Turing and Turing-Hopf bifurcations for a reaction diffusion equation with nonlocal advection

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Abstract

In this paper we study the stability and the bifurcation properties of the positive interior equilibrium for a reaction-diffusion equation with nonlocal advection. Under rather general assumption on the nonlocal kernel we first study the local well posedness of the problem in suitable fractional spaces and we obtain stability results for the homogeneous steady-state. As a special case, we obtain that "standard" kernels such as Gaussian, Cauchy, Laplace and triangle, will lead to stability. Next we specify the model with a given step function kernel and investigate two types of bifurcations, namely Turing bifurcation and Turing-Hopf bifurcation. In fact, we prove that a single scalar equation may display this two types of bifurcations with the dominant wave number as large as we want. Moreover, similar instabilities can also be observed by using a bi-modal kernel. The resulting complex spatio-temporal dynamics are illustrated by numerical simulations.

Key words: Nonlocal reaction-diffusion-advection equation, equilibria stability, Turing bifurcation, Turing-Hopf bifurcation.

MSC: 35K55, 35B32, 35B35

1 Introduction

In this work we consider the following one-dimensional nonlocal reaction-diffusion-advection equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(u\mathbf{v}) + f(u),\ t > 0,\ x \in \mathbb{R}. \tag{1.1}$$

Here $\varepsilon \geq 0$ denotes the viscosity parameter. The velocity field $\mathbf{v}$ is derived from pressure P, where

$$\mathbf{v} = -\frac{\partial P}{\partial x} \text{ with } P(t,x) = (\rho * u(t,.))(x) = \int_{\mathbb{R}} \rho(x-y)u(t,y)dy. \tag{1.2}$$

In the above equation we assume that the pressure P follows nonlocal Darcy law with the kernel $\rho \in L^1(\mathbb{R})$ and the symbol $*$ denotes the convolution product on $\mathbb{R}$. Hence with this closure equation, (1.1) reads as a reaction-diffusion problem with a nonlocal advection term.

In this article Problem (1.1)-(1.2) is supplemented with an initial data $u(0,x) = u_0(x)$ that is assumed to be $2L-$periodic with some given and fixed value $L > 0$. In that periodic setting, the above

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problem re-writes as the following equation posed on $(-L, L)$ with periodic boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}\left(u \dfrac{\partial}{\partial x}(K \circ u)\right) + f(u),\ t > 0,\ x \in (-L, L), \\ \\ u(0, x) = u_0(x),\ x \in (-L, L), \end{cases} \tag{1.3}$$

wherein the kernel $K \in L^1_{\text{per}}(-L, L)$ is defined by

$$K(x) = 2L \sum_{k \in \mathbb{Z}} \rho(x + 2Lk),\ x \in \mathbb{R}. \tag{1.4}$$

Here and in the sequel of this article the symbol $\circ$ denotes the convolution product on the torus $\mathbb{R}/(2L\mathbb{Z})$ i.e.,

$$(g \circ h)(x) = \frac{1}{2L} \int_{-L}^{L} g(x - y) h(y) dy,\ \forall g, h \in L^1_{\text{per}}(-L, L),$$

while for any $p \in [1, \infty]$, we shall also make use of the notation $L^p_{\text{per}}(-L, L)$ to denote the usual Lebesgue spaces of $2L-$periodic functions on $\mathbb{R}$.

System (1.1)-(1.2) (or the periodic equation (1.3)) appears in the mathematical modelling of cell population dynamics. It allows to model the motion of cells by taking into account interactions through cell-cell communication, but also the proliferation of cells and cell cycle through the active part of the equation, namely the function $f = f(u)$.

The nonlinear operator responsible for the motion of cells, denoted by $M(u)$ and defined by

$$M(u) = \frac{\partial}{\partial x}\left[\varepsilon \frac{\partial u}{\partial x} + u \frac{\partial}{\partial x}(\rho * u)\right]$$

was proposed and studied by several authors in the literature. With zero viscosity term $\varepsilon = 0$, this operator has been obtained by Oelschläger in [25] as a suitable limit of interacting system of particles. We also refer to Morale, Capasso and Oelschläger [24] for the derivation of the above operator with a viscosity term. The nonlinear operator $M(u)$ has also been introduced in crowd dynamics and we refer to the survey paper of Bernoff and Topaz in [1] and the references therein.

Some properties of the equation $\dfrac{\partial u}{\partial t} = M(u)$ without viscosity has been studied for instance in [4, 20, 28] (see also the references cited therein). We also refer to Burger and Di Francesco [5] and the references therein for a study of a slightly different equation including nonlinear diffusion.

The nonlinear equation (1.1) has been considered by Ducrot and Magal in [12] with the zero viscosity $\varepsilon = 0$. The authors mostly considered the case of logistic nonlinearity function $f = f(u)$, and most importantly, they considered a specific class of kernel function ρ. More specifically, the aforementioned work deals with the global asymptotic behaviour of the problem for kernels with positive Fourier transform. In this work, this situation roughly corresponds to the stability case (see Remark 2.10 below).

As already mentioned, in the context of cell population dynamics, the function f models the process of cell proliferation. Instead of considering the reaction term as logistic type, we shall make use of the function derived by Ducrot et al. in [11]. Hence throughout this article we use the following specific form for this function f

$$f(u) = \frac{bu}{1 + \gamma u} - \mu u,\ b > 0,\ \mu > 0,\ \gamma > 0. \tag{1.5}$$

This specific form will allow us to make use of explicit computations in our analysis and to use the parameter $\gamma > 0$ as a bifurcation parameter. In the context of cell population dynamics, this nonlinear function takes account of the cell division and exit rate through the parameter b and μ respectively.

The saturation part due to the parameter $\gamma > 0$ reflects the cell cycle and more precisely the dormant phase. We refer to [11] for more details on the modelling issues.

The aim of this article is to study stability and pattern formation for Problem (1.1)-(1.2) or more specifically for its $2L-$periodic counterpart (1.3) through bifurcation analysis methods. Roughly speaking, a scalar and local reaction-diffusion equation typically does not exhibit pattern formation, which is the result of suitable comparison arguments. However as far as nonlocal interaction are concerned, the application of comparison arguments may fail and more complex dynamical behaviours may occur.

Oscillations due to nonlocal interactions has already been observed and studied. We refer for instance to Fiedler and Polácik [14] for a nice work in this direction. Here, we shall discuss the existence of complex asymptotic behaviour of the solutions of (1.1)-(1.2) (or (1.3)) close to the positive homogeneous stationary state and, our discussion will be strongly related with some properties (expressed in term of Fourier transform) of the kernel function ρ arising in the nonlocal advection term.

One type of kernel function that is of particular interest is a step function of the form

$$\rho(x) = \rho_{\eta,s}(x) = \frac{1}{2\eta}\chi_{[-1,1]}\left(\frac{x-s}{\eta}\right),\ x \in \mathbb{R}, \tag{1.6}$$

for some scaling parameter $\eta > 0$, a shift $s \in \mathbb{R}$ and where $\chi_{[-1,1]}$ denotes the characteristic function of the interval $[-1, 1]$, that is

$$\chi_{[-1,1]}(x) = \begin{cases} 1, & \text{if } x \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

As it will be seen in this article, this kernel may destabilize the positive homogeneous steady-state yielding Turing instabilities and the existence of spatially heterogeneous steady-state and, more surprisingly, it may also lead to spatio-temporal regime through Turing-Hopf bifurcation.

Using this kernel, one may observe that the solution of (1.1)-(1.2) is – at least formally – solution of the following active nonlocal Burger equation with viscosity

$$\frac{\partial u}{\partial t} = \varepsilon\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}Q[u] + f(u),\ t > 0,\ x \in \mathbb{R},$$

wherein Q denotes the quadratic nonlocal operator

$$Q[u(t,.)](x) = u(t,x)\frac{u(t,x-\eta+s) - u(t,x+\eta+s)}{2\eta},\ x \in \mathbb{R}.$$

As already mentioned above our goal in this article is to study the stability of the positive homogeneous equilibrium for Problem (1.3) and to provide a bifurcation analysis when it destabilizes. The stability condition is studied using a rather general and possibly non smooth kernel function ρ. Our bifurcation analysis is performed using the more specific kernel ρ proposed in (1.6) above involving the two parameters $\eta > 0$ and $s \in \mathbb{R}$. Here, this specific choice of the kernel ρ is used to compute explicitly the bifurcation property of the system.

This work is organized as follows. In Section 2 we reformulate (1.3) as an abstract parabolic Cauchy problem. From this we are able to study the local well posedness of Problem (1.3) and to study the stability properties of equilibrium state through spectral analysis. In Section 3 we study bifurcations at the positive equilibrium when it becomes spectrally unstable. Moreover, we prove, using the kernel ρ proposed in (1.6), that Turing and Turing-Hopf bifurcations may occur yielding complex spatio-temporal dynamics. We conclude this paper by a short discussion on (1.3) without vital dynamics, namely $f(u) = 0$ and its connection with the porous medium equation.

2 Semiflow and stability

2.1 Spectral analysis

In this section, we consider Problem (1.3). We assume that $\varepsilon > 0,\ L > 0$ are given and fixed. Next recalling the definition of the function f in (1.5) we assume that

$$\gamma > 0,\ b > 0,\ \mu > 0 \text{ and } b - \mu > 0.$$

In that case Problem (1.3) has a unique positive homogeneous steady state given by

$$u_e := \frac{b-\mu}{\gamma\mu} > 0. \tag{2.1}$$

In this section we first study some spectral properties of the – formally – linearized problem at the above positive equilibrium. Then we turn to the stability analysis. The linearized problem in the state space $L^2_{\text{per}}(-L, L)$ reads as follows

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2}{\partial x^2}\left(\varepsilon v + u_e(K \circ v(t,\cdot))(x)\right) + f'(u_e)v(t,x), & t > 0, x \in (-L, L), \\ v(t,-L) = v(t,L),\ \partial_x v(t,-L) = \partial_x v(t,L), & t > 0, \end{cases} \tag{2.2}$$

where

$$f'(u_e) = \frac{\mu(\mu - b)}{b} < 0.$$

To handle this problem we define the linear operator $\mathcal{A} : D(\mathcal{A}) \subset L^2_{\text{per}}(-L, L) \to L^2_{\text{per}}(-L, L)$ as follows

$$\begin{cases} D(\mathcal{A}) = H^2_{\text{per}}(-L, L), \\ \mathcal{A}\phi = \varepsilon\phi'' + u_e\left(K \circ \phi''\right). \end{cases} \tag{2.3}$$

Here recall that the kernel $K \in L^1_{\text{per}}(-L, L)$. To analyze this operator we shall make use of Fourier analysis. To that aim we shall also use of the notation $\langle ., . \rangle$ to denote the inner product in $L^2_{\text{per}}(-L, L; \mathbb{C})$ defined by

$$\langle f, g\rangle = \frac{1}{2L}\int_{-L}^{L} \overline{f(x)}g(x)dx,\ \forall f, g \in L^2_{\text{per}}(-L, L).$$

The corresponding norm on $L^2_{\text{per}}(-L, L)$ is denoted by $\|.\|_0$. We also introduce the Hilbert basis $\left\{e_n : x \to e^{i\pi\frac{nx}{L}}\right\}_{n\in\mathbb{Z}}$ on $L^2_{\text{per}}(-L, L)$ as well as, for each function $\varphi \in L^1_{\text{per}}(-L, L; \mathbb{C})$, its Fourier coefficients $\{c_n(\varphi)\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ defined by

$$c_n(\varphi) = \langle e_n, \varphi\rangle = \frac{1}{2L}\int_{-L}^{L} \varphi(x)e^{-i\pi\frac{nx}{L}}dx,\ \text{for any } n \in \mathbb{Z}. \tag{2.4}$$

Recall that using the above notation, the map $\varphi \mapsto \{c_n(\varphi)\}_{n\in\mathbb{Z}}$ is an isometry from $L^2_{\text{per}}(-L, L; \mathbb{C})$ – endowed with the norm $\|.\|_0$ – onto $l^2(\mathbb{Z}; \mathbb{C})$.

We now describe the spectrum of the operator $\mathcal{A}$ defined above.

Proposition 2.1. *Recalling that $K \in L^1_{\text{per}}(-L, L)$, the spectrum of the linear operator $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, consists in point spectrum and one has*

$$\sigma(\mathcal{A}) = \left\{\lambda_n := -\left(\frac{n\pi}{L}\right)^2\left[\varepsilon + u_e c_n(K)\right],\ n \in \mathbb{Z}\right\},$$

and the corresponding eigenvectors are given by $\mathcal{A}e_n = \lambda_n e_n$ for all $n \in \mathbb{Z}$.

In addition, for each $\lambda \in \rho(\mathcal{A}) := \mathbb{C} \setminus \sigma(\mathcal{A})$, the resolvent set of $\mathcal{A}$, and each $f \in L^2_{\text{per}}(-L, L)$, one has

$$(\lambda - \mathcal{A})^{-1}f = \sum_{n\in\mathbb{Z}} \frac{c_n(f)}{\lambda - \lambda_n}e_n.$$

Remark 2.2. *The key observation in the above lemma is the fact that since $K \in L^1_{\text{per}}(-L, L)$ we have by the Riemann-Lebesgue lemma*

$$\lim_{|n| \to +\infty} c_n(K) = 0.$$

Therefore the results for purely diffusive systems (i.e. whenever $K = 0$) can be extended to the above class of linear operators.

Recalling the definition of the kernel K in (1.4), one may notice that the Fourier coefficients $c_n(K)$ can be expressed using the Fourier transform of the kernel ρ in the original model (1.3). In the one dimensional setting, this relationship reads as follows

$$c_n(K) = \widehat{\rho}\left(\frac{n}{2L}\right),\ n \in \mathbb{Z} \text{ where } \widehat{\rho}(\xi) = \int_{\mathbb{R}} \rho(x) e^{-2i\pi x\xi} dx. \tag{2.5}$$

Proof. Let us first observe that for each $n \in \mathbb{Z}$ one has:

$$\left(\mathcal{A}e_n\right)(x) = -\left(\frac{n\pi}{L}\right)^2 \left[\varepsilon + \frac{u_e}{2L} \int_{-L}^{L} K(y) e^{-\frac{in\pi}{L}y} dy\right] e_n(x).$$

As a consequence one obtains $\mathcal{A}e_n = \lambda_n e_n$ for all $n \in \mathbb{Z}$, that is $\{\lambda_n, n \in \mathbb{Z}\} \subset \sigma_p(\mathcal{A})$, the point spectrum of $\mathcal{A}$.

Now we claim that:

Claim 2.3. *Let $\lambda \in \mathbb{C} \setminus \{\lambda_n,\ n \in \mathbb{Z}\}$ be given. Then for each $f \in L^2_{\text{per}}(-L, L)$ there exists a unique $u_f \in H^2_{\text{per}}(-L, L)$ such that*

$$\left(\lambda - \mathcal{A}\right) u_f = f,$$

and that the linear map $f \mapsto u_f$ is continuous on $L^2_{\text{per}}(-L, L)$ into $H^2_{\text{per}}(-L, L)$ and it is given by

$$u_f = \sum_{n \in \mathbb{Z}} \frac{c_n(f)}{\lambda - \lambda_n} e_n.$$

Note that this claim ensures that

$$\mathbb{C} \setminus \{\lambda_n,\ n \in \mathbb{Z}\} \subset \rho(\mathcal{A}),$$

which implies

$$\sigma_p(\mathcal{A}) \subset \sigma(\mathcal{A}) \subset \{\lambda_n,\ n \in \mathbb{Z}\},$$

and this completes the first part of the proposition. Note also that the explicit formula for the resolvent operator also follows from the above claim.

To prove this claim recall that the space $H^2_{\text{per}}(-L, L)$ can be re-written using the Fourier coefficients as follows:

$$H^2_{\text{per}}(-L, L) = \left\{\varphi \in L^2_{\text{per}}(-L, L) :\ \sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^2 |c_n(\varphi)|^2 < \infty\right\},$$

and the norm $\|.\|_2$ on $H^2_{\text{per}}(-L, L)$, defined by:

$$\|\varphi\|_2^2 = \sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^2 |c_n(\varphi)|^2,\ \forall \varphi \in H^2_{\text{per}}(-L, L),$$

is equivalent to the usual $H^2_{\text{per}}(-L, L)-$norm. Using this characterization we are now able to complete the proof of the above claim.

Proof of Claim 2.3: Let $\lambda \in \mathbb{C} \setminus \{\lambda_n,\ n \in \mathbb{Z}\}$ be given. Let $f \in L^2_{\text{per}}(-L,L)$ be given. Assume first that there exists $u = u_f \in D(\mathcal{A})$ such that

$$(\lambda - \mathcal{A})u = f.$$

Then we get

$$\langle e_n, (\lambda - \mathcal{A})u \rangle = \langle e_n, f \rangle\,,\ \forall n \in \mathbb{Z}.$$

However, since for each $n \in \mathbb{Z}$, one has

$$\langle e_n, (\lambda - \mathcal{A})u \rangle = (\lambda - \lambda_n)\, c_n(u),$$

one obtains that

$$c_n(u) = \frac{c_n(f)}{\lambda - \lambda_n},\ \forall n \in \mathbb{Z}.$$

As a consequence, the solution is unique as soon as it exists.

On the other hand consider the sequence $\left\{F_n(\lambda) := \frac{c_n(f)}{\lambda - \lambda_n}\right\}_{n \in \mathbb{Z}}$, that is well defined since $\lambda \neq \lambda_n$ for all $n \in \mathbb{Z}$. Since $K \in L^1_{\text{per}}(-L,L)$, one can use the Riemann-Lebesgue lemma to get that $c_n(K) \to 0$ as $|n| \to \infty$, so that

$$\lambda_n \sim -\varepsilon \left(\frac{n\pi}{L}\right)^2 \text{ as } |n| \to +\infty.$$

Hence the sequence $\left\{\frac{1+n^2}{\lambda - \lambda_n}\right\}_{n \in \mathbb{Z}}$ is bounded. As a consequence one gets

$$\sum_{n=-\infty}^{\infty} \left|1+n^2\right|^2 |F_n(\lambda)|^2 \le \sup_{n \in \mathbb{Z}} \left|\frac{1+n^2}{\lambda - \lambda_n}\right|^2 \sum_{n=-\infty}^{\infty} |c_n(f)|^2 \le \sup_{n \in \mathbb{Z}} \left|\frac{1+n^2}{\lambda - \lambda_n}\right|^2 \|f\|_0^2. \tag{2.6}$$

As a consequence the function $u = u_f$ defined by

$$u_f = \sum_{n \in \mathbb{Z}} F_n(\lambda) e_n,$$

satisfies:

$$u_f \in H^2_{\text{per}}(-L,L) \text{ and } (\lambda - \mathcal{A})u_f = f.$$

Summarizing the above arguments we have obtained that for each $f \in L^2_{\text{per}}(-L,L)$ the function $u_f \in H^2_{\text{per}}(-L,L)$ is the unique solution of $(\lambda - \mathcal{A})u_f = f$. Furthermore (2.6) ensures that

$$\left\|(\lambda - \mathcal{A})^{-1} f\right\|_2^2 \le \sup_{n \in \mathbb{Z}} \left|\frac{1+n^2}{\lambda - \lambda_n}\right|^2 \|f\|_0^2,\ \forall f \in L^2_{\text{per}}(-L,L),$$

that completes the proof of the claim.

■

Remark 2.4. *As a corollary of the above proposition and more precisely of the resolvent formula, one obtains the following estimate:*
For each $\lambda \in \rho(\mathcal{A})$ one has:

$$\|(\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(L^2_{\text{per}}(-L,L))} \le \sup_{n \in \mathbb{Z}} \frac{1}{|\lambda - \lambda_n|}. \tag{2.7}$$

Now observe that, since $c_n(K) \to 0$ as $|n| \to \infty$, one has

$$\lim_{|n| \to +\infty} \frac{\text{Im}\lambda_n}{\text{Re}\lambda_n} = \lim_{|n| \to +\infty} \frac{u_e \text{Im}\{c_n(K)\}}{\epsilon + u_e \text{Re}\{c_n(K)\}} = 0.$$

Hence, since $|\lambda_n|$ is bounded from above, for each $a > 0$ large enough there exists $\phi_a \in \left(0, \frac{\pi}{2}\right)$ and $0 < k_a < a$ large enough such that $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \overline{\Sigma_a}$ wherein $\Sigma_a \subset \mathbb{C}$ is defined by

$$\Sigma_a = \left\{ z = a + re^{i\theta} \in \mathbb{C} : \; r > k_a \text{ and } |\pi - \theta| < \phi_a \right\}.$$

One concludes from the above estimate that the linear operator $\mathcal{A}$ is a sectorial operator in $L^2_{\text{per}}(-L, L)$.

Using the above proposition we now focus on the stability analysis of the homogeneous steady state u_e (defined in (2.1)) of Problem (1.3). To that aim we need to strengthen our assumption for the kernel $K \in L^1_{\text{per}}(-L, L)$. More precisely we assume that

Assumption 2.5. *There exists $\nu \in (0, 1]$ such that the convolution kernel $K \in L^1_{\text{per}}(-L, L)$ satisfies*

$$\sup_{n\in\mathbb{Z}} \left(|n|^\nu \, |c_n(K)| \right) < \infty.$$

Using the above assumption we shall re-write (1.3) as an abstract Cauchy problem involving a sectorial operator and suitable fractional spaces. To reach this goal, let us introduce the scale of Hilbert spaces H^s_{per} for $s \in \mathbb{R}$ by

$$H^s_{per}(-L, L) = \left\{ \varphi \in L^2_{\text{per}}(-L, L) : \sum_{n\in\mathbb{Z}} \left(1 + |n|^2\right)^s |c_n(\varphi)|^2 < \infty \right\}.$$

These spaces are endowed with the inner product $\langle ., . \rangle_s$ defined by

$$\langle \varphi, \psi \rangle_s = \sum_{n\in\mathbb{Z}} \left(1 + |n|^2\right)^s \overline{c_n(\varphi)} c_n(\psi).$$

We denote by $\|\varphi\|_s := \sqrt{\langle \varphi, \varphi \rangle_s}$ the corresponding norm. Beside, we denote that $L^2_{\text{per}}(-L, L) = L^2_{\text{per}}$ and with norm $\|.\|_0$.

Now define the sectorial operator $A : D(A) \subset L^2_{\text{per}} \to L^2_{\text{per}}$ by

$$D(A) = H^2_{\text{per}}(-L, L) \text{ and } A = -I + \varepsilon \frac{\partial^2}{\partial x^2}.$$

Next observe (see [17, 34]) that for all $s \in \mathbb{R}$ one has

$$(-A)^s = \sum_{n=-\infty}^{\infty} \left[1 + \varepsilon \left(\frac{n\pi}{L} \right)^2 \right]^s c_n(.) e_n,$$

so that $H^{2s}_{\text{per}} = D\left((-A)^s\right)$ and the norm $\|.\|_{2s}$ is equivalent to the graph norm $\|(-A)^s.\|$ on H^{2s}_{per}. Moreover, noticing the norm of $D((-A)^s) = H^{2s}_{\text{per}}$ is equivalent to the norm on $H^s(-L, L)$ (See [31, p.50]). Thus, for the simplicity of notation, we denote $H^s := H^s_{\text{per}}(-L, L)$ for any $s > 0$ and we choose $H^{2-\nu}$ as our state space, therefore $H^{2-\nu} \hookrightarrow C_{\text{per}}([-L, L])$ is a continuous embedding if $0 < \nu \leq 1$ where $C_{\text{per}}([-L, L])$ denotes the space of the continuous $2L$-periodic functions endowed with the uniform norm $\| \cdot \|_\infty$. In the sequel we shall also use the notation H^0 to denote L^2_{per}.

2.2 Existence of a semiflow in some fractional space

In this section we shall re-write Problem (1.3) as an abstract Cauchy problem involving a sectorial linear operator and prove that it generates a maximal semiflow in a suitable fractional space, namely $H^{2-\nu}$ where the parameter $\nu \in (0, 1]$ is defined in Assumption 2.5. To that aim we first need to prove the following lemma.

Lemma 2.6. *Let Assumption 2.5 be satisfied. Then the bilinear map*

$$B:\ (\varphi,\psi)\mapsto \frac{d}{dx}\left(\varphi\frac{d}{dx}K\circ\psi\right),$$

is continuous from $H^1\times H^{2-\nu}$ to L^2_{per}.

Proof. Let φ and ψ be two $2L-$periodic smooth functions. Then one has

$$B(\varphi,\psi)=\varphi'\left(K\circ\psi'\right)+\varphi\left(K\circ\psi''\right).$$

Hence we get

$$\begin{aligned}\|B(\varphi,\psi)\|_0 &\le \|\varphi'\|_0\|K\circ\psi'\|_0+\|\varphi\|_{L^\infty}\|K\circ\psi''\|_0\\ &\le \|\varphi'\|_0\|K\|_{L^1}\|\psi'\|_0+\|\varphi\|_{L^\infty}\|K\circ\psi''\|_0.\end{aligned}$$

Recalling that $N=1$, due to Sobolev embedding one has $H^1\hookrightarrow L^\infty$ and, there exists some constant $C_1>0$ (that does not depend on φ and ψ) such that

$$\|B(\varphi,\psi)\|_0\le \|\varphi\|_1\|K\|_{L^1}\|\psi\|_1+C_1\|\varphi\|_1\|K\circ\psi''\|_0.$$

It remains to estimate the last term. To that aim note that

$$\begin{aligned}\|K\circ\psi''\|_0^2 &= \sum_{n=-\infty}^{\infty}\left(\frac{n\pi}{L}\right)^4|c_n(K)|^2|c_n(\psi)|^2=\left(\frac{\pi}{L}\right)^4\sum_{n=-\infty}^{\infty}|n|^{2\nu}|c_n(K)|^2\left(|n|^{2-\nu}\right)^2|c_n(\psi)|^2\\ &\le \left(\frac{\pi}{L}\right)^4\left(\sup_{n\in\mathbb{Z}}|n|^{\nu}|c_n(K)|\right)^2\sum_{n=-\infty}^{\infty}\left(1+|n|^{2-\nu}\right)^2|c_n(\psi)|^2\\ &\le C_2^2\|\psi\|_{2-\nu}^2 \text{ with } C_2=\left(\frac{\pi}{L}\right)^2\left(\sup_{n\in\mathbb{Z}}|n|^{\nu}|c_n(K)|\right).\end{aligned}$$

As a consequence of the above estimates and since $\nu\in(0,1]$, so that $H^{2-\nu}\subset H^1$, one obtains that for any smooth periodic functions

$$\|B(\varphi,\psi)\|_0\le\left[\|K\|_{L^1}+C_1C_2\right]\|\varphi\|_1\|\psi\|_{2-\nu}.$$

This completes the proof of the lemma using a usual density argument. ∎

Using the above lemma we can re-write (1.3) as an abstract Cauchy problem. Recall that $H^{2-\nu}\subset H^1\subset C_{\text{per}}([-L,L])$ with continuous embedding. We also modify the reaction term f on the negative real line. We consider $\tilde f(u)$ that coincide with the formula (1.5) when $u\ge 0$ and $\tilde f$ is C^∞ on $\mathbb{R}$. Hence the map $F:H^{2-\nu}\to L^2_{\text{per}}$ defined by

$$F(\varphi)(x)=B(\varphi,\varphi)(x)+\tilde f(\varphi(x))+\varphi(x),\ \forall x\in(-L,L),$$

is smooth. Problem (1.3) re-writes in the space $H^{2-\nu}$ as the following abstract Cauchy problem:

$$\begin{cases}\dfrac{du(t)}{dt}=Au(t)+F(u(t)),\ \text{for } t\ge 0,\\ u(0)=u_0\in H^{2-\nu}.\end{cases}\tag{2.8}$$

A function $u\in C([0,\tau],H^{2-\nu})$ is called a *mild solution* of the equation (2.8) on $[0,\tau]$, if

$$u(t)=e^{At}u(0)+\int_0^t e^{A(t-s)}F(u(s))ds,\forall t\in[0,\tau].\tag{2.9}$$

Before going further let us recall the following definition.

Definition 2.7. *Let τ (maximal time of existence) be a map from a Banach space X into $(0, +\infty]$ and let U be a map from $D_\tau := \{(t, u) \in [0, +\infty) \times X : 0 \leq t < \tau(u)\}$ into X.*
Set $U(t)u := U(t, u), \forall (t, u) \in D_\tau$. We say that (U, τ) is a maximal semiflow on X if the following properties are satisfied:

i) $U(t)U(s)u = U(t + s)u, \forall t, s \in [0, \tau(u))$ with $t + s < \tau(u)$ and $u \in X$;
ii) $U(0)u = u$ for all $u \in X$;
iii) $\tau(U(s)u) = s + \tau(u)$ for any $u \in X$ and $s \in [0, \tau(u))$;
iv) if $\tau(u) < \infty$ then

$$\lim_{t \nearrow \tau(u)} \|U(t)u)\|_X = \infty.$$

The existence of a maximal semiflow for (2.8) is based on the fact that the map $F : H^{2-\nu} \to L^2_{\mathrm{per}}$ is smooth enough and Lipschitz continuous on bounded sets and the following estimate

$$\sup_{t \in [0,T]} \| \int_0^t e^{A(t-s)} \varphi(s) ds \|_{2-\nu} \leq C T^{\nu/2} \sup_{t \in [0,T]} \|\varphi(t)\|_{L^2_{\mathrm{per}}},$$

for any $\varphi \in C([0, T], L^2_{\mathrm{per}})$ where C is some constant.

By using the above estimation we follow the same idea as in Cazenave and Haraux [7, Chapter 5], Lunardi [21, Theorem 7.1.3 (i) p.260 and Proposition 7.1.9 (i) p.267] and Magal and Ruan [22, 23] to derive the following theorem.

