Hopf bifurcation for a maturity structured population dynamic model

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Abstract

This article is devoted to investigate some dynamical properties of a structured population dynamic model with random walk on $(0, +\infty)$. This model has a nonlinear and nonlocal boundary condition. We reformulate the problem as an abstract non-densely defined Cauchy problem, and use integrated semigroup theory to study such a partial differential equation. Moreover, a Hopf bifurcation theorem is given for this model.

Key words. Hopf bifurcation, random walk, integrated semigroups.

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1 Introduction

In this paper we investigate a Hopf bifurcation for the following system describing a random walk process on $(0, +\infty)$:

$$\begin{cases} \partial_t u^+(t,x) + c^+ \partial_x u^+(t,x) = -\sigma^+ u^+(t,x) + \sigma^- u^-(t,x) - \mu u^+(t,x), x \geq 0, \\ \partial_t u^-(t,x) - c^- \partial_x u^-(t,x) = \sigma^+ u^+(t,x) - \sigma^- u^-(t,x) - \mu u^-(t,x), \ x \geq 0, \\ c^+ u^+(t,0) = c^- u^-(t,0) + \alpha f(\int_0^{+\infty} \gamma(x) \left(u^+(t,x) + u^-(t,x)\right) dx), \\ \left(u^+(0,.), u^-(0,.)\right) = \left(u_0^+, u_0^-\right) \in L^1_+ \left((0,+\infty), \mathbb{R}\right) \times L^1_+ \left((0,+\infty), \mathbb{R}\right), \end{cases} \tag{1.1}$$

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where $c^+ > 0$, $c^- > 0$, $0 \leq \sigma^+ < \sigma^-$, $\mu > 0$ and

$$\gamma \in L^\infty_+ ((0, +\infty), \mathbb{R}).$$

In model (1.1), the variable x is maturity of the individuals (i.e. the ability of individuals to reproduce). Depending on the context, in population dynamics the maturity can be measured by using the age of individuals, but also their size or their weight. We refer to Arino [5], Arino and Sanchez [6], Calsina and Saldana [10], Calsina and Sanchón [11], Webb [56], and Ackleh and Deng [1] (and references therein) for studies on size-structured models in the context of ecology and cell population dynamics. In this article we are interested in the maturity measured by size or the weight of individuals.

At the individual level the growth in maturity can be viewed as a stochastic process. The point in such a model is to combine both the stochastic growth in maturity and the reproduction $\gamma(x)$. The combination of this two processes will influence the dynamical properties of the population. The constant $c^+ > 0$ (respectively $c^- > 0$) is the speed at which the maturity increases (respectively decays) at the individual level. The function $x \to u^+(t, x)$ (respectively $x \to u^-(t, x)$) is the density of population with growing maturity (respectively with decreasing maturity) at time t. The growing velocity is c^+, and the decaying velocity is $-c^-$. The multiplicative terms $\sigma^+ > 0$ and $\sigma^- > 0$ are called turning rates in the context of random walk (see [29]). This means that, individuals pass alternatively from the growing speed c^+ to the decaying speed $-c^-$ (and conversely). The time spent by an individual in the u^+-class (respectively u^--class) follows an exponential law with mean $1/\sigma^+$ (respectively $1/\sigma^-$). In other words, once individuals are born the maturity grows as a succession of increases and decreases with eventually an advantage or a disadvantage driven by the parameters σ^+ and σ^-, and/or driven by the speeds c^+ and $-c^-$. The mortality of individuals is described by the parameters μ. The total density of population is

$$u(t, x) := u^+(t, x) + u^-(t, x),$$

that is to say that for each $x_1, x_2 \in [0, +\infty)$ with $x_1 < x_2$, the quantity

$$\int_{x_1}^{x_2} u(t, x)dx$$

is the number of individuals with maturity x in between x_1 and x_2 at time t.

As far as we know the model (1.1) has not been considered in the context of population dynamics, while it seems very natural to introduce stochastic fluctuations between individuals to describe their growth in maturity. One may observe that when $c^- = 0$, the u^--class corresponds to a resting phase or a non growing phase. Models with resting phase have been studied in the context of cell population dynamic to describe the quiescences of cells. We refer to Gyllenberg and Webb [28], and Dyson, Villella-Bressan and Webb [21, 22] (and references therein) for results on this subject. In this article we will focus on the case $c^- > 0$ (the case $c^- = 0$ will be investigated elsewhere). In this case, the maturity can be understood as the weight of individuals, since at the individuals level the maturity can increase and decrease. The maturity may then stochastically fluctuate due to availability of food, or environmental fluctuations (temperature, etc...). The case $c^- > 0$ corresponds mathematically to a random walk process. Since the work of Kac [35], random walk model have been extensively used in the context of population dynamics. We refer to Hadeler [29] and references therein for a nice survey. For example, random walk model has been used to model chemotaxis phenomenon Hillen [30, 31, 32], or more recently to study pattern formation Eftimie et al. [23, 24, 25]. We also refer to Chalub et al. [14], Bellomo et al. [7] for more results going in that direction.

In usual size structured model (see [56] and references therein), given a group of individuals with the same size $x_0 \in (0, +\infty)$ at a given time t_0, all the individuals of this group will also have the same size in the future. The goal of model (1.1) is to describe the fact that given a group of individuals with the same maturity x_0 at a given time t_0, their maturities are likely to be different after a period of time. Indeed, due to the exchanges between the class u^+ and the class u^-, and since the time spent in the class u^+ and the class u^- is stochastic, the model (1.1) allows to describe such a phenomenon. So given a group of individuals with the same maturity $x_0 \in (0, +\infty)$ at a given time, their maturities will be distributed around some mean value after a period of time (i.e. the density of population will not be concentrated at one point in the future). This type of phenomenon has been previously considered in [15] by using a diffusion process. Here our goal is to reconsider this problem by using a random walk process.

In order to understand further the model (1.1), we now consider some special cases. First, by integrating system (1.1) with respect to x, we obtain

$$\frac{d \int_0^{+\infty} u(t,x)dx}{dt} = \alpha f \left(\int_0^{+\infty} \gamma(x) u(t,x)dx \right) - \mu \int_0^{+\infty} u(t,x)dx.$$

It follows that $\alpha f \left(\int_0^{+\infty} \gamma(x) u(t,x)dx \right)$ is the flux of new born individuals at time t. Through this article, we will assume that $f(x)$ is a Ricker's type birth function [46, 47], defined by

$$f(x) := x \exp(-\xi x),$$

for some constant $\xi > 0$. This type of birth function has been commonly used in the literature, to take into account some limitation of births when the population increases. The function $\gamma(x)$ takes into account the minimal size necessary for individuals to reproduce. When $\gamma \equiv 1$, then the total number of individuals $U(t) := \int_0^{+\infty} u(t,x)dx$ satisfies the following ordinary differential equation

$$\begin{cases} \dfrac{dU(t)}{dt} = \alpha f(U(t)) - \mu U(t), \forall t \geq 0, \\ U(0) = U_0 \geq 0. \end{cases}$$

Hence, the positive equilibrium (when it exists) is globally asymptotically stable, and no oscillations can occur around the positive equilibrium. This first shows that the oscillating properties strongly dependent on the specific choice of the function $\gamma(x)$.

Set

$$U^-(t) := \int_0^{+\infty} u^-(t,x)dx.$$

When the turning rate $\sigma^+ = 0$, we obtain

$$\begin{aligned} \frac{dU^-(t)}{dt} &= \int_0^{+\infty} c^- \partial_x u^-(t,x)dx - (\sigma^- + \mu)U^-(t) \\ &= -c^- u^-(t,0) - (\sigma^- + \mu)U^-(t) \leq -(\sigma^- + \mu)U^-(t), \end{aligned}$$

hence

$$U^-(t) \leq e^{-(\sigma^- + \mu)t} U^-(0) \to 0 \text{ as } t \to +\infty.$$

Thus when $\sigma^+ = 0$, the dynamical properties of system (1.1) are captured by the following system

$$\begin{cases} \partial_t u^+(t,x) + c^+ \partial_x u^+(t,x) = -\mu u^+(t,x), \\ c^+ u^+(t,0) = \alpha f(\int_0^{+\infty} \gamma(x) u^+(t,x)dx), \\ u^+(0,\cdot) = u_0^+ \in L^1_+(0,+\infty). \end{cases}$$

After the change of variable $x = c^+ a$ and $v(t,a) = u^+(t, c^+ a)$, the above equations corresponds exactly to the age-structured model considered in [43]. In this case, we can derive an Hopf bifurcation theorem under certain assumptions for the function $\gamma(x)$. One can note that the assumptions in [43] made on $\gamma(x)$ are only needed to analyze the characteristic equation. The same will be true for the present article.

Due to the large number of parameters, the mathematical analysis of model (1.1) is difficult in general. Here we will make some simplifying assumptions in order to obtain an approximation of the model (with convection and diffusion) presented in [15]. In what follows, we will make the following assumption.

Assumption 1.1 *We assume that*

$$c^+ = c^- := c > 0, \text{ and } 0 < \sigma^+ < \sigma^-.$$

Throughout the paper we will use the following notations

$$\eta_0 := \frac{\sigma^+ + \sigma^-}{2} \text{ and } \sigma_0 := \frac{\sigma^- - \sigma^+}{2},$$

which is equivalent to

$$\sigma^\pm = \eta_0 \mp \sigma_0.$$

Then the above assumption on σ^+ and σ^- are equivalent to

$$\eta_0 > \sigma_0 > 0.$$

Under Assumption 1.1, system (1.1) can be considered as a formal approximation of the following reaction diffusion equation

$$\begin{cases} \partial_t u(t,x) = \varepsilon^2 \partial_x^2 u(t,x) - \rho \partial_x u(t,x) - \mu u(t,x), \\ -\varepsilon^2 \partial_x u(t,0) + \rho u(t,0) = \alpha f(\int_0^{+\infty} \gamma(x) u(t,x) dx), \\ u(0,.) = u_0 := u_0^+ + u_0^- \in L^1_+ \left((0,+\infty), \mathbb{R}\right). \end{cases} \tag{1.2}$$

To be more precise, fix the mortality $\mu > 0$ and assume

$$\lim_{\eta_0 \to +\infty} \frac{c^2}{2(\eta_0 + \mu)} = \varepsilon^2 > 0 \quad \text{and} \quad \lim_{\eta_0 \to +\infty} \frac{c\sigma_0}{\eta_0 + \mu} = \rho > 0, \tag{1.3}$$

which means that when $\eta_0 \to +\infty$, both of c and σ_0 increase as a positive constant times $\sqrt{\eta_0}$. We introduce

$$v(t,x) = c\left(u^+(t,x) - u^-(t,x)\right).$$

With our notations, we have

$$v(t,0) = \alpha f(\int_0^{+\infty} \gamma(x) u(t,x) dx), \tag{1.4}$$

$$\partial_t u(t,x) + \partial_x v(t,x) = -\mu u(t,x), \tag{1.5}$$

$$\partial_t v(t,x) + c^2 \partial_x u(t,x) = -2c\left(\sigma^+ u^+ - \sigma^- u^-\right) - \mu v(t,x). \tag{1.6}$$

Assuming that $u(t,x)$ and $v(t,x)$ are $C^1\left([0,+\infty)^2, \mathbb{R}\right)$, differentiating (1.5) with respect to t and (1.6) with respect to x, then by eliminating $\partial^2_{xt} v(t,x)$ and note that

$$\partial_x v(t,x) = -\mu u(t,x) - \partial_t u(t,x), \tag{1.7}$$

we obtain the following telegraph equation

$$\begin{aligned} &\partial_t^2 u(t,x) + 2(\eta_0 + \mu)\partial_t u(t,x) \\ &= c^2 \partial_x^2 u(t,x) - 2c\sigma_0 \partial_x u(t,x) - (2\eta_0 + \mu)\mu u(t,x). \end{aligned}$$

For fixed $\mu > 0$, letting $\eta_0 \to +\infty$ and assuming (1.3) be satisfied, we formally obtain the following limit equation:

$$\partial_t u(t,x) = \varepsilon^2 \partial_x^2 u(t,x) - \rho \partial_x u(t,x) - \mu u(t,x). \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\partial_x v(t,x) = -\mu u(t,x) - \partial_t u(t,x) = -\varepsilon^2 \partial_x^2 u(t,x) + \rho \partial_x u(t,x). \tag{1.9}$$

Integrating both sides of (1.9) with respect to x over $(0,+\infty)$ and assuming that $v(t,.) \in W^{1,1}(0,+\infty), u(t,.) \in W^{2,1}(0,+\infty)$, we obtain

$$v(t,0) = -\varepsilon^2 \partial_x u(t,0) + \rho u(t,0). \tag{1.10}$$

The following boundary condition is derived by putting (1.10) into (1.4)

$$-\varepsilon^2 \partial_x u(t,0) + \rho u(t,0) = \alpha f\left(\int_0^{+\infty} \gamma(x) u(t,x) dx\right).$$

From the above computation, we can conclude that the distribution of the population $u(t,x)$ of our model (1.1) formally approximates to the solution of reaction diffusion system (1.2).

