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# Optimal Control of Harvesting in a Nonlinear Elliptic System Arising from Population Dynamics

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## 1. INTRODUCTION

The aim of this article is to study an optimal control problem for a system of nonlinear elliptic equations with Dirichlet boundary conditions,

$$\begin{aligned} -\Delta u(x) &= a(x)v(x) - c(x)u(x) - eu(x)(u(x)+v(x)), \quad x \in \Omega, \\ -\Delta v(x) &= b(x)u(x) - d(x)v(x) - fv(x)(u(x)+v(x)), \quad x \in \Omega, \\ u(x) &= v(x) = 0, \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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where $\Omega$ is a bounded and regular domain in $\mathbb{R}^n$. This system arises from population dynamics where it models the steady-state solutions of the corresponding nonlinear evolution problem [3]. Here, functions $u$ and $v$ represent two interacting subpopulations of the same species living in $\Omega$. More precisely, $u$ means the concentration of the adult population and $v$ means the concentration of the young one. Functions $a$ and $b$ show, respectively, the rate of young which become adults and the rate of young produced by adults. Moreover, functions $c$ and $d$ reflect the result of harvesting a portion of the population and they play the role of control. The constants $e$ and $f$ measure the crowding effect and the competition between $u$ and $v$. The Laplacian operator shows the diffusive character of $u$ and $v$ within $\Omega$, and the boundary condition in (1.1) may be interpreted as the condition that the populations $u$ and $v$ may not stay on $\partial\Omega$.

We define the class of admissible controls,

$$C_{\delta_1} \times C_{\delta_2} = \{(c, d) \in L^\infty(\Omega) \times L^\infty(\Omega) : 0 \le c \le \delta_1, 0 \le d \le \delta_2\},$$

where $\delta_1$ and $\delta_2$ are fixed positive constants. Under certain assumptions (see hypotheses [H1] and [H2] below), system (1.1) will have, for each given $(c, d) \in C_{\delta_1} \times C_{\delta_2}$, a unique coexistence state (a solution $(u, v)$ with both components nonnegative and nontrivial) denoted by $(u_{c,d}, v_{c,d})$, and we will be interested in maximizing the payoff functional $J$: $C_{\delta_1} \times C_{\delta_2} \to \mathbb{R}$, defined by

$$J(c, d) = \int_\Omega \left[\lambda u_{c,d}(x)c(x) - c^2(x) + \mu v_{c,d}(x)d(x) - d^2(x)\right], \quad (1.2)$$

where $\lambda$ and $\mu$ are fixed positive constants. This functional represents the difference between economic revenue and cost. The positive constants $\lambda$ and $\mu$ describe, respectively, the quotient between the price of the species and the cost of the control ([3, 9, 13]).

The article is organized as follows. Section 2 gives the existence and uniqueness of coexistence states of system (1.1). In our opinion, the results and the methods of this section (mainly, the proof of uniqueness by using convexity arguments) may be of interest, in addition to the control problem considered in this article. In Section 3, we prove the existence of an optimal control. Moreover, the optimal controls are characterized in terms of the optimality system, which is the state system coupled with the adjoint one. This may be used to prove, when the parameters $\lambda$ and $\mu$ are sufficiently small, the uniqueness and approximation to the optimal control.

Related problems were considered in [7, 16] (Dirichlet boundary conditions) and [15] (Neumann boundary conditions), where the authors consider the case of a scalar equation. Systems of equations, with a similar payoff functional, were studied in [3, 9, 13]. In [3], the authors study a problem like (1.1) but with Neumann boundary conditions. Here, we consider the case of Dirichlet boundary conditions, which seems to be different in many aspects from the Neumann one (see [7, 15, 14]). Moreover, the current article offers definite improvements on the article [3]. For example, we simplify the proof about the uniqueness of coexistence states of (1.1) with respect to that given in [3]. In addition, we delete some of the hypotheses used in [3] to obtain the optimality conditions (see Theorem 3.3 below), which is very important for the applications. In the proofs, moreover of some standard techniques of control theory, we use different methods and results on elliptic problems (upper and lower solutions notion, strong maximum principle) and strictly convex operators in ordered Banach spaces. Finally, the corresponding parabolic control problem will be treated elsewhere [12].

## 2. EXISTENCE AND UNIQUENESS OF COEXISTENCE STATES

The following assumptions are made throughout the article,

[H1] $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $a, b, c, d \in L^\infty_+(\Omega)$, $e, f \in \mathbb{R}^+$,

where $L^\infty_+(\Omega) = \{g \in L^\infty(\Omega) : g \geq 0\}$.

DEFINITION. A pair of functions $(u, v) \in H^1_0(\Omega) \cap L^\infty(\Omega)$, ($H^1_0(\Omega)$ is the usual Sobolev space), is a weak solution of system (1.1) provided

$$\begin{aligned} \int_\Omega \nabla u(x)\, \nabla\varphi(x)\, dx &= \int_\Omega \big[a(x)v(x) - c(x)u(x) \\ &\qquad - eu(x)(u(x) + v(x))\big]\,\varphi(x)\, dx, \\ \int_\Omega \nabla v(x)\, \nabla\varphi(x)\, dx &= \int_\Omega \big[b(x)u(x) - d(x)v(x) \\ &\qquad - fv(x)(u(x) + v(x))\big]\,\varphi(x)\, dx \end{aligned} \tag{2.1}$$

for all $\varphi \in H^1_0(\Omega)$. It follows that each weak solution $(u, v)$ of (1.1) belongs to $C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$ [8]. By a coexistence state of system (1.1) we mean a solution $(u, v)$ of (1.1) with both components nonnegative and nontrivial.

Also, for each $g \in L^\infty(\Omega)$, we denote $\underline{g} = \operatorname{ess\,inf} g$, $\bar{g} = \operatorname{ess\,sup} g$. Finally, for every $q \in L^\infty(\Omega)$, $\rho_1(q)$ is the principal eigenvalue of the eigenvalue problem,

$$-\Delta u + qu = \rho u, \qquad \text{in } \Omega, \qquad u = 0, \qquad \text{on } \partial\Omega.$$

It is known [10] that $\rho_1(q)$ is simple and it verifies the variational characterization,

$$\rho_1(q) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + \int_\Omega q|u|^2}{\int_\Omega |u|^2}.$$

We denote by $\varphi_1(q)$ the unique eigenfunction associated to $\rho_1(q)$ verifying $\varphi_1(q) > 0$ in $\Omega$, $\|\varphi_1(q)\|_{L^\infty(\Omega)} = 1$. Let us note that, for every constant $\delta \in \mathbb{R}$, one has $\varphi_1(\delta) = \varphi_1(0)$, $\rho_1(\delta) = \rho_1(0) + \delta$.

The main result of this section is the next theorem where we suppose that the positive constants $\delta_1, \delta_2, e, f$ and the functions $a, b$ are given. Then, under some additional hypotheses, we prove the existence and the uniqueness of coexistence states of system (1.1) for each $(c, d) \in C_{\delta_1} \times C_{\delta_2}$. Basically, these additional hypotheses are of two types. On the one hand, the quantity $\underline{a}\underline{b}$ must be greater than a positive constant which depends on $\delta_1$, $\delta_2$, and $\Omega$. On the other hand, the constants $\bar{a}, \bar{b}$ must be, respectively, smaller than the other two positive constants which depend on $\underline{a}$, $\underline{b}$, $e$, and $f$.

THEOREM 2.1. *Let $\delta_i > 0$, for $i = 1, 2$, be two positive numbers. Assume* [H1] *and*

$$[\text{H2}] \quad \underline{a}\underline{b} > \rho_1(\delta_1)\rho_1(\delta_2), \qquad \bar{a} \le \underline{a}(1 + \Gamma), \bar{b} \le \underline{b}(1 + \Gamma^{-1}),$$

*where $\Gamma = \underline{a}f/\underline{b}e$.*

*Then, for each $(c, d) \in C_{\delta_1} \times C_{\delta_2}$, there exists a unique coexistence state $(u, v)$ of system* (1.1). *Moreover, $(u, v) \in [0, \underline{a}/e]_{L^\infty(\Omega)} \times [0, \underline{b}/f]_{L^\infty(\Omega)}$, i.e., $0 \le u \le \underline{a}/e$, $0 \le v \le \underline{b}/f$.*

*Proof.* The proof is divided into two parts. We first prove the existence of coexistence states by using the upper–lower solution method. For the uniqueness, we combine different results and techniques such as the strong maximum principle for elliptic operators [8] and some ideas related to the uniqueness of fixed points for order convex maps in ordered Banach spaces [2, 11].

More precisely, the existence of coexistence states is based on Theorem 2.3 of [6]. In fact, the functions,

$$u_* = \nu\tau\varphi_1(0), \qquad u^* = \frac{\underline{a}}{e}, \qquad v_* = \tau\varphi_1(0), \qquad v^* = \frac{\underline{b}}{f} \tag{2.2}$$

are a system of upper−lower solutions for system (1.1), where the constant $\nu$ is chosen to satisfy

$$\frac{\rho_1(\delta_2)}{\underline{b}} < \nu < \frac{\underline{a}}{\rho_1(\delta_1)}, \tag{2.3}$$

and $\tau$ is a sufficiently small positive real number. To see this, from [H2] we obtain

$$0 \geq \bar{a}\frac{\underline{b}}{f} - \underline{a}\left(\frac{\underline{b}}{f} + \frac{\underline{a}}{e}\right).$$

Therefore,

$$0 \geq (a(x) - \underline{a})\frac{\underline{b}}{f} - \frac{\underline{a}^2}{e},$$

which trivially implies

$$0 \geq (a(x) - \underline{a})\frac{\underline{b}}{f} - c(x)\frac{\underline{a}}{e} - \frac{\underline{a}^2}{e}.$$

But then,

$$0 \geq (a(x) - \underline{a})v(x) - c(x)\frac{\underline{a}}{e} - \frac{\underline{a}^2}{e}$$

for each $v \in [v_*, v^*]$, which we rewrite in the form,

$$-\Delta u^* \geq a(x)v(x) - c(x)u^* - eu^*(u^* + v) \tag{2.4}$$

for each $v \in [v_*, v^*]$.

Also, from (2.3) we obtain that, if $\tau$ is sufficiently small, then

$$\nu\rho_1(\delta_1) = \nu(\rho_1(0) + \delta_1) \leq a(x) - e\nu\tau\varphi_1(x) - e\nu^2\tau\varphi_1(x),$$

which trivially implies

$$\nu\rho_1(0) \leq a(x) - e\nu\tau\varphi_1(x) - c(x)\nu - e\nu^2\tau\varphi_1(x)$$

for each $c \in C_{\delta_1}$.

Since the function $\varphi_1 \equiv \varphi_1(0)$ is strictly positive in $\Omega$, we deduce from the previous expression,

$$\nu\tau\rho_1(0)\varphi_1(x) \le \tau\varphi_1(x)(a(x) - e\nu\tau\varphi_1(x)) - c(x)\nu\tau\varphi_1(x) - e\nu^2\tau^2\varphi_1^2(x).$$

Then,

$$\nu\tau\rho_1(0)\varphi_1(x) \le v(x)(a(x) - e\nu\tau\varphi_1(x)) - c(x)\nu\tau\varphi_1(x) - e\nu^2\tau^2\varphi_1^2(x)$$

for each $v \in [v_*, v^*]$.

Lastly, as $-\Delta u_* = \rho_1(0)u_*$, we obtain

$$-\Delta u_* \le a(x)v(x) - c(x)u_* - eu_*(u_* + v) \tag{2.5}$$

for each $v \in [v_*, v^*]$.

We can proceed in the same way, proving

$$-\Delta v^* \ge b(x)u - d(x)v^* - fv^*(u + v^*) \tag{2.6}$$

for each $u \in [u_*, u^*]$, and

$$-\Delta v_* \le b(x)u - d(x)v_* - fv_*(u + v_*) \tag{2.7}$$

for each $u \in [u_*, u^*]$.

The inequalities (2.4)–(2.7) prove that functions $u_*, u^*, v_*, v^*$ are a system of upper–lower solutions for system (1.1), concluding that (1.1) has a coexistence state $(u, v) \in [u_*, u^*] \times [v_*, v^*]$. ∎

*Remark.* Note that the positive constants $\nu$ and $\tau$ may be chosen independently from $(c, d) \in C_{\delta_1} \times C_{\delta_2}$.

Next, we prove that under hypotheses [H1] and [H2], (1.1) has a unique coexistence state. This purpose will be carried out with the help of some previous lemmas. The first one refers to some properties satisfied by the nonnegative solutions of the scalar equations associated to (1.1).

LEMMA 2.2. 1. *Given $v \in C(\overline{\Omega})$ (continuous functions defined on $\overline{\Omega}$), $0 \le v \le \underline{b}/f$, there exists a unique nonnegative weak solution $P(v) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solving the problem,*

$$\begin{aligned} -\Delta u &= a(x)v - c(x)u - eu(u + v), \qquad \text{in } \Omega, \\ u &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.8}$$

*This unique solution $P(v) \in C^{1,\alpha}(\overline{\Omega})$, $\forall \alpha \in (0,1)$, $0 \le P(v) \le \underline{a}/e$, and $v \equiv 0 \Leftrightarrow P(v) \equiv 0$. Moreover, if $v \ne 0$ and $t \in (0,1)$, then $P(tv) - tP(v) > 0$ in $\Omega$. Also, if $v_2 \ge v_1$ then $P(v_2) \ge P(v_1)$.*

2. *Given* $u \in C(\overline{\Omega})$, $0 \le u \le \underline{a}/e$, *there exists a unique nonnegative weak solution* $Q(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *solving the problem*,

$$\begin{aligned} -\Delta v = b(x)u - d(x)v - fv(u+v), \qquad &\text{in } \Omega,\\ v = 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.9}$$

*This unique solution* $Q(u) \in C^{1,\alpha}(\overline{\Omega})$, $\forall \alpha \in (0,1)$, $0 \le Q(u) \le \underline{b}/f$, *and* $u \equiv 0 \Leftrightarrow Q(u) \equiv 0$. *Moreover*, *if* $u \ne 0$ *and* $t \in (0,1)$, *then* $Q(tu) - tQ(u) > 0$ *in* $\Omega$. *Also*, *if* $u_2 \ge u_1$ *then* $Q(u_2) \ge Q(u_1)$.

*Proof.* Let $v \in C(\overline{\Omega})$, $0 \le v \le \underline{b}/f$, be given. Then, as in the proof of Theorem 2.1 and by using [H2], it is easily checked that $u_* = 0$, $u^* = \underline{a}/e$ are, respectively, subsolution and supersolution for problem (2.8).

We can apply Theorem 1 in [5] to obtain functions $w_* \le w^*$ (both belonging to $C^{1,\alpha}(\overline{\Omega})$), respectively, minimal and maximal solutions of (2.8) between $u_*$ and $u^*$. Moreover,

$$\begin{aligned} &-\Delta(w^* - w_*) + ev(w^* - w_*)\\ &\qquad = c(x)(w_* - w^*) + e(w_* - w^*)(w_* + w^*) \le 0. \end{aligned}$$

Therefore, the maximum principle for elliptic operators implies $w^* \le w_*$ and consequently $w^* = w_*$. This concludes that, in the interval $[u_*, u^*]$, (2.8) has a unique solution. But, let us observe that, instead of $u^*$ we may choose any sufficiently positive large constant as a supersolution for (2.8). Moreover, from regularity theory, we have that any nonnegative weak solution of (2.8) must be bounded. This proves that given $v$, (2.8) has a unique nonnegative weak solution. On the other hand,

$$-\Delta P(v) + c(x)P(v) + eP(v)(P(v) + v) = a(x)v,$$

in $\Omega$. It follows from the maximum principle that $v \equiv 0 \Leftrightarrow P(v) \equiv 0$ and that if $v \ne 0$ then $P(v) > 0$ in $\Omega$ and $\partial P(v)/\partial n < 0$, on $\partial\Omega$, where $\partial/\partial n$ denotes the directional derivative with respect to the outward-pointing normal on $\partial\Omega$.

Moreover, if $M$ is a fixed positive number, $t \in (0,1)$ and $v \ne 0$, then by definition of operator $P$, we have

$$\begin{aligned} &-\Delta(P(tv) - tP(v)) + M(P(tv) - tP(v))\\ &\qquad = (P(tv) - tP(v))(-c(x) - eP(tv) - etP(v) + M)\\ &\qquad\qquad + eP^2(v)(t - t^2) + etP(v)v\\ &\qquad \ge (P(tv) - tP(v))(-c(x) - eP(tv) - etP(v) + M). \end{aligned} \tag{2.10}$$

Also, it is easily seen that $tP(v)$ is a strict subsolution of (2.8) for the function $tv$, i.e.,

$$-\Delta(tP(v)) < a(x)tv - c(x)tP(v) - etP(v)(tP(v) + v), \qquad \text{in } \Omega.$$

This implies $P(tv) \geq tP(v)$ in $\Omega$. Hence, if $M$ is sufficiently large, from (2.10) and again by the maximum principle, $P(tv) - tP(v) > 0$ in $\Omega$.

Finally, if $v_2 \geq v_1$, the function $P(v_1)$ is a subsolution of (2.8) for the function $v_2$, since $a(x) - eP(v_1) \geq a(x) - \underline{a} \geq 0$, and this implies

$$-\Delta P(v_1) \leq a(x)v_2 - c(x)P(v_1) - e(P(v_1))(P(v_1) + v_2), \qquad \text{in } \Omega.$$

Therefore, $P(v_1) \leq P(v_2)$.

By analogous considerations it is possible to prove the properties of the operator $Q$. ▮

In the following lemma, we show that (1.1) has a unique coexistence state in the interval $[0, \underline{a}/e] \times [0, \underline{b}/f]$.

LEMMA 2.3. 1. *If* $(u, v)$ *is a coexistence state of* (1.1), *such that* $0 \leq u \leq \underline{a}/e$, $0 \leq v \leq \underline{b}/f$, *then* $u = P(v)$ *and* $v = Q(u)$. *Consequently*, $(QP)(v) = v$.

2. *If* $v \in C(\overline{\Omega})\backslash\{0\}$ *is such that* $0 \leq v \leq \underline{b}/f$, *and* $(QP)(v) = v$, *then the pair* $(P(v), v)$ *is a coexistence state of* (1.1).

3. *There exists a unique* $v \in C(\overline{\Omega})\backslash\{0\}$, $0 \leq v \leq \underline{b}/f$, *such that* $(QP)(v) = v$.

*Proof.* The first and second part are a trivial consequence of the definition of the operators $P$ and $Q$. We will prove the last one. To see this, let $v_i \in C(\overline{\Omega})\backslash\{0\}$, $0 \leq v_i \leq \underline{b}/f$, be such that $(QP)(v_i) = v_i$, $i = 1, 2$. It follows from the strong maximum principle that there exists a positive number $s$ such that $v_1 \geq sv_2$ in $\Omega$ (see the proof of the previous lemma, where it is shown that $v_2 > 0$ in $\Omega$ and $\partial v_2/\partial n < 0$ on $\partial\Omega$). Moreover, if $s$ is any positive number such that $v_1 > sv_2$ in $\Omega$, then there is $\epsilon > 0$ verifying $v_1 > (s + \epsilon)v_2$ in $\Omega$ (see Lemma 5.3 in [1]). Let us define $s_0 = \sup\{s > 0 : v_1 \geq sv_2 \text{ in } \Omega\}$. If $s_0 < 1$, then, by Lemma 2.2, we have

$$v_1 = (QP)(v_1) \geq (QP)(s_0 v_2) \geq Q(s_0 P(v_2)) > s_0(QP)(v_2) = s_0 v_2, \qquad \text{in } \Omega.$$

This is a contradiction with the definition of $s_0$. Hence, we must have $s_0 \geq 1$ and as a consequence, $v_1 \geq v_2$ in $\Omega$. Analogously, $v_1 \leq v_2$. ▮

LEMMA 2.4. *If* $(u, v)$ *is any coexistence state of* (1.1), *then* $0 \leq u \leq \underline{a}/e$ *and* $0 \leq v \leq \underline{b}/f$.

*Proof.* Let $(u, v)$ be any coexistence state of (1.1) and let $(u_0, v_0)$ be the coexistence state of (1.1) which belongs to $[0, \underline{a}/e] \times [0, \underline{b}/f]$. Then,

$$-\Delta v + d(x)v + fv(u+v) = b(x)u, \qquad \text{in } \Omega$$

It follows from [H2] and the maximum principle that $v > 0$ in $\Omega$ and $\partial v/\partial n < 0$, on $\partial\Omega$. As in the previous lemma, there exists a positive number $t$ such that $v_0 \geq tv$ in $\Omega$. Also, if $t$ is any positive number such that $v_0 > tv$ in $\Omega$, then there exists $\epsilon > 0$ verifying $v_0 > (t + \epsilon)v$ in $\Omega$. Let us define $t_0 = \sup\{t > 0 : v_0 \geq tv \text{ in } \Omega\}$. If $t_0 < 1$, then by Lemma 2.2, we have

$$v_0 = (QP)(v_0) \geq (QP)(t_0 v) \geq Q(t_0 P(v)) > t_0(QP)(v) = t_0 v, \qquad \text{in } \Omega.$$

This is a contradiction with the definition of $t_0$. Hence, we must have $t_0 \geq 1$ and as a consequence, $v_0 \geq v$ in $\Omega$. Similarly, $u_0 \geq u$ in $\Omega$.

Now, by using three previous lemmas we deduce, under [H1] and [H2], the uniqueness of coexistence states of (1.1).

## 3. THE OPTIMAL CONTROL PROBLEM

In this section, we prove, under the hypotheses [H1] and [H2], the existence of an optimal control, which is characterized in terms of the optimality system. To see this, we previously show some differentiability properties of the payoff functional with respect to the control.

THEOREM 3.1. *Under the assumptions* [H1] *and* [H2], *there exists an optimal control*, *i.e.*, *there is* $(c, d) \in C_{\delta_1} \times C_{\delta_2}$ *such that*

$$J(c, d) = \sup_{(g,h) \in C_{\delta_1} \times C_{\delta_2}} J(g, h)$$

*Proof.* Since the state variables (solutions of (1.1)) and the admissible controls (functions of $C_{\delta_1} \times C_{\delta_2}$) are bounded, there exists a maximizing sequence $\{(g_n, h_n)\}$ such that

$$\lim_{n \to \infty} J(g_n, h_n) = \sup_{(g,h) \in C_{\delta_1} \times C_{\delta_2}} J(g, h).$$

Let $(u_n, v_n)$ be the unique coexistence state of (1.1) for $(c, d) = (g_n, h_n)$, $\forall n \in \mathbb{N}$. Then, there exists a subsequence, again denoted by $(g_n, h_n)$, such that

$$(g_n, h_n) \rightharpoonup (g^*, h^*), \qquad \text{weakly in } L^2(\Omega) \times L^2(\Omega),$$

$$(u_n, v_n) \to (u^*, v^*), \qquad \text{strongly in } H_0^1(\Omega) \times H_0^1(\Omega).$$

Passing to the limit in the $(u_n, v_n)$ system, we have that $(u^*, v^*)$ is the unique coexistence state of (1.1) associated with $(g^*, h^*)$ (see the included remark in the proof of Theorem 2.1). Then, by using the lower semicontinuity of $L^2(\Omega)$ norm with respect to weak convergence, and the definition of functional $J$, we have

$$J(g^*, h^*) \geq \sup_{(g,h)\in C_{\delta_1}\times C_{\delta_2}} J(g,h),$$

which concludes $(g^*, h^*)$ is an optimal control that maximizes the functional $J$ on $C_{\delta_1} \times C_{\delta_2}$.

Next, we study the differentiability properties of the solutions of (1.1) with respect to the controls.

LEMMA 3.2. *Let us assume* [H1] *and* [H2]. *Then if* $(c,d) \in C_{\delta_1} \times C_{\delta_2}$ *and* $(g,h) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *are such that* $(c,d) + \beta(g,h) \in C_{\delta_1} \times C_{\delta_2}$ *for* $\beta > 0$ *small, then*

$$\frac{u_\beta - u}{\beta} \to \xi, \qquad \text{in } H_0^1(\Omega),$$
$$\frac{v_\beta - v}{\beta} \to \eta, \qquad \text{in } H_0^1(\Omega), \tag{3.1}$$

*as* $\beta \to 0$, *where* $(u,v)$ *is the unique coexistence state of* (1.1) *for* $(c,d)$, $(u_\beta, v_\beta)$ *is the unique coexistence state of* (1.1) *for* $(c+\beta g, d+\beta h)$, *and* $(\xi, \eta)$ *is the unique solution of the linear system*,

$$\begin{aligned} -\Delta\xi + [c + e(2u+v)]\,\xi - (a - eu)\eta &= -gu, \qquad \text{in } \Omega,\\ -\Delta\eta + [d + f(u+2v)]\,\eta - (b - fv)\,\xi &= -hv, \qquad \text{in } \Omega,\\ \xi = \eta &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

*Proof.* It follows from a result of Sweers ([17, Theorem 1.1]) that (3.2) has a unique solution $(\xi, \eta)$. To see this, we may rewrite (3.2) in the form,

$$Lw = Hw + k, \qquad \text{in } \Omega; \qquad w = 0, \qquad \text{on } \partial\Omega, \tag{3.3}$$

where

$$w = \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

$$L = \begin{pmatrix} -\Delta + [c + e(2u+v)] & 0 \\ 0 & -\Delta + [d + f(u+2v)] \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & a - eu \\ b - fv & 0 \end{pmatrix},$$

$$k = \begin{pmatrix} -gu \\ -hv \end{pmatrix}.$$

Now, all the hypotheses of Theorem 1.1 in [17] are easily checked, provided that there is a positive strict supersolution of (3.3) with $k = 0$. But, if $(u, v)$ is the unique coexistence state of (1.1) for $(c, d)$, then, rewriting (1.1) as

$$\begin{pmatrix} -\Delta u + [c + e(2u+v)]u \\ -\Delta v + [d + f(u+2v)]v \end{pmatrix} = \begin{pmatrix} 0 & a - eu \\ b - fv & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} eu(u+v) \\ fv(u+v) \end{pmatrix}$$

and denoting $z = \binom{u}{v}$ we obtain $(L - H)z \geq 0$, $(L - H)z \neq 0$, in $\Omega$, which proves that $z = \binom{u}{v}$ is a positive strict supersolution of (3.3) with $k = 0$. On the other hand, it is easily seen that the pair $(\xi_\beta, \eta_\beta)$ defined as $\xi_\beta = (u_\beta - u)/\beta$, $\eta_\beta = (v_\beta - v)/\beta$, satisfies the problem,

$$\begin{aligned} -\Delta \xi_\beta + \left[c + e(u + u_\beta + v)\right] \xi_\beta - (a - eu_\beta)\eta_\beta &= -gu_\beta, \\ &\quad \text{in } \Omega, \\ -\Delta \eta_\beta + \left[d + f(u + v_\beta + v)\right] \eta_\beta - (b - fv_\beta)\xi_\beta &= -hv_\beta, \\ &\quad \text{in } \Omega, \\ \xi_\beta = \eta_\beta = 0, \qquad &\text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Moreover, there exists $C > 0$, such that

$$\left|(\xi_\beta, \eta_\beta)\right|_{L^\infty(\Omega)} \leq C, \tag{3.5}$$

independent of $\beta$, where $|(\xi_\beta, \eta_\beta)|_{L^\infty(\Omega)} = |\xi_\beta|_{L^\infty(\Omega)} + |\eta_\beta|_{L^\infty(\Omega)}$.

In fact, if this is not true, then there is a sequence $\{\beta_n\} \to 0$ verifying $|(\xi_{\beta_n}, \eta_{\beta_n})|_{L^\infty(\Omega)} \to \infty$. Also, it follows that $(\alpha_n, \gamma_n) \equiv (\xi_{\beta_n}, \eta_{\beta_n})/|(\xi_{\beta_n}, \eta_{\beta_n})|_{L^\infty(\Omega)}$ is a solution of the problem,

$$\begin{aligned} -\Delta \alpha_n + \left[c + e(u + u_{\beta_n} + v)\right] \alpha_n - (a - eu_{\beta_n})\gamma_n & \\ = -g \frac{u_{\beta_n}}{\left|(\xi_{\beta_n}, \eta_{\beta_n})\right|_{L^\infty(\Omega)}} \qquad &\text{in } \Omega, \\ -\Delta \gamma_n + \left[d + f(u + v_{\beta_n} + v)\right] \gamma_n - (b - fv_{\beta_n})\alpha_n & \\ = -h \frac{v_{\beta_n}}{\left|(\xi_{\beta_n}, \eta_{\beta_n})\right|_{L^\infty(\Omega)}} \qquad &\text{in } \Omega, \\ \alpha_n = \gamma_n = 0, \qquad &\text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

Since $(\alpha_n, \gamma_n), (u_{\beta_n}, v_{\beta_n})$ are bounded in $(L^\infty(\Omega))^2$, using elliptic estimates, we have that $(\alpha_n, \gamma_n)$ is bounded in $W^{2,p}(\Omega)$ for $p \in (1, \infty)$. Therefore, there exists a subsequence, again denoted by $(\alpha_n, \gamma_n)$, such that $(\alpha_n, \gamma_n) \to (\alpha, \gamma)$, where $(\alpha, \gamma)$ is the unique solution of the problem,

$$
\begin{aligned}
-\Delta\alpha + [c + e(2u + v)]\alpha - (a - eu)\gamma &= 0, \qquad \text{in } \Omega,\\
-\Delta\gamma + [d + f(u + 2v)]\gamma - (b - fv)\alpha &= 0, \qquad \text{in } \Omega,\\
\alpha = \gamma = 0, \qquad &\text{on } \partial\Omega.
\end{aligned}
$$

But Sweer's theorem implies $(\alpha, \gamma) \equiv (0, 0)$, which is a contradiction with the property $\|(\alpha, \gamma)\|_{L^\infty(\Omega)} = 1$.