Theorem 2.8 (**Existence of the unique maximal semiflow**)**.** *The abstract Cauchy problem (2.8) generates a unique maximal semiflow on $H^{2-\nu}$. This means for each $u_0 \in H^{2-\nu}$, we can find a map $\tau : H^{2-\nu} \to (0, +\infty]$ (maximal time of existence) and a map $U : D_\tau \to H^{2-\nu}$ where*

$$D_\tau := \{(t, u_0) \in [0, +\infty) \times H^{2-\nu} : 0 \leq t < \tau(u_0)\}.$$

such that there exists a unique mild solution $U(\cdot)u_0 \in C([0, \tau(u_0)), H^{2-\nu})$. Moreover, for every $\hat{\tau} < \tau(u_0)$, there exist two constants $r > 0$ and $K > 0$ such that if $\|u_0 - \widehat{u}_0\|_{2-\nu} \leq r$, then $\tau(\widehat{u}_0) > \hat{\tau}$ and

$$\|U(t)u_0 - U(t)\widehat{u}_0\|_{2-\nu} \leq K \|u_0 - \widehat{u}_0\|_{2-\nu}, \forall t \in [0, \hat{\tau}].$$

2.3 Stability and instability of u_e

In this section we discuss the linear stability and instability of the stationary state u_e by using the abstract Cauchy problem formulation described in the previous section. Towards that purpose, we shall make use of Theorem 5.1.2 and 5.1.3 in the monograph of Henry [17] to deal with the stability and instability of the stationary state u_e. Within this framework the – local – stability and instability of u_e relies on the spectrum of the linear operator $A + F'(u_e)$ that reads as $\mathcal{A} + f'(u_e)$. The spectrum of this linear operator has been fully described in Section 2 and one has:

$$\sigma(A + F'(u_e)) = \left\{ -\left(\frac{n\pi}{L}\right)^2 [\varepsilon + u_e c_n(K)] - \frac{\mu(b - \mu)}{b},\ n \in \mathbb{Z} \right\}. \tag{2.10}$$

As a consequence one obtains the following result:

Theorem 2.9. *Suppose Assumption 2.5 is satisfied. Then the following statements hold true:*

(i) If

$$-\left(\frac{n\pi}{L}\right)^2 [\varepsilon + u_e \operatorname{Re}(c_n(K))] - \frac{\mu(b - \mu)}{b} < 0,\ \forall n \in \mathbb{Z},$$

then u_e is a locally – exponentially – stable homogeneous steady state of (2.8) in a neighbourhood of u_e in $H^{2-\nu}$. Here $c_n(K)$ is the Fourier coefficient defined by (2.5).

(ii) If there exists $n \in \mathbb{Z}$ such that

$$-\left(\frac{n\pi}{L}\right)^2 [\varepsilon + u_e \operatorname{Re}(c_n(K))] - \frac{\mu(b-\mu)}{b} > 0,$$

then u_e is an unstable stationary state of (2.8) in $H^{2-\nu}$.

Remark 2.10. *Using the first statement (i) in the above result, note that if $\operatorname{Re}(c_n(K)) \geq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$, then the spectrum is contained in the left complex half plane and u_e is locally stable.*

Due to the above remark and by using the explicit computations of the Fourier transform coupled with Remark 2.2, one obtains the following corollary showing that "standard" kernels lead to stability.

Corollary 2.11 (Local stability for "standard" kernels). *Standard kernel functions, ρ, such as Gaussian, Cauchy, Laplace and triangle law respectively defined by the following forms*

$$x \mapsto e^{-\pi x^2},\ x \mapsto \frac{2}{1+4\pi^2 x^2},\ x \mapsto \pi e^{-2\pi|x|} \text{ and } x \mapsto \rho_{\text{triangle}}(x),$$

lead to the local stability of the interior equilibrium u_e. Here the function ρ_{triangle} is defined by

$$\rho_{\text{triangle}}(x) = \begin{cases} 1+x, & x \in [-1,0), \\ 1-x, & x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma shows that the positive equilibrium is locally exponentially stable whenever the parameter $\gamma > 0$ is large enough.

Theorem 2.12 (Local stability for $\gamma \gg 1$). *Let Assumption 2.5 be satisfied. Let $\varepsilon > 0$, with $b > \mu \geq 0$. Then there exists $\gamma_0 = \gamma_0(\varepsilon, b, \mu) > 0$ such that when $\gamma \geq \gamma_0$ the homogeneous steady state $u_e = (b-\mu)/(\gamma\mu)$ of the equation (2.8) is locally exponentially stable. i.e.,*

$$\operatorname{Re}(\lambda_n + f'(u_e)) = -\left(\frac{n\pi}{L}\right)^2 [\varepsilon + u_e \operatorname{Re}(c_n(K))] - \frac{\mu(b-\mu)}{b} < 0,\ \forall n \in \mathbb{Z}.$$

Proof. We denote u_e as $u_e(\gamma)$ indicating u_e is dependent on the parameter γ. For the moment, we choose $\gamma \geq \bar{\gamma}$ for a fixed $\bar{\gamma} > 0$, therefore $u_e(\gamma)$ is bounded above. For any K satisfy the Assumption 2.5, we have $c_n(K) \to 0$ as $|n| \to \infty$. Therefore, for any $\varepsilon > 0$ fixed, there exists a n_0 such that

$$\inf_{n \geq n_0} \{\varepsilon + u_e(\gamma) \operatorname{Re}(c_n(K))\} \geq 0. \tag{2.11}$$

Notice if we increase γ, the equation (2.11) still holds. Thus for the finite set $\{0, 1, \ldots, n_0\}$ one can easily deduce

$$\lim_{\gamma\to\infty} \max_{n \in \{0,1,\ldots,n_0\}} \left\{-\left(\frac{n\pi}{L}\right)^2 [\varepsilon + u_e(\gamma) \operatorname{Re}(c_n(K))]\right\} < 0.$$

■

3 Bifurcation analysis

In this section we investigate pattern formation for Problem (1.3). Our study is based on bifurcation analysis and we will show that with a suitable choice for the model parameters and with an appropriate kernel function, Turing bifurcation and Turing-Hopf bifurcation can occur.

In this section, we always fix a specific kernel function $\rho = \rho_{\eta,s}$ as in (1.6). The corresponding $2L-$periodic kernel (see (1.4)) is denoted by $K = K_{\eta,s}$. This choice of the above kernel function is motivated by Remark 2.10. Indeed the Fourier coefficients of $K_{\eta,s}$ can be explicitly computed and they read as follows (see Remark 2.2):

$$c_n\left(K_{\eta,s}\right) = \widehat{\rho_{\eta,s}}\left(\frac{n}{2L}\right) = \frac{\sin\left(n\eta\pi/L\right)}{n\eta\pi/L} e^{-i\frac{n\pi s}{L}},\ \forall n \in \mathbb{Z}. \tag{3.1}$$

As we can see, the real part of the Fourier coefficients that changes signs will lead to the instability of the system. Note also that, with such kernel, namely (1.6), Assumption 2.5 is satisfied so that the results of the previous section holds true with such a choice.

3.1 Turing bifurcation

Throughout this subsection we consider Problem (1.3) with the kernel $\rho_{\eta,0}$ defined above in (1.6) with $s = 0$. We shall focus on the existence of Turing bifurcation for this problem.

We denote by $\mathcal{A}_\eta$ the linear operator defined in (2.3) with the kernel $K = K_{\eta,0}$ associated to the step function $\rho_{\eta,0}$ (see definition (1.4)). Next lemma describes that a proper choice of parameters can lead to spectral Turing bifurcation singularities.

Lemma 3.1. *Let $k_0 \in \mathbb{N}\backslash\{0\}$ and $\eta_0 > 0$ such that $L/(2\eta_0) \in \mathbb{N}$. Then there exists a pair of parameters $\varepsilon_0 > 0$ and $\gamma_0 > 0$, such that the eigenvalues $\lambda_n + f'(u_e) =: \widehat{\lambda}_n$ of the linear operator $\mathcal{A}_{\eta_0} + f'(u_e)$ satisfy*

$$\widehat{\lambda}_{\pm n_0} = 0,\ \widehat{\lambda}_n < 0,\ \textit{for any } n \in \mathbb{Z}\backslash\{\pm n_0\},$$

with $n_0 = \frac{L}{2\eta_0}(-1 + 4k_0) \in \mathbb{N} \setminus \{0\}$. *In other words one can choose k_0 as large as we want, and set*

$$\varepsilon_0 = \frac{4\mu(b-\mu)}{b}\left(\frac{\eta_0}{-\pi + 4k_0\pi}\right)^2, \qquad \gamma_0 = \frac{b}{4(\mu\eta_0)^2}(-\pi + 4k_0\pi)$$

such that $\widehat{\lambda}_{n_0} (= \widehat{\lambda}_{-n_0})$ is the only zero eigenvalue of multiplicity two while the other eigenvalues are negative.

Remark 3.2. *Note that since the kernel $\rho_{\eta,0}$ is symmetric (hence is $K_{\eta,0}$) then $c_n(K_{\eta,0}) = c_{-n}(K_{\eta,0})$ for all $n \in \mathbb{Z}$. As a consequence $\widehat{\lambda}_n = \widehat{\lambda}_{-n}$ for all $n \in \mathbb{Z}$ and, with the notations of the above lemma one has*

$$\ker\left(\widehat{\lambda}_{n_0} - \mathcal{A}_{\eta_0}\right) = \operatorname{span}\left(x \mapsto \cos\left(\frac{n_0\pi x}{L}\right),\ x \mapsto \sin\left(\frac{n_0\pi x}{L}\right)\right).$$

Proof. Our proof is divided in two steps. We first provide parameter conditions that ensure the existence of a unique pair (due to symmetry) of dominant eigenvalues and then we describe conditions for the dominant eigenvalue to be zero.

First step: Existence and uniqueness of a pair of dominant eigenvalues:

Set $u_e(\gamma) = \frac{b-\mu}{\gamma\mu}$. Then the eigenvalues of $\mathcal{A}_\eta + f'(u_e(\gamma))$ reads as follows (recall here that $s = 0$ in this subsection)

$$\widehat{\lambda}_n = -\left(\frac{n\pi}{L}\right)^2\left[\varepsilon + u_e(\gamma)\frac{\sin(n\eta\pi/L)}{n\eta\pi/L}\right] - \frac{\mu(b-\mu)}{b}, \forall n \in \mathbb{Z},$$

Due to symmetry we only consider $n \in \mathbb{N}$ and we set $\alpha = \varepsilon/\eta^2$, $\beta = (b-\mu)/(\gamma\mu\eta^2)$ and

$$\phi(x) := -\alpha x^2 - \beta x \sin x.$$

By using the above notations, we can re-write the eigenvalues $\widehat{\lambda}_n$ as

$$\widehat{\lambda}_n = \phi\left(\frac{n\eta\pi}{L}\right) - \frac{\mu(b-\mu)}{b}. \tag{3.2}$$

The function ϕ is of transcendental type and it is not easy to consider the maximum directly. Thus we re-write $\phi(x)$ as follows

$$\phi(x) = -\alpha \left(x + \frac{\beta \sin x}{2\alpha}\right)^2 + \frac{\beta^2 \sin^2 x}{4\alpha} \leq \frac{\beta^2}{4\alpha},$$

and ϕ reaches its maximum $\frac{\beta^2}{4\alpha}$ if and only if

$$x = -\frac{\beta}{2\alpha} \sin x, \ \sin^2 x = 1. \tag{3.3}$$

If we assume $x > 0$, the above equation has an unique solution which satisfies

$$x = \frac{\beta}{2\alpha}, \quad \sin x = -1.$$

Therefore, fix $k_0 \in \mathbb{N}\backslash\{0\}$ arbitrarily large and we choose γ and ε such that the product $\gamma\varepsilon$ satisfies

$$\frac{\beta}{2\alpha} \equiv \frac{b-\mu}{2\mu(\gamma\varepsilon)} = -\frac{\pi}{2} + 2k_0\pi. \tag{3.4}$$

With such a choice, (3.3) is satisfied and thus $\phi(\frac{b-\mu}{2\gamma\mu\varepsilon}) = \sup_{x\geq 0} \phi(x)$. Next note that

$$\frac{n\eta\pi}{L} = -\frac{\pi}{2} + 2k_0\pi \Longleftrightarrow n = L/(2\eta)(-1+4k_0).$$

Hence choosing $n_0 = L/(2\eta_0)(-1+4k_0) \in \mathbb{N} \setminus \{0\}$ one has

$$\frac{n_0\eta_0\pi}{L} = \frac{b-\mu}{2\gamma\mu\varepsilon} = -\frac{\pi}{2} + 2k_0\pi \Longleftrightarrow n_0 = \frac{L}{2\eta_0}(-1+4k_0). \tag{3.5}$$

By (3.2) and (3.5) one deduces

$$\widehat{\lambda}_{n_0} = \phi(\frac{n_0\eta_0\pi}{L}) - \frac{\mu(b-\mu)}{b} \text{ and } \widehat{\lambda}_{n_0} > \max_{n\in\mathbb{N}\backslash\{n_0\}} \widehat{\lambda}_n. \tag{3.6}$$

Second step: The dominant eigenvalue is zero.

To complete the proof of the lemma we have to fix ε and γ such $\widehat{\lambda}_{n_0} = 0$ keeping in mind that the product $\varepsilon\gamma$ is already fixed by (3.4).

In order to ensure that $\widehat{\lambda}_{n_0} = 0$ is the unique zero eigenvalue we fix $\gamma_0 > 0$ such that

$$\frac{b-\mu}{2\gamma_0\mu\eta_0^2}\left(-\frac{\pi}{2} + 2k_0\pi\right) \equiv \phi(\frac{n_0\eta\pi}{L}) = \frac{\mu(b-\mu)}{b}. \tag{3.7}$$

Hence $\varepsilon_0 > 0$ is fixed by (3.4) and we obtain that $\widehat{\lambda}_{n_0} = 0$ and $\widehat{\lambda}_n < 0$, for any $n \in \mathbb{N}\backslash\{n_0\}$. This completes the proof of the lemma. ■

Now we will show the configuration of the parameters described above induces a Turing bifurcation using γ as a bifurcation parameter, that will lead to the existence of a spatially heterogeneous stationary state. To that aim we fix $k_0 \in \mathbb{N} \setminus \{0\}$, $\eta_0 > 0$, $\varepsilon_0 > 0$ and $\gamma_0 > 0$ as in Lemma 3.1 as such that $n_0 \in \mathbb{N} \setminus \{0\}$. Next we re-write the stationary equation associated to (2.8) by shifting the positive homogeneous steady state to 0. By setting $w := u - u_e(\gamma)$ we obtain the following stationary equation

$$0 = \mathcal{H}(w,\gamma) := Aw + \tilde{F}(w,\gamma), \quad w \in H^2, \tag{3.8}$$

where $A : D(A) \to L^2_{\text{per}}$ is the sectorial operator defined by

$$Aw = -w + \varepsilon_0 w'', \tag{3.9}$$

while $\tilde{F} : H^{2-\nu} \times (0, +\infty) \to L^2_{\text{per}}$ is defined by

$$\tilde{F}(w,\gamma) = \frac{b-\mu}{\gamma\mu}\left(K_{\eta_0} \circ w\right)'' + B(w,w) + \left(\frac{\mu^2}{b+\gamma\mu w} - \mu\right) w + w. \tag{3.10}$$

Therefore $\tilde{F}(0,\gamma) = 0$ for any $\gamma \in (0, +\infty)$ and

$$\partial_w \tilde{F}(0,\gamma)\cdot\tilde{w} = \frac{b-\mu}{\gamma\mu}\left(K_{\eta_0}\circ\tilde{w}\right)'' - \frac{\mu(b-\mu)}{b}\tilde{w} + \tilde{w}.$$

Next the linear operator $\partial_w \mathcal{H}(0,\gamma) = \mathcal{A}_{\eta_0} + f'(u_e(\gamma))$ and its spectrum is given by

$$\sigma\left(\partial_w\mathcal{H}(0,\gamma)\right) = \left\{\widehat{\lambda}_n(\gamma) = -\left(\frac{n\pi}{L}\right)^2\left[\varepsilon_0 + \frac{b-\mu}{\gamma\mu}c_n\left(K_{\eta_0}\right)\right] - \frac{\mu(b-\mu)}{b}, n\in\mathbb{Z}\right\}.$$

Due to Lemma 3.1 and the choice of the parameters we know that

$$\widehat{\lambda}_{\pm n_0}(\gamma_0) = 0 \text{ and } \widehat{\lambda}_n(\gamma_0) < 0,\ \forall n \neq \pm n_0. \tag{3.11}$$

Furthermore by the continuity of the eigenvalues with respect to the parameter γ, there exists $\delta_0 > 0$ small enough such that

$$\widehat{\lambda}_n(\gamma) < -\delta_0 < 0 : n \neq \pm n_0,\ \forall \gamma \in (\gamma_0 - \delta_0, \gamma_0 + \delta_0), \tag{3.12}$$

together with

$$\frac{d\widehat{\lambda}_{n_0}(\gamma_0)}{d\gamma} = \left(\frac{n_0\pi}{L}\right)^2 \frac{b-\mu}{\gamma_0^2\mu} c_{n_0}(K_{\eta_0}) < 0. \tag{3.13}$$

Now in order to investigate the existence of non trivial branch of solutions for (3.8) and provide a simple proof, we shall overcome the difficulty coming from the zero eigenvalue of multiplicity two (see Remark 3.2) by working on the close subspace of symmetric functions. To that aim let us consider for $s \in \mathbb{R}$, the closed subspace $H^s_\sharp$ defined by

$$H^s_\sharp = \left\{\varphi \in H^s :\ \varphi(-x) = \varphi(x),\ a.e.\ x \in (-L, L)\right\}.$$

Note that the above spaces can also be characterized using the symmetry of the Fourier coefficients as follows:

$$H^s_\sharp = \left\{\varphi \in H^s :\ c_n(\varphi) = c_{-n}(\varphi),\ \forall n \in \mathbb{Z}\right\}.$$

Using the above set of notations, we now state our Turing bifurcation result.