The existence of Hopf bifurcation has been studied for system (1.2) in [43] whenever $\varepsilon = 0$, and in [15] whenever $\varepsilon > 0$ and $\rho > 0$ (which is a parabolic system). The goal of this article is to extend the analysis to study the existence of Hopf bifurcation for hyperbolic system (1.1). This work is based on the Hopf bifurcation theorem proved in [39, Theorem 2.4] for semilinear non-densely defined abstract Cauchy problems.

We would like to point out that there are few results concerning the existence of non-trivial periodic solutions in the context of age/size structured models. We refer to Cushing [17, 18], Prüss [45], Swart [50], Kostova and Li [38], Bertoni [8], Magal and Ruan [44]. It is believed that such periodic solutions in age structured models are induced by Hopf bifurcations (Castillo-Chavez et al. [13], Inaba [33, 34], Zhang et al. [57]).

Hopf bifurcation analysis has been considered for various classes of partial differential equations in Amann [2], Crandall and Rabinowitz [16], Da Prato and Lunardi [19], Guidotti and Merino [27], Koch and Antman [37], Sandstede and Scheel [48], and Simonett [49]. However, since there is a nonlinear and nonlocal boundary condition in our model (1.1), their results and techniques do not apply to (1.1).

The paper is organized as follows. In Section 2, we reformulate (1.1) as a non-densely defined Cauchy problem, and the existence and uniqueness of the semiflow generated by (1.1) are investigated. The existence of the positive equilibrium is studied in Section 3. In Section 4, we linearize system (1.1) at the positive equilibrium, investigate the spectral properties of the linearized equation, and give the characteristic equation. The stability of the system is considered in Section 5. In Section 6, the Hopf bifurcation is studied when α is considered as the bifurcation parameter. In Section 6.1, we study the existence of purely imaginary eigenvalues of the characteristic equation. The transversality condition is studied in Section 6.2, and an Hopf bifurcation theorem is presented in Section 6.3. Finally, in Section 7 we summarize the results of the paper, and some bifurcation diagrams are presented, as well as some numerical simulations of (1.1).

2 Preliminary

In this section we follow the approach used by Thieme [51] for age structured model. We also refer to Arendt [3], Thieme [52], Kellermann and Hieber [36], the book of Arendt et al. [4] and Magal and Ruan [41, 42, 43] for details on the theory of integrated semigroups. Firstly, we rewrite the system (1.1) as a non-densely defined abstract Cauchy problem. Consider the Banach space

$$X = \mathbb{R} \times L^1(0,+\infty) \times L^1(0,+\infty)$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi_+ \\ \varphi_- \end{pmatrix} \right\| = |\alpha| + \|\varphi_+\|_{L^1(0,+\infty)} + \|\varphi_-\|_{L^1(0,+\infty)}.$$

The positive cone of X is

$$X_+ = \mathbb{R}_+ \times L^1_+(0,+\infty) \times L^1_+(0,+\infty).$$

We introduce the linear operator $A : D(A) \subset X \to X$ defined by

$$A \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} -c\varphi_+(0) + c\varphi_-(0) \\ -c\varphi'_+ \\ c\varphi'_- \end{pmatrix}$$

with

$$D(A) = \{0\} \times W^{1,1}(0,+\infty) \times W^{1,1}(0,+\infty).$$

It is easy to see that A is non-densely defined since

$$\overline{D(A)} = \{0\} \times L^1(0,+\infty) \times L^1(0,+\infty) \neq X.$$

Set

$$X_0 := \overline{D(A)} \text{ and } X_{0+} := X_0 \cap X_+.$$

Let $A_0 : D(A_0) \subset X_0 \to X_0$ be the part of A in X_0, which is defined by

$$A_0 x = Ax, \forall x \in D(A_0),$$

with

$$D(A_0) := \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\}.$$

It is readily checked that

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} \in \{0\} \times W^{1,1}(0,+\infty)^2 : -c\varphi_+(0) + c\varphi_-(0) = 0 \right\}.$$

Define the linear operator $L : \overline{D(A)} \to \overline{D(A)}$ by

$$L \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma^+\varphi_+ + \sigma^-\varphi_- - \mu\varphi_+ \\ \sigma^+\varphi_+ - \sigma^-\varphi_- - \mu\varphi_- \end{pmatrix}$$

and the map $F : \overline{D(A)} \to X$ by

$$F \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} \alpha f(\int_0^{+\infty} \gamma(x) \left(\varphi_+(x) + \varphi_-(x)\right) dx) \\ 0 \\ 0 \end{pmatrix}.$$

By identifying $\begin{pmatrix} u^+(t,.) \\ u^-(t,.) \end{pmatrix}$ to $v(t) = \begin{pmatrix} 0 \\ u^+(t,.) \\ u^-(t,.) \end{pmatrix}$, under Assumption 1.1, we rewrite the problem (1.1) as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + Lv(t) + F(v(t)), \text{ for } t \geq 0, \text{ and } v(0) = \begin{pmatrix} 0 \\ u_0^+ \\ u_0^- \end{pmatrix} \in \overline{D(A)}. \tag{2.1}$$

Since the range of L is include in X_0, $(A+L)_0$ the part of $A+L : D(A) \subset X \to X$ is defined by

$$(A + L)_0 x = (A + L)x, \forall x \in D((A + L)_0),$$

with

$$D((A + L)_0) = D(A_0).$$

Lemma 2.1 *The linear operator A is Hille-Yosida operator. More precisely, the resolvent set $\rho(A)$ of A satisfies*

$$(0, +\infty) \subset \rho(A),$$

and

$$\left\| (\lambda I - A)^{-n} \right\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}, \forall \lambda > 0, \forall n \geq 1.$$

Moreover for each $\lambda > 0$ we have

$$(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix}$$

$$\Leftrightarrow$$

$$\begin{cases} \varphi_+(x) = c^{-1} e^{-\frac{\lambda}{c} x} \left[\int_0^{+\infty} e^{-\frac{\lambda}{c} l} \psi_-(l) dl + \alpha \right] + c^{-1} \int_0^x e^{-\frac{\lambda}{c}(x-l)} \psi_+(l) dl, \\ \varphi_-(y) = c^{-1} \int_y^{+\infty} e^{-\frac{\lambda}{c}(l-y)} \psi_-(l) dl. \end{cases}$$

Furthermore A_0 is the infinitesimal generator of $\{T_{A_0}(t)\}_{t \geq 0}$ the strongly continuous semigroup of bounded linear operator on X_0 defined by

$$T_{A_0}(t) \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} 0 \\ T_{A_0}(t)_+ (\varphi_+, \varphi_-) \\ T_{A_0}(t)_- (\varphi_-) \end{pmatrix}$$

where

$$T_{A_0}(t)_- (\varphi_-)(x) = \varphi_-(x + ct),$$

and

$$T_{A_0}(t)_+ (\varphi_+, \varphi_-)(x) = \begin{cases} \varphi_+(x - ct), & \textit{if } x \geq ct, \\ \varphi_-(ct - x), & \textit{if } x < ct. \end{cases}$$

In what follows, for $z \in \mathbb{C}$, $\sqrt{z}$ denotes the principal branch of the general multi-valued function $z^{1/2}$. The branch cut is the negative real axis and the argument of z, denoted by $\arg(z)$, is π on the upper margin of the branch cut. So if $z = \rho e^{i\theta}$, $\theta \in (-\pi, \pi)$, $\rho > 0$, then $\sqrt{z} = \sqrt{\rho} e^{i\theta/2}$. In what follows, we will use the following notation:

$$\Omega := \{\lambda \in \mathbb{C} : Re(\lambda) > -\mu\}, \tag{2.2}$$

and

$$\Lambda := (\sigma_0)^2 + (\lambda + \mu)^2 + 2\eta_0(\lambda + \mu), \ \forall \lambda \in \mathbb{C}. \tag{2.3}$$

We now establish some inequalities which will be used in the following.

Lemma 2.2 *Let Assumption 1.1 be satisfied. Then for each $\lambda \in \Omega$, we have*

$$Re(\lambda) + \mu + \eta_0 > Re\sqrt{\Lambda} > \sigma_0 + Re(\lambda) + \mu.$$

Proof. Set

$$\sqrt{\Lambda} = a + ib, \text{ with } a > 0.$$

Then

$$Re(\Lambda) = a^2 - b^2 = (\sigma_0)^2 + (Re(\lambda) + \mu)^2 - (Im(\lambda))^2 + 2\eta_0 (Re(\lambda) + \mu), \tag{2.4}$$

$$Im(\Lambda) = 2ab = 2(Re(\lambda) Im(\lambda) + \mu Im(\lambda) + \eta_0 Im(\lambda)). \tag{2.5}$$

Since by assumption $\eta_0 > \sigma_0 > 0$, so by using (2.4) we obtain

$$a^2 < (Re(\lambda) + \mu + \eta_0),$$

thus

$$Re\sqrt{\Lambda} < Re(\lambda) + \mu + \eta_0.$$

From (2.5) we deduce that

$$Im(\lambda) = \frac{ab}{Re(\lambda) + \mu + \eta_0},$$

and then from (2.4) we obtain

$$a^2 = (\sigma_0)^2 + (Re(\lambda) + \mu)^2 - \left(\frac{ab}{Re(\lambda) + \mu + \eta_0}\right)^2 + 2\eta_0 (Re(\lambda) + \mu) + b^2,$$

or equivalently

$$a^2 = \left(1 + \left(\frac{b}{Re(\lambda) + \mu + \eta_0}\right)^2\right)^{-1} \times \left((\sigma_0)^2 + (Re(\lambda) + \mu)^2 + 2\eta_0 (Re(\lambda) + \mu) + b^2\right).$$

For $\lambda \in \Omega$, we have

$$a^2 - (\sigma_0 + Re\,(\lambda) + \mu)^2 = \left(1 + \left(\frac{b}{Re\,(\lambda) + \mu + \eta_0}\right)^2\right)^{-1}$$
$$\times \left(2\,(\eta_0 - \sigma_0)\,(Re\,(\lambda) + \mu) + b^2 \left(1 - \left(\frac{\sigma_0 + Re\,(\lambda) + \mu}{Re\,(\lambda) + \mu + \eta_0}\right)^2\right)\right) > 0,$$

$$(Re\,(\lambda) + \mu + \eta_0)^2 - a^2 = \left(1 + \left(\frac{b}{Re\,(\lambda) + \mu + \eta_0}\right)^2\right)^{-1} \times \left((\eta_0)^2 - (\sigma_0)^2\right) > 0,$$

and the result follows. ■

In the following lemma, we summarize some properties of $A + L$, and we obtain an explicit formula for the resolvent of $A + L$.

Lemma 2.3 *The linear operator $A + L : D(A) \subset X \to X$ is Hille-Yosida operator. We have the following inclusion*

$$\Omega \subset \rho\,(A + L) = \rho\,((A + L)_0)\,,$$

$$\left\|(\lambda I - (A + L))^{-n}\right\|_{\mathcal{L}(X)} \leq \frac{1}{(\lambda + \mu)^n}, \forall \lambda > -\mu, \forall n \geq 1,$$

and

$$(\lambda I - (A + L))^{-1} X_+ \subset X_+, \forall \lambda > -\mu \text{ (large enough).}$$

Moreover $\left\{T_{(A+L)_0}(t)\right\}_{t \geq 0}$ the strongly continuous semigroup of bounded linear operator on X_0 generated by $(A + L)_0$ satisfies

$$\omega_0\,((A + L)_0) = \lim_{t \to +\infty} \frac{\ln\left(\left\|T_{(A+L)_0}(t)\right\|\right)}{t} \leq -\mu. \tag{2.6}$$

Furthermore, for each $\lambda \in \Omega$ we have the following explicit formula:

$$(\lambda I - (A + L))^{-1} \begin{pmatrix} \alpha \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix}$$

is equivalent to

$$\varphi_+(x) = \frac{1}{\frac{c}{\sigma^-}(\zeta_1 - \zeta_2)} \int_x^{+\infty} e^{\zeta_1 (x-s)} \left(\frac{\psi_+(s)}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right) + \frac{\psi_-(s)}{c}\right) ds$$
$$+ e^{\zeta_2 x} c_2 + \frac{1}{\frac{c}{\sigma^-}(\zeta_1 - \zeta_2)} \int_0^x e^{\zeta_2 (x-s)} \left(\frac{\psi_+(s)}{\sigma^-}\left(\zeta_1 + \frac{\lambda + \sigma^+ + \mu}{c}\right) + \frac{\psi_-(s)}{c}\right) ds,$$

$$
\begin{aligned}
\varphi_-(x) = {} & \frac{c}{\sigma^-}\left(\zeta_1 + \frac{\lambda+\sigma^+ +\mu}{c}\right) \\
& \times \frac{1}{\frac{c}{\sigma^-}(\zeta_1-\zeta_2)} \int_x^{+\infty} e^{\zeta_1(x-s)} \left(\frac{\psi_+(s)}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right) + \frac{\psi_-(s)}{c}\right) ds \\
& + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right) \\
& \times \left(e^{\zeta_2 x} c_2 + \frac{1}{\frac{c}{\sigma^-}(\zeta_1-\zeta_2)} \int_0^x e^{\zeta_2(x-s)} \left(\frac{\psi_+(s)}{\sigma^-}\left(\zeta_1 + \frac{\lambda+\sigma^+ +\mu}{c}\right) + \frac{\psi_-(s)}{c}\right) ds\right),
\end{aligned}
$$

where

$$
\zeta_1 := \frac{\sigma_0 + \sqrt{\Lambda}}{c},\ \zeta_2 := \frac{\sigma_0 - \sqrt{\Lambda}}{c}, \tag{2.7}
$$

$$
c_1 := \frac{1}{\frac{c}{\sigma^-}(\zeta_1-\zeta_2)} \int_0^{+\infty} e^{-\zeta_1 s} \left(\frac{\psi_+(s)}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right) + \frac{\psi_-(s)}{c}\right) ds, \tag{2.8}
$$

$$
c_2 := \frac{\frac{\alpha}{c} + \left(-1 + \frac{c}{\sigma^-}\left(\zeta_1 + \frac{\lambda+\sigma^+ +\mu}{c}\right)\right) c_1}{1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right)}. \tag{2.9}
$$

Proof. The proof is straightforward. ■

By using the results in Thieme [51], Magal [40], and Magal and Ruan [42], we have the following theorem.