Finally, (3.4), (3.5), again elliptic estimates and the uniqueness of solutions of (3.2), imply that $(\xi_\beta, \eta_\beta) \to (\xi, \eta)$ in $H_0^1(\Omega)$ when $\beta \to 0$. ∎

Next, an optimal control is characterized in terms of the optimality system. This result is very important to obtain an explicit characterization of the unique optimal control and some of its qualitative properties.

THEOREM 3.3. *Let us assume* [H1] *and* [H2]. *Then, any optimal control* $(c, d) \in C_{\delta_1} \times C_{\delta_2}$ *may be expressed in the form*,

$$
\begin{aligned}
c &= \min\left\{\frac{\lambda}{2}u(1 - r)^+, \delta_1\right\}, \qquad \text{in } \Omega,\\
d &= \min\left\{\frac{\mu}{2}v(1 - s)^+, \delta_2\right\}, \qquad \text{in } \Omega,
\end{aligned}
\tag{3.7}
$$

*where* $(u, v)$ *is the corresponding solution of* (1.1) *and* $(r, s)$ *is the unique solution of the adjoint system*,

$$
\begin{aligned}
-\Delta r + [c + e(2u + v)]r - \frac{\mu}{\lambda}(b - fv)s &= c, \qquad \text{in } \Omega,\\
-\Delta s + [d + f(u + 2v)]s - \frac{\lambda}{\mu}(a - eu)r &= d, \qquad \text{in } \Omega,\\
r = s = 0, \qquad &\text{on } \partial\Omega.
\end{aligned}
\tag{3.8}
$$

*Proof.* We begin by proving that system (3.8) has a unique solution $(r, s)$ which belongs to $W^{2,p}(\Omega)$, for any $p$ sufficiently large. To see this, we may observe that $(r, s)$ is a solution of (3.8) if and only if $(R, S) \equiv (\lambda r, \mu s)$ is a solution of

$$
\begin{aligned}
-\Delta R + [c + e(2u + v)]R - (b - fv)S &= \lambda c, \qquad \text{in } \Omega,\\
-\Delta S + [d + f(u + 2v)]S - (a - eu)R &= \mu d, \qquad \text{in } \Omega,\\
R = S = 0, \qquad &\text{on } \partial\Omega.
\end{aligned}
\tag{3.9}
$$

System (3.9) is similar to system (3.2), but it must be remembered that just to prove the existence of solutions of (3.2) we need the existence of a positive strict supersolution for the homogeneous system. The existence of this supersolution is not clear now, since the terms $(a - eu)$ and $(b - fv)$ were interchanged. We overcome this difficulty by using another approach based on some elementary notions of linear functional analysis. To see this, let us define (for $p > n$) the operator $A$: $L^p(\Omega) \times L^p(\Omega) \to W^{2,p}(\Omega) \times W^{2,p}(\Omega)$, by $A(g,h) = (U,V)$, where $(U,V)$ is the unique solution of the system,

$$\begin{aligned} -\Delta U + [c + e(2u + v)]U &= g, \qquad \text{in } \Omega, \\ -\Delta V + [d + f(u + 2v)]V &= h, \qquad \text{in } \Omega, \\ U = V &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.10}$$

Then, $(R,S)$ is a solution of (3.9) if and only if

$$(I - AB)(R,S) = A(\lambda c, \mu d), \tag{3.11}$$

where $AB(R,S) = A((b - fv)S, (a - eu)R)$.

Now, let us observe that $A(\lambda c, \mu d) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and that the operator $AB$: $C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is linear and compact (see [17]). Therefore, (3.11) will have a solution if the operator $I - AB$: $C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ has a trivial kernel [4]. But this fact is easily checked since if $(R,S)$ satisfies $(I - AB)(R,S) = (0,0)$, then,

$$\begin{aligned} -\Delta R + [c + e(2u + v)]R - (b - fv)S &= 0, \qquad \text{in } \Omega, \\ -\Delta S + [d + f(u + 2v)]S - (a - eu)R &= 0, \qquad \text{in } \Omega, \\ R = S &= 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.12}$$

Moreover, if $(\xi, \eta)$ is the unique solution of the system,

$$\begin{aligned} -\Delta\xi + [c + e(2u + v)]\xi - (a - eu)\eta &= g, \qquad \text{in } \Omega, \\ -\Delta\eta + [d + f(u + 2v)]\eta - (b - fv)\xi &= h, \qquad \text{in } \Omega, \\ \xi = \eta &= 0, \qquad \text{on } \partial\Omega \end{aligned} \tag{3.13}$$

then,

$$\int_\Omega (\xi 0 + \eta 0) = \int_\Omega (gR + hS) \tag{3.14}$$

for any $(g,h) \in L^p(\Omega) \times L^p(\Omega)$. This proves that $(R,S) = (0,0)$.

The remaining part of the proof is standard. In fact, a similar reasoning may be seen in [3, 7]. However, for clarity of the exposition, we sketch the main ideas. Let $(c,d) \in C_{\delta_1} \times C_{\delta_2}$ be an optimal control and $(g,h) \in$

$L^{\infty}(\Omega)$ so that $(c + \beta g, d + \beta h) \in C_{\delta_1} \times C_{\delta_2}$ as $\beta \to 0^+$. Then,

$$J(c + \beta g, d + \beta h) \leq J(c, d).$$

Dividing by $\beta$, letting $\beta \to 0^+$, and using Lemma 3.2, we have

$$\int_\Omega (\lambda \xi c + \mu \eta d + \lambda u g + \mu v h - 2cg - 2\,dh) \leq 0, \tag{3.15}$$

where $(\xi, \eta)$ is the unique solution of (3.2). Now, multiplying in (3.8) the first equation by $\lambda \xi$, the second equation by $\mu \eta$, multiplying in (3.2) the first equation by $\lambda r$, the second equation by $\mu s$, integrating and subtracting both expressions, we obtain

$$\int_\Omega (\lambda c \xi + \mu\, d \eta + g u \lambda r + h v \mu s) = 0. \tag{3.16}$$

Combining (3.15) and (3.16), we deduce

$$\int_\Omega (-g u \lambda r - h v \mu s + \lambda u g + \mu v h - 2cg - 2\,dh) \leq 0. \tag{3.17}$$

If $h \equiv 0$, the previous relation is

$$\int_\Omega g(u \lambda(-r + 1) - 2c) \leq 0. \tag{3.18}$$

By a standard control argument concerning the sign of the variation $g$ depending on the size of $c$, we obtain the desired characterization of $c$. A similar reasoning may be done for $d$ (see [3]). ▮

COROLLARY 3.4. *Assume* [H1], [H2], *and* $\lambda$, $\mu$ *sufficiently small. Then, any optimal control* $(c, d) \in C_{\delta_1} \times C_{\delta_2}$ *may be expressed in the form,*

$$c = \frac{\lambda}{2} u(1 - r)^+, \qquad d = \frac{\mu}{2} v(1 - s)^+, \tag{3.19}$$

*where* $(u, v, r, s)$ *satisfies the optimality system*,

$$
\begin{aligned}
-\Delta u &= v(a - eu) - \frac{\lambda}{2} u^2 (1 - r)^+ - eu^2, \\
-\Delta v &= u(b - fv) - \frac{\mu}{2} u^2 (1 - s)^+ - fv^2, \\
-\Delta r &+ \left[ \frac{\lambda}{2} u(1 - r)^+ + e(2u + v) \right] r - \frac{\mu}{\lambda}(b - fv) s \\
&= \frac{\lambda}{2} u(1 - r)^+, \\
-\Delta s &+ \left[ \frac{\mu}{2} v(1 - s)^+ + f(u + 2v) \right] s - \frac{\lambda}{\mu}(a - eu) r \\
&= \frac{\mu}{2} v(1 - s)^+, \\
u > 0, \quad v > 0 \quad &\textit{in } \Omega; \qquad u = v = r = s = 0, \qquad \textit{on } \partial\Omega.
\end{aligned}
\tag{3.20}
$$

*Remarks*. 1. The previous expression for the optimal controls may be used for deducing some of their qualitative properties. For instance, under the hypotheses of the previous theorem, all the optimal controls $(c, d)$ must belong to the space $C(\overline{\Omega}) \times C(\overline{\Omega})$.

2. It would be possible to prove, by using similar ideas to those used in [3, 7], that under the hypotheses of the previous theorem, the operator,

$$T: C_{\delta_1} \times C_{\delta_2} \to C_{\delta_1} \times C_{\delta_2}$$

defined as

$$T(c, d) = \left( \frac{\lambda}{2} u(1 - r), \frac{\mu}{2} v(1 - s) \right),$$

where $(u, v)$ is the unique coexistence state of (1.1) and $(r, s)$ is the unique solution of (3.8), is contractive. In fact, for proving that the mapping $(c, d) \to (u, v)$ is Lipschitz, it is sufficient to consider in Lemma 3.2, instead of $(\xi_\beta, \eta_\beta)$, the functions $Du_{(c,d)}(g, h)$ and $Dv_{(c,d)}(g, h)$, the directional derivatives of $u$ and $v$, at $(c, d)$ in the direction $(g, h)$, and then to show that these derivatives are uniformly bounded.

The contractive character of $T$ may be used to demonstrate that when the parameters $\lambda$ and $\mu$ are sufficiently small, the optimal control is unique. Moreover, this gives an iterative scheme which provides a sequence of functions converging to the unique optimal control. This treatment would be essentially different from that presented in [3], where the concave character of the functional $J$ was established to obtain the uniqueness of the optimal control.

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