x(t)
0
50
100
150
200
250
300
350
400
t



0.3
0.2
0.1
0
-0.1
-0.2
y(t)
0
50
100
150
200
250
300
350
400
t

Homoclinic orbit for a second order logistic equation

Consider the equation

$$x''(t) = -x(t)(1 - x(t)), \forall t \geq 0, \text{ with } x(0) = x_0 \text{ and } x'(0) = x_0'. \quad (17)$$

This equation can be regarded as the following system

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t)(1 - x(t)) \end{cases} \quad (18)$$

and the equilibria of (18) are

$$(x, y) = (0, 0) \text{ and } (x, y) = (1, 0).$$

The system (18) is a Hamiltonian system which corresponds to a special case of the Bogdanov-Takens normal form (see [2] and [4]). The terminology Hamiltonian means that we have

$$x(1-x)x' = -yy'.$$

Therefore we obtain the following conservation property along the trajectories

$$\frac{x(t)^2}{2} - \frac{x(t)^3}{3} = -\frac{y(t)^2}{2} + h,$$

with a constant of integration $h \in \mathbb{R}$.

To study the trajectories passing through the points $y_0 = 0$ and x_0 between 0 and 1, we obtain the following condition for h

$$x_0^2 \left(\frac{1}{2} - \frac{x_0}{3} \right) = h.$$

The function $x \to \frac{x^2}{2} - \frac{x^3}{3}$ is increasing between 0 and 1. Therefore h varies between 0 (for $x = 0$) and $\frac{1}{6}$ (for $x = 1$).
We deduce that the orbit passing through the point $(x, 0)$ (with $x \in [0, 1]$) satisfies

$$y = \pm \sqrt{2 \left[h - \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \right]},$$

where x takes some appropriate values in order for the quantity below the square root to be positive.

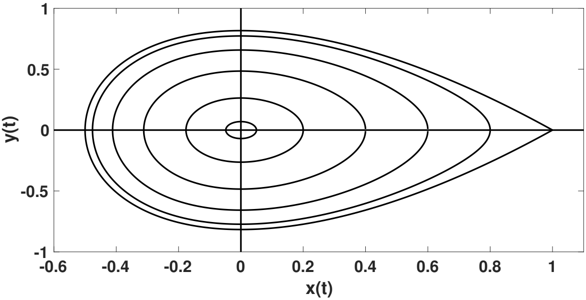


Figure: *We plot the complete orbits* $(x(t), y(t))$ *passing respectively through* $x = 0.05, 0.2, 0.4, 0.6, 0.8$ *and* $y = 0$*. The last orbit is a homoclinic orbit passing through* $(x, y) = (-0.5, 0)$ *and the alpha limit set as well as the omega limit set is the equilibrium* $(x, y) = (1, 0)$*.*

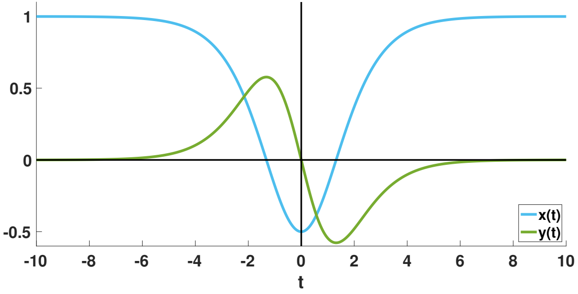


Figure: *We plot the homoclinic orbits* $t \to (x(t), y(t))$ *which is the solution passing through* $(x, y) = (-0.5, 0)$*. We observe numerically the convergence to* $(1, 0)$ *when the time* t *goes to* $\pm\infty$*.*

Beverton and Holt discrete time model

The model of Beverton and Holt [1] was introduced in the context of fisheries in 1957. This model is the following

$$N(t+1) = \frac{\beta N(t)}{1+\alpha N(t)}, \forall t \geq 0, \text{ with } N(0) = N_0 \geq 0, \tag{19}$$

where $N(t)$ is the number of individuals, $\beta > 0$ is the growth rate of the population and the term $1/(1+\alpha N(t))$ (with $\alpha \geq 0$) describes the competition for food, cannibalism effect (indeed the adult fish eat the larvea when they reproduce) etc... The competition occurs between individuals of the same species. Therefore this effect is usually called *intra-specific competition*.