Theorem 2.4 ***(Existence)*** *There exists a unique continuous semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that $\forall x \in X_{0+}$, $t \to U(t)x$ is the unique integrated solution of*

$$
\frac{dU(t)x}{dt} = (A+L)\,U(t)x + F(U(t)x), U(0) = x,
$$

or equivalently,

$$
U(t)x = x + (A+L)\int_0^t U(l)x dl + \int_0^t F\,(U(l)x)\,dl, \forall t \geq 0.
$$

3 Positive equilibrium

Now we consider the positive equilibrium of (2.1).

Lemma 3.1 ***(Equilibrium)*** *There exists a positive equilibrium of the system (1.1) (or (2.1)) if and only if*

$$
R_0 := \alpha\chi > 1,
$$

where

$$\chi := \frac{1 + \frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0 + \frac{\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)} \int_0^{+\infty} \gamma\left(x\right) e^{(\zeta_2)_0 x} dx.$$

Moreover, when it exists, the positive equilibrium is unique and is given by $\bar{v} = \begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix}$ *with*

$$\begin{aligned} &\overline{u}^+(x) = e^{(\zeta_2)_0 x}\left(c_2\right)_0, \\ &\overline{u}^-(x) = \tfrac{c}{\sigma^-}\left(\left(\zeta_2\right)_0 + \tfrac{\sigma^+ + \mu}{c}\right) \times \left(e^{(\zeta_2)_0 x}\left(c_2\right)_0\right), \end{aligned}$$

where

$$\left(\zeta_2\right)_0 := \frac{\sigma_0 - \sqrt{\Lambda_0}}{c}, \tag{3.1}$$

$$\Lambda_0 := \left(\sigma_0\right)^2 + \mu^2 + 2\eta_0\mu, \tag{3.2}$$

and

$$\left(c_2\right)_0 := \frac{1}{c\left(1 - \frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)} \frac{\ln R_0}{\xi\chi}. \tag{3.3}$$

Proof. It is obvious that

$$\begin{aligned} & A\begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix} + L\begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix} + F\begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix} = 0 \\ \Leftrightarrow & \begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix} = \left(-\left(A + L\right)\right)^{-1}\begin{pmatrix} \alpha f\left(\int_0^{+\infty} \gamma\left(x\right)\left(\overline{u}^+(x) + \overline{u}^-(x)\right) dx\right) \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

According to the explicit formula of the resolvent of $A + L$ obtained in Lemma 2.3, taking $\lambda = 0$, we have

$$\begin{pmatrix} 0 \\ \overline{u}^+ \\ \overline{u}^- \end{pmatrix} = \left(-\left(A + L\right)\right)^{-1}\begin{pmatrix} \alpha f\left(\int_0^{+\infty} \gamma\left(x\right)\left(\overline{u}^+(x) + \overline{u}^-(x)\right) dx\right) \\ 0 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{aligned} &\overline{u}^+(x) = e^{(\zeta_2)_0 x}\left(c_2\right)_0, \\ &\overline{u}^-(x) = \tfrac{c}{\sigma^-}\left(\left(\zeta_2\right)_0 + \tfrac{\sigma^+ + \mu}{c}\right) \times \left(e^{(\zeta_2)_0 x}\left(c_2\right)_0\right), \end{aligned}$$

where

$$\left(\zeta_2\right)_0 = \frac{\sigma_0 - \sqrt{\Lambda_0}}{c},$$

and

$$\left(c_2\right)_0 = \frac{\alpha f\left(\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx\right)}{c\left(1-\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)}. \tag{3.4}$$

Since

$$\begin{aligned}
&\overline{u}^+(x)+\overline{u}^-(x)\\
&=\left(1+\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)\times\left(e^{(\zeta_2)_0 x}\left(c_2\right)_0\right)\\
&=e^{(\zeta_2)_0 x}\frac{1+\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)}{c\left(1-\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)}\alpha f\left(\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx\right),
\end{aligned} \tag{3.5}$$

it follows that

$$\overline{u}^++\overline{u}^-\neq 0 \text{ iff } \int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx\neq 0.$$

Integrating both sides of (3.5) after multiplying by $\gamma\left(x\right)$, we have

$$\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx=\chi\alpha f\left(\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx\right).$$

So we obtain

$$1=\chi\alpha\exp\left(-\xi\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx\right),$$

which is equivalent to

$$\int_0^{+\infty}\gamma\left(x\right)\left(\overline{u}^+(x)+\overline{u}^-(x)\right)dx=\frac{\ln\left(\chi\alpha\right)}{\xi}. \tag{3.6}$$

Substituting (3.6) into (3.4) yields

$$\left(c_2\right)_0=\frac{1}{c\left(1-\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)}\frac{\ln R_0}{\xi\chi},$$

and the result follows. ■

4 Linearized equation and spectral properties

From now on, we set the positive equilibrium $\bar{v}=\begin{pmatrix}0\\ \overline{u}^+\\ \overline{u}^-\end{pmatrix}$, where $\overline{u}^+$ and $\overline{u}^-$ are given in Lemma 3.1 with $R_0>1$.

The linearized system of (2.1) around $\bar{v}$ is

$$\frac{dv(t)}{dt} = (A + L)\, v(t) + DF(\bar{v})v(t) \text{ for } t \geq 0, v(t) \in X_0, \tag{4.1}$$

where

$$DF(\bar{v}) \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = \begin{pmatrix} \eta(\alpha) \int_0^{+\infty} \gamma(x) \left(\varphi_+(x) + \varphi_-(x)\right) dx \\ 0 \\ 0 \end{pmatrix},$$

with

$$\eta(\alpha) = \alpha f' \left(\int_0^{+\infty} \gamma(x) \left(\overline{u}^+(x) + \overline{u}^-(x)\right) dx \right).$$

Note that

$$f'(x) = e^{-\xi x}(1 - \xi x).$$

Using (3.6) it follows that

$$\eta(\alpha) = \frac{\alpha}{R_0}(1 - \ln R_0) = \frac{1}{\chi}\left(1 - \ln\left(\alpha\chi\right)\right).$$

This Cauchy problem (4.1) corresponds to the following system of linear hyperbolic partial differential equations

$$\begin{cases} \partial_t u^+(t,x) + c\partial_x u^+(t,x) = -\sigma^+ u^+(t,x) + \sigma^- u^-(t,x) - \mu u^+(t,x), \\ \partial_t u^-(t,x) - c\partial_x u^-(t,x) = \sigma^+ u^+(t,x) - \sigma^- u^-(t,x) - \mu u^-(t,x), \\ cu^+(t,0) = cu^-(t,0) + \eta(\alpha) \int_0^{+\infty} \gamma(x) \left(u^+(t,x) + u^-(t,x)\right) dx, \\ \left(u^+(0,.), u^-(0,.)\right) = \left(u_0^+, u_0^-\right) \in L^1_+\left((0,+\infty), \mathbb{R}\right) \times L^1_+\left((0,+\infty), \mathbb{R}\right). \end{cases} \tag{4.2}$$

Next we study the spectral properties of the linearized equation (4.2). To simplify the notation, we define $B_\alpha : D(B_\alpha) \subset X \to X$ as

$$B_\alpha x = Ax + Lx + DF(\bar{v})x \text{ with } D(B_\alpha) = D(A),$$

and we consider $(B_\alpha)_0$ the part of B_α in $\overline{D(A)}$, then

$$\left(B_\alpha\right)_0 x = B_\alpha x = Ax + Lx + DF(\bar{v})x$$

with

$$D\left((B_\alpha)_0\right) = \left\{ \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} \in 0 \times W^{1,1}(0,+\infty) \times W^{1,1}(0,+\infty) : \right.$$
$$\left. c\varphi_+(0) - c\varphi_-(0) = \eta(\alpha) \int_0^{+\infty} \gamma(x) \left(\varphi_+(x) + \varphi_-(x)\right) dx \right\}.$$

In the following lemma we mainly obtain an explicit formula for the resolvent of B_α. In what follows, this formula will be used to compute the order of the eigenvalue of B_α.

Lemma 4.1 *For each $\lambda \in \Omega = \{\lambda \in \mathbb{C} : Re(\lambda) > -\mu\}$, we have*

$$\lambda \in \rho(B_\alpha) \Leftrightarrow \Delta(\alpha, \lambda) \neq 0,$$

with

$$\Delta(\alpha, \lambda) := 1 - \eta(\alpha) \frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)} \int_0^{+\infty} \gamma(x) e^{\zeta_2 x} dx.$$

where

$$\zeta_2 := \frac{\sigma_0 - \sqrt{\Lambda}}{c}, \text{ and } \Lambda := (\sigma_0)^2 + (\lambda + \mu)^2 + 2\eta_0(\lambda + \mu).$$

Moreover, we have the following explicit formula:

$$(\lambda I - B_\alpha)^{-1} \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix}$$

$$\Leftrightarrow$$

$$\begin{aligned}
\varphi_+(x) &= \omega_1(x) + \omega_2(x) + \omega_3(x) + \Delta(\alpha, \lambda)^{-1} e^{\zeta_2 x} \tfrac{[\varsigma + \eta(\alpha) W(\lambda)]}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)}, \\
\varphi_-(x) &= \tfrac{c}{\sigma^-}\left(\zeta_1 + \tfrac{\lambda+\sigma^++\mu}{c}\right)\omega_1(x) + \tfrac{c}{\sigma^-}\left(\zeta_2 + \tfrac{\lambda+\sigma^++\mu}{c}\right)(\omega_2(x) + \omega_3(x)) \\
&\quad + \Delta(\alpha, \lambda)^{-1} \tfrac{c}{\sigma^-}\left(\zeta_2 + \tfrac{\lambda+\sigma^++\mu}{c}\right) e^{\zeta_2 x} \tfrac{[\varsigma + \eta(\alpha) W(\lambda)]}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)},
\end{aligned}$$

where

$$\begin{aligned}
\omega_1(x) &:= \tfrac{1}{\frac{c}{\sigma^-}(\zeta_1 - \zeta_2)} \int_x^{+\infty} \left(\tfrac{\psi_+(s)}{\sigma^-}\left(\zeta_2 + \tfrac{\lambda+\sigma^++\mu}{c}\right) + \tfrac{\psi_-(s)}{c}\right) e^{\zeta_1(x-s)} ds, \\
\omega_2(x) &:= \tfrac{1}{\frac{c}{\sigma^-}(\zeta_1 - \zeta_2)} \int_0^{x} \left(\tfrac{\psi_+(s)}{\sigma^-}\left(\zeta_1 + \tfrac{\lambda+\sigma^++\mu}{c}\right) + \tfrac{\psi_-(s)}{c}\right) e^{\zeta_2(x-s)} ds, \\
\omega_3(x) &:= e^{\zeta_2 x} \tfrac{-1 + \frac{c}{\sigma^-}\left(\zeta_1 + \frac{\lambda+\sigma^++\mu}{c}\right)}{1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)} c_1, \\
W(\lambda) &:= \int_0^{+\infty} \gamma(x) \left(\left(1 + \tfrac{c}{\sigma^-}\left(\zeta_1 + \tfrac{\lambda+\sigma^++\mu}{c}\right)\right)\omega_1(x)\right. \\
&\quad \left. + \left(1 + \tfrac{c}{\sigma^-}\left(\zeta_2 + \tfrac{\lambda+\sigma^++\mu}{c}\right)\right)(\omega_2(x) + \omega_3(x))\right) dx,
\end{aligned}$$

and c_1, ζ_1, ζ_2 are defined in Lemma 2.3.

Proof. Since $\lambda \in \Omega$, by Lemma 2.3, the linear operator $(\lambda I - A - L)$ is invertible. It follows that

$$\lambda I - B_\alpha \text{ is invertible} \Leftrightarrow I - DF(\bar{v})(\lambda I - A - L)^{-1} \text{ is invertible}$$

and

$$(\lambda I - B_\alpha)^{-1} = (\lambda I - A - L)^{-1}\left[I - DF(\bar{v})(\lambda I - A - L)^{-1}\right]^{-1}.$$

We start by computing $\left[I - DF(\bar{v})\left(\lambda I - A - L\right)^{-1}\right]^{-1}$. So we consider the system

$$\left[I - DF(\bar{v})\left(\lambda I - A - L\right)^{-1}\right] \begin{pmatrix} \widehat{\varsigma} \\ \widehat{\varphi_+} \\ \widehat{\varphi_-} \end{pmatrix} = \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} \widehat{\varsigma} \\ \widehat{\varphi_+} \\ \widehat{\varphi_-} \end{pmatrix} - \begin{pmatrix} \eta(\alpha) \int_0^{+\infty} \gamma(x) \left(\varphi_+(x) + \varphi_-(x)\right) dx \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix},$$

where

$$\begin{pmatrix} 0 \\ \varphi_+ \\ \varphi_- \end{pmatrix} = (\lambda I - A - L)^{-1} \begin{pmatrix} \widehat{\varsigma} \\ \widehat{\varphi_+} \\ \widehat{\varphi_-} \end{pmatrix}.$$

It follows that

$$\begin{aligned} \widehat{\varphi_+} &= \psi_+, \\ \widehat{\varphi_-} &= \psi_-, \end{aligned}$$

and it remains to consider

$$\widehat{\varsigma} - \eta(\alpha) \int_0^{+\infty} \gamma(x) \left(\varphi_+(x) + \varphi_-(x)\right) dx = \varsigma, \tag{4.3}$$

where the formula of $\varphi_+(x)$ and $\varphi_-(x)$ can be obtained by Lemma 2.3

$$\begin{aligned} \varphi_+(x) =& \omega_1(x) + \omega_2(x) + \omega_3(x) + e^{\zeta_2 x} \frac{\widehat{\varsigma}}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right)\right)}, \\ \varphi_-(x) =& \frac{c}{\sigma^-}\left(\zeta_1 + \frac{\lambda + \sigma^+ + \mu}{c}\right)\omega_1(x) + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right)\left(\omega_2(x) + \omega_3(x)\right) \\ &+ \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right) e^{\zeta_2 x} \frac{\widehat{\varsigma}}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right)\right)}. \end{aligned}$$

Hence, (4.3) is equivalent to

$$\widehat{\varsigma} - \widehat{\varsigma}\eta(\alpha) \frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda + \sigma^+ + \mu}{c}\right)\right)} \int_0^{+\infty} \gamma(x) e^{\zeta_2 x} dx = \eta(\alpha) W(\lambda) + \varsigma.$$

We deduce that $\left[I - DF(\bar{v})\left(\lambda I - A - L\right)^{-1}\right]$ is invertible if and only if $\Delta(\alpha, \lambda) \neq 0$. Moreover, we have

$$\left[I - DF(\bar{v})\left(\lambda I - A - L\right)^{-1}\right]^{-1} \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \widehat{\varsigma} \\ \widehat{\varphi_+} \\ \widehat{\varphi_-} \end{pmatrix}$$

is equivalent to

$$\begin{aligned} \widehat{\varphi_+} &= \psi_+, \\ \widehat{\varphi_-} &= \psi_-, \\ \widehat{\varsigma} &= \Delta(\alpha, \lambda)^{-1} \left[\eta(\alpha) W(\lambda) + \varsigma \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} (\lambda I - B_\alpha)^{-1} \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix} &= (\lambda I - A - L)^{-1} \left[I - DF(\bar{v}) (\lambda I - A)^{-1} \right]^{-1} \begin{pmatrix} \varsigma \\ \psi_+ \\ \psi_- \end{pmatrix} \\ &= (\lambda I - A - L)^{-1} \begin{pmatrix} \widehat{\varsigma} \\ \widehat{\varphi_+} \\ \widehat{\varphi_-} \end{pmatrix} \\ &= (\lambda I - A - L)^{-1} \begin{pmatrix} \Delta(\alpha, \lambda)^{-1} \left[\varsigma + \eta(\alpha) W(\lambda) \right] \\ \psi_+ \\ \psi_- \end{pmatrix}, \end{aligned}$$

and by Lemma 2.3, the result follows. ■

By using the above explicit formula for the resolvent of B_α we obtain the following lemma.

Lemma 4.2 *If $\lambda_0 \in \sigma(B_\alpha) \cap \Omega$, then λ_0 is a simple eigenvalue of B_α if and only if*

$$\frac{d\Delta(\alpha, \lambda_0)}{d\lambda} \neq 0.$$

Since $DF(\bar{v})$ is a bounded linear operator, and $(A + L)$ is a Hille-Yosida operator, it follows that $B_\alpha = A + L + DF(\bar{v})$ is a Hille-Yosida operator. Consequently, $(B_\alpha)_0$ is the infinitesimal generator of a strongly continuous semigroup $\left\{ T_{(B_\alpha)_0}(t) \right\}_{t \geq 0}$ on X_0.

Lemma 4.3 *The essential growth rate of $\left\{ T_{(B_\alpha)_0}(t) \right\}_{t \geq 0}$ satisfies the following estimation:*

$$\omega_{0,ess}\left((B_\alpha)_0 \right) \leq -\mu.$$

Proof. Since $DF(\bar{v})$ is compact, and $\omega_{0,ess}\left((A + L)_0 \right) \leq \omega_0\left((A + L)_0 \right) \leq -\mu$ (see Lemma 2.3), by using the result in Thieme [53] or Ducrot, Liu and Magal [20, Theorem 1.2], we obtain $\omega_{0,ess}\left((B_\alpha)_0 \right) \leq \omega_{0,ess}\left((A + L)_0 \right) \leq -\mu$. ■

Lemma 4.4 *We have*

$$\sigma((B_\alpha)_0) \cap \Omega = \sigma_p((B_\alpha)_0) \cap \Omega = \left\{ \lambda \in \Omega : \Delta(\alpha, \lambda) = 0 \right\}.$$

Proof. By Lemma 4.3 and results on spectral theory (see Webb [54, 55], Engel and Nagel [26]) we have

$$\sigma((B_\alpha)_0) \cap \Omega = \sigma_p((B_\alpha)_0) \cap \Omega.$$

Then by Lemma 4.1, the result follows. ■

5 Local stability

This section is devoted to the local stability of the positive steady state $\overline{v}$. Recall that this positive equilibrium $\overline{v}$ exists and is unique if and only if $R_0 > 1$. Since the essential growth rate $\omega_{0,ess}\left((B_\alpha)_0\right) < 0$, we apply the local stability result proved in Thieme [51] or in Magal and Ruan [42]. So it is sufficient to prove that all the eigenvalues of B_α have a strictly negative real part.

Lemma 5.1 *If $R_0 > 1$, then $\lambda = 0$ is not a root of the characteristic equation $\Delta(\alpha, \lambda) = 0$.*

Proof. Since

$$\begin{aligned}\Delta(\alpha, 0) &= 1 - \eta(\alpha)\frac{1 + \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)}\int_0^{+\infty} \gamma(x)e^{(\zeta_2)_0 x}dx \\ &= 1 - \eta(\alpha)\chi \\ &= 1 - \frac{1}{\chi}\left(1 - \ln\left(\alpha\chi\right)\right) \times \chi \\ &= \ln\left(\alpha\chi\right) = \ln R_0 > 0,\end{aligned}$$

the result follows. ■

Lemma 5.2 *If λ is a root of the characteristic equation such that*

$$Re(\lambda) \geq 0,$$

then we have

$$Re\left(\zeta_2\right) < \left(\zeta_2\right)_0,$$

and

$$Re\left(\sqrt{\Lambda}\right) > \sqrt{\Lambda_0} + Re\left(\lambda\right) \geq \sqrt{\Lambda_0}.$$

Proof. Set

$$\sqrt{\Lambda} = a + ib, \text{ with } a > 0.$$

By the proof of Lemma 2.2, it follows that

$$ab = Re\left(\lambda\right)Im\left(\lambda\right) + \mu Im\left(\lambda\right) + \eta_0 Im\left(\lambda\right). \tag{5.1}$$

$$\begin{aligned}a^2 = &\left(1 + \left(\frac{b}{Re\left(\lambda\right) + \mu + \eta_0}\right)^2\right)^{-1} \\ &\times \left(\left(\sigma_0\right)^2 + \left(Re\left(\lambda\right) + \mu\right)^2 + 2\eta_0\left(Re\left(\lambda\right) + \mu\right) + b^2\right).\end{aligned}$$

Hence

$$a^2 - \left(\sqrt{\Lambda_0} + Re\,(\lambda)\right)^2 = \left(1 + \left(\tfrac{b}{Re(\lambda)+\mu+\eta_0}\right)^2\right)^{-1} \times \left(2Re\,(\lambda)\left(\mu+\eta_0-\sqrt{\Lambda_0}\right) + \left(1 - \tfrac{\left(\sqrt{\Lambda_0}+Re(\lambda)\right)^2}{(Re(\lambda)+\mu+\eta_0)^2}\right) b^2\right). \tag{5.2}$$

If $Re\,(\lambda) = 0$ and $b = 0$, then by using (5.1), we have $Im\,(\lambda) = 0$, so $\lambda = 0$, which is impossible by Lemma 5.1. Thus if $Re\,(\lambda) \geq 0$, we have

$$(Re\,(\lambda))^2 + b^2 > 0.$$

Consequently, from (5.2) we obtain

$$a^2 - \left(\sqrt{\Lambda_0} + Re\,(\lambda)\right)^2 > 0,$$

i.e.,

$$Re\left(\sqrt{\Lambda}\right) > \sqrt{\Lambda_0} + Re\,(\lambda) \geq \sqrt{\Lambda_0}.$$

Since

$$\zeta_2 = \frac{\sigma_0 - \sqrt{\Lambda}}{c} \text{ and } (\zeta_2)_0 = \frac{\sigma_0 - \sqrt{\Lambda_0}}{c},$$

we deduce that

$$Re\,(\zeta_2) < (\zeta_2)_0\,,$$

which completes the proof. ∎

The main result of this section is the following theorem. One may observe that the condition obtained here is similar to the one obtained in [15, Theorem 5.3].

Theorem 5.3 *If*

$$1 < R_0 \leq e^2,$$

then the positive equilibrium $\bar{v}$ of the system (1.1) is locally asymptotically stable.

Proof. We consider the characteristic equation

$$\Delta(\alpha,\lambda) = 1 - \eta(\alpha)\frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)}\int_0^{+\infty}\gamma(x)e^{\zeta_2 x}dx = 0.$$

Since

$$Re\left(1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right) = \frac{-Re\left(\sqrt{\Lambda}\right) + Re\,(\lambda) + 2\eta_0 + \sigma_0 + \mu}{\sigma^-} > 0,$$

the above characteristic equation is equivalent to .

$$\frac{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)}{\left(1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)} = \eta(\alpha)\int_0^{+\infty}\gamma(x)e^{\zeta_2 x}dx. \tag{5.3}$$

By Lemma 5.2, if $Re\,(\lambda) \geq 0$, we must have

$$Re\,(\zeta_2) < (\zeta_2)_0\,.$$

Then it is derived from (5.3) that

$$\begin{aligned}
\left|\frac{c\left(1-\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right)}{\left(1+\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right)}\right| &= \left|\eta(\alpha)\int_0^{+\infty}\gamma(x)e^{\zeta_2 x}dx\right| \\
&\leq |\eta(\alpha)|\int_0^{+\infty}\gamma(x)e^{Re\zeta_2 x}dx \\
&< |\eta(\alpha)|\int_0^{+\infty}\gamma(x)e^{(\zeta_2)_0 x}dx \\
&= |\eta(\alpha)|\,\chi\frac{c\left(1-\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)\right)}{1+\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)}.
\end{aligned}$$

We claim that if $Re\,(\lambda) \geq 0$, then

$$\left|\frac{c\left(1-\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right)}{\left(1+\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right)}\right| > \frac{c\left(1-\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)\right)}{1+\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)}. \tag{5.4}$$

Since $\Lambda_0 = \sigma^2 + \mu^2 + 2\eta_0\mu$,

$$1+\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right) = 1+\frac{\eta_0+\mu-\sqrt{\Lambda_0}}{\sigma^-} > 0,$$

$$1-\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right) = 1-\frac{\eta_0+\mu-\sqrt{\Lambda_0}}{\sigma^-} = \frac{\sigma_0+\sqrt{\Lambda_0}-\mu}{\sigma^-} > 0,$$

inequality (5.4) is satisfied if

$$\begin{aligned}
&\left|1-\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right|^2\left(1+\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)\right)^2 \\
&> \left|1+\frac{c}{\sigma^-}\left(\zeta_2+\frac{\lambda+\sigma^++\mu}{c}\right)\right|^2\left(1-\frac{c}{\sigma^-}\left((\zeta_2)_0+\frac{\sigma^++\mu}{c}\right)\right)^2,
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
&\left(\sigma_0+Re\left(\sqrt{\Lambda}\right)-Re\,(\lambda)-\mu\right)^2\left(\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu\right)^2 \\
&\quad+\left(Im\left(\sqrt{\Lambda}\right)-Im\,(\lambda)\right)^2\left(\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu\right)^2 \\
&> \left(\sigma_0-Re\left(\sqrt{\Lambda}\right)+Re\,(\lambda)+2\eta_0+\mu\right)^2\left(\sigma_0+\sqrt{\Lambda_0}-\mu\right)^2 \\
&\quad+\left(Im\left(\sqrt{\Lambda}\right)-Im\,(\lambda)\right)^2\left(\sigma_0+\sqrt{\Lambda_0}-\mu\right)^2.
\end{aligned} \tag{5.5}$$

Since $\eta_0+\mu>\sqrt{\Lambda_0}$, we obtain

$$\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu>\sigma_0+\sqrt{\Lambda_0}-\mu>0.$$

It follows that

$$\begin{aligned}&\left(Im\left(\sqrt{\Lambda}\right)-Im\left(\lambda\right)\right)^2\left(\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu\right)^2\\ &\geq\left(Im\left(\sqrt{\Lambda}\right)-Im\left(\lambda\right)\right)^2\left(\sigma_0+\sqrt{\Lambda_0}-\mu\right)^2.\end{aligned}$$

Then (5.5) will be proved if

$$\begin{aligned}&\left(\sigma_0+Re\left(\sqrt{\Lambda}\right)-Re\left(\lambda\right)-\mu\right)^2\left(\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu\right)^2\\ &>\left(\sigma_0-Re\left(\sqrt{\Lambda}\right)+Re\left(\lambda\right)+2\eta_0+\mu\right)^2\left(\sigma_0+\sqrt{\Lambda_0}-\mu\right)^2.\end{aligned}$$

Since $p^2>q^2\Leftrightarrow(p+q)(p-q)>0$, after some simplifications, the above inequality is equivalent to,

$$\begin{aligned}&\left[\left(\sigma_0+Re\left(\sqrt{\Lambda}\right)-Re\left(\lambda\right)-\mu\right)\left(\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu\right)\right.\\ &+\left.\left(\sigma_0-Re\left(\sqrt{\Lambda}\right)+Re\left(\lambda\right)+2\eta_0+\mu\right)\left(\sigma_0+\sqrt{\Lambda_0}-\mu\right)\right]\\ &\times 2\left(\eta_0+\sigma_0\right)\left(Re\left(\sqrt{\Lambda}\right)-Re\left(\lambda\right)-\sqrt{\Lambda_0}\right)>0.\end{aligned}\tag{5.6}$$

Remember that $\Lambda_0=(\sigma_0)^2+\mu^2+2\eta_0\mu$ and $\eta_0>\sigma_0>0$, we have

$$\sigma_0-\sqrt{\Lambda_0}+2\eta_0+\mu>\eta_0+\mu-\sqrt{\Lambda_0}>0,\tag{5.7}$$

$$\sigma_0+\sqrt{\Lambda_0}-\mu>0.\tag{5.8}$$

By Lemma 5.2 we have

$$2\left(\eta_0+\sigma_0\right)\left(Re\left(\sqrt{\Lambda}\right)-Re\left(\lambda\right)-\sqrt{\Lambda_0}\right)>0.\tag{5.9}$$

By using again Lemma 5.2 and (5.8) we also have

$$\sigma_0+Re\left(\sqrt{\Lambda}\right)-Re\left(\lambda\right)-\mu>\sqrt{\Lambda_0}+\sigma_0-\mu>0.\tag{5.10}$$

By Lemma 2.2 we have $Re\left(\lambda\right)+\mu+\eta_0-Re\left(\sqrt{\Lambda}\right)>0$, and by noting that $\eta_0>\sigma_0>0$, we obtain

$$\sigma_0-Re\left(\sqrt{\Lambda}\right)+Re\left(\lambda\right)+2\eta_0+\mu>Re\left(\lambda\right)+\eta_0+\mu-Re\left(\sqrt{\Lambda}\right)>0.\tag{5.11}$$

It follows from (5.7)-(5.11) that (5.6) is satisfied. Consequently, inequality (5.4) is satisfied. Now we observe that if

$$|\eta(\alpha)|\,\chi\frac{c\left(1-\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)}{1+\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)}\leq\frac{c\left(1-\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)\right)}{1+\frac{c}{\sigma^-}\left(\left(\zeta_2\right)_0+\frac{\sigma^++\mu}{c}\right)},$$

i.e.,

$$|\eta(\alpha)|\,\chi \leq 1, \tag{5.12}$$

there will be no roots of the characteristic equation with non-negative real part. Since

$$\eta(\alpha) = \frac{\alpha}{R_0}(1 - \ln R_0) = \frac{1}{\chi}\left(1 - \ln R_0\right),$$

the above inequality (5.12) is equivalent to

$$|\ln R_0 - 1| \leq 1,$$

and the result follows. ∎

6 Hopf bifurcation

In this section, regarding α as the bifurcation parameter we will study the existence of Hopf bifurcation by using the Hopf bifurcation theory developed by Liu, Magal and Ruan [39, Theorem 2.4]. In order to apply this Hopf bifurcation theorem to system (2.1), we need to show the four following properties:

(i) the essential growth rate of $(B_\alpha)_0$ is strictly negative;

(ii) the existence of a unique pair of purely imaginary eigenvalues of B_{α_0} for some fixed parameter α_0;

(iii) the purely imaginary eigenvalues of B_{α_0} are simple;

(iv) the transversality condition.

Notice that $\omega_{0,ess}\left((B_\alpha)_0\right) < 0$ has already been proved in Lemma 4.3. So we only need to focus on the three last properties.

By Theorem 5.3 we already know that the positive equilibrium $\bar{v}$ of the system (1.1) is locally asymptotically stable if

$$1 < R_0 \leq e^2,$$

that is

$$\alpha \in \left(\widehat{\alpha}_0, \widehat{\alpha}_1\right],$$

where

$$R_0 = \alpha\chi,\ \widehat{\alpha}_0 := \frac{1}{\chi} \text{ and } \widehat{\alpha}_1 := \frac{e^2}{\chi}.$$

So we will study the existence of bifurcation value in $\left(\widehat{\alpha}_1, +\infty\right)$. Recall the characteristic equation

$$0 = \Delta(\alpha, \lambda) = 1 - \eta(\alpha)\frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ +\mu}{c}\right)\right)}\int_0^{+\infty} \gamma(x)e^{\zeta_2 x}dx, \tag{6.1}$$

where

$$
\begin{aligned}
\zeta_2 &= \frac{\sigma_0 - \sqrt{\Lambda}}{c}, \quad \Lambda = (\sigma_0)^2 + (\lambda+\mu)^2 + 2\eta_0(\lambda+\mu), \\
(\zeta_2)_0 &= \frac{\sigma_0 - \sqrt{\Lambda_0}}{c}, \ \Lambda_0 = (\sigma_0)^2 + \mu^2 + 2\eta_0\mu, \ \sigma^{\mp} = \eta_0 \pm \sigma_0, \\
\eta(\alpha) &= \frac{\alpha}{R_0}(1 - \ln R_0) = \frac{1}{\chi}\left(1 - \ln(\alpha\chi)\right), \\
\chi &= \frac{1 + \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)} \int_0^{+\infty} \gamma(x)\, e^{(\zeta_2)_0 x} dx.
\end{aligned}
$$

It is clear that

$$\eta(\alpha) < 0 \text{ for } \alpha > \widehat{\alpha}_1.$$

6.1 Existence and uniqueness of purely imaginary eigenvalues

This subsection is devoted to the existence and the uniqueness of purely imaginary roots of the characteristic equation (6.1) under the following assumption on the function γ.

Assumption 6.1 *Assume that*

$$\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x)$$

for $\beta > 0, n \in \mathbb{N}, \tau > 0,$ *or* $\beta = 0, n = 0, \tau > 0.$

Let Assumption 6.1 be satisfied. In order to obtain the explicit formula for the characteristic equation, we first give the following computation

$$
\begin{aligned}
\int_0^{+\infty} \gamma(x)\, e^{\zeta_2 x} dx &= e^{\beta\tau} \int_\tau^{+\infty} (x-\tau)^n e^{(\zeta_2-\beta)x} dx \\
&= e^{\beta\tau} \int_0^{+\infty} s^n e^{(\zeta_2-\beta)(s+\tau)} ds \\
&= -e^{\beta\tau} e^{(\zeta_2-\beta)\tau} \int_{-\infty}^0 \left(\frac{l}{\zeta_2-\beta}\right)^n e^l \frac{1}{\zeta_2-\beta} dl \\
&= \frac{-e^{\zeta_2\tau}}{(\zeta_2-\beta)^{n+1}} \int_0^{+\infty} (-1)^n x^n e^{-x} dx \\
&= \frac{(-1)^{n+1} e^{\zeta_2\tau} n!}{(\zeta_2-\beta)^{n+1}} = \frac{n! e^{\zeta_2\tau}}{(\beta-\zeta_2)^{n+1}}.
\end{aligned}
$$

So the characteristic equation (6.1) becomes

$$0 = 1 - \eta(\alpha) \frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^+ + \mu}{c}\right)\right)} \frac{n! e^{\zeta_2\tau}}{(\beta-\zeta_2)^{n+1}}.$$

If we set

$$\sqrt{\Lambda} = a + ib \text{ with } a > 0,$$

by noting that $\sigma^{\pm} = \eta_0 \mp \sigma_0$ and the expression of ζ_2, we arrive at the following characteristic equation

$$1 - n!c^n \eta(\alpha) \frac{2\eta_0 + \sigma_0 - a - ib + \lambda + \mu}{\sigma_0 + a + ib - \lambda - \mu} \frac{\exp\left((\sigma_0 - a - ib)\,\tau c^{-1}\right)}{(c\beta - \sigma_0 + a + ib)^{n+1}} = 0. \tag{6.2}$$

Now we are in the position to look for purely imaginary roots $\lambda = \omega i$ with $\omega > 0$ in (6.2).

Since

$$\sqrt{\Lambda} = \sqrt{(\sigma_0)^2 + (i\omega + \mu)^2 + 2\eta_0(i\omega + \mu)} = a + ib \text{ with } a > 0,$$

by the proof of Lemma 2.2 we obtain $b > 0$,

$$a = a(b) := \left(1 + \left(\frac{b}{\mu + \eta_0}\right)^2\right)^{-\frac{1}{2}} \left((\sigma_0)^2 + \mu^2 + 2\eta_0\mu + b^2\right)^{\frac{1}{2}}, \tag{6.3}$$

and

$$\omega = \omega(b) := \frac{a(b)b}{\mu + \eta_0}, \tag{6.4}$$

that is to say that both of a and ω are functions of b. So in order to obtain $\lambda = i\omega$ with $\omega > 0$, it only remains to determine b in (6.2).