Theorem 3.3 (Turing bifurcation). *Suppose $\eta_0, \varepsilon_0, \gamma_0$ and $n_0 \in \mathbb{N}$ are given as in Lemma 3.1 such that (3.11), (3.12) and (3.13) are satisfied. Then $(0, \gamma_0)$ is a bifurcation point for the stationary equation $\mathcal{H}(w,\gamma) = 0$ with $w \in H^2_\sharp$ in the sense that there exist $\sigma_0 > 0$ and a unique C^1-curve $(\gamma, \psi) : (-\sigma_0, \sigma_0) \to \mathbb{R} \times Z_\sharp$ such that*

$$\begin{cases} \mathcal{H}\left(\sigma\cos\left(\frac{n_0\pi\cdot}{L}\right) + \psi(\sigma), \gamma(\sigma)\right) = 0, \\ \gamma(0) = \gamma_0, \psi(0) = \psi'(0) = 0, \end{cases},\ \forall\sigma \in (-\sigma_0, \sigma_0). \tag{3.14}$$

Herein $Z_\sharp \subset H^2_\sharp$ denotes the closed subspace defined by $Z_\sharp = \{\varphi \in H^2_\sharp : \int_{-L}^{L}\varphi(x)\cos\left(\frac{n_0\pi x}{L}\right)dx = 0\}$. Furthermore, there is a neighbourhood $\mathcal{V}$ of $(0,\gamma_0)$ in $H^2_\sharp \times (0,\infty)$ such that

$$\mathcal{H}^{-1}(0) \cap \mathcal{V} = \{(0,\gamma) : \gamma \in (\mu,\infty)\} \cup \{(se_{n_0} + \psi(s), \gamma(s)) : |s| < \delta_0\}.$$

The proof of this theorem given below is based on the implicit function theorem. The idea of the proof goes back to Crandall and Rabinowitz [8]. Here we closely follow the proof of Theorem 13.4 of the monograph of Smoller [30].

Remark 3.4. *One can observe that, due to the translation invariance of (1.3), the above result allows us to obtain a non-symmetric family of heterogeneous stationary states for the equation $\mathcal{H}(w,\gamma)=0$ in a neighbourhood of $(0,\gamma_0)$. Indeed, with the notations of the above theorem, for each $\sigma\in(-\sigma_0,\sigma_0)$ and each $\tau\in\mathbb{R}$ one has*

$$\mathcal{H}\left(w_\sigma(\cdot+\tau,\gamma(\sigma)\right)=0,$$

wherein we have set $w_\sigma(x)=\cos\left(\frac{n_0\pi}{L}x\right)+\psi(\sigma)(x)$, $x\in[-L,L]$. Furthermore from the numerical experiments provided in Figure 3, for each σ, the family $\{w_\sigma(\cdot+\tau)\}_{\tau\in\mathbb{R}}$ seems to be orbitally stable.

Proof. To prove this result, we shall apply the implicit function theorem on a space of symmetric function such that the eigenspace associated to the zero eigenvalue is one dimensional. To that aim we consider the map

$$\begin{aligned}\mathcal{G}(\sigma,z,\gamma)&:=\mathcal{H}\left(\sigma\cos\left(\frac{n_0\pi\cdot}{L}\right)+\sigma z,\gamma\right)\\&=A\left(\sigma\cos\left(\frac{n_0\pi\cdot}{L}\right)+\sigma z\right)+\tilde{F}\left(\sigma\cos\left(\frac{n_0\pi\cdot}{L}\right)+\sigma z,\gamma\right),\end{aligned}$$

that is of the class C^2 and is defined on a small neighbourhood of $(0,0,\gamma_0)\in\mathbb{R}\times Z\times(0,+\infty)$. Here $Z:=\{\varphi\in H^2:\int_{-L}^{L}\varphi(x)\cos\left(\frac{n_0\pi x}{L}\right)dx=0\}$. As a consequence we fix $r>0$ small enough and we consider the map $\mathcal{G}$ as defined from $(-r,r)\times B_Z(0,r)\times(\gamma_0-r,\gamma_0+r)$ with value in H^0. Here $B_Z(0,r)\subset H^2$ denotes the open ball in the Banach space Z with center 0 and radius r small enough. Now observe that, since the kernel $K_{\eta_0,0}$, is symmetric with respect to 0, the nonlinear operator $\mathcal{G}$ satisfies

$$\mathcal{G}(\sigma,\varphi,\gamma)\in H^0_\sharp,\ \text{for all }|\sigma|<r,\ \varphi\in B_Z(0,r)\cap H^2_\sharp\text{ and }|\gamma_0-\gamma|<r.$$

As a consequence we consider the map $\mathcal{G}_\sharp(\sigma,z,\gamma)=\mathcal{G}(\sigma,z,\gamma)$ defined from $(-r,r)\times B_{Z_\sharp}(0,r)\times(\gamma_0-r,\gamma_0+r)$ with values in $H^0_\sharp$ with $B_{Z_\sharp}(0,r)=B_Z(0,r)\cap H^2_\sharp$. As already mentioned it is a smooth map, namely of the class C^2, on this open set and it furthermore satisfies

$$\mathcal{G}_\sharp(0,z,\gamma)=0,\ \forall(\gamma,z)\in(\gamma_0-r,\gamma_0+r)\times B_{Z_\sharp}(0,r).$$

Now to prove our result we consider the C^1 map $\mathcal{F}_\sharp$ defined from $(-r,r)\times B_{Z_\sharp}(0,r)\times(\gamma_0-r,\gamma_0+r)$ into $H^0_\sharp$ by

$$\mathcal{F}(\sigma,z,\gamma)=\begin{cases}\frac{1}{\sigma}\mathcal{G}_\sharp\left(\sigma,z,\gamma\right)&\text{if }\sigma\neq0\\ \\ \partial_\sigma\mathcal{G}_\sharp\left(0,z,\gamma\right)&\text{if }\sigma=0.\end{cases}$$

Let us observe that one has

$$\mathcal{F}(0,0,\gamma_0)=\partial_\sigma\mathcal{G}_\sharp(0,0,\gamma_0)=\partial_w\mathcal{H}(0,\gamma_0)\cdot\mathrm{Re}\left(e_{n_0}\right)=\mathrm{Re}\left(\widehat{\lambda}_{n_0}(\gamma_0)e_{n_0}\right)=0.$$

Hence to prove our result we shall apply the implicit function theorem for the C^1-function $\mathcal{F}$ in the neighbourhood of the point $(\sigma,z,\gamma)=(0,0,\gamma_0)$. Thus to complete the proof of the theorem, it is sufficient to prove that the partial derivative operator $\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0)$ is a linear isomorphism from $Z_\sharp\times\mathbb{R}$ onto $H^0_\sharp$. To check the invertibility of $\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0)$ first note that one has

$$\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0)\cdot(z,\gamma)=\partial_w\mathcal{H}(0,\gamma_0)\cdot z+\gamma\widehat{\lambda}'_{n_0}(\gamma_0)\mathrm{Re}\ \left(e_{n_0}\right).$$

Let $h\in H^0_\sharp$ be given and let us solve the equation

$$\text{Find }(z,\gamma)\in Z_\sharp\times\mathbb{R}\text{ such that }\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0)\cdot(z,\gamma)=h.$$

However writing $h=\sum_{n\in\mathbb{Z}}h_ne_n$ with $(h_n)\in l^2(\mathbb{Z};\mathbb{C})$ and $h_n=h_{-n}$ for all $n\in\mathbb{Z}$ and projecting the above equation on the Hilbert basis (e_n) the above equation re-writes as follows:

$$\begin{cases}\widehat{\lambda}_n(\gamma_0)z_n=h_n\ \forall n\in\mathbb{Z}\setminus\{\pm n_0\},\\ \gamma\widehat{\lambda}'_{\pm n_0}(\gamma_0)=h_{\pm n_0}.\end{cases}$$

Herein $z_n \in \mathbb{C}$ denotes the projection of z on e_n. Since $\widehat{\lambda}'_{n_0}(\gamma_0) \neq 0$, $\widehat{\lambda}_n(\gamma) = \widehat{\lambda}_{-n}(\gamma)$ and $h_n = h_{-n}$ this yields

$$z_n = z_{-n},\ \forall n \in \mathbb{Z}\backslash\{\pm n_0\},$$
$$z_n = \frac{h_n}{\widehat{\lambda}_n(\gamma_0)}\ \forall n \in \mathbb{N} \setminus \{n_0\} \text{ and } \gamma = \frac{h_{n_0}}{\widehat{\lambda}'_{n_0}(\gamma_0)}.$$

Hence the above equation has at most one solution in $Z_\sharp \times \mathbb{R}$. Furthermore since $\widehat{\lambda}_n(\gamma_0) \sim -n^2 \frac{\pi^2 \varepsilon_0}{L^2}$ as $|n| \to \infty$, the vector (z, γ) defined by

$$z = \sum_{n \in \mathbb{Z}\backslash\{\pm n_0\}} \frac{h_n}{\widehat{\lambda}_n(\gamma_0)} e_n,\ \gamma = \frac{h_{n_0}}{\widehat{\lambda}'_{n_0}(\gamma_0)},$$

satisfies $z \in Z_\sharp$ and $\gamma \in \mathbb{R}$ as well as $\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0) \cdot (z,\gamma) = h$. Hence $\partial_{(z,\gamma)}\mathcal{F}(0,0,\gamma_0)$ is invertible from $Z_\sharp \times \mathbb{R}$ into $H^0_\sharp$. As a consequence, the implicit function theorem ensures the existence of a C^1-map $\sigma \mapsto (\phi(\sigma), \gamma(\sigma)) \in Z_\sharp \times \mathbb{R}$ defined in some neighbourhood of $\sigma = 0$, denoted by $(-\sigma_0, \sigma_0)$ for some $\sigma_0 > 0$, such that

$$\gamma(0) = \gamma_0 \text{ and } \phi(0) = 0,$$

and for all $\sigma \in (-\sigma_0, \sigma_0)$,

$$\begin{cases} z \in Z_\sharp,\ \|z\|_{H^2} \le r,\ |\gamma - \gamma_0| \le r, \\ \mathcal{G}(\sigma, z, \gamma) = 0 \end{cases} \quad \Leftrightarrow\ z = \phi(\sigma) \text{ and } \gamma = \gamma(\sigma).$$

This completes the proof of the theorem by setting $\psi(\sigma) = \sigma\phi(\sigma) \in Z_\sharp$. ∎

We complete this section by the following stability result for the symmetric and spatially heterogeneous bifurcation branch. Let us notice that since the kernel $K_{\eta_0,0}$ is symmetric, the nonlinear maximal semiflow provided in Theorem 2.8 leaves the space $H^{2-\nu}_\sharp$ invariant. We denote the semiflow restricted on $H^{2-\nu}_\sharp$ by $U_\sharp(t)(\cdot)$. Next by using the results in [17, Theorem 6.3.2 p.178] and incorporating (3.11), (3.12) and (3.13), we obtain the following stability results of the bifurcated solution.

Theorem 3.5. *Let η_0, ε_0, γ_0 and n_0 be parameters as in Lemma 3.1 such that (3.11), (3.12) and (3.13) are satisfied. Then there exists $r > 0$ small enough and a nontrivial equilibrium $u_\gamma = u_\gamma(x) \in H^2_\sharp$ for $\gamma \in (\gamma_0 - r, \gamma_0 + r)$ such that it is unstable with respect to $U_\sharp$ (in $H^{2-\nu}_\sharp$) if $\gamma > \gamma_0$ but asymptotically stable for $\gamma < \gamma_0$.*

Remark 3.6. *As the Lemma 3.1 shows, $\phi(x) - \frac{\mu(b-\mu)}{b} = 0$ will be the curve above which the bifurcation occurs. Re-writing it explicitly reads as follows*

$$-\frac{\varepsilon}{\eta^2}x^2 - \frac{b-\mu}{\gamma\mu\eta^2}x\sin x = \frac{\mu(b-\mu)}{b}.$$

Therefore, for any fixed b, μ, η_0 and L with $\frac{L}{2\eta_0} \in \mathbb{N}$. For $n \ge 0$, regarding ε as a function of γ^{-1}, the curves

$$\epsilon = -\frac{b-\mu}{\mu}\frac{\sin(n\eta\pi/L)}{n\eta\pi/L}\gamma^{-1} - \left(\frac{n\pi}{L}\right)^{-2}\frac{\mu(b-\mu)}{b} =: H_n(\gamma^{-1}),$$

determines the stability region of the system. In fact, the spatially homogeneous steady state $u_e = u_e(\gamma)$ is locally stable in the region above all the curves $H_n(\gamma^{-1})$ for $n = \frac{L}{2\eta_0}(-1+4k)$ with $k \in \mathbb{N} \setminus \{0\}$.

We continue this section with numerical experiments of (1.3) with the kernel $\rho_{\eta_0,0}$ (as defined in (1.6)). To that aim we fix the parameter values

$$L = 2,\ b = 1.5,\ \mu = 1.2 \text{ and } \eta = 1. \tag{3.15}$$

Note that $\frac{L}{2\eta} = 1$ so that the condition $\frac{L}{2\eta} \in \mathbb{N}$ is satisfied.

Figure 1 depicts the stability region and the different bifurcation curves corresponding to different $k = 1, 2, ..., 10$.

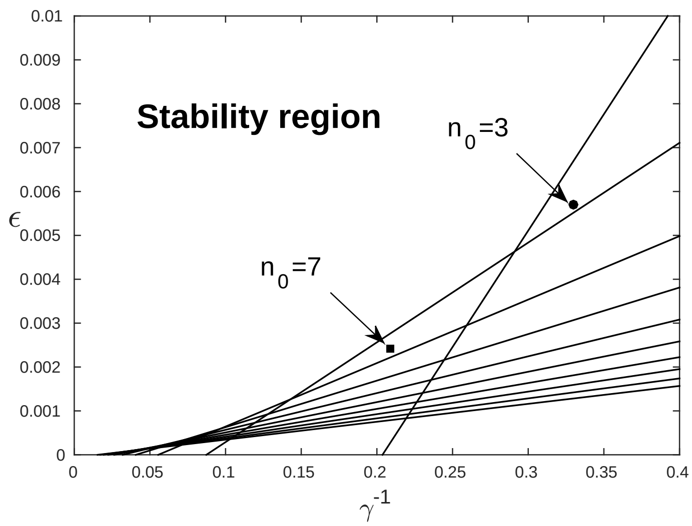

Figure 1: *Plots of the curves* $\varepsilon = H_n(\gamma^{-1})$ *with* $n = -1 + 4k$ *and* $k = 1, 2, \ldots, 10$. *They are straight lines and their slopes is decreasing with respect to* n*, hence to* k*. The stability region of the homogeneous steady state is above all these curves.*

Our numerical experiments are concerned with the behaviour of the nonlinear system (1.3) with the parameter (3.15). The choice of the parameter ε and γ are given in Table 3.17. These choices of parameters are presented in Figure 1 by the circle and square dots respectively. Both situations correspond to the instability of the homogeneous steady state and more deeply, both of these situations correspond to a unique pair of unstable eigenvalues $\widehat{\lambda}_{\pm n_0}$ for $n_0 = 3$ and 7 respectively. The results of the simulations are presented in Figure 2. Here we use the following Gaussian type function

$$u_0(x) = \frac{0.05}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{(2\sigma^2)}}, \; x \in [-2, 2], \tag{3.16}$$

with $\sigma = 0.2$ as initial distribution. The simulations show that the instability of the homogeneous stationary state will give rise to a stable symmetric stationary pattern solutions. As we can observe from Figure 2, the dominant wave number of the solutions of the nonlinear equation (1.3) is exactly in accordance with the index n_0 where $\widehat{\lambda}_{\pm n_0}$ is the unique pair positive eigenvalues of the linear operator $\mathcal{A} + f'(u_e)$.

$$\begin{aligned} \text{First configuration:} \quad (\varepsilon, \gamma) = (0.0056, \, 3.03)\,, \\ \text{Second configuration:} \quad (\varepsilon, \gamma) = (0.0023, \, 4.80)\,. \end{aligned} \tag{3.17}$$

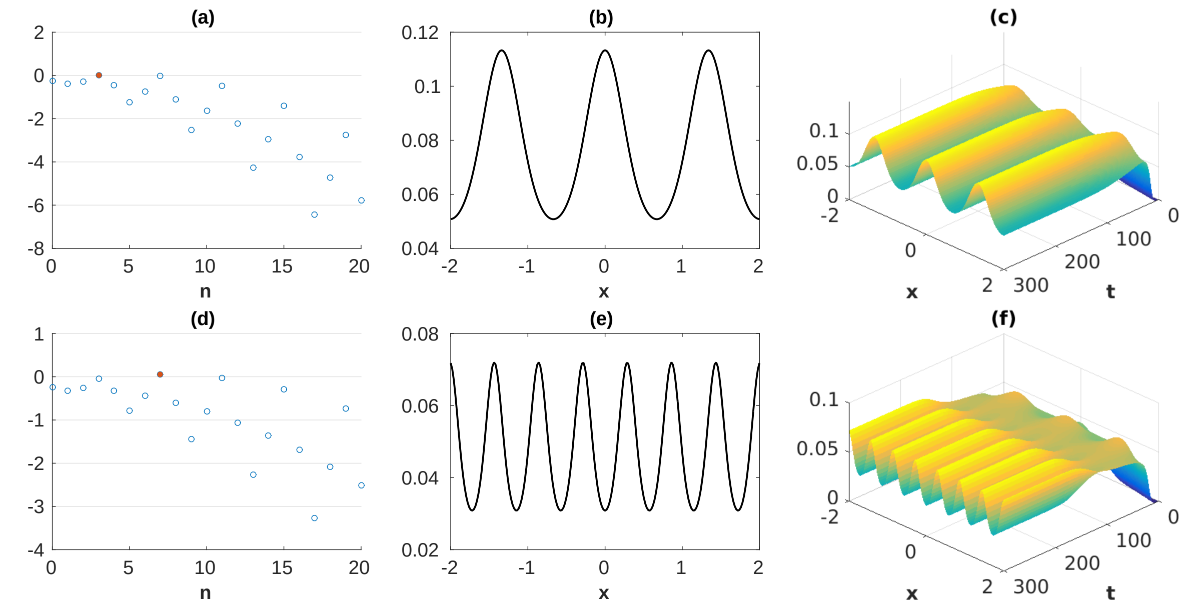

Figure 2: *Simulations* (1.3) *with parameters values* (3.15) *and* (3.17). *The upper row corresponds to* $(\varepsilon, \gamma) = (0.0056, 3.03)$ *while the bottom row to* $(\varepsilon, \gamma) = (0.0023, 4.80)$. *Figure* $(a), (d)$ *describe the spectrum of the linearized equation at the homogeneous steady state;* $(b), (e)$ *present the spatial distributions of the solution at a given large time* $T = 300$*; and,* $(c), (f)$ *present the spatio-temporal evolution of the solutions. The initial distribution is given in* (3.16) *with* $\sigma = 0.2$.

As we have mentioned in Remark 3.4, the symmetric heterogeneous steady state is translation invariant. Therefore, given a non-symmetric initial profile, we present the spatio-temporal evolution of the solution as in Figure 3 and the simulation indicates the solution will converge to a non-symmetric heterogeneous steady state. Therefore the family of steady states $\{w_\sigma(\cdot + \tau)\}_{\tau \in \mathbb{R}}$ should be orbitally stable.

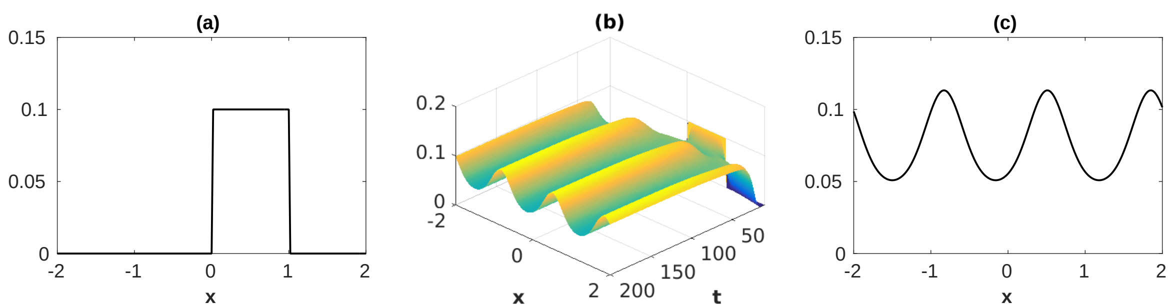

Figure 3: *Choosing parameters values as in* (3.15) *and the first configuration in* (3.17) *we obtain the above figures. Figure (a) presents the given non-symmetric initial value, figure (b) presents the spatio-temporal evolution of the solution and figure (c) presents the solution at a large time* $T = 200$ *when it is mostly stabilized close to a suitable shift of the symmetric stationary state. The other parameters are the same as in Figure 2 for the wave number* $n_0 = 3$.

3.2 Turing-Hopf bifurcation

In this section we continue the bifurcation analysis of Problem (1.3) by using the kernel function $\rho_{\eta,s}$ defined in (1.6). Here we shall vary the shift parameter $s \in \mathbb{R}$ which will lead us to what we call Turing-Hopf bifurcation and the existence of spatially heterogeneous time periodic solutions.

The reason that we call it Turing-Hopf bifurcation is based on the fact that by choosing the parameters of the system properly, it admits a Hopf bifurcation such that:
1) We can find some mode n_0, as large as we want, such that the periodic orbit is tangent to the eigenfunction e_{n_0};
2) It consists in the first Hopf bifurcation, which means that the equilibrium is passing from a stable to an unstable situation, by playing on the Hopf bifurcation parameter.

Let us mention that the first bifurcation is of particular interest in practice since this is the bifurcation that can be observed numerically.

As already mentioned that we will work on Problem (1.3) with the kernel $K_{\eta,s}$, the corresponding $2L-$periodic kernel associated to $\rho_{\eta,s}$ in (1.6), for some well chosen parameter $\eta > 0$ and $s \in (0, L]$. The corresponding linearized operator at the equilibrium $u_e = u_e(\gamma)$ is denoted by $\mathcal{A}_{\eta,s} + f'(u_e(\gamma))$.

Note that introducing the shift parameter s implies that the step function is no longer symmetric so that the eigenvalues of $\mathcal{A}_{\eta,s}+f'(u_e(\gamma))$ can take complex – non real – values. In the next lemma we shall prove a result rather similar to the one stated in Lemma 3.1 with complex 'dominant' eigenvalues. More precisely, choosing the shifting parameter $s = \eta$, we shall prove that one can choose a mode $n_0 \geq 1$ such that the eigenvalues $\widehat{\lambda}_{n_0}$ and $\widehat{\lambda}_{-n_0}$ are a unique pair of purely imaginary eigenvalues satisfying the transversality condition with respect to the bifurcation parameter γ while the other eigenvalues have negative real part.

Lemma 3.7. *Let $k_0 \in \mathbb{N}\backslash\{0\}$ be given and fix $s = \eta_0$ with $L/(4\eta_0) \in \mathbb{N}$. Let us denote by $\widehat{\lambda}_n(\gamma)$ the sequence of eigenvalues of $\mathcal{A}_{\eta_0,\eta_0} + f'(u_e(\gamma))$. Then there exist $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that*

$$\overline{\widehat{\lambda}_{n_0}(\gamma_0)} = \widehat{\lambda}_{-n_0}(\gamma_0),$$
$$\mathrm{Re}(\widehat{\lambda}_{\pm n_0}(\gamma_0)) = 0,\ \mathrm{Im}(\widehat{\lambda}_{n_0}(\gamma_0)) > 0,\quad \frac{d\mathrm{Re}(\widehat{\lambda}_{n_0})(\gamma_0)}{d\gamma} \neq 0,$$

and

$$\sigma\left(\mathcal{A}_{\eta_0,\eta_0} + f'(u_e(\gamma_0))\right) \cap i\mathbb{R} = \left\{\widehat{\lambda}_{n_0}(\gamma_0), \widehat{\lambda}_{-n_0}(\gamma_0)\right\},$$

with $n_0 = \frac{L}{4\eta_0}(-1 + 4k_0) \in \mathbb{N}\backslash\{0\}$. Moreover, we have

$$\mathrm{Re}(\widehat{\lambda}_n(\gamma_0)) < 0,\ \textit{for any } n \in \mathbb{Z}\backslash\{\pm n_0\}.$$

Proof. As mentioned above we set $s = \eta$. Hence recalling (3.1), the eigenvalues of $\mathcal{A}_{\eta,\eta} + f'(u_e(\gamma))$ take the following form, for any $n \in \mathbb{Z}$,

$$\mathrm{Re}(\widehat{\lambda}_n(\gamma)) = -\left(\frac{n\pi}{L}\right)^2\left(\varepsilon + u_e(\gamma)\frac{\sin(2n\eta\pi/L)}{2n\eta\pi/L}\right) - \frac{\mu(b-\mu)}{b},$$
$$\mathrm{Im}(\widehat{\lambda}_n(\gamma)) = \left(\frac{n\pi}{L}\right)^2\frac{\sin^2(n\eta\pi/L)}{n\eta\pi/L}u_e(\gamma).$$

Note that one has
$$\overline{\widehat{\lambda}_n(\gamma)} = \widehat{\lambda}_{-n}(\gamma),\ \forall n \in \mathbb{Z},\ \gamma > 0.$$

Let $k_0 \geq 1$ and $\eta_0 > 0$ such that $L(4\eta_0)^{-1} \in \mathbb{N}$ be given. Then using the same arguments as the ones in the proof of Lemma 3.1 one can find $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that

$$\mathrm{Re}(\widehat{\lambda}_{\pm n_0}(\gamma_0)) = 0,\ \frac{d\mathrm{Re}(\widehat{\lambda}_{n_0})(\gamma_0)}{d\gamma} \neq 0,$$
$$\mathrm{Re}(\widehat{\lambda}_n(\gamma_0)) < 0,\ \text{for any } n \in \mathbb{Z}\backslash\{\pm n_0\},$$

with $n_0 = \frac{L}{4\eta_0}(-1+4k_0)$, that is

$$\frac{2n_0\eta_0\pi}{L} \equiv \frac{b-\mu}{2\mu\gamma_0\varepsilon_0} = -\frac{\pi}{2} + 2k_0\pi. \tag{3.18}$$

To complete the proof of the lemma, it remains to check that $\operatorname{Im}(\widehat{\lambda}_{n_0}(\gamma_0)) > 0$. However simple computations yield

$$\operatorname{Im}(\widehat{\lambda}_{n_0}(\gamma_0)) = \left(\frac{n_0\pi}{L}\right)^2 \frac{\sin^2(-\frac{\pi}{4}+k_0\pi)}{n_0\eta_0\pi/L} u_e(\gamma_0) = \left(\frac{n_0\pi}{L}\right)\frac{1}{2\eta_0}u_e(\gamma_0) > 0.$$

And, this complete the proof of the lemma. ■

The spectral configuration discussed in the above lemma will allow us to state the following Hopf bifurcation result for the evolution problem

$$\frac{dw(t)}{dt} = Aw(t) + \tilde{F}(w(t),\gamma), \tag{3.19}$$

wherein we have set, as in the previous subsection, $w(t) = u(t) - u_e(\gamma)$, the linear operator A and the function $\tilde{F}$ are defined in (3.9) and (3.10) respectively.

In order to discuss our Hopf bifurcation theorem we first discuss the existence of a center manifold reduction for the above problem. To that aim, we fix $k_0 \geq 1$ and $\eta_0 > 0$ as in the previous lemma and let $\varepsilon_0 > 0$, $\gamma_0 > 0$ and $n_0 \geq 1$ be the parameter provided by this lemma. Next we include the parameter γ into the state space and we consider the the following problem

$$\frac{d}{dt}\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix} = L\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix} + R\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix},$$

wherein we have set

$$L = \begin{pmatrix} \left(A + \partial_w \tilde{F}(0,\gamma_0)\right) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\left(H^2\times\mathbb{R}, H^0\times\mathbb{R}\right),$$

and

$$R\begin{pmatrix} w \\ \gamma \end{pmatrix} = \begin{pmatrix} \tilde{F}(w,\gamma) - \partial_w\tilde{F}(0,\gamma_0)w \\ 0 \end{pmatrix}.$$

The function R is defined and of the class C^∞ from a neighbourhood $\mathcal{V} \subset H^2\times\mathbb{R}$ of $(w,\gamma) = (0,\gamma_0)$ into $H^0\times\mathbb{R}$. Note also that R satisfies $R\begin{pmatrix} 0 \\ \gamma_0 \end{pmatrix} = 0$ and $DR\begin{pmatrix} 0 \\ \gamma_0 \end{pmatrix} = 0$.

Next, the spectral configuration described in Lemma 3.7 ensures that

$$\sigma(L)\cap i\mathbb{R} = \left\{0, \widehat{\lambda}_{n_0}(\gamma_0), \overline{\widehat{\lambda}_{n_0}(\gamma_0)}\right\},$$

while $\sigma(L)\cap\{z\in\mathbb{C}:\ \operatorname{Re} z > 0\} = \emptyset$.

Note that the center space $\mathcal{E}_c$ is generated by $(e_{n_0},0)$, $(e_{-n_0},0)$ and $(0,1)$.

Moreover because of the resolvent estimate (2.7) and due to the spectral configuration described in Lemma 3.7, there exist $\omega_0 > 0$ and $M > 0$ such that

$$\left\|(i\omega - L)^{-1}\right\|_{\mathcal{L}(H^0\times\mathbb{R})} \leq \frac{M}{|\omega|}, \text{ for all } \omega\in\mathbb{R} \text{ such that } |\omega| > \omega_0,$$

Now since $H^2\times\mathbb{R}$ and $H^0\times\mathbb{R}$ are both Hilbert spaces, Theorem 2.20 in [15] applies and ensures the existence of smooth center manifold. Applying Hopf bifurcation theorem (see for instance [16]), this center manifold reduction allows us to obtain the Hopf bifurcation result.