Semiflow: Recall that the semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is defined by

$$U(t)\left(N_0\right) = T^t\left(N_0\right), \forall t \geq 0, \forall N_0 \geq 0,$$

where $T\left(N_0\right) = \frac{\beta N(t)}{1+\alpha N(t)}$ and T^t is defined by induction as follows

$$T^{t+1}\left(N_0\right) = T\left(T^t\left(N_0\right)\right) = T^t\left(T\left(N_0\right)\right), \forall t \geq 0,$$

and

$$T^0\left(N_0\right) = N_0.$$

Equilibria: It is readily checked that 0 is always an equilibrium of (19). The non-zero equilibrum satisfies

$$\frac{\beta}{1+\alpha\overline{N}} = 1 \Leftrightarrow \overline{N} = \frac{\beta-1}{\alpha}.$$

Dissipation property: We follow the construction of the Liapunov function by Fisher and Goh [3]. Define

$$V(N(t)) = |N(t) - \overline{N}|.$$

Then

$$\begin{aligned} V(N(t+1)) &= \left|N(t+1) - \overline{N}\right| = \left|\frac{\beta N(t)}{1+\alpha N(t)} - \frac{\beta - 1}{\alpha}\right| \\ &= \left|\frac{\alpha\beta N(t) - (\beta - 1)(1+\alpha N(t))}{\alpha(1+\alpha N(t))}\right| \\ &= \left|\frac{\alpha N(t) - (\beta - 1)}{\alpha(1+\alpha N(t))}\right|, \end{aligned}$$

and we obtain

$$V(N(t+1)) = \frac{1}{(1+\alpha N(t))} V(N(t)). \tag{20}$$

We deduce that if we define $\{U(t)\}_{t\in\mathbb{N}}$ the discrete time semiflow generated by (19) for each $t \in \mathbb{N}$,

$$\begin{cases} V(U(t)N_0) < V(N_0), & \text{if } N_0 > 0, \text{ and } N_0 \neq \overline{N}, \\ V(U(t)N_0) = V(N_0), & \text{if } N_0 = 0, \text{ or } N_0 = \overline{N}. \end{cases} \tag{21}$$

Theorem 8.4

The discrete time semiflow generated by (19) *is bounded dissipative on* $[0, \infty)$. *More precisely, for each* $\gamma_0 > \overline{N}$ *the subset*

$$B_{\gamma_0} = \{x \geq 0 : V(x) \leq \gamma_0\}$$

is a bounded absorbing subset.

Proof. We have for each $N_0 > 0$, with $V(N_0) \geq \overline{N}$,

$$V(U(t)N_0) < V(N_0), \forall t \geq 0,$$

the result follows. □

Remark 8.5

One may observe that this model can be derived from the logistic equation (11) *by using the following semi-implicit scheme*

$$\frac{N(t+\Delta t)-N(t)}{\Delta t}=N(t)(\lambda-\chi N(t+\Delta t))$$

which is equivalent to

$$N(t+\Delta t)=\frac{(1+\Delta t\lambda)N(t)}{1+\Delta t\chi N(t)}.$$

The behavior of the system (19) *is analogous to the behavior of the solution of the logistic equation* (11) *(see Figure 9).*

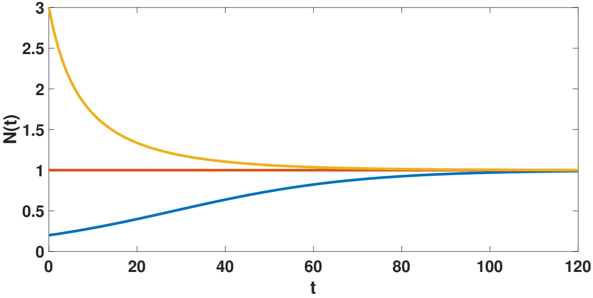


Figure: *In this figure we plot three solutions of equation* (19) *with* $\alpha = 0.05$ *small and* $\lambda = 1 + \alpha$ *(similarly to the above approximation of the logistic equation). Then we choose three initial value* $N_0 = 0.2$ *(blue curve),* $N_0 = 1$ *(green curve) and* $N_0 = 3$ *(red curve). The green curve corresponds to the equilbrium* $N = 1$*. The second member of* (19) *being (concave) monotone increasing no other behavior can be observed whenever* $\lambda > 1$*.*

Ricker model: chaotic behavior

The first population dynamics model introduced in the context of fisheries to describe the reproduce of salmons in Canada was written by Ricker [8, 9] in 1954. The model is the following

$$N(t+1) = \beta N(t)\exp(-N(t)),\ \forall t \geq 0, \text{ with } N(0) = N_0 \geq 0, \tag{22}$$

where $\beta > 0$ is the growth rate of the population and the term $\exp(-N(t))$ (with $\alpha \geq 0$) describes the intra-specific competition.

Semiflow: Recall that the semiflow $\{U(t)\}_{t\in\mathbb{N}}$ is defined by

$$U(t)(N_0) = T^t(N_0), \forall t \geq 0, \forall N_0 \geq 0,$$

where $T(N_0) = \beta N_0 \exp(-N_0)$ and T^t is defined by induction as follows

$$T^{t+1}(N_0) = T\left(T^t(N_0)\right) = T^t(T(N_0)), \forall t \geq 0,$$

and

$$T^0(N_0) = N_0.$$

Equilibria: It is readily checked that 0 is always an equilibrium of (22). The non-zero equilibrum satisfies

$$\beta \exp(-\overline{N}) = 1 \Leftrightarrow \overline{N} = \ln(\beta).$$

Dissipation property: We follow the construction of the Liapunov function by Fisher and Goh [3]. Define

$$V(N(t)) = \left(N(t) - \overline{N}\right)^2.$$

Then

$$\begin{aligned} V(N(t+1)) - V(N(t)) &= \left(N(t+1) - \overline{N}\right)^2 - \left(N(t) - \overline{N}\right)^2 \\ &= \left(\beta N(t)\exp(-N(t)) - \overline{N}\right)^2 - \left(N(t) - \overline{N}\right)^2 \\ &= (\beta N(t)\exp(-N(t)))^2 - N(t)^2 \\ &\quad -2\beta N(t)\exp(-N(t))\overline{N} + 2N(t)\overline{N} \\ &= [\beta\exp(-N(t)) - 1]\,[\beta\exp(-N(t)) + 1]\,N(t)^2 \\ &\quad -2N(t)\overline{N}\,[\beta\exp(-N(t)) - 1] \end{aligned}$$

and we obtain

$$V(N(t+1)) - V(N(t)) = h(N(t)) \left[\beta \exp(-N(t)) - 1\right], \tag{23}$$

where

$$\begin{aligned} h(N(t)) &= \left\{\left[\beta \exp(-N(t)) + 1\right] N(t) - 2\overline{N}\right\} N(t) \\ &= \left\{N(t)\beta \exp(-N(t)) - \overline{N} + N(t) - \overline{N}\right\} N(t) \\ &= \left\{N(t) \exp(\overline{N} - N(t)) + \left[N(t) - 2\overline{N}\right]\right\} N(t) \end{aligned} \tag{24}$$

Proposition 8.6

Assume that $\beta \in (1, e^2]$. *We deduce that if we define* $\{U(t)\}_{t\in\mathbb{N}}$ *the discrete time semiflow generated by* (22) *for each* $t \in \mathbb{N}$,

$$V(U(t)N_0) \leq V(N_0), \forall t \in \mathbb{N}, \forall N_0 \geq 0. \quad (25)$$

Proof. Let us prove that

$$h(N(t)) \leq 0, \forall N \leq \overline{N}$$

and

$$h(N(t)) \geq 0, \forall N \geq \overline{N}.$$

Indeed, we have

$$h(0) = 0, \text{ and } h(\overline{N}) = 0, \text{ and } h(N) > 0, \forall N \geq 2\overline{N}.$$

Consider $N \in (0, 2\overline{N})$ and $N \neq \overline{N}$. Then $h(N) = 0$ if

$$\begin{aligned} & N(t)\exp(\overline{N} - N(t)) + \left[N(t) - 2\overline{N}\right] = 0 \\ \Leftrightarrow \quad & \exp(\overline{N} - N(t)) = 2\frac{\overline{N}}{N(t)} - 1 \\ \Leftrightarrow \quad & \overline{N} - N(t) = \ln\left(2\frac{\overline{N}}{N(t)} - 1\right) \end{aligned}$$

which is equivalent to

$$\overline{N} = \frac{\ln\left(2\frac{\overline{N}}{N(t)} - 1\right)}{1 - \frac{N(t)}{\overline{N}}}. \tag{26}$$

First case: $N \in (0, \overline{N})$**:** Set $y = 1/(1 - N(t)/\overline{N}) > 1$. Then (26) gives

$$\begin{aligned} \overline{N} \quad &= y\left\{\ln(1+1/y) - \ln(1-1/y)\right\} = y\left\{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^{-n}}{n} + \sum_{n=1}^{\infty}\frac{y^{-n}}{n}\right\} \\ &= 2\sum_{n=0}^{\infty}\frac{y^{-2n}}{2n+1} > 2. \end{aligned}$$

Second case: $N \in (\overline{N}, 2\overline{N})$**:** Set $y = 1/(N(t)/\overline{N} - 1) > 1$. Then (26) gives

$$\overline{N} \quad = y \left\{ \ln(1 + 1/y) - \ln(1 - 1/y) \right\} > 2.$$

Finally $\beta \in (1, e^2]$ implies $\overline{N} \in (0, 2]$ we conclude that

$$h(N) \neq 0 \text{ and } N \in (0, 2\overline{N}) \Rightarrow N = \overline{N},$$

and the proof is completed. □

Theorem 8.7

Assume that $\beta \in (1, e^2]$. The discrete time semiflow generated by (22) is bounded dissipative on $[0, \infty)$. More precisely, for each $\gamma_0 > \overline{N}$ the subset

$$B_{\gamma_0} = \{x \geq 0 : V(x) \leq \gamma_0\}$$

is bounded absorbing subset.

Proof. Let $U(t)$ be the discrete time semiflow generated by (22). We have

$$V(U(t)0) = V(0) = \overline{N},$$

and

$$V(U(t)\overline{N}) = V(\overline{N}) = 0,$$

and for each $N_0 \in (0, +\infty) \setminus \left\{\overline{N}\right\}$,

$$V(U(t)N_0) < V(N_0), \forall t \in \mathbb{N},$$

the result follows. □

Remark 8.8

An Hopf bifurcation for maps occurs at $\beta = e^2$ *(i.e. the derivative of the map* $x \to \beta x e^{-x}$*) crosses* -1 *when* β *crosses* e^{-2}*). It means that periodic orbit with period 2 with appear. Then see Figure 10 and Figure 11 periodic orbits of period 4, 8, etc... appear. That is often called a period-doubling bifurcation. Chaos appears when* β *becomes even larger and gives rise to a periodic orbit of period 3.*

Assume that $\beta > e^2$. We can use the fact that $x \to \beta x \exp(-x)$ is bounded to derive the dissipation property. Indeed,

$$(\beta x \exp(-x))' = \beta [1 - x] \exp(-x).$$

Therefore the map $x \to \beta x \exp(-x)$ reaches a maximum at $x = 1$, and this maximum is βe^{-1}. Define

$$W(N) = \max(N, \beta e^{-1}).$$

Let $U(t)$ be the discrete time semiflow generated by (22). Then we have

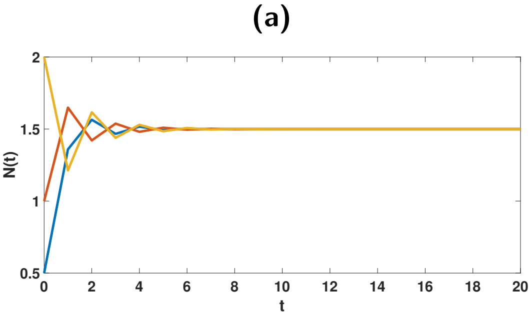
$$W(U(t)N_0) \leq W(N_0), \forall t \geq 0, \forall N_0 \geq 0.$$

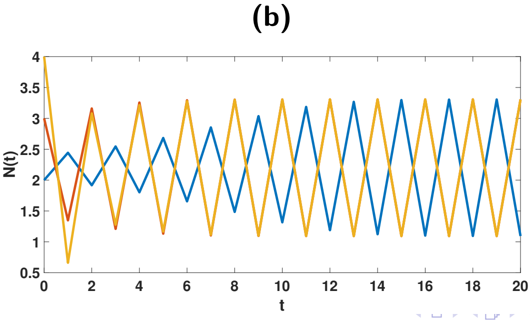
Theorem 8.9

Assume that $\beta \geq 0$. *The discrete time semiflow generated by* (22) *is bounded dissipative on* $[0, \infty)$.

Remark 8.10

Then by using a simple first order approximation $\exp(N(t)) = 1 + N(t) + h.o.t$*, we can regard the model of Beverton and Holt as a first order approximation of the Ricker model. Nevertheless the behavior of this two systems is very different.*


(a)
2
1.5
1
0.5
N(t)
0
2
4
6
8
10
12
14
16
18
20
t


(b)
4
3.5
3
2.5
2
1.5
1
0.5
N(t)
0
2
4
6
8
10
12
14
16
18
20
t

When the parameter β is large enough and a periodic orbit of period 3 appears, the system becomes chaotic (see Figures 10 and 11). In other words, period three implies chaos (Li and Yorke [6]).

This is the so called chaos of Sharkovskii [11, 12]. The result of Sharkovskii was rediscovered by Li and Yorke [6]. Both results of Sharkovskii [11, 12] and Li and Yorke show that when the system has a periodic solution of period 3 then periodic orbits of all periods exists. Therefore the system has infinitely many periodic orbits. This could serve as first step to characterize the chaos. But Li and Yorke explored the dependency with respect to the initial condition in the following sense.

Theorem 8.11 (Li and Yorke [6])

Let $T : I \to I$ be a continuous map on some interval $I \subset \mathbb{R}$. Assume that there exists $N_0 \geq 0$ such that

$$T^3 (N_0) \leq N_0 < T (N_0) < T^2 (N_0)$$

or

$$T^3 (N_0) \geq N_0 > T (N_0) > T^2 (N_0) .$$

Then

(i) *For each integer $k = 1, 2, \ldots$ there exists a periodic point in I having the period k.*

Theorem 8.12 (Li and Yorke [6])

(ii) *There exists an uncountable set $S \subset I$ (containing no periodic points), which satisfies the following conditions:*

(a) *For every $p, q \in S$ with $p \neq q$,*

$$\limsup_{t \to \infty} |T^t(p) - T^t(q)| > 0,$$

and

$$\liminf_{t \to \infty} |T^t(p) - T^t(q)| = 0.$$

(b) *For every $p \in S$ and every periodic point $q \in I$,*

$$\limsup_{t \to \infty} |T^t(p) - T^t(q)| > 0.$$

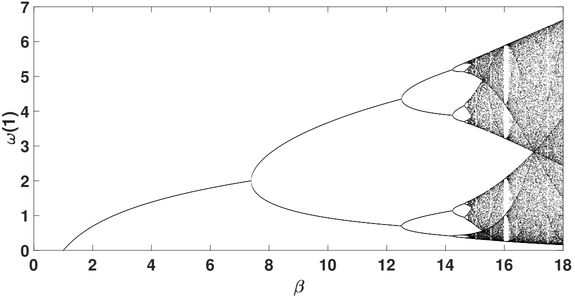


Figure: *In this figure we plot for each β varying from 0 to 18 the omega limit set $\omega(1)$ of the solution of* (22) *starting from the initial value $N_0 = 1$.*

In Figure 11, we observe a cascade of bifurcations. The branch of single point corresponds to a positive equilibrium, which first bifurcate to a 2 periodic orbit. One may notice that the positive equilibrium exists for each β but becomes unstable after the bifurcation. Then we observe a period doubling bifurcation, we period orbits of period 2, 4, 8, etc. Periodic orbit of period 3 appears only when the parameter β becomes large enough, the chaos in the sense of Li and Yorke [6] appear.

Lorentz system: chaos and dissipation properties

The system of Lorenz [7] was introduced in his paper in 1963 as taken from Saltzman (1962, [10]) as a minimalist model of thermal convection in a box. This system is the following

$$\begin{cases} x' = \sigma(y - x), \\ y' = x(\rho - z) - y, \\ z' = xy - \beta z, \end{cases} \tag{27}$$

where $\sigma > 0$, $\rho > 0$, and $\beta > 0$, and with initial value

$$x(0) = x_0, y(0) = y_0, \text{ and } z(0) = z_0.$$

In Figures 12-13, we plot two solutions of system (27) with the parameters values

$$\sigma = 10, \rho = 28, \text{ and } \beta = 8/3.$$

In Figures 12-13, we observe that starting from two very close initial values gives some very different trajectories. In Figure 13, the solution starting from the green dot (respectively from the black dot) end up at the yellow dot (respectively at the red dots) at time $t = 19$. So we can visualize the fact that changing a little bit the initial value may have a large impact on the trajectory.

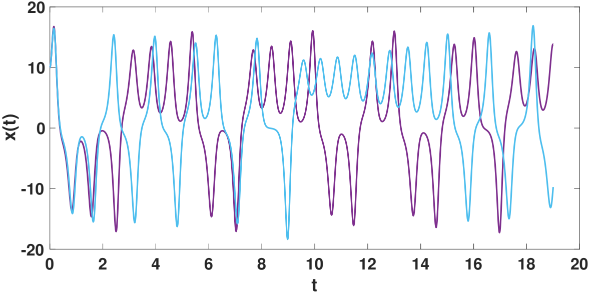
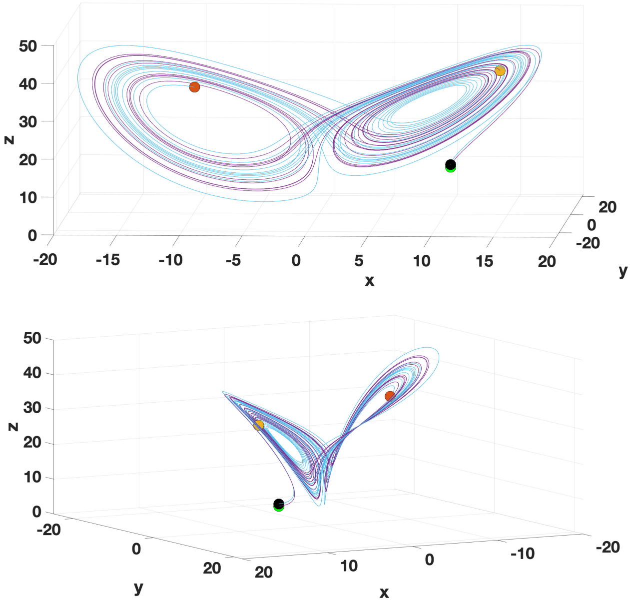


Figure: *In this figure, we plot the first component* $t \to x(t)$ *of two solutions starting from the initial value from* $[x_0, y_0, z_0] = [10, 10, 10]$ *(light blue curve) and from* $[x_0, y_0, z_0] = [10, 10, 10.7]$ *(purple curve).*


50
40
30
20
10
0
z
-20
-15
-10
-5
0
5
10
15
20
x
20
0
-20
y
50
40
30
20
10
0
z
-20
0
20
y
20
10
0
-10
-20
x

Lorenz looks at his system as a simplified model for weather prediction. He is assuming that one loop corresponds to the calm weather and the other loop to tornados. The orbit generated by a slight disturbance of the initial distribution of the system looks very different. The shape of the omega limit set is very similar for each solution. The chaotic nature of this system is due to the "fractal" structure of the attractor in between the loops.

The frequency at which a solution passes from one loop to the next turns to be similar for each solution. Lorentz already mentioned this phenomenon during his presentation at the American Association for the Advancement of Science in 1972. During his presentation he says that:

Over the years minuscule disturbances neither increase nor decrease the frequency of occurrences of various weather events such as tornados; the most they may do is to modify the sequences in which they occur.

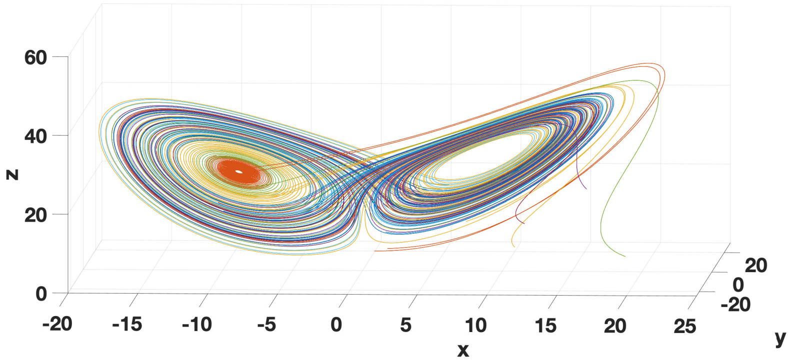


Figure: *In this figure, we use 20 randomly chosen initial distributions.*

It turns out that the Lorenz attractor is an attractor in the sense developed in the Chapter 2. That is to say that, it is a compact invariant subset which attracts one of its neighborhood at each point of $\mathbb{R}^3$. The existence such a global attractor is a consequence of the dissipation of the system. So we now investigate the dissipation of the Lorentz system.

For the Lorenz system, the dissipativity can be studied by using the results of Leonov [5]. Let us consider

$$V(x, y, z) = \frac{1}{\sigma} x^2 + y^2 + z^2. \tag{28}$$

Then

$$\frac{dV(x(t), y(t), z(t))}{dt} = 2x(y - x) + 2yx(\rho - z) - 2y^2 + 2xyz - 2\beta z^2$$

and after simplification we obtain

$$\frac{dV(x(t), y(t), z(t))}{dt} = -2\left[x^2 + y^2 + \beta z^2\right] + 2(\rho + 1)xy$$

but $2xy = x^2 + y^2 - (x - y)^2$ therefore

$$\frac{dV(x(t), y(t), z(t))}{dt} = -(1 - \rho)x^2 - (1 - \rho)y^2 - 2\beta z^2 - (\rho + 1)(x - y)^2.$$

Lemma 8.13

Assume that $\rho \in (0,1)$*. Then each solution of system* (27) *converges to* $0_{\mathbb{R}^3}$*. More precisely, we have the following estimation of the solution:*

$$V(x(t), y(t), z(t)) \leq e^{-\delta t} V(x_0, y_0, z_0), \forall t \geq 0, \tag{29}$$

where

$$\delta := \min\left\{(1-\rho)\sigma, (1-\rho), 2\beta\right\}.$$

Proof. By choosing the above constant $\delta > 0$, we have

$$\frac{dV(x(t), y(t), z(t))}{dt} \leq -\delta V(x(t), y(t), z(t))$$

and the result follows from the differential form of Gronwall's lemma (see Chapter 8 in Volume I). □

Theorem 8.14

Assume that $\rho \in (0, 1)$. *Then the semiflow* $U(t)$ *generated by the system* (27) *is bounded dissipative.*

Proof. Let $\gamma_0 > 0$ be fixed. Let us define

$$B_{\gamma_0} = \left\{ (x, y, z) \in \mathbb{R}^3 : V(x, y, z) \leq \gamma_0 \right\}.$$

Then for each bounded subset $B \subset \mathbb{R}^3$ we have

$$\gamma := \sup_{(x,y,z) \in B} V(x, y, z) < \infty.$$

Let $t_0 = t_0(B) > 0$ such that

$$e^{-\delta t_0} \gamma \leq \gamma_0,$$

then by using the definition of B_{γ_0}, we deduce that

$$U(t)B \subset B_{\gamma_0}, \forall t \geq t_0,$$

and the proof is completed. □

Next we consider the functional

$$W(x,y,z) = y^2 + (z-\rho)^2. \tag{30}$$

Then

$$\begin{aligned}\frac{dW(x(t),y(t),z(t))}{dt} &= 2yx(\rho - z) - 2y^2 + 2(z-\rho)\,(xy - \beta z) \\ &= 2yx(\rho - z) - 2y^2 + 2xy(z-\rho) - 2\beta z(z-\rho)\end{aligned}$$

hence

$$\frac{dW(x(t),y(t),z(t))}{dt} = -2y^2 - 2\beta z(z-\rho).$$

Lemma 8.15

Let $\lambda \in (0, 2\min(1, \beta))$. *Then*

$$\frac{dW(x(t), y(t), z(t))}{dt} \leq -\lambda W(x(t), y(t), z(t)) + \chi, \tag{31}$$

where

$$\chi := [2\beta - \lambda] \left(\frac{\beta\rho}{\lambda - 2\beta} \right)^2.$$

Proof. By using the fact that $\lambda < 2$ we deduce that

$$\begin{aligned}
\frac{dW(x(t),y(t),z(t))}{dt} \quad & +\lambda W(x(t),y(t),z(t)) \\
& = -2y^2 - 2\beta z(z-\rho) + \lambda y^2 + \lambda (z-\rho)^2 \\
& \leq [\lambda - 2\beta]\,(z-\rho)^2 - 2\beta\rho(z-\rho) \\
& = [\lambda - 2\beta]\left\{(z-\rho) - \frac{\beta\rho}{\lambda - 2\beta}\right\}^2 \\
& \quad - [\lambda - 2\beta]\left(\frac{\beta\rho}{\lambda - 2\beta}\right)^2 .
\end{aligned}$$

and since $\lambda < 2\beta$ the inequality (31) follows. □

Theorem 8.16

Let $\lambda \in (0, 2\min(1,\beta))$*. Then, for each* $t \geq 0$*,*

$$W(x(t), y(t), z(t)) \leq e^{-\lambda t} W(x_0, y_0, z_0) + \int_0^t e^{-\lambda(t-s)} \chi ds,$$

and

$$\limsup_{t \to +\infty} W(x(t), y(t), z(t)) \leq \frac{\chi}{\lambda}.$$

Moreover the semiflow $U(t)$ *generated by the Lorentz system* (27) *is bounded dissipative.*

Proof. The proof is left as an exercise. (**Hint:** Start first with the two last components y and z use the same arguments than in the proof of Theorem 8.14. Then use the comparison principle for the x-equation to derive the dissipativity of the full system). □

Thank you for listening

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