By Lemma 2.2, we have $\mu + \eta_0 - a > 0$ and $a - \sigma_0 - \mu > 0$, which imply

$$b - \omega = b - \frac{ab}{\mu + \eta_0} = \frac{\mu + \eta_0 - a}{\mu + \eta_0} b > 0,$$

$$2\eta_0 + \sigma_0 + \mu - a > 0,$$

$$c\beta - \sigma_0 + a > 0.$$

Let

$$\begin{aligned}
(2\eta_0 + \sigma_0 + \mu - a) - i\,(b - \omega) \quad &: \quad = r_1(b)e^{i\theta_1(b)}, \\
(a + \sigma_0 - \mu) + i\,(b - \omega) \quad &: \quad = r_2(b)e^{i\theta_2(b)}, \\
c\beta - \sigma_0 + a + ib \quad &: \quad = r_3(b)e^{i\theta 3(b)}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
r_1(b) &= \sqrt{(2\eta_0 + \sigma_0 + \mu - a)^2 + (b - \omega)^2}, \\
r_2(b) &= \sqrt{(a + \sigma_0 - \mu)^2 + (b - \omega)^2}, \\
r_3(b) &= \sqrt{(c\beta - \sigma_0 + a)^2 + b^2}, \\
\theta_1(b) &= \arctan \frac{-b + \omega}{2\eta_0 + \sigma_0 + \mu - a}, \\
\theta_2(b) &= \arctan \frac{b - \omega}{a + \sigma_0 - \mu}, \\
\theta_3(b) &= \arctan \frac{b}{c\beta - \sigma_0 + a}.
\end{aligned}
\tag{6.5}
$$

Hence it follows from (6.2) that

$$
1 - n!c^n \eta(\alpha) \frac{r_1(b)e^{i\theta_1(b)}}{r_2(b)e^{i\theta_2(b)}} \frac{\exp\left((\sigma_0 - a - ib)\,\tau c^{-1}\right)}{(r_3(b))^{n+1}\, e^{i(n+1)\theta_3(b)}} = 0,
$$

i.e.,

$$
1 = n!c^n \eta(\alpha) \frac{r_1(b) \exp\left((\sigma_0 - a)\,\tau c^{-1}\right)}{r_2(b)\,(r_3(b))^{n+1}} \exp i(\theta_1(b) - \theta_2(b) - (n+1)\,\theta_3(b) - \frac{b\tau}{c}).
$$

Since $\eta(\alpha) < 0$ for $\alpha > \widehat{\alpha}_1$, taking norm of both sides of the above equation we deduce that

$$
1 = -n!c^n \eta(\alpha) \frac{r_1(b)}{r_2(b)} \frac{\exp\left((\sigma_0 - a)\,\tau c^{-1}\right)}{(r_3(b))^{n+1}},
$$

and consequently we have

$$
1 = -\exp i(\theta_1(b) - \theta_2(b) - (n+1)\,\theta_3(b) - \frac{b\tau}{c}).
$$

It follows that

$$
\eta(\alpha) = -\frac{r_2(b)\,(r_3(b))^{n+1}}{n!r_1(b)c^n} \exp\left((-\sigma_0 + a)\,\tau c^{-1}\right) \tag{6.6}
$$

and

$$
\theta_1(b) - \theta_2(b) - (n+1)\,\theta_3(b) - \frac{b\tau}{c} = -\pi - 2k\pi, k \in \mathbb{Z}.
$$

Because

$$
\eta(\alpha) = \frac{\alpha}{R_0}(1 - \ln R_0) = \frac{1}{\chi}\left(1 - \ln(\alpha\chi)\right),
$$

combining with (6.6) we obtain

$$
\alpha = \frac{1}{\chi} \exp\left(1 + \chi \frac{r_2(b)\,(r_3(b))^{n+1}}{n!r_1(b)c^n} \exp\left((-\sigma_0 + a)\,\tau c^{-1}\right)\right),
$$

where

$$\chi = \frac{1 + \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)} \frac{n! e^{(\zeta_2)_0 \tau}}{(\beta - (\zeta_2)_0)^{n+1}}. \tag{6.7}$$

Through the above computation we obtain the following proposition to show the existence of purely imaginary eigenvalues.

Proposition 6.2 *Let Assumptions 1.1 and 6.1 be satisfied. Then the characteristic equation (6.1) has a pair of purely imaginary solutions $\pm i\omega$, with $\omega > 0$ if and only if there exists $b > 0$ satisfying*

$$\Theta(b) := \theta_1(b) - \theta_2(b) - (n+1)\,\theta_3(b) - \frac{b\tau}{c} = -\pi - 2k\pi, \tag{6.8}$$

for some $k \in \mathbb{Z}$ and

$$\alpha = \alpha(b) := \frac{1}{\chi} \exp\left(1 + \chi \frac{r_2(b)\,(r_3(b))^{n+1}}{n! r_1(b) c^n} \exp\left((-\sigma_0 + a(b))\,\tau c^{-1}\right)\right), \tag{6.9}$$

where $a(b), \omega(b), r_1(b), r_2(b), r_3(b), \theta_1(b), \theta_2(b), \theta_3(b), \chi$ are defined in (6.3)-(6.5) and (6.7). Moreover, there exists a sequence $b_k \to +\infty$ as $k \to +\infty$, $k \in \mathbb{N}$ satisfying (6.8), and the characteristic equation (6.1) admits at least one pair of purely imaginary solution $\pm i\omega_k := \pm i\omega(b_k)$ for $\alpha_k := \alpha(b_k)$.

Proof. It is sufficient to prove that there exists a sequence b_k satisfying (6.8) and $b_k \to +\infty$ as $k \to +\infty$. Clearly, $\Theta(0) = 0$, $\Theta(+\infty) = -\infty$, and Θ is a continuous function with respect to b. Then for any $k \in \mathbb{N}$ there exists some $b_k > 0$ such that (6.8) is satisfied, and $b_k \to +\infty$ as $k \to +\infty$. ∎

In what follows, we will consider the case for $b_k \to +\infty$. Firstly, we give the following lemma which can be easily proved.

Lemma 6.3 *Let*

$$a = \left(1 + \left(\frac{b}{\mu + \eta_0}\right)^2\right)^{-\frac{1}{2}} \left((\sigma_0)^2 + \mu^2 + 2\eta_0\mu + b^2\right)^{\frac{1}{2}},$$

and

$$\omega = \frac{ab}{\mu + \eta_0}.$$

Then we have

$$
\begin{aligned}
&\lim_{b\to+\infty} a = \mu + \eta_0, \\
&\lim_{b\to+\infty} \frac{da}{db} = \lim_{b\to+\infty} b\frac{da}{db} = \lim_{b\to+\infty} b^2\frac{da}{db} = 0, \quad \lim_{b\to+\infty} b^3\frac{da}{db} = \left((\eta_0)^2 - (\sigma_0)^2\right)(\mu+\eta_0), \\
&\lim_{b\to+\infty} (b-\omega) = 0, \quad \lim_{b\to+\infty} b(b-\omega) = \frac{(\eta_0)^2 - (\sigma_0)^2}{2}, \\
&\lim_{b\to+\infty} \frac{d\omega}{db} = 1, \quad \lim_{b\to+\infty} b\left(1 - \frac{d\omega}{db}\right) = 0, \\
&\lim_{b\to+\infty} b^2\frac{d^2a}{db^2} = 0, \quad \lim_{b\to+\infty} b^2\frac{d^2\omega}{db^2} = 0.
\end{aligned}
$$

Next we give the following lemma to show that under Assumption 6.1, for any given $\alpha > 0$ large enough, there exists at most one pair of purely imaginary solutions of the characteristic equation.

Lemma 6.4 ***(Uniqueness)*** *Let Assumptions 1.1 and 6.1 be satisfied. Then for each $\alpha > 0$ large enough, $\Delta(\alpha, i\omega_1) = \Delta(\alpha, i\omega_2) = 0$ with $\omega_1, \omega_2 > 0$ implies $\omega_1 = \omega_2$.*

Proof. By Proposition 6.2, we know that if $\Delta(\alpha, i\omega) = 0$, then we have the following expression for α

$$
\alpha = \frac{1}{\chi}\exp\left(1 + \chi\frac{r_2(b)\,(r_3(b))^{n+1}}{n!r_1(b)c^n}\exp\left((-\sigma_0 + a)\,\tau c^{-1}\right)\right),
$$

where

$$
\chi = \frac{1 + \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left((\zeta_2)_0 + \frac{\sigma^+ + \mu}{c}\right)\right)} \frac{n!e^{(\zeta_2)_0\tau}}{(\beta - (\zeta_2)_0)^{n+1}}
$$

which is independent of ω, and $r_1(b), r_2(b), r_3(b), a, b$ can be seen as functions of ω. To be more precise, we have the following formulae for $r_1(b), r_2(b), r_3(b), a$ and b

$$
\begin{aligned}
r_1(b) &= \sqrt{(2\eta_0 + \sigma_0 + \mu - a)^2 + (b-\omega)^2}, \\
r_2(b) &= \sqrt{(a + \sigma_0 - \mu)^2 + (b-\omega)^2}, \\
r_3(b) &= \sqrt{(c\beta - \sigma_0 + a)^2 + b^2}, \\
a &= \left(1 + \left(\frac{b}{\mu+\eta_0}\right)^2\right)^{-\frac{1}{2}} \left((\sigma_0)^2 + \mu^2 + 2\eta_0\mu + b^2\right)^{\frac{1}{2}}, \\
b &= \frac{\mu+\eta_0}{a}\omega.
\end{aligned}
$$

In order to prove this lemma, we will prove that $\frac{d\alpha(b)}{db} > 0$ for all $b \geq 0$ large enough, and $\frac{d\omega(b)}{db} > 0, \forall b \geq 0$. So one deduce that we can construct a map $\omega \to \alpha(\omega)$. Moreover for each $\omega > 0$ large enough, we will prove that

$$\frac{d\alpha(\omega)}{d\omega} > 0,$$

and the result will follows.
Step 1: We claim that

$$\frac{d\alpha(b)}{db} > 0, \text{ for } b \text{ large enough},$$

i.e.,

$$\frac{d}{db}\left(\frac{r_2(b)\left(r_3(b)\right)^{n+1}}{r_1(b)c^n}\exp\left(\frac{a\tau}{c}\right)\right) > 0, \text{ for } b \text{ large enough}. \tag{6.10}$$

Indeed, we have

$$\frac{d}{db}\left(\frac{r_2(b)\left(r_3(b)\right)^{n+1}}{r_1(b)c^n}\exp\left(\frac{a\tau}{c}\right)\right) = \left(r_3(b)\right)^n\left(\exp\left(\frac{a\tau}{c}\right)\right)c^{-n} \times A,$$

where

$$\begin{aligned} A :=& \left(r_1(b)\right)^{-2}\left(\left(\frac{d}{db}r_2(b)\right)r_3(b)r_1(b) + (n+1)\,r_2(b)\left(\frac{d}{db}r_3(b)\right)r_1(b) - r_2(b)r_3(b)\left(\frac{d}{db}r_1(b)\right)\right) \\ &+ \left(r_1(b)\right)^{-1}r_2(b)r_3(b)\frac{\tau}{c}\frac{da}{db}. \end{aligned}$$

Thus in order to prove (6.10) we only need to show that $A > 0$ for b large enough. By Lemma 6.3 we have

$$\begin{aligned} a \to& \quad \mu + \eta_0, \text{ as } b \to +\infty, \\ r_1(b) \text{ and } r_2(b) \to& \quad \eta_0 + \sigma_0, \text{ as } b \to +\infty, \qquad (6.11)\\ r_3(b) \to& \quad +\infty, \text{ as } b \to +\infty, \\ \frac{b}{r_3(b)} \to& \quad 1, \text{ as } b \to +\infty, \qquad (6.12) \end{aligned}$$

and a basic calculation leads to

$$\begin{aligned} \frac{d}{db}r_1(b) =& \frac{1}{r_1(b)}\left((2\eta_0 + \sigma_0 + \mu - a)\left(-\frac{da}{db}\right) + (b - \omega)\left(1 - \frac{d\omega}{db}\right)\right) \qquad (6.13)\\ &\to 0, \text{ as } b \to +\infty, \\ \frac{d}{db}r_2(b) =& \frac{1}{r_1(b)}\left((a + \sigma_0 - \mu)\frac{da}{db} + (b - \omega)\left(1 - \frac{d\omega}{db}\right)\right) \to 0, \text{ as } b \to +\infty, \\ \frac{d}{db}r_3(b) =& \frac{1}{r_3(b)}\left((c\beta - \sigma_0 + a)\frac{da}{db} + b\right) \to 1, \text{ as } b \to +\infty. \qquad (6.14) \end{aligned}$$

Because

$$\begin{aligned}
&\lim_{b\to+\infty}\left(\frac{d}{db}r_2(b)\right)(r_3(b)) = \lim_{b\to+\infty}\frac{\frac{d}{db}r_2(b)}{(r_3(b))^{-1}} = \lim_{b\to+\infty}\frac{\frac{d^2}{db^2}r_2(b)}{-(r_3(b))^{-2}\,\frac{d}{db}r_3(b)}\\
= &\lim_{b\to+\infty}\left(\frac{r_3(b)}{b}\right)^2\left(\frac{d}{db}r_3(b)\right)^{-1}\frac{\frac{d}{db}r_1(b)}{(r_1(b))^2}\\
&\times\left((a+\sigma_0-\mu)\left(b^2\frac{da}{db}\right)+(b(b-\omega))\left(b\left(1-\frac{d\omega}{db}\right)\right)\right)\\
&-\lim_{b\to+\infty}\left(\frac{r_3(b)}{b}\right)^2\frac{1}{r_1(b)}\left(\frac{d}{db}r_3(b)\right)^{-1}\\
&\times\left(b^2\left(\frac{da}{db}\right)^2+(a+\sigma_0-\mu)\,b^2\frac{d^2a}{db^2}+\left(b\left(1-\frac{d\omega}{db}\right)\right)^2-(b-\omega)\left(b^2\frac{d^2\omega}{db^2}\right)\right),
\end{aligned}$$

then by Lemma 6.3, (6.11), (6.12), (6.13) and (6.14) we obtain that

$$\lim_{b\to+\infty}\left(\frac{d}{db}r_2(b)\right)(r_3(b)) = 0.$$

Similarly, we can deduce that

$$\lim_{b\to+\infty}\left(\frac{d}{db}r_1(b)\right)(r_3(b)) = 0.$$

Therefore, we have

$$\begin{aligned}
\lim_{b\to+\infty}A &= \lim_{b\to+\infty}\left((r_1(b))^{-2}\left((n+1)\,r_2(b)\left(\frac{d}{db}r_3(b)\right)r_1(b)\right)+(r_1(b))^{-1}\,r_2(b)\frac{r_3(b)}{b}\frac{\tau}{c}\left(b\frac{da}{db}\right)\right)\\
&= \lim_{b\to+\infty}(r_1(b))^{-2}\left((n+1)\,r_2(b)\left(\frac{d}{db}r_3(b)\right)r_1(b)\right) = n+1>0,
\end{aligned}$$

which completes the proof for this step.
Step 2: By using the limits (6.11) and (6.12) obtained in Step 1, and by using the expression for α (i.e. (6.9)) we deduce that

$$\alpha\to+\infty,\ \text{as } b\to+\infty.$$

Step 3: We prove

$$\frac{d\omega}{db}>0, \forall b\geq 0.$$

Since $\omega = \frac{ab}{\mu+\eta_0}$ and $a = \left(1+\left(\frac{b}{\mu+\eta_0}\right)^2\right)^{-\frac{1}{2}} \left((\sigma_0)^2 + \mu^2 + 2\eta_0\mu + b^2\right)^{\frac{1}{2}}$, we have

$$\begin{aligned}
\frac{d\omega}{db} &= \frac{d}{db}\left(\frac{ab}{\mu+\eta_0}\right) \\
&= \frac{a}{\mu+\eta_0} + \frac{b}{\mu+\eta_0} \times \frac{da}{db} \\
&= \frac{a}{\mu+\eta_0} + \frac{b}{\mu+\eta_0} \times \frac{b}{a\left(1+\left(\frac{b}{\mu+\eta_0}\right)^2\right)^2} \frac{(\eta_0)^2 - (\sigma_0)^2}{(\mu+\eta_0)^2} > 0,
\end{aligned}$$

and the result in Step 3 follows.

Step 4: Finally by using the formula $\omega(b) = \frac{a(b)b}{\mu+\eta_0}$, and since $a(b) \to \mu + \eta_0$, as $b \to +\infty$, we deduce that

$$\omega(0) = 0, \text{ and } \lim_{b\to+\infty} \omega(b) = +\infty,$$

and the proof is completed. ∎

6.2 Transversality condition

The aim of this section is to prove a transversality condition for model (1.1) under Assumptions 1.1 and 6.1.

Lemma 6.5 *If* $\alpha > \widehat{\alpha}_1$, $\lambda \in \Omega$ *and* $\Delta(\alpha, \lambda) = 0$, *then*

$$\frac{\partial \Delta(\alpha, \lambda)}{\partial \alpha} < 0.$$

Proof. We have

$$\Delta(\alpha, \lambda) = 1 - \eta(\alpha) \frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)} \int_0^{+\infty} \gamma(x) e^{\zeta_2 x} dx,$$

$$\eta(\alpha) = \frac{1}{\chi}\left(1 - \ln(\alpha\chi)\right),$$

where χ is independent of α, then

$$\frac{d\eta(\alpha)}{d\alpha} = -\frac{1}{\alpha\chi},$$

$$\frac{\partial \Delta(\alpha, \lambda)}{\partial \alpha} = -\frac{d\eta(\alpha)}{d\alpha} \frac{1 + \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)}{c\left(1 - \frac{c}{\sigma^-}\left(\zeta_2 + \frac{\lambda+\sigma^++\mu}{c}\right)\right)} \int_0^{+\infty} \gamma(x) e^{\zeta_2 x} dx.$$

Since $\Delta(\alpha, \lambda) = 0$, we obtain

$$\frac{\partial \Delta(\alpha, \lambda)}{\partial \alpha} = -\frac{d\eta(\alpha)}{d\alpha} \times \frac{1}{\eta(\alpha)} = \frac{1}{\alpha\left(1 - \ln\left(\alpha\chi\right)\right)}.$$

Moreover, if $\alpha > \widehat{\alpha}_1$, then $\frac{\partial \Delta(\alpha,\lambda)}{\partial \alpha} < 0$. ■

Lemma 6.6 *Let Assumptions 1.1 and 6.1 be satisfied. For each $k \geq 0$ large enough, let $\lambda_k = i\omega_k$, $\omega_k > 0$ be the purely imaginary root of the characteristic equation associated to $\alpha_k > 0$ (provided in Proposition 6.2), then we have*

$$Re \frac{\partial \Delta(\alpha_k, i\omega_k)}{\partial \lambda} > 0.$$

Proof. Under Assumption 6.1, substituting $\sigma^{\pm} = \eta_0 \mp \sigma_0$ in the expression of the characteristic equation, we have

$$\Delta(\alpha, \lambda) = 1 - \eta(\alpha) \frac{2\eta_0 + c\zeta_2 + \lambda + \mu}{c\left(2\sigma_0 - c\zeta_2 - \lambda - \mu\right)} \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}}.$$

After a simple of computation, we obtain

$$\begin{aligned} \frac{\partial \Delta(\alpha, \lambda)}{\partial \lambda} &= -\eta(\alpha) \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}} \frac{d}{d\lambda} \frac{2\eta_0 + c\zeta_2 + \lambda + \mu}{c\left(2\sigma_0 - c\zeta_2 - \lambda - \mu\right)} \\ &\quad - \eta(\alpha) \frac{2\eta_0 + c\zeta_2 + \lambda + \mu}{c\left(2\sigma_0 - c\zeta_2 - \lambda - \mu\right)} \frac{d}{d\lambda} \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}}, \end{aligned}$$

$$\frac{d}{d\lambda} \frac{2\eta_0 + c\zeta_2 + \lambda + \mu}{c\left(2\sigma_0 - c\zeta_2 - \lambda - \mu\right)} = \frac{\left(c\frac{d\zeta_2}{d\lambda} + 1\right)\left(2\eta_0 + 2\sigma_0\right)}{c\left(2\sigma_0 - c\zeta_2 - \lambda - \mu\right)^2},$$

$$\frac{d}{d\lambda} \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}} = \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}} \left(\tau + \frac{n+1}{\beta - \zeta_2}\right) \frac{d\zeta_2}{d\lambda},$$

$$\frac{d\zeta_2}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\sigma_0 - \sqrt{\left(\sigma_0\right)^2 + \left(\lambda + \mu\right)^2 + 2\eta_0(\lambda + \mu)}}{c}\right) = -\frac{\lambda + \mu + \eta_0}{c\sqrt{\Lambda}}.$$

If $\Delta(\alpha, i\omega) = 0$, we deduce

$$1 = \eta(\alpha) \frac{2\eta_0 + c\zeta_2 + i\omega + \mu}{c\left(2\sigma_0 - c\zeta_2 - i\omega - \mu\right)} \frac{n! e^{\zeta_2 \tau}}{\left(\beta - \zeta_2\right)^{n+1}}.$$

It follows that

$$\begin{aligned} \frac{\partial \Delta(\alpha, i\omega)}{\partial \lambda} &= \frac{\left(-\frac{i\omega + \mu + \eta_0}{\sqrt{\Lambda}} + 1\right)\left(2\eta_0 + 2\sigma_0\right)}{\left(-\sigma_0 - \sqrt{\Lambda} + i\omega + \mu\right)\left(2\eta_0 + \sigma_0 - \sqrt{\Lambda} + i\omega + \mu\right)} \\ &\quad + \left(\tau + \frac{c\left(n+1\right)}{c\beta - \sigma_0 + \sqrt{\Lambda}}\right)\left(\frac{i\omega + \mu + \eta_0}{c\sqrt{\Lambda}}\right), \end{aligned}$$

where

$$\Lambda = (\sigma_0)^2 + (i\omega + \mu)^2 + 2\eta_0 (i\omega + \mu).$$

Set

$$\sqrt{\Lambda} = a + ib, \text{ with } a > 0.$$

Through the proof of Lemma 2.2, we have

$$a^2 = \left(1 + \left(\frac{b}{\mu + \eta_0}\right)^2\right)^{-1} \left((\sigma_0)^2 + \mu^2 + 2\eta_0\mu + b^2\right)$$

and

$$\omega = \frac{ab}{\mu + \eta_0},$$

and by Proposition 6.2, we obtain that

$$b \to +\infty, \text{ as } k \to +\infty.$$

It follows that

$$\begin{array}{rcl} \omega & \to & +\infty, \text{ as } b \to +\infty, \\ a & \to & \mu + \eta_0, \text{ as } b \to +\infty. \end{array}$$

We have

$$\begin{aligned} & \left(\tau + \frac{c(n+1)}{c\beta - \sigma_0 + \sqrt{\Lambda}}\right)\left(\frac{i\omega + \mu + \eta_0}{c\sqrt{\Lambda}}\right) \\ = \; & \frac{\tau}{c} \frac{a(\mu + \eta_0) + b\omega + i(a\omega - b(\mu + \eta_0))}{a^2 + b^2} \\ & + (n+1) \frac{c\beta - \sigma_0 + a - ib}{(c\beta - \sigma_0 + a)^2 + b^2} \frac{a(\mu + \eta_0) + b\omega + i(a\omega - b(\mu + \eta_0))}{a^2 + b^2}, \end{aligned}$$

and

$$\begin{aligned} & \frac{\left(-\frac{i\omega + \mu + \eta_0}{\sqrt{\Lambda}} + 1\right)(2\eta_0 + 2\sigma_0)}{\left(-\sigma_0 - \sqrt{\Lambda} + i\omega + \mu\right)\left(2\eta_0 + \sigma_0 - \sqrt{\Lambda} + i\omega + \mu\right)} = \frac{2\eta_0 + 2\sigma_0}{a^2 + b^2} \\ & \times \frac{[a(a - (\mu + \eta_0)) + b(b - \omega)] + i(b - \omega)(a + \mu + \eta_0)}{\left((-\sigma_0 - a + \mu)(2\eta_0 + \sigma_0 - a + \mu) - (\omega - b)^2\right) + i2(\omega - b)(\mu + \eta_0 - a)}. \end{aligned}$$

Hence

$$\begin{aligned} & Re\left[\left(\tau + \frac{c(n+1)}{c\beta - \sigma_0 + \sqrt{\Lambda}}\right)\left(\frac{i\omega + \mu + \eta_0}{c\sqrt{\Lambda}}\right)\right] \\ = \; & \frac{\tau}{c} \frac{a(\mu + \eta_0) + b\frac{ab}{\mu + \eta_0}}{a^2 + b^2} \\ & + (n+1) \frac{(c\beta - \sigma_0 + a)\left(a(\mu + \eta_0) + b\frac{ab}{\mu + \eta_0}\right) + b\left(a\frac{ab}{\mu + \eta_0} - b(\mu + \eta_0)\right)}{\left((c\beta - \sigma_0 + a)^2 + b^2\right)(a^2 + b^2)} \\ \to \; & \frac{\tau}{c}, \text{ as } b \longrightarrow +\infty. \end{aligned}$$

By Lemma 6.3, we have

$$
\begin{aligned}
& Re\frac{\left(-\frac{i\omega+\mu+\eta_0}{\sqrt{\Lambda}}+1\right)(2\eta_0+2\sigma_0)}{\left(-\sigma_0-\sqrt{\Lambda}+i\omega+\mu\right)\left(2\eta_0+\sigma_0-\sqrt{\Lambda}+i\omega+\mu\right)} \\
= \quad & \frac{2\eta_0+2\sigma_0}{a^2+b^2} \\
& \times\left[\left((-\sigma_0-a+\mu)(2\eta_0+\sigma_0-a+\mu)-(\omega-b)^2\right)^2+4(\omega-b)^2(\mu+\eta_0-a)^2\right]^{-1} \\
& \times\left[(a(a-(\mu+\eta_0))+b(b-\omega))\left((-\sigma_0-a+\mu)(2\eta_0+\sigma_0-a+\mu)-(\omega-b)^2\right)\right. \\
& \left.-2(b-\omega)^2(a+\mu+\eta_0)(\mu+\eta_0-a)\right] \\
\to \quad & 0,\ \text{as } b\longrightarrow+\infty.
\end{aligned}
$$

Thus we deduce that

$$
\lim_{b\to+\infty}\mathrm{Re}\frac{\partial\Delta(\alpha,i\omega)}{\partial\lambda}=\frac{\tau}{c}>0,
$$

and the result follows. ■

Now, we are in position to derive the transversality condition.

Theorem 6.7 ***(Transversality condition)*** *Let Assumptions 1.1 and 6.1 be satisfied. For each $k\geq 0$ large enough, let $\lambda_k=i\omega_k$, $\omega_k>0$ be the purely imaginary root of the characteristic equation associated to $\alpha_k>0$ (defined in Proposition 6.2), then there exists $\rho_k>0$ (small enough) and a C^1-map $\widehat{\lambda_k}:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ such that*

$$
\widehat{\lambda_k}(\alpha_k)=i\omega_k,\ \Delta(\alpha,\widehat{\lambda_k}(\alpha))=0,\ \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k),
$$

satisfying the transversality condition

$$
\mathrm{Re}\frac{d\widehat{\lambda_k}(\alpha_k)}{d\alpha}>0.
$$

Proof. By Lemma 6.6 we can use the implicit function theorem around each $(\alpha_k,i\omega_k)$ provided by Proposition 6.2 for each $k\geq 0$ sufficiently large, and obtain that there exists $\rho_k>0$ and a C^1-map $\widehat{\lambda_k}:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ such that

$$
\widehat{\lambda_k}(\alpha_k)=i\omega_k,\ \Delta(\alpha,\widehat{\lambda_k}(\alpha))=0,\ \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k).
$$

Moreover, we have

$$
\frac{\partial\Delta\left(\alpha,\widehat{\lambda_k}(\alpha)\right)}{\partial\alpha}+\frac{\partial\Delta\left(\alpha,\widehat{\lambda_k}(\alpha)\right)}{\partial\lambda}\frac{d\widehat{\lambda_k}(\alpha)}{d\alpha}=0,\ \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k).
$$

It follows that

$$\frac{d\widehat{\lambda_k}(\alpha)}{d\alpha} = -\frac{1}{\frac{\partial\Delta\left(\alpha,\widehat{\lambda_k}(\alpha)\right)}{\partial\lambda}} \frac{\partial\Delta\left(\alpha,\widehat{\lambda_k}(\alpha)\right)}{\partial\alpha}, \ \forall\alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

By using Lemma 6.5, we deduce that $\forall\alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k)$

$$\mathrm{Re}\frac{d}{d\alpha}\widehat{\lambda_k}(\alpha) > 0 \Leftrightarrow \mathrm{Re}\left(\frac{\partial\Delta\left(\alpha,\widehat{\lambda_k}(\alpha)\right)}{\partial\lambda}\right) > 0.$$

In particular, we have

$$\mathrm{Re}\frac{d}{d\alpha}\widehat{\lambda_k}(\alpha_k) > 0 \Leftrightarrow \mathrm{Re}\left(\frac{\partial\Delta\left(\alpha_k, i\omega_k\right)}{\partial\lambda}\right) > 0.$$

By Lemma 6.6, the result follows. ■

6.3 Hopf bifurcations

For α large enough, the existence of a unique pair of pure imaginary eigenvalues of B_α and transversality condition have been obtained by Proposition 6.2, Lemma 6.4 and Theorem 6.7 in subsections 6.1 and 6.2. Moreover, according to Lemma 4.2, the simplicity of these eigenvalues follows directly from Lemma 6.6. Remember that $\omega_{0,ess}\left((B_\alpha)_0\right) < 0$, has been obtained in Lemma 4.3. Now all the four aspects mentioned at the beginning of this section have been studied. Hence by using the Hopf bifurcation Theorem proved in [39, Theorem 2.4], we obtain the following Hopf bifurcation result.

Theorem 6.8 (Hopf Bifurcation) *Let Assumptions 1.1 and 6.1 be satisfied. Then there exists $k_0 \in \mathbb{N}$ (large enough) such that for each $k \geq k_0$, the number α_k (defined in Proposition 6.2) is a Hopf Bifurcation point for system (1.1) parametrized by α, around the equilibrium point $\bar{v}$ given in Lemma 3.1.*

7 Summary and numerical simulations

In this section, we first summarize the main results of this paper, then we will present some numerical simulations.

First, we proved that the positive equilibrium exists if and only if

$$R_0 := \alpha\chi > 1.$$

Moreover when it exists, the positive equilibrium is unique. Then we have investigated the local asymptotic behavior around the positive equilibrium. Namely, we proved that:

(a) The positive equilibrium is locally asymptotic stable if $1 < R_0 \leq e^2$.

(b) Hopf bifurcation may occur whenever

$$\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x),$$

for $\beta > 0, n \in \mathbb{N}, \tau > 0,$ *or* $\beta = 0, n = 0, \tau > 0.$

Then regarding α as a bifurcation parameter, we obtain an infinite number of Hopf bifurcation points α_k. More precisely, Proposition 6.2 we obtain a sequence b_k going to $+\infty$ and satisfying (6.8), and the bifurcation points are given by

$$\alpha_k = \frac{1}{\chi} \exp\left(1 + \chi \frac{r_2(b_k)\,(r_3(b_k))^{n+1}}{n! r_1(b_k) c^n} \exp\left((-\sigma_0 + a_k)\,\tau c^{-1}\right)\right),$$

where $a_k, \chi, r_1(b_k), r_2(b_k), r_3(b_k)$ are defined as before (see section 6).

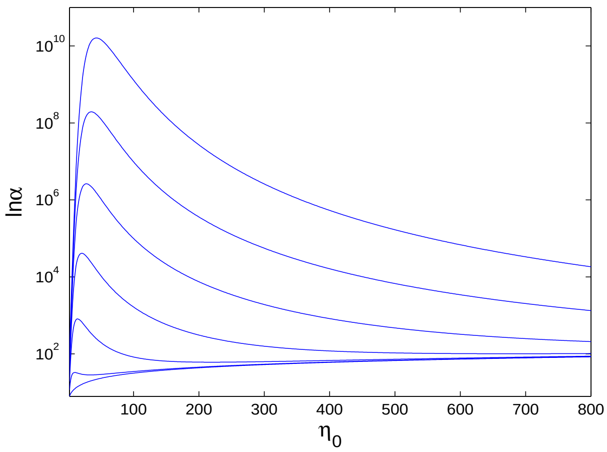

Figure 1: In this figure, we plot some bifurcation curves given by (6.8) and (6.9) in the $(\eta_0, \ln\alpha)$-plane for $\tau = 2$, $\mu = 1$, $c = 1$, $n = 1$, $\beta = 3$ and $\sigma_0 = 1$.

Figure 1 allows to understand the influence of η_0 on the bifurcation points. It can be observed that when k is large the bifurcation parameter α grows extremely fast with respect to some relatively small interval of η_0.

Now we can also compare the system (1.1) with system (1.2). So we let η_0 goes to infinity and we assume that (1.3) is satisfied. Set

$$c = \varepsilon\sqrt{2(\eta_0 + \mu)} \text{ and } \sigma_0 = \frac{1}{\varepsilon}\sqrt{\frac{\eta_0 + \mu}{2}}.$$

Then in Figure 2 we consider ε and α as the parameters of the system, we draw the bifurcation diagrams in the (ε, α)-plane with large value of η_0. Formally this provides an approximation of the bifurcation diagram obtained for system (1.2) in [15].

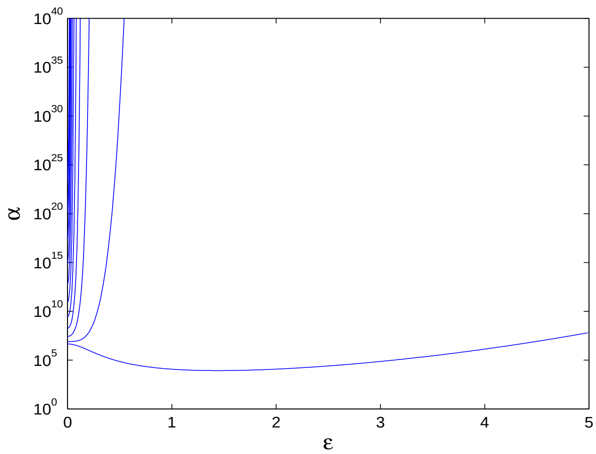

Figure 2: In this figure, we plot some bifurcation curves given by (6.8) and (6.9) with $c = \varepsilon\sqrt{2(\eta_0 + \mu)}$ and $\sigma_0 = \frac{1}{\varepsilon}\sqrt{\frac{\eta_0+\mu}{2}}$ in the (ε, α)-plane for $\tau = 2$, $\mu = 5$, $n = 1$, $\beta = 0.1$, $\eta_0 = 10^5$.

Next we provide some numerical simulations of the solutions of system (1.1), in order to show that the theoretical results can be observed numerically. In Figures 3, 4, and 5, we fix $\gamma(x) = (x - 0.75)\, e^{-0.1\times(x-0.75)} 1_{[0.75,1.5)}(x), \sigma_0 = 0.5, c = 1, \mu = 0.05$, and $\xi = 0.5$, and the parameter α varies from 40 to 80.

In Figure 3 and Figure 4 we plot the total number of individuals growing in maturity (respectively decaying in maturity) (i.e. the L^1 norm of $u^+(t, .)$ respectively the L^1 norm of $u^-(t, .)$). We observe that when α increases, the system passes from a stable to an oscillating regimen.

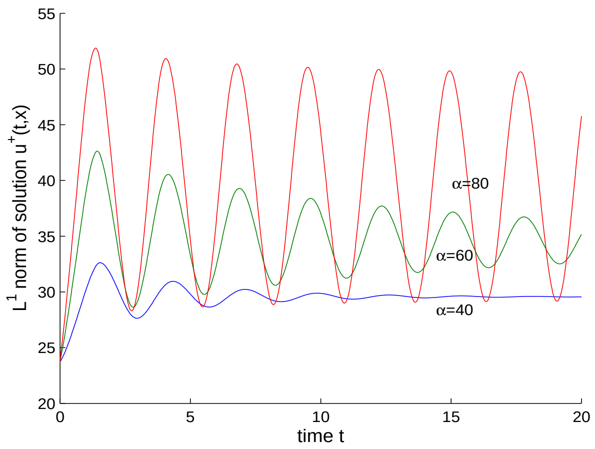

Figure 3: Graph of the evolution of the L^1-norm of the solutions $u^+(t,x)$ in function of time.

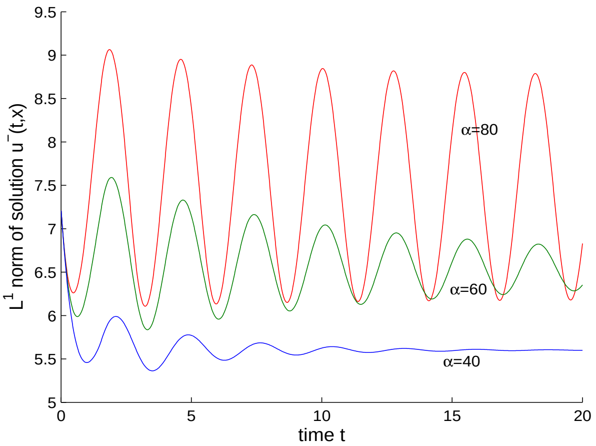

Figure 4: Graph of the evolution of the L^1-norm of the solutions $u^-(t,x)$ in function of time.

To conclude this article, we plot the distribution $u^+(t,.)$ and $u^-(t,.)$ of the system in maturity in Figure 5. We can see that the largest oscillations take place for the small maturity and decay rapidly when the maturity increases. Intuitively, if the oscillations can persist for large maturity values then the system can produce some undamped oscillation, namely some periodic solutions. Actually, the problem is much more delicate then that, and one needs a rigorous mathematical analysis to understand the influence of the parameters on the oscillating properties of the system.

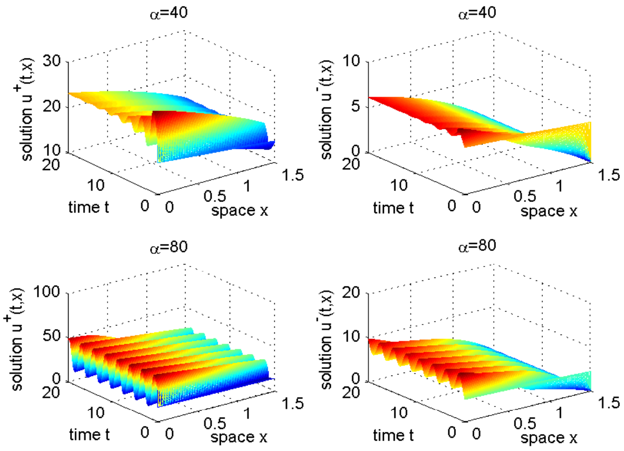

Figure 5: Surface of solutions $u^{+}(t,x), u^{-}(t,x)$

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