Theorem 3.8 (**Hopf Bifurcation**). *Let $k_0 \geq 1$ and $\eta_0 > 0$ be given such that $L/(4\eta_0) \in \mathbb{N}$. Let $\varepsilon_0 > 0$ and $\gamma_0 > 0$ be the parameters provided by Lemma 3.7. There exist $\sigma^* > 0$, two smooth functions $\sigma \mapsto \gamma(\sigma)$ and $\sigma \mapsto \omega(\sigma)$ defined on $(0, \sigma^*)$ such that for all $\sigma \in (0, \sigma^*)$ the equation*

$$\frac{dw(t)}{dt} = Aw(t) + \tilde{F}\left(w(t), \gamma(\sigma)\right),\ t \in \mathbb{R},$$

has a non trivial $\omega(\sigma)-$time periodic solution $w(t)$. Furthermore one has

$$\gamma(\sigma) = \gamma_0 + O(\sigma^2),\ \ \omega(\sigma) = \frac{2\pi}{\operatorname{Im} \widehat{\lambda}_{n_0}(\gamma_0)} + O(\sigma^2) \text{ as } \sigma \to 0.$$

Remark 3.9. *The stability of the bifurcated periodic solution is studied in Appendix 5.1 by using the center manifold reduction and the study of the normal form. The stability of the Turing bifurcation presented in the previous section can also be investigated by using similar computations.*

Remark 3.10. *Another proof for the existence of the Hopf Bifurcation in our case can be found in the work by Crandall and Rabinowitz [9] by using the implicit function theorem. In fact, in our case the sectorial operator $A_{\eta_0,\eta_0} + \partial_w \tilde{F}(0, \gamma_0)$ satisfies $\sigma\left(A_{\eta_0,\eta_0} + \partial_w \tilde{F}(0, \gamma_0)\right) \cap i\mathbb{R} = \left\{\widehat{\lambda}_{n_0}(\gamma_0), \widehat{\lambda}_{-n_0}(\gamma_0)\right\}$ and the eigenvalues are simple. Moreover, the operator*

$$(\lambda - (A_{\eta_0,\eta_0} + \partial_w \tilde{F}(0, \gamma_0)))^{-1} : H^0 \to H^0$$

is compact for any λ in the resolvent set. Therefore, the hypothesis $(HL), (Hf)$ and $(H\beta)$ in [9] are satisfied and Theorem 1.11 in this aforementioned work ensures the existence and uniqueness of the Hopf bifurcation in a small neighbourhood of $(0, \gamma_0) \in H^{2-\nu} \times \mathbb{R}_+$.

We continue this section by numerical experiments for System (1.3) with the kernel (1.6). To that aim, we fix the following parameter sets

$$b = 1.5,\ \mu = 1.2,\ L = 2,\ \eta = \eta_0 = s = 0.5. \tag{3.20}$$

Note that one has $L/(4\eta_0) = 1 \in \mathbb{N}$.

We consider two situations, close to the Turing-Hopf bifurcation point described above, that correspond to the parameters

$$\begin{aligned} \text{First configuration:}\quad (\varepsilon, \gamma) &= (0.0023,\ 4.8)\,, \\ \text{Second configuration:}\quad (\varepsilon, \gamma) &= (0.00084,\ 8.4)\,. \end{aligned} \tag{3.21}$$

With such choices the spectrum configuration reads as follows: $\widehat{\lambda}_{\pm n_0}$ are the only eigenvalues with positive real part and all the other eigenvalues have negative real parts. Moreover $\operatorname{Re}(\widehat{\lambda}_{\pm n_0})$ is close to zero and $\operatorname{Im}(\widehat{\lambda}_{\pm n_0}) \neq 0$. This holds true for $n_0 = 7$ and $n_0 = 11$ respectively for the two parameter sets (ε, γ) mentioned above.

With the parameters described above and equipped with the same initial data as the one use in Figure 2 (see (3.16)), the spatio-temporal evolution for the solutions of (1.3) is presented in Figure 4 for the two parameter configurations in (3.21).

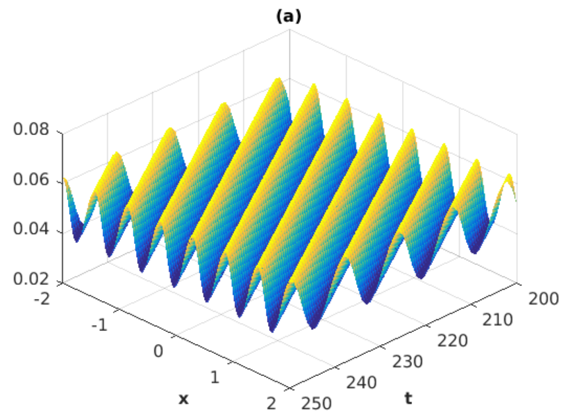

Figure 4: *In this figure we fix the parameter values as in (3.20) and (3.21). We observe a spatio-temporal evolution of the solutions corresponding in (a) (respectively in (b)) to the first configuration (respectively the second configuration) of the parameters in (3.21).*

These simulations show that the solutions takes the form of a periodic wave train solution. Heuristically the first order approximation of the bifurcated solutions take the form

$$\begin{aligned} u(t,x) \approx & u_e + \mathrm{Re}\left(a(\gamma)e^{i\omega t}e_{n_0}(x)\right) + h.o.t, \quad \text{for some constant } a(\gamma) \in \mathbb{C}\setminus\{0\} \\ = & u_e + |a(\gamma)|\cos\left(\omega t + \frac{n_0\pi}{L}x + \varphi\right) + h.o.t, \end{aligned} \tag{3.22}$$

where $\omega \approx \mathrm{Im}(\widehat{\lambda}_{n_0})$ and $\varphi \in \mathbb{R}$ is a phase number while the amplitude $|a(\gamma)|$ of the oscillating solution depends on the bifurcation parameter γ. Moreover from the normal form reduction provided in Appendix 5.1 we have $|a(\gamma)| \sim a_* \sqrt{|\gamma - \gamma_0|}$ for some constant a_* when $0 < \gamma - \gamma_0 \ll 1$ or $0 < \gamma_0 - \gamma \ll 1$ depending on the nature (supercritical or subcritical) of the Hopf bifurcation. Therefore, the expression in (3.22) roughly explains the spatio-temporal pattern observed in Figure 4. Moreover the numerical comparison of the wave lengths of the solutions in Figure 4 and the above expression are in accordance.

We continue this section by exploring an other type of kernel function ρ. And we show that instabilities, and more precisely Turing-Hopf bifurcation, may also occur for some bi-modal kernel. Here as an example, we consider two identical Gaussian functions, one shifted to the left and one shifted to the right, namely

$$\rho_2(x) = \frac{1}{2}\left(G(x+s_1) + G(x-s_2)\right), \text{ with } G(x) := e^{-\pi x^2},$$

and wherein s_1 and s_2 are two positive parameters. Here we restrict ourselves to a numerical exploration of Problem (1.1)-(1.2) with such a kernel function. However similar analytical results as the ones presented above can be obtained for this bi-modal example (see Remark 3.11 below). Note also that such a choice of multi-modal kernel is biologically relevant when we consider the preferred sensing radius of a certain type of cell.
When kernel ρ_2 is considered, the Fourier transform of this kernel can be calculated explicitly and we have

$$\widehat{\rho_2}(\xi) = \frac{1}{2}e^{-\pi\xi^2}\left[e^{2i\pi s_1\xi} + e^{-2i\pi s_2\xi}\right]. \tag{3.23}$$

Therefore, according to (2.10), the real part and imaginary part of the eigenvalues for the system (1.3) are given as follows

$$
\begin{aligned}
\operatorname{Re}(\lambda_n) &= -\left(\frac{n\pi}{L}\right)^2 \left[\epsilon + u_e \operatorname{Re}\left(\widehat{\rho_2}\left(\frac{n}{2L}\right)\right)\right] - \frac{\mu(b-\mu)}{b}, \\
\operatorname{Im}(\lambda_n) &= -\left(\frac{n\pi}{L}\right)^2 u_e \operatorname{Im}\left(\widehat{\rho_2}\left(\frac{n}{2L}\right)\right), \ n \in \mathbb{Z}.
\end{aligned}
\tag{3.24}
$$

In the following simulation, we fix parameters b, μ and L as in (3.20) while we take

$$
(\varepsilon, \gamma) = (0.01,\, 0.2)\,, \quad s_1 = 0.4,\ s_2 = 0.3. \tag{3.25}
$$

By choosing the above parameters one can check that when $n = \pm 4$, λ_n is the only pair of eigenvalues which has positive real part and we plot the distribution of the eigenvalues on the complex plane in Figure 5 (a). Using the same initial distribution in Figure 2, the numerical simulation of (1.3) with kernel ρ_2 is presented in Figure 5 (b).

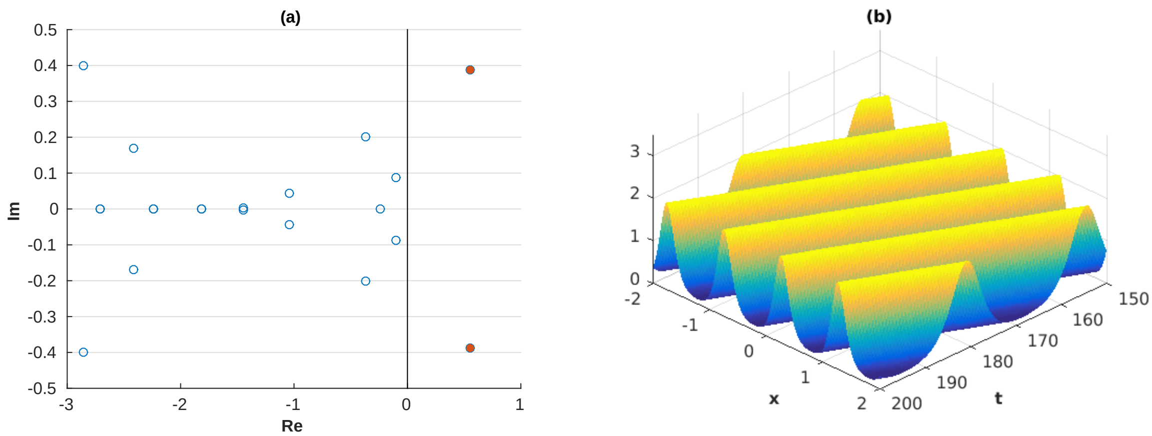

Figure 5: *In this figure we fix the parameter values as in* (3.21) *and* (3.25)*. In Figure* (*a*) *we plot the eigenvalues of the linearized equation by* (3.24) *in the complex plane for* $n = -10, -9, .., 9, 10$*. By choosing the parameters in configuration* (3.25)*, there is only one pair of eigenvalues, namely* $\lambda_{\pm 4}$*, with a positive real part (see the filled dots). We observe a corresponding spatio-temporal evolution of the solutions in* (*b*)*. The simulation shows the bi-modal kernels can also lead to instability.*

Remark 3.11. *If we take* γ *as a bifurcation parameter and choose appropriate parameters* ε*,* s_1 *and* s_2*, a similar spectral analysis as in Lemma 3.7 for the linearized equation with kernel* ρ_2 *can be performed by using the explicit formula in* (3.23) *and* (3.24)*. And one may use similar arguments as the ones developed for the proof of Theorem 3.8 to prove the existence of a Hopf bifurcation for the bi-modal case.*

4 Conclusion and Discussion

In this article we discussed some dynamical properties of Problem (1.3). Depending on the kernel function ρ, we are able first to discuss the stability and instability of the unique homogeneous positive steady state. A bifurcation analysis has been performed to understand emerging complex patterns when the positive homogeneous steady state becomes unstable. With a symmetric step function kernel, Turing bifurcation of equilibrium may occur. As a result we obtain the existence of a stable branch of spatially heterogeneous steady states. More surprisingly when this symmetry is broken by shifting the step function, the homogeneous steady state may undergo what we have called Turing-Hopf bifurcation yielding the existence of a branch of spatially heterogeneous and time periodic solutions.

It is also interesting to recognize the complexity raised by the nonlinear and nonlocal diffusion compare to nonlinear but local diffusion equation. As we can see from our bifurcation analysis, when ε goes to 0, rich dynamical behaviours emerge from the model (1.3). This is also true without vital dynamic term, *i.e.* whenever $f = 0$.

The case of zero viscosity, i.e., $\varepsilon = 0$, is also of particular interest. From the spectral analysis of the operator $\mathcal{A}$ with kernel $\rho_{\eta,s}$, we can expect that the frequencies of oscillating solutions will become higher if the viscosity coefficient becomes small. This may be due to the increasing number of positive eigenvalues in such case. Moreover, we point out other kernels with their Fourier transform changing signs will present the similar complex dynamics as the one observed for the step function kernel when $\varepsilon \ll 1$ is small enough. To illustrate this issue we consider the C^∞ kernel

$$\rho_\sharp(x) = c_0 e^{\frac{1}{x^2-1}} \chi_{(-1,1)}(x), \tag{4.1}$$

where the constant c_0 is defined by $c_0 := 1/\int_{-1}^{1} \rho_\sharp(x)dx$. We furthermore denote by $\rho_{\eta,\sharp}(x) := \frac{1}{\eta}\rho_\sharp(\frac{x}{\eta})$ the function $\rho_\sharp$ with scaling parameter $\eta > 0$. However, unlike the step kernel, the Fourier coefficient of $\rho_{\eta,\sharp}$ does not have an explicit form, here we give in Figure 6 a numerical illustration of the following map

$$n \longmapsto -\left(\frac{n\pi}{L}\right)^2 \widehat{\rho}\left(\frac{n}{2L}\right), \ n \in \mathbb{Z}$$

for step function kernel $\rho = \rho_{\eta,0}$ and $\rho = \rho_{\eta,\sharp}$. Note that $\widehat{\rho}_\eta\left(\frac{n}{2L}\right) = \widehat{\rho}\left(\frac{n\eta}{2L}\right)$. These numerical illustrations are performed with the fixed values $L = 4$ and $\eta = 0.8$. By the symmetry of Fourier coefficients, here we plot these maps for $n = 0, 1, 2, \dots, 50$.

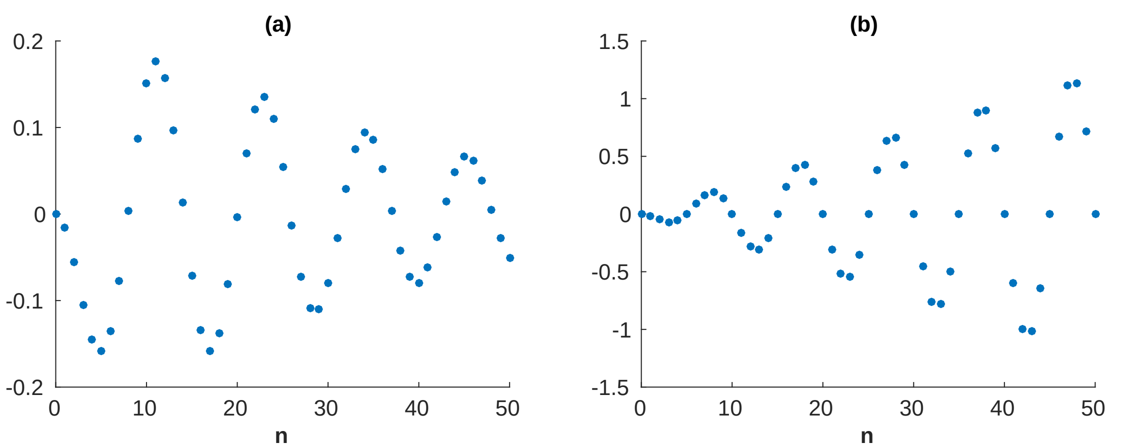

Figure 6: *In this figure we plot of* $n \mapsto -\left(\frac{n\pi}{L}\right)^2 \widehat{\rho}\left(\frac{n\eta}{2L}\right)$ *(with* $n = 0, 1, \dots, 50$*). The Figure* (*a*) *and* (*b*) *correspond respectively to the smooth function* $\rho = \rho_{\eta,\sharp}$ *defined in* (4.1) *and to the step function* $\rho = \rho_{\eta,0}$ *defined in* (1.6)*. In both cases we fix* $L = 4$ *and* $\eta = 0.8$*. The blue dots corresponds to the eigenvalues of the linear operator* $\mathcal{A}$ *whenver* $\varepsilon = 0$*. As we can see in* (*a*)*, a smooth kernel can also leads to an infinite number of positive eigenvalues. Also the Figure (b) should be compared to the Figure* 3 (*a*) *and* (*d*) *in which* $\varepsilon > 0$ *plays an crucial role to get only one positive eigenvalue.*

The existence of positive Fourier coefficients will result in the essential difference between nonlinear diffusion and nonlocal diffusion. Notice when η is small, we have , at least formally, $\partial_x(\rho_\eta * u(t,\cdot)) \approx$

$\partial_x u(t,\cdot)$ so that (1.3) with $f(u)=0$, with $\varepsilon \ll 1$ and $\eta \ll 1$ should be a good approximation of porous medium equation

$$\partial_t u(t,x) = \partial_x \left(u\partial_x u(t,x)\right), \quad x\in\mathbb{R},\ t>0. \tag{4.2}$$

To explore numerically the connexion between (1.3) with $f(u)=0$ and (4.2) we consider the so-called Barenblatt solution to equation (4.2) that is defined as

$$u_{\mathbf{B}}(t,x,C) = t^{-1/3}\max(C-k|x|^2t^{-2/3},0), \tag{4.3}$$

where $C>0$ denotes any positive constant (see for instance [33]). In the numerical experiments below we fix $C=0.1$ and we fix the scaling parameter for the kernel function $\eta=0.8$. In the sequel we shall make use of the notation ρ_0 and $\rho_\sharp$ to denote $\rho_{\eta,0}$ and $\rho_{\eta,\sharp}$ respectively. To go further we also introduce the viscosity threshold associated to the kernel ρ

$$\varepsilon_0 := -u^*\min_{n\in\mathbb{N}}\left\{\widehat{\rho}\left(\frac{n\eta}{2L}\right)\right\},$$

wherein we have set $u^*=\frac{1}{2L}\int_{-L}^{L}u_0(x)dx$, the total mass of u_0 in $[-L,L]$.

Next we select the initial distribution $u_0(x) := u_{\mathbf{B}}(T_1,x,0.1)$ for $T_1=3$ and $x\in[-4,4]$. The simulation starts from time T_1 to time $T_2=5,10$ and 50 respectively. With such an initial data one has $u^*=0.0258$ while, we can obtain the threshold values for two kernels

$$\varepsilon_0 = \begin{cases} 0.0025 \text{ for } \rho=\rho_0,\\ 0.0056 \text{ for } \rho=\rho_\sharp.\end{cases}$$

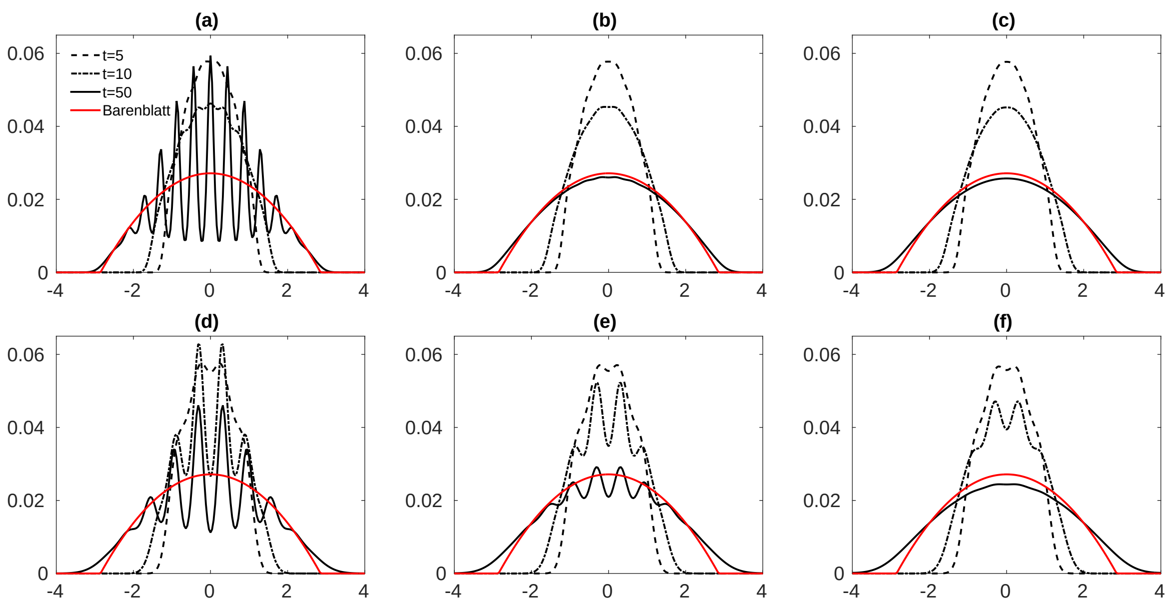

Figure 7: *Numerical simulations of* (1.3) *for the two kernels* $\rho_\sharp$ *and* ρ_0 *with initial data* $u_0(x) := u_{\boldsymbol{B}}(3,x,0.1)$ *and without reaction term (i.e.* $f=0$*). We plot the solutions at time* $T_2=5,10,50$ *in each sub-figure and we compare the simulation at the final time* $T=50$ *with Barenblatt solution* $u_{\boldsymbol{B}}(50,x,0.1)$ *(red cuvres). Figures (a)-(c) on the top correspond to the solutions with kernel* $\rho_\sharp$ *and viscosity coefficient* $\varepsilon=0.0015, 0.0025$ *and* 0.0035 *respectively; while figures (d)-(f) in the bottom correspond to the kernel* ρ_0 *and the viscosity* $\varepsilon=0.0046, 0.0056$ *and* 0.0066 *respectively.*

Remark 4.1. *The numerical experiments in Figure 7 are performed so that the space step Δx is chosen rather small in order to overcome some difficulties linked with the high concentration of the kernel (due to the scaling parameter η). We choose $\Delta x \leq 0.1\eta$ so that we set a mesh with more than 20 points in the interval $[-\eta, \eta]$. The numerical method is discussed in the appendix.*

As we can see from Figure 7, when η and ε are small, the nonlocal model (1.3) with the above two kernels does not provide a good approximation of the solution of the nonlinear diffusion (4.2). As we can see from the Figure 7, when $\varepsilon = \varepsilon_0 - 10^{-3}$, the solutions with both kernels $\rho_\sharp$ and ρ_0 in Figure (a) and (d) differ remarkably from the Barenblatt solution of the porous medium equation. While when we set $\varepsilon = \varepsilon_0 + 10^{-3}$, the simulations (c) and (f) at time $t = 50$ are relatively good approximation of Barenblatt solution.

5 Appendix

5.1 Reduced system

In this subsection, we give a brief calculation of the center manifold reduction as a supplement of Theorem 3.8. Recall our equation reads as follows

$$\frac{d}{dt}\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix} = L\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix} + R\begin{pmatrix} w(t) \\ \gamma(t) \end{pmatrix},$$

wherein we have set

$$L = \begin{pmatrix} \left(A + \partial_w \tilde{F}(0,\gamma_0)\right) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\left(H^2 \times \mathbb{R}, H^0 \times \mathbb{R}\right),$$

and

$$R\begin{pmatrix} w \\ \gamma \end{pmatrix} = \begin{pmatrix} \tilde{F}(w,\gamma) - \partial_w \tilde{F}(0,\gamma_0)w \\ 0 \end{pmatrix}.$$

Recall also that $\tilde{F}$ and B are defined by

$$\tilde{F}(w,\gamma) = \frac{b-\mu}{\gamma\mu}(K \circ w)'' + B(w,w) + \left(\frac{\mu^2}{b+\gamma\mu w} - \mu\right)w + w, \quad B(w,w) = \frac{d}{dx}\left(w\frac{d}{dx}K \circ w\right),$$

and we define

$$G(w,\gamma) = \tilde{F}(w,\gamma) - \partial_w \tilde{F}(0,\gamma_0)w.$$

Moreover, we also define $\hat{A} = A + \partial_w \tilde{F}(0,\gamma_0)$ and let us observe that $\hat{A}e_n = \lambda_n(\gamma_0)e_n$ for all $n \in \mathbb{Z}$. Recall that the framework of 3.8 implies that we have, for some $n_0 \geq 1$,

$$\lambda_{\pm n_0}(\gamma_0) \in i\mathbb{R}, \quad \mathrm{Re}(\lambda_n(\gamma_0)) < 0, \ \forall n \neq \pm n_0.$$

To perform our center manifold reduction we will need the following computations:

- $K \circ e_n = c_n(K)e_n$ for all $n \in \mathbb{Z}$
- $B(e_n, e_m) = -\left(\frac{\pi}{L}\right)^2 c_m(K)m(m+n)e_{m+n}$ for all $(n,m) \in \mathbb{Z}^2$.

Define the center space $\mathcal{E}^c = \text{span}\ (e_{\pm n_0}) \times \mathbb{R}$ and the stable space $\mathcal{E}^s = \text{span}\ (e_{\pm n_0})^\perp \times \{0\}$ where $\text{span}\,(e_{\pm n_0})$ denotes the vector space spanned by eigenfunctions $e_{\pm n_0}$ while $\text{span}\,(e_{\pm n_0})^\perp$ denotes its orthogonal space for the $L^2(-L,L)-$inner product. We denote by $\tilde{\Psi} : \mathcal{E}^c \to \mathcal{E}^s$ the local center manifold and in the sequel we will make use of the following notation

$$\tilde{\Psi}(x^c,\gamma) = (\Psi(x^c,\gamma), 0) \in \mathcal{E}^s, \text{ for } (x_c,\gamma) \in \mathcal{E}^c \text{ close to } (0,\gamma_0),$$

and $x^c = x_- e_{-n_0} + x_+ e_{n_0}$. Since the center manifold is smooth (here C^∞) we re-write it as follows:

$$\Psi(x^c,\gamma) = \sum_{n\neq \pm n_0} \Psi_n(x^c,\gamma) e_n = \sum_{n\neq \pm n_0} P_n(x^c,\gamma)e_n + O\left(\left((\|x_c\| + |\gamma-\gamma_0|\right)^3\right) \text{ in } H^2,$$

where, for each $n \in \mathbb{Z}\setminus\{\pm n_0\}$, $P_n(x^c,\gamma)$ is homogeneous polynomial of degree 2 for the variables x_-, x_+ and $(\gamma-\gamma_0)$. For notational simplicity we also denote by $P_{\pm n_0}(x^c,\gamma)$ the first order polynomials

$$P_{-n_0}(x^c,\gamma) = x_- \text{ and } P_{n_0}(x^c,\gamma) = x_+.$$

Note that since the center manifold is real valued, one has

$$x_+ = \overline{x}_- \text{ and } \Psi_{-n}(x^c,\gamma) = \overline{\Psi_n(x^c,\gamma)},\ \forall n \neq \pm n_0.$$

To compute the – center manifold – reduced system, let us introduce the center and stable projectors Π^c and Π^s as follows:

$$\Pi^c\varphi = \sum_{n=\pm n_0} c_n(\varphi)e_n \text{ and } \Pi^s\varphi = \sum_{n\neq \pm n_0} c_n(\varphi)e_n,$$

as well as the center and stable part of the linear operator $\hat{A}$, respectively denoted by $\hat{A}^c$ and $\hat{A}^s$ and defined by

$$\hat{A}^c\varphi = \sum_{n=\pm n_0} c_n(\varphi)\lambda_n(\gamma_0)e_n \text{ and } \hat{A}^s\varphi = \sum_{n\neq \pm n_0} \lambda_n(\gamma_0)c_n(\varphi)e_n.$$

Next the reduced system reads as

$$\begin{cases} \dfrac{dx^c(t)}{dt} = \hat{A}^c x^c(t) + \Pi^c G\left(x^c(t) + \Psi\left(x^c(t),\gamma(t)\right),\gamma(t)\right), \\ \dfrac{d\gamma(t)}{dt} = 0, \end{cases} \tag{5.1}$$

and the center manifold satisfies the following equation in the neighbourhood of $(x^c,\gamma) = (0,,\gamma_0)$:

$$\partial_{x^c}\Psi(x^c,\gamma)\left[A^c x^c + \Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)\right] = A^s\Psi(x^c,\gamma) + \Pi^s G(x^c + \Psi(x^c,\gamma),\gamma). \tag{5.2}$$

Our goal is to obtain a Taylor expansion up to order 3 of the above reduced system. To that aim we shall first compute a Taylor expansion of $\Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)$ and $\Pi^s G(x^c + \Psi(x^c,\gamma),\gamma)$ respectively up to order 3 and 2. To do so, first note that for $\|w\|$ small enough and γ close to γ_0 one has the series expansion

$$\tilde{F}(w,\gamma) = \frac{b-\mu}{\gamma\mu}(K\circ w)'' + B(w,w) + w(1-\mu) + \frac{\mu^2}{b}\sum_{p=0}^{\infty}\frac{\gamma^p\mu^p w^{p+1}}{b^p},$$

and

$$\partial_w\tilde{F}(0,\gamma_0)w = \frac{b-\mu}{\gamma_0\mu}(K\circ w)'' + w(1-\mu) + \frac{\mu^2}{b}w.$$

As a consequence one has, for all w and $|\gamma-\gamma_0|$ small enough,

$$G(w,\gamma) = \frac{b-\mu}{\mu}\frac{\gamma_0-\gamma}{\gamma_0\gamma}(K\circ w)'' + B(w,w) + \frac{\mu^2}{b}\sum_{p=1}^{\infty}\frac{\gamma^p\mu^p w^{p+1}}{b^p}.$$

Hence this leads us to the following order 3 Taylor expansion

$$\begin{aligned} G(w,\gamma) =& \frac{b-\mu}{\gamma_0\mu}\frac{(2\gamma_0-\gamma)(\gamma_0-\gamma)}{\gamma_0^2}(K\circ w)'' + B(w,w) + \frac{\mu^3\gamma_0}{b^2}w^2 \\ &+ \frac{\mu^3(\gamma-\gamma_0)}{b^2}w^2 + \frac{\mu^4\gamma_0^2}{b^3}w^3 + O((|w| + |\gamma-\gamma_0|)^4). \end{aligned}$$

Now choosing the following form for w

$$w = x^c + \Psi(x^c,\gamma) = x_- e_{-n_0} + x_+ e_{n_0} + \sum_{n\neq \pm n_0} P_n(x^c,\gamma) + O\left((\|x^c\| + |\gamma-\gamma_0|)^3\right) \text{ in } H^2,$$

yields

$$\begin{aligned}(K\circ w)'' = &-\left(\left(\frac{n_0\pi}{L}\right)^2 c_{-n_0}(K) x_- e_{-n_0} + \left(\frac{n_0\pi}{L}\right)^2 c_{n_0}(K) x_+ e_{n_0}\right)\\ &- \sum_{n\neq\pm n_0}\left(\frac{n\pi}{L}\right)^2 c_n(K) P_n(x^c,\gamma) e_n + O\left((\|x^c\| + |\gamma-\gamma_0|)^3\right) \text{ in } H^0,\end{aligned}$$

and

$$\begin{aligned}B(w,w) = &\sum_{m,n\in\mathbb{Z}^2} P_n(x^c,\gamma) P_m(x^c,\gamma) B(e_n,e_m)\\ = &-\sum_{m,n} P_n(x^c,\gamma) P_m(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_m(K) m(m+n) e_{m+n} + O\left((\|x^c\| + |\gamma-\gamma_0|)^4\right) \text{ in } H^0,\end{aligned}$$

Now, we calculate those terms of $B(w,w)$ up to order 2, which are given by

$$\text{order 2}\begin{cases} -x_+^2\left(\frac{\pi}{L}\right)^2 c_{n_0}(K) 2n_0^2 e_{2n_0}, & n=m=n_0;\\ -x_-^2\left(\frac{\pi}{L}\right)^2 c_{-n_0}(K) 2n_0^2 e_{-2n_0}, & n=m=-n_0;\\ 0, & n=n_0, m=-n_0; \text{ or } n=-n_0, m=n_0.\end{cases}$$

For further normal form computation, we list all possible situations of order 3 of $\Pi^c B(w,w)$, that the components along the vectors e_{n_0} and e_{-n_0}. They reads as follows

$$\text{order 3}\begin{cases} 0, & n=n_0,\ m=0;\\ -x_+ P_{-2n_0}(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{-2n_0}(K) 2n_0^2 e_{-n_0}, & n=n_0,\ m=-2n_0;\\ 0, & n=-n_0,\ m=0;\\ -x_- P_{2n_0}(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{2n_0}(K) 2n_0^2 e_{n_0}, & n=-n_0,\ m=2n_0;\\ -x_+ P_0(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{n_0}(K) n_0^2 e_{n_0}, & n=0,\ m=n_0;\\ x_+ P_{-2n_0}(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{n_0}(K) n_0^2 e_{-n_0}, & n=-2n_0,\ m=n_0;\\ -x_- P_0(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{-n_0}(K) n_0^2 e_{-n_0}, & n=0,\ m=-n_0;\\ x_- P_{2n_0}(x^c,\gamma)\left(\frac{\pi}{L}\right)^2 c_{-n_0}(K) n_0^2 e_{n_0}, & n=2n_0,\ m=-n_0;\end{cases}$$

Finally, we compute the term $\Pi^c w^2$ and $\Pi^c w^3$. To that aim, note that one has

$$\begin{aligned}w^2 = &\left(x_- e_{-n_0} + x_+ e_{n_0} + \sum_{n\neq\pm n_0} P_n(x^c,\gamma) e_n\right)^2\\ = &x_+^2 e_{2n_0} + 2x_+ x_- e_0 + x_-^2 e_{-2n_0} + (x_- e_{-n_0} + x_+ e_{n_0})\sum_{n\neq\pm n_0} P_n(x^c,\gamma) e_n + O((\|x^c\| + |\gamma-\gamma_0|)^4).\end{aligned}$$

therefore

$$\Pi^c w^2 = (x_+ P_0(x^c,\gamma) + x_- P_{2n_0}(x^c,\gamma))\, e_{n_0} + (x_+ P_{-2n_0}(x^c,\gamma) + x_- P_0(x^c,\gamma))\, e_{-n_0} + O((\|x^c\| + |\gamma-\gamma_0|)^4).$$

Next, one has

$$w^3 = (x_- e_{-n_0} + x_+ e_{n_0})^3 + O((\|x^c\| + |\gamma-\gamma_0|)^4),$$

so that we get

$$\Pi^c w^3 = 3x_+^2 x_- e_{n_0} + 3x_+ x_-^2 e_{-n_0} + O((\|x^c\| + |\gamma-\gamma_0|)^4).$$

Coupling the above computations allows us to compute a Taylor expansion up to order 3 for the quantity $\Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)$ and more precisely we get

$$\begin{aligned}
&\Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)\\
=&\frac{b-\mu}{\gamma_0\mu}\frac{(2\gamma_0-\gamma)(\gamma-\gamma_0)}{\gamma_0^2}\left(\frac{n_0\pi}{L}\right)^2 (c_{-n_0}(K) x_- e_{-n_0} + c_{n_0}(K) x_+ e_{n_0})\\
&+\left(\frac{\pi}{L}\right)^2 \left(x_- P_{2n_0}(x^c,\gamma) c_{-n_0}(K) n_0^2 - x_- P_{2n_0}(x^c,\gamma) c_{2n_0}(K) 2n_0^2 - x_+ P_0(x^c,\gamma) c_{n_0}(K) n_0^2\right) e_{n_0}\\
&+\left(\frac{\pi}{L}\right)^2 \left(x_+ P_{-2n_0}(x^c,\gamma) c_{n_0}(K) n_0^2 - x_+ P_{-2n_0}(x^c,\gamma) c_{-2n_0}(K) 2n_0^2 - x_- P_0(x^c,\gamma) c_{-n_0}(K) n_0^2\right) e_{-n_0}\\
&+\frac{\mu^3\gamma_0}{b^2}(x_+ P_0(x^c,\gamma) + x_- P_{2n_0}(x^c,\gamma)) e_{n_0} + \frac{\mu^3\gamma_0}{b^2}(x_+ P_{-2n_0}(x^c,\gamma) + x_- P_0(x^c,\gamma)) e_{-n_0}\\
&+\frac{\mu^4\gamma_0^2}{b^3}\left(3x_+^2 x_- e_{n_0} + 3x_+ x_-^2 e_{-n_0}\right) + O((\|x^c\| + |\gamma-\gamma_0|)^4) \text{ in } H^0.
\end{aligned} \tag{5.3}$$

Similarly, we also obtain a Taylor expansion for the quantity $\Pi^s G(x^c + \Psi(x^c,\gamma),\gamma)$ up to order 2 as follows,

$$\begin{aligned}
\Pi^s G(x^c + \Psi(x^c,\gamma),\gamma) =& 2\left(\frac{n_0\pi}{L}\right)^2 \left(c_{n_0}(K) x_+^2 e_{2n_0} + c_{-n_0}(K) x_-^2 e_{-2n_0}\right)\\
&+\frac{\mu^3\gamma_0}{b^2}\left(x_+^2 e_{2n_0} + 2x_+ x_- e_0 + x_-^2 e_{-2n_0}\right) + O((\|x^c\| + |\gamma-\gamma_0|)^3).
\end{aligned}$$

We now plug the above Taylor expansion into (5.4) to identify the polynomials P_n needed to obtain a Taylor expansion up to order 3 of the reduced system.

First note that the left-hand side of (5.2) can be rewritten as

$$\begin{aligned}
&\partial_{x^c}\Psi(x^c,\gamma)\left[A^c x^c + \Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)\right]\\
=&\partial_{x^c}\Psi(x^c,\gamma)\left[\lambda_{n_0}(\gamma_0) x_+ e_{n_0} + \lambda_{-n_0}(\gamma_0) x_- e_{-n_0} + h.o.t. \geq 2\right]\\
=&\lambda_{n_0}(\gamma_0) x_+ \partial_{x^c}\Psi(x^c,\gamma) e_{n_0} + \lambda_{-n_0}(\gamma_0) x_- \partial_{x^c}\Psi(x^c,\gamma) e_{-n_0} + h.o.t. \geq 3
\end{aligned} \tag{5.4}$$

where $h.o.t. \geq 2$ (resp. 3) means those terms with order greater than 2 (resp. 3). And similarly, the right-hand side of (5.2) can be re-written as

$$\begin{aligned}
&A^s\Psi(x^c,\gamma) + \Pi^s G(x^c + \Psi(x^c,\gamma),\gamma)\\
=&\sum_{n\neq\pm n_0} \lambda_n(\gamma_0) P_n(x^c,\gamma) e_n - 2\left(\frac{n_0\pi}{L}\right)^2 \left(c_{n_0}(K) x_+^2 e_{2n_0} + c_{-n_0}(K) x_-^2 e_{-2n_0}\right)\\
&+\frac{\mu^3\gamma_0}{b^2}\left(x_+^2 e_{2n_0} + 2x_+ x_- e_0 + x_-^2 e_{-2n_0}\right) + h.o.t. \geq 3\\
=&\sum_{n\neq\pm n_0} \lambda_n(\gamma_0) P_n(x^c,\gamma) e_n + C_0 x_+ x_- e_0 + C_{2n_0} x_+^2 e_{2n_0} + C_{-2n_0} x_-^2 e_{-2n_0} + h.o.t. \geq 3,
\end{aligned} \tag{5.5}$$

wherein we have set

$$C_0 = 2\frac{\mu^3\gamma_0}{b^2},\ C_{2n_0} = -2\left(\frac{n_0\pi}{L}\right)^2 c_{n_0}(K) + \frac{\mu^3\gamma_0}{b^2},\ C_{-2n_0} = -2\left(\frac{n_0\pi}{L}\right)^2 c_{-n_0}(K) + \frac{\mu^3\gamma_0}{b^2}. \tag{5.6}$$

According to (5.3) we only need to compute those terms when $n = 0, \pm 2n_0$. Next since (5.4) and (5.5) are equal, identifying the terms of order 2 yields

$$\lambda_{n_0}(\gamma_0)x_+\frac{\partial}{\partial x_+}P_n(x^c,\gamma) + \lambda_{-n_0}(\gamma_0)x_-\frac{\partial}{\partial x_-}P_n(x^c,\gamma) = \lambda_n(\gamma_0)P_n(x^c,\gamma) + Q_n(x^c) \text{ for } n = 0, \pm 2n_0,$$

where we have defined

$$Q_0(x^c) = C_0 x_+ x_-,\ Q_{2n_0}(x^c) = C_{2n_0}x_+^2,\ Q_{-2n_0}(x^c) = C_{-2n_0}x_-^2.$$

Recalling that P_n are homogeneous polynomials of degree 2 with respect to the three variables x_-, x_+ and $(\gamma - \gamma_0)$, obtains that

$$\begin{aligned} P_0(x^c,\gamma) &= -\frac{C_0}{\lambda_0(\gamma_0)}x_+x_-, \\ P_{2n_0}(x^c,\gamma) &= -\frac{C_{2n_0}}{\lambda_{2n_0}(\gamma_0)}x_+^2, \\ P_{-2n_0}(x^c,\gamma) &= -\frac{C_{-2n_0}}{\lambda_{-2n_0}(\gamma_0)}x_-^2, \end{aligned}$$

where the constants C_0, $C_{\pm 2m_0}$ are defined in (5.6). Finally substituting the above expression into the Taylor expansion of $\Pi^c G(x^c + \Psi(x^c,\gamma),\gamma)$ yields the following reduced system up to order 3,

$$\begin{cases} \dfrac{dx_+(t)}{dt} = \left[i\omega + a(\gamma)\right]x_+ + x_- x_+^2\beta + h.o.t \geq 4, \\ x_-(t) = \overline{x}_+(t), \\ \dfrac{d\gamma(t)}{dt} = 0. \end{cases}$$

Here we have set $\lambda_{n_0}(\gamma_0) = i\omega$,

$$a(\gamma) = \left(\frac{n_0\pi}{L}\right)^2 \frac{b-\mu}{\gamma_0\mu} c_{n_0}(K)\frac{(2\gamma_0-\gamma)(\gamma-\gamma_0)}{\gamma_0^2},$$

and

$$\begin{aligned} \beta =& \frac{3\gamma_0^2\mu^4}{b^3} + \frac{2\gamma_0\mu^3\left(\pi^2b^2n_0^2c_{n_0}(K) - \gamma_0\mu^3L^2\right)}{b^4L^2\lambda_0(\gamma_0)} \\ &+ \frac{\left(2\pi^2b^2n_0^2c_{n_0}(K) - \gamma_0\mu^3L^2\right)\left(\pi^2b^2n_0^2\left(c_{-n_0}(K) - 2c_{2n_0}(K)\right) + \gamma_0\mu^3L^2\right)}{b^4L^4\lambda_{2n_0}(\gamma_0)}. \end{aligned}$$

The first equation in the above system turns out to be the Poincaré normal form. It allows us to study the stability of the bifurcated periodic solution. To that aim observe that

$$\mathrm{Re}(a(\gamma)) = \left(\frac{n_0\pi}{L}\right)^2 \varepsilon_0 \frac{(2\gamma_0-\gamma)(\gamma-\gamma_0)}{\gamma_0^2},$$

so that $\mathrm{Re}(a(\gamma)) > 0$ for $\gamma > \gamma_0$ and negative for $\gamma < \gamma_0$ but close to γ_0. The stability of the bifurcating period orbit is fully by the sign of the real part of the β. However we are not able to conclude about this sign. To summarize the Hopf bifurcation at γ_0 is:

1. **supercritical** if $\mathrm{Re}\,\beta < 0$, namely the origin is stable for $\gamma < \gamma_0$ and unstable for $\gamma > \gamma_0$. Moreover when $\gamma > \gamma_0$ the system has a stable limit cycle. Here the circular limit cycle has a radius proportional to $\sqrt{\gamma - \gamma_0}$.

2. **subcritical** if $\mathrm{Re}\,\beta > 0$, namely the origin is stable for $\gamma < \gamma_0$ and unstable when $\gamma > \gamma_0$. Moreover when $\gamma < \gamma_0$ the system has an unstable limit cycle, with a radius proportional to $\sqrt{\gamma_0 - \gamma}$.

5.2 Numerical Scheme

The numerical method used is based on upwind scheme. We refer to [13, 19] for more results about this subject. We briefly illustrate our numerical scheme in this section: the approximation of the convolution term is as follows

$$(K \circ u(t,\cdot))(x) = \int_{[-L,L]} u(t,y)K(x-y)dy \approx \sum_j K(x-x_j)u(t,x_j)\Delta x.$$

In addition, we define

$$l_i^n := \sum_j K(x_i - x_j)u(t_n, x_j)\Delta x, \tag{5.7}$$

for $i = 1, 2, ..., M,\ n = 0, 1, 2, ..., N$. We use the numerical scheme as illustrated in [18] to deal with the nonlocal convection term and the scheme reads as follows

$$\begin{aligned} &\frac{u_i^{n+1} - u_i^n}{\Delta t} = \varepsilon \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + \frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x}, \\ &i = 1, 2, ..., M,\ n = 0, 1, 2, ..., N \end{aligned} \tag{5.8}$$

where

$$F_{i+\frac{1}{2}}^n = \begin{cases} g_{i+\frac{1}{2}}^n u_i^{n+1}, & \text{if } g_{i+\frac{1}{2}}^n \geq 0 \\ g_{i+\frac{1}{2}}^n u_{i+1}^{n+1}, & \text{if } g_{i+\frac{1}{2}}^n < 0, \end{cases} \quad i = 0, 1, 2, \cdots, M. \tag{5.9}$$

with

$$g_{i+\frac{1}{2}}^n = \frac{l_{i+1}^n - l_i^n}{\Delta x},\ i = 0, 1, 2, \cdots, M.$$

By the periodic boundary condition, one has $g_{\frac{1}{2}}^n = g_{M+\frac{1}{2}}^n$ and $u_0^n = u_M^n, u_1^n = u_{M+1}^n$, therefore,

$$F_{M+\frac{1}{2}}^n = F_{\frac{1}{2}}^n = \begin{cases} g_{\frac{1}{2}}^n u_0^{n+1}, & \text{if } g_{\frac{1}{2}}^n \geq 0 \\ g_{\frac{1}{2}}^n u_1^{n+1}, & \text{if } g_{\frac{1}{2}}^n < 0. \end{cases}$$

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