PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

A SHORT PROOF FOR HOPF BIFURCATION IN GURTIN-MACCAMY'S POPULATION DYNAMICS MODEL

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ABSTRACT. In this paper, we provide a short proof for the Hopf bifurcation theorem in the Gurtin-MacCamy's population dynamics model. Here we use the Crandall and Rabinowitz's approach, based on the implicit function theorem. Compared with previous methods, here we require the age-specific birth rate to be slightly smoother (roughly of bounded variation), but we have a huge gain for the length of the proof.

1. Age-Structured Models

We consider the existence of periodic solutions for the following equation,

$$(1.1)\qquad \begin{cases}(\partial_t+\partial_a)u(t,a)=-m\,u(t,a), \quad a\in(0,+\infty),\\ u(t,0)=f\left(\nu,\int_0^\infty b(a)u(t,a)da\right),\\ u(0,\cdot)=\psi\in L^1_+((0,\infty),\mathbb{R}),\end{cases}$$

where $m>0$ and $b\in L^\infty_+(0,\infty)$ are mortality rate and fertility rate of the population respectively, the nonlinear function $f\in C^2(\mathbb{R}\times\mathbb{R},\mathbb{R})$ describes the birth limitations for the population while $\nu\in\mathbb{R}$ is regarded as a bifurcation parameter. Here $u(t,a)$ denotes the density of a population at time t with age a. This equation is referred as the *Gurtin-MacCamy's age-structured equation* and was introduced in its nonlinear form by Gurtin and MacCamy in [8] to study temporal evolution of biological populations.

The existence of nontrivial periodic solution induced by Hopf bifurcation has been observed in various specific age-structured models (Cushing [3, 4], Prüss [16], Swart [17], Kostava and Li [11], Bertoni [1], Magal and Ruan [14]). In this paper we shall use the implicit function theorem to establish the Hopf bifurcation theorem that is used to obtain the existence of nontrivial periodic solutions of the age-structured model (1.1), that is, a nontrivial periodic solution bifurcated from the equilibrium of (1.1) when the bifurcation parameter ν takes some critical values.

For two dimensional ordinary differential equations Crandall and Rabinowitz [2] requires less regularity (C^2-right hand side) than the standard result by Hale and Kocack [9] (which requires C^3-right hand side) and Hassard, Kazarinoff and Wan [10] (which requires C^4-right hand side). Here we assume that the function f is only C^2 which corresponds to the regularity imposed by Crandall and Rabinowitz [2] for their result applied to ordinary differential equations. Such a regularity assumption has been mentioned already in Liu, Magal and Ruan [12, see Remark 2.5]. In the context of partial differential equations Crandall and Rabinowitz [2] original theorem only applies to parabolic PDE since their proof strongly uses the fact that the semigroup is generated by a sectorial operator $A:D(A)\subset X\to X$ on a Banach space X and verifies that the map

Key words and phrases. Age structure; Population dynamics; Hopf bifurcation.

$t \to e^{At}x$ (with $x \in X$) is of class C^1 on $(0,\infty)$. Here we are working with hyperbolic operator, therefore such a property is not satisfied. Nevertheless by exploiting the special structure of the system, and imposing some extra regularity for the birth function $a \to b(a)$ (i.e. b has bounded variation on $[0,\infty)$) we are still able to apply Crandall and Rabinowitz's ideas.

Note that the previous result by Magal and Ruan [14] and Liu, Magal and Ruan [12] only assume that the birth function b belongs to $L^\infty(0,\infty)$. Here the fact that $a \to b(a)$ has bounded variation on each $[0,\infty)$, means that $a \to b(a)$ has a finite number of discontinuity point on each bounded interval on $[0,\infty)$ and that $b(a)$ is continuous in between two successive discontinuity points. Such an assumption is sufficient for most practical examples. Finally this paper is closely related to the work of Cushing [3] in which he considered an equation with age and delay at birth. In [3], the function $a \to b(a)$ is assumed to be of class C^1 which is stronger than our bounded variation assumption.

This paper is organized as follows. In Section 2, we give the well-posedness result of (1.1). In Section 3, we provide the assumptions for our Hopf bifurcation theorem, while Section 4 is devoted to state and prove the Hopf bifurcation theorem.

2. Well-posedness

Set

$$X = \mathbb{R} \times L^1((0,\infty),\mathbb{R}) \text{ and } X_0 = \{0_\mathbb{R}\} \times L^1((0,\infty),\mathbb{R}).$$

Assume that X is endowed with the product norm

$$\|x\| = |\alpha| + \|\psi\|_{L^1((0,\infty),\mathbb{R})}, \quad \forall x = \begin{pmatrix}\alpha\\ \psi\end{pmatrix} \in X.$$

Consider the linear operator $A : D(A) \subset X \to X$ given by

$$A\begin{pmatrix}0\\ \psi\end{pmatrix} = \begin{pmatrix}-\psi(0)\\ -\psi' - m\psi\end{pmatrix}$$

with

$$D(A) = \{0_\mathbb{R}\} \times W^{1,1}((0,\infty),\mathbb{R}).$$

Recall that X_0 is the closure of $D(A)$ in X. In addition, note that A_0, the part of A in X_0, generates a C_0-semigroup of bounded linear operators, denoted by $\{T_{A_0}(t)\}_{t\geq 0}$ and explicitly given by

$$T_{A_0}(t)\begin{pmatrix}0\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \widehat{T}_{A_0}(t)\psi\end{pmatrix},$$

wherein we have set, for all $t \geq 0$ and $\psi \in L^1((0,\infty),\mathbb{R})$,

$$\widehat{T}_{A_0}(t)(\psi)(a) = \begin{cases} e^{-m\,t}\psi(a-t), & \text{if } \ a \geq t,\\ 0, & \text{if } \ a \leq t.\end{cases}$$

Moreover A generates an integrated semigroup of X, denoted by $\{S_A(t)\}_{t\geq 0}$ and defined, for $t \geq 0$ by

$$S_A(t)\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ L(t)\alpha + \int_0^t \widehat{T}_{A_0}(s)\psi ds\end{pmatrix}, \ \forall \begin{pmatrix}\alpha\\ \psi\end{pmatrix} \in X,$$

where

$$L(t)(\alpha)(a) = \begin{cases} 0, & \text{if } \ a \geq t,\\ e^{-m\,a}\alpha, & \text{if } \ a \leq t.\end{cases}$$

Define the map $H : \mathbb{R} \times X_0 \to X$ by

$$H\left(\nu, \begin{pmatrix} 0 \\ \psi \end{pmatrix}\right) = \begin{pmatrix} f\left(\nu, \int_0^\infty b(a)\psi(a)da\right) \\ 0 \end{pmatrix}.$$

Then by identifying $u(t)$ with $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, problem (1.1) can be rewritten as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H(\nu, v(t)), \quad t \geq 0, \quad v(0) = y \in X_0. \tag{2.1}$$

Since f is Lipschitz continuous on bounded sets, the general results proved in Magal and Ruan [13, see section 5] or [15, Chapter 5] ensure that the Cauchy problem (2.1) generated a maximal semiflow (with eventually some blowup), denoted below by $\{U_\nu(t)\}_{t\geq 0}$.

3. Assumptions

Regularity of the birth function: Let us recall some definition and properties of the so called bounded variation functions. Let $F : [0, \infty) \to \mathbb{R}$ be some function. For each $a > 0$ define

$$T_F(a) := \sup\left\{ \sum_{j=1}^{n} |F(a_j) - F(a_{j-1})| : n \in \mathbb{N}, 0 = a_0 < \cdots < a_n = a \right\} \in [0, \infty],$$

where the supremum is taken over all finite strictly increasing sequences in $[0, a]$.

Definition 3.1. *A function $F : [0, \infty) \to \mathbb{R}$ is said to be of bounded variation on $[0, \infty)$ if*

$$\sup_{a>0} T_F(a) < \infty.$$

In that case the function $a \to T_F(a)$ is bounded and increasing on $[0, \infty)$.

Let $F : [0, \infty) \to \mathbb{R}$ be a right continuous function of bounded variation on $[0, \infty)$, then $a \to T_F(a)$ is also right continuous and according to Folland [7, Theorem 3.29], there exists a unique Borel measure μ_F such that

$$\mu_F((a, b]) = F(b) - F(a), \text{ for all } a, b \in [0, \infty), \text{ with } a < b.$$

Furthermore, its total variation $|\mu_F|$ is the positive and bounded Borel measure associated to the right continuous and increasing function $a \to T_F(a)$.

Next let us recall the integration by parts formula proved by Folland [7, Theorem 3.36] as well. If $G : [0, \infty) \to \mathbb{R}$ is of class C^1 then for all $0 \leq a < b$, one has

$$\int_a^b G(s)\mu_F(ds) = [G(b)F(b-) - G(a)F(a+)] - \int_a^b G'(s)F(s)ds. \tag{3.1}$$

We now make a set of assumptions on the fertility rate b.

Assumption 3.2. *Assume that the function $\chi(a) := b(a)e^{-m\,a}$ satisfies the two following properties*

(i) *Assume that $\chi \in L^1(0, \infty)$ with*

$$\int_0^\infty \chi(a)da = 1 \Leftrightarrow \int_0^\infty b(a)e^{-m\,a}da = 1.$$

(ii) *We assume that the function $\tau : a \to a\chi(a)$ is right continuous and of bounded variation on $[0, \infty)$. We let μ_τ be the unique Borel measure associated to τ.*

Remark 3.3. *Note that when $b \in L^\infty(0,\infty)$, for each integer $n \in \mathbb{N}$ one has*

$$a \to a^n \chi(a) = a^n b(a) e^{-m\,a} \in L^1((0,\infty),\mathbb{R}).$$

Equilibria: Recall that $\begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} \in D(A)$ is an equilibrium of the semiflow $\{U_\nu(t)\}_{t\geq 0}$ if and only if

$$\begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} \in D(A) \quad \text{and} \quad A\begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} + H\left(\nu, \begin{pmatrix} 0 \\ \bar{u} \end{pmatrix}\right) = 0.$$

As a consequence a positive equilibrium is given by

$$\bar{u}(a) = \overline{w}_\nu e^{-m\,a},\ a \geq 0$$

where $\overline{w}_\nu > 0$ becomes a solution of the following equation

$$\overline{w}_\nu = f\left(\nu, \overline{w}_\nu\right).$$

Our next assumption is concerned with the existence of such equilibrium point and its regularity with respect to the parameter ν.

Assumption 3.4. *We assume that there exists an open interval I such that for each $\nu \in I$ there exists a constant solution $\overline{w}_\nu \in \mathbb{R}$ of the equation*

$$\overline{w}_\nu = f(\nu, \overline{w}_\nu).$$

We assume further that the map $\nu \to \overline{w}_\nu$ is continuously differentiable on the interval I. In the sequel we set

$$\bar{v}_\nu = \begin{pmatrix} 0 \\ \bar{u}_\nu \end{pmatrix} \quad \text{with} \quad \bar{u}_\nu(a) = \overline{w}_\nu e^{-m\,a}, \quad \forall \nu \in I.$$

In the following we will use the notation $\mathcal{L}(Y,Z)$ to denote the space of the linear bounded operators from Y to Z where Y and Z are two Banach spaces. Define for $\nu \in I$ the linear operator $B_\nu : D(B_\nu) \subset X \to X$ as follows,

$$D(B_\nu) = D(A) \text{ and } B_\nu x = Ax + \partial_v H(\nu, \bar{v}_\nu)x,\ \forall x \in D(B_\nu),$$

wherein ∂_v corresponds to the partial derivative of $H(\nu, v)$ with respect to v. The bounded linear operator $\partial_v H(\nu, \bar{v}_\nu) \in \mathcal{L}(X_0, X)$ is defined by

$$\partial_v H(\nu, \bar{v}_\nu)\begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_w f\left(\nu, \overline{w}_\nu\right) \int_0^\infty b(l)\varphi(l)dl \\ 0_{L^1} \end{pmatrix},\ \forall \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} \in X_0.$$

Herein ∂_w denotes the partial derivative of $f = f(\nu, w)$ with respect to w.

By using the result of Ducrot, Liu and Magal [6], the essential growth rate of the semigroup generated by $(B_\nu)_0$, the part of B_ν in the closure of its domain, satisfies

$$\omega_{0,ess}((B_\nu)_0) \leq -m < 0.$$

The following result follows from [15, Theorem 4.3.27, Lemma 4.4.2, Theorem 4.4.3-(ii)] to which we refer the reader for a proof and more details.

Lemma 3.5. *The spectrum of B_ν in the half plane*

$$\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -m\}$$

contains only isolated eigenvalues which are poles of the resolvent of B_ν.

Recall that the characteristic function, describing the spectrum of B_ν in Ω, is obtained by computing the resolvent of B_ν as presented in Liu, Magal and Ruan [12]. We define the **characteristic function** for $\nu \in I$ and $\lambda \in \Omega$ as follows

$$\Delta(\nu,\lambda) := 1 - \partial_w f\left(\nu, \overline{w}_\nu\right) \int_0^{\infty} b(l) e^{-(m+\lambda)l} dl.$$

Recall that the resolvent set $\rho(A)$ of A contains Ω and for each $\lambda \in \Omega$ the resolvent of A is defined by the following formula

$$(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$\Leftrightarrow \varphi(a) = e^{-(\lambda+m)a}\alpha + \int_0^a e^{-(\lambda+m)(a-s)}\psi(s)ds.$$

We now recall some result already presented (in a more general framework in the Section 5.2 in [12])

Lemma 3.6. *For each $\nu \in I$ the resolvent set $\rho(B_\nu)$ of B_ν satisfies*

$$\lambda \in \rho(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu,\lambda) \neq 0,$$

or equivalently the spectrum $\sigma(B_\nu) := \mathbb{C} \setminus \rho(B_\nu)$ of B_ν satisfies

$$\lambda \in \sigma(B_\nu) \cap \Omega \Leftrightarrow \Delta(\nu,\lambda) = 0.$$

Moreover one has

$$(\lambda I - B_\nu)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$\Leftrightarrow \varphi(a) = e^{-(\lambda+m)a}\alpha_1 + \int_0^a e^{-(\lambda+m)(a-s)}\psi(s)ds,$$

where

$$\alpha_1 := \Delta(\nu,\lambda)^{-1} \left[\alpha - \partial_w f\left(\nu, \overline{w}_\nu\right) \int_0^{\infty} b(a) \int_0^a e^{-(\lambda+m)(a-s)}\psi(s)dsda\right].$$

Proof. For each $\lambda \in \Omega$ the linear operator $\lambda I - B_\nu$ is invertible if and only if the linear operator is invertible

$$I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}.$$

In that case we have

$$(\lambda I - B_\nu)^{-1} = (\lambda I - A)^{-1} \left[I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}\right]^{-1}.$$

Computation of the inverse of $I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}$: We have

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix} = \left[I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}\right] \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha - \partial_w f\left(\nu, \overline{w}_\nu\right) \int_0^{\infty} b(a) \left[e^{-(\lambda+m)a}\alpha + \int_0^a e^{-(\lambda+m)(a-s)}\ \varphi(s)ds\right] da \\ \varphi \end{pmatrix}.$$

Therefore $I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}$ is invertible (for $\lambda \in \Omega$) if and only if $\Delta(\nu,\lambda) \neq 0$ and one has

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \left[I - \partial_v H(\nu, \bar{v}_\nu) \left(\lambda I - A\right)^{-1}\right]^{-1} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} \Delta(\nu,\lambda)^{-1} \left[\widehat{\alpha} - \partial_w f\left(\nu, \overline{w}_\nu\right) \int_0^{\infty} b(a) \int_0^a e^{-(\lambda+m)(a-s)}\widehat{\varphi}(s)dsda\right] \\ \widehat{\varphi} \end{pmatrix}.$$

The result follows. □

Assumption 3.7. *There exists $\nu_0 \in I$ and $\omega_0 > 0$ such that the following properties are satisfied*

(i)
$$\Delta(\nu_0, \omega_0 i) = 0.$$

(ii) *Crandall and Rabinowitz's condition*
$$\Delta(\nu_0, k\,\omega_0 i) \neq 0, \forall k \in \mathbb{N} \text{ and } k \neq 1.$$

(iii) *Simplicity of the eigenvalue $\omega_0 i$*
$$\partial_\lambda \Delta(\nu_0, \omega_0 i) \neq 0,$$
which is also equivalent to
$$\partial_w f\,(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty b(l)\, l\, e^{-(m+\omega_0 i)l} dl \neq 0.$$

(iv) *Transversality condition*
$$\text{Re}\left(\partial_\nu \Delta(\nu_0, \omega_0 i) \times \overline{\partial_\lambda \Delta(\nu_0, \omega_0 i)}\right) \neq 0.$$

We can observe that by combining (i) and (iii) and by using the implicit function theorem there exists a branch $\lambda : (\nu_0 - \eta, \nu_0 + \eta) \subset I \to \mathbb{C}$ with some $\eta > 0$ small enough such that for each $\nu \in (\nu_0 - \eta, \nu_0 + \eta)$, $\lambda(\nu) = \alpha(\nu) + i\omega(\nu)$ and $\overline{\lambda(\nu)} = \alpha(\nu) - i\omega(\nu)$ satisfying solution of

$$\Delta(\nu, \lambda(\nu)) = 0 \tag{3.2}$$

and
$$\lambda(\nu_0) = i\omega_0.$$

Moreover by combining (iii)-(iv) we deduce that the transversality condition is satisfied. Namely, one has
$$\text{Re}\,\frac{d\lambda(\nu_0)}{d\nu} \neq 0.$$

Moreover by using the property (iii) in Lemma 5.8 in [12] we also deduce that $i\omega_0$ is a simple eigenvalue of B_{ν_0} since
$$\lim_{\lambda \to \lambda_0} \frac{\Delta(\nu_0, \lambda)}{\lambda - \lambda_0} \neq 0 \Leftrightarrow \partial_\lambda \Delta(\nu_0, \lambda_0) \neq 0. \quad (\text{with } \lambda_0 = \lambda(\nu_0)).$$

The condition (ii) avoids to assume that the purely imaginary spectrum is reduced to a single pair of purely imaginary eigenvalues. Such a condition has been introduced in the Crandall and Rabinowitz's proof [2].

4. Main Result

In this section we state the main theorem of this paper. The following result is inspired by Crandall and Rabinowitz [2].

Theorem 4.1. *Let Assumptions 3.2, 3.4 and 3.7 be satisfied. Then there exist a constant $\delta > 0$ and two C^1 maps, $s \to \nu(s)$ from $(-\delta, \delta)$ to $\mathbb{R}$ and $s \to \omega(s)$ from $(-\delta, \delta)$ to $\mathbb{R}$ such that for each $s \in (-\delta, \delta)$ there exists a $2\pi/\omega(s)-$periodic solution $u(s)$ of class of C^1 which is a solution of (1.1) with the parameter $\nu = \nu(s)$. Moreover, the branch of periodic orbit is bifurcating from ν_0 at $\nu = \nu_0$, that is to say that*
$$\nu(0) = \nu_0, \quad \omega(0) = \omega_0.$$

Proof. (Proof of Theorem 4.1) Up to time rescaling we can assume, without loss of generality, that $\omega_0 = 1$. Observe that Assumption 3.7-(i) implies that (1.1) linearized about $u = \bar{u}_{\nu_0}$ for $\nu = \nu_0$ has nontrivial $2\pi-$periodic solutions. We now seek nontrivial $2\pi/\omega-$periodic solutions of (1.1) with ω close to 1 and (u, ν) close to $(\bar{u}_{\nu_0}, \nu_0)$.

Solving (1.1) along the characteristic line $t - a =$constant, one obtains

$$u(t, a) = u(t - a, 0)e^{-m\,a},\ t \in \mathbb{R}, a > 0.$$

Thus $v = v(t)$ given by

$$v(t) = u(t, 0),$$

satisfies the renewal equation

$$v(t) = f\left(\nu, \int_0^\infty b(a)v(t - a)e^{-m\,a}da\right),\ t \in \mathbb{R}.$$

Setting

$$w(t) = v\left(\frac{t}{\omega}\right),$$

yields the following equation for the $2\pi-$periodic function $w = w(t)$

$$\begin{aligned} w(t) &= v\left(\tfrac{t}{\omega}\right) = f\left(\nu, \textstyle\int_0^\infty b(a)v(t/\omega - a)e^{-m\,a}da\right) \\ &= f\left(\nu, \textstyle\int_0^\infty b(a)w(t - \omega a)e^{-m\,a}da\right),\ t \in \mathbb{R}. \end{aligned}$$

Recalling the definition of χ in Assumption 3.2 and using the change of the variable $l = \omega a$ in the integral lead to the following equation for $w = w(t)$

$$w(t) = f\left(\nu, \int_0^\infty \frac{1}{\omega}\chi\left(\frac{l}{\omega}\right)w(t - l)dl\right),\ t \in \mathbb{R}. \tag{4.1}$$

Now the existence of nontrivial $2\pi/\omega-$periodic solution of (1.1) becomes equivalent to the one of nontrivial $2\pi-$ periodic solution of (4.1). Next we shall apply the implicit function theorem to investigate the existence of nontrivial $2\pi-$periodic solution of (4.1) for ν close to ν_0.

Let $C_{2\pi}(\mathbb{R})$ be the Banach space of the continuous $2\pi-$periodic functions. Define the map $F : \mathbb{R}^2 \times C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ by

$$F(\omega, \nu, x)(t) = x(t) - f\left(\nu, \int_0^\infty \frac{1}{\omega}\chi\left(\frac{l}{\omega}\right)x(t - l)dl\right), \quad \forall(\omega, \nu, x) \in \mathbb{R}^2 \times C_{2\pi}(\mathbb{R}).$$

We now aim at investigating the zeros of the equation

$$F(\omega, \nu, x) = 0,$$

for (ω, ν, x) close to $(1, \nu_0, \overline{w}_{\nu_0})$ using the implicit function theorem. To do so, we need first to verify the smoothness of

$$\int_0^\infty \frac{1}{\omega}\chi\left(\frac{l}{\omega}\right)x(t - l)dl = \int_0^\infty \chi(l)x(t - \omega l)dl,$$

with respect to ω.

The first main step of this proof is the following lemma.

Lemma 4.2. *The map $G : \mathbb{R} \times C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ defined by*

$$G(\omega, x)(t) = \int_0^\infty \chi(l)x(t - \omega l)dl,\ t \in \mathbb{R},$$

is continuously differentiable with respect to $\omega \in \mathbb{R}$ *and its partial derivative with respect to* ω*, denoted by* $\partial_\omega G$*, is given*

$$\partial_\omega G(\omega, x)(\cdot) = \int_0^\infty x(\cdot - \omega l)\mu_\tau(dl), \ \forall (\omega, x) \in \mathbb{R} \times C_{2\pi}(\mathbb{R}).$$

Herein τ *and* μ_τ *are defined in Assumption 3.2-(ii).*

Proof. First observe that using Fubini theorem, for any $x \in C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})$ and $\omega \in \mathbb{R}$ we have, for any $t \in \mathbb{R}$,

$$\begin{aligned} \int_0^\omega \int_0^\infty -x'(t-\sigma l)\, l\, \chi(l)\, dl\, d\sigma &= \int_0^\infty \int_0^\omega -l x'(t-\sigma l)\, d\sigma\, \chi(l)\, dl \\ &= \int_0^\infty \left[x(t-\sigma l)\right]_{\sigma=0}^{\sigma=\omega} \chi(l)\, dl \\ &= \int_0^\infty \chi(l) x(t-\omega l) dl - x(t) \int_0^\infty \chi(l) dl, \end{aligned}$$

so that, since $\int_0^\infty \chi(l)dl = 1$ (see Assumption 3.2-(i)), we get

$$\int_0^\omega \int_0^\infty -x'(t-\sigma l) l \chi(l) dl d\sigma = \int_0^\infty \chi(l) x(t-\omega l) dl - x(t), \tag{4.2}$$

that is for all $(\omega, x) \in \mathbb{R} \times \left(C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})\right)$ and $t \in \mathbb{R}$

$$G(\omega, x)(t) = x(t) - \int_0^\omega \int_0^\infty x'(t-\sigma l)\tau(l) dl d\sigma.$$

We deduce that for all $x \in C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})$ the map $\omega \to G(\omega, x) = \int_0^\infty \chi(l) x(\cdot - \omega l) dl$ is of class C^1 and

$$\frac{d}{d\omega} \int_0^\infty \chi(l) x(t-\omega l) dl = -\int_0^\infty x'(t-\omega l) l \chi(l) dl. \tag{4.3}$$

Moreover by using again the formula (4.2) we get

$$\int_\omega^{\omega+\varepsilon} \int_0^\infty -x'(t-\sigma l) l \chi(l) dl d\sigma = \int_0^\infty \chi(l) x(t-(\omega+\varepsilon) l) dl - \int_0^\infty \chi(l) x(t-\omega l) dl.$$

hence

$$\int_\omega^{\omega+\varepsilon} \frac{d}{d\sigma} \int_0^\infty \chi(l) x(t-\sigma l) dl d\sigma = \int_0^\infty \chi(l) x(t-(\omega+\varepsilon) l) dl - \int_0^\infty \chi(l) x(t-\omega l) dl. \tag{4.4}$$

By using the integration by parts formula (3.1) and (4.3), we obtain for all $(\omega, x) \in \mathbb{R} \times \left(C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R})\right)$

$$\partial_\omega G(\omega, x)(t) = \frac{d}{d\omega} \int_0^\infty \chi(l) x(t-\omega l) dl = -\int_0^\infty x'(t-\omega l) l \chi(l) dl = \int_0^\infty x(t-\omega l)\mu_\tau(dl).$$

Then using Assumption 3.2-(ii) we infer from the above equality that

$$\|\partial_\omega G(\omega, x)\|_{C_{2\pi}(\mathbb{R})} \le \|x\|_{C_{2\pi}(\mathbb{R})} |\mu_\tau|((0,\infty)) < \infty, \forall (\omega, x) \in \mathbb{R} \times C^1(\mathbb{R}) \cap C_{2\pi}(\mathbb{R}),$$

wherein $|\mu_\tau|((0,\infty))$ is nothing but the variation of $\tau(a)$ on $(0,\infty)$. That is supremum over all the subdivision $0 = t_0 < t_1 < t_2 < \ldots < t_n = M$ of

$$|\mu_\tau|((0, M]) = \sup_{0=t_0<t_1<t_2<\ldots<t_n=M} \sum_{i=0}^n |\tau(t_{i+1}) - \tau(t_i)|,$$

and

$$|\mu_\tau|((0,\infty)) = \lim_{M\to+\infty} |\mu_\tau|((0, M]).$$

We now define $L_\omega \in \mathcal{L}\left(C_{2\pi}(\mathbb{R})\right)$ by

$$L_\omega(x)(t) := \int_0^\infty x(t-\omega l)\mu_\tau(dl).$$

Now the result follows by using the (4.4) and the density of $C^1(\mathbb{R})\cap C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$, since it implies that

$$\int_\omega^{\omega+\varepsilon} L_\sigma(x)(t)d\sigma = \int_0^\infty \chi(l)x(t-(\omega+\varepsilon)l)dl - \int_0^\infty \chi(l)x(t-\omega l)dl.$$

It remains to prove that the map $(\omega,x)\to\partial_\omega G(\omega,x)=\int_0^\infty x(\cdot-\omega l)\mu_\tau(dl)$ is continuous from $\mathbb{R}\times C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. To see this fix (ω_1,x_1) in $\mathbb{R}\times C_{2\pi}(\mathbb{R})$ and observe that for all $(\omega,x)\in\mathbb{R}\times C_{2\pi}(\mathbb{R})$ one has:

$$\partial_\omega G(\omega,x)-\partial_\omega G(\omega_1,x_1)=\int_0^\infty\left[x(\cdot-\omega l)-x_1(\cdot-\omega_1 l)\right]\mu_\tau(dl)=\mathrm{J}_1+\mathrm{J}_2,$$

wherein we have set

$$\mathrm{J}_1 := \int_0^\infty \left[x(\cdot-\omega l)-x_1(\cdot-\omega l)\right]\mu_\tau(dl),\ \mathrm{J}_2 := \int_0^\infty \left[x_1(\cdot-\omega l)-x_1(\cdot-\omega_1 l)\right]\mu_\tau(dl).$$

We first observe that

$\|\mathrm{J}_1\|_{C_{2\pi}(\mathbb{R})} \leq \|x-x_1\|_{C_{2\pi}(\mathbb{R})}|\mu_\tau|((0,\infty))\to 0$ uniformy for $\omega\in\mathbb{R}$ as $\|x-x_1\|_{C_{2\pi}(\mathbb{R})}\to 0$.

On the other hand fix $\varepsilon>0$ and since one has

$$|\mu_\tau|((M,\infty)) := |\mu_\tau|((0,\infty)) - |\mu_\tau|((0,M])\to 0 \text{ as } M\to\infty,$$

choose $M>0$ large enough so that $2\|x_1\|_{C_{2\pi}(\mathbb{R})}|\mu_\tau|((M,\infty))\leq\varepsilon$. With such a choice we split J_2 as follows $\mathrm{J}_2=\mathrm{I}_1+\mathrm{I}_2$ with

$$\mathrm{I}_1 := \int_0^M \left[x_1(\cdot-\omega l)-x_1(\cdot-\omega_1 l)\right]\mu_\tau(dl),\ \mathrm{I}_2 := \int_M^\infty \left[x_1(\cdot-\omega l)-x_1(\cdot-\omega_1 l)\right]\mu_\tau(dl).$$

Hence we get

$$\|\mathrm{I}_2\|_{C_{2\pi}(\mathbb{R})} \leq 2\|x_1\|_{C_{2\pi}(\mathbb{R})}|\mu_\tau|((M,\infty))\leq\varepsilon,$$

and, since x_1 is continuous and 2π-periodic, it is uniformly continuous on $\mathbb{R}$. Thus for all ω is sufficiently close to ω_1, we have

$$\|\mathrm{I}_1\|_{C_{2\pi}(\mathbb{R})} \leq \sup_{0\leq l\leq M}|x_1(t-\omega l)-x_1(t-\omega_1 l)||\mu_\tau|((0,M])\leq\varepsilon.$$

As a consequence $\partial_\omega G$ is continuous and the proof is completed. □

Computation of the derivatives of F: One can calculate the following derivatives directly,

$$\partial_x F(\omega,\nu,\overline{w}_\nu)(x)(t) = x(t) - \partial_w f(\nu,\overline{w}_\nu)\int_0^\infty \chi(l)x(t-\omega l)dl, \tag{4.5}$$

$$\partial_\nu\partial_x F(\omega,\nu,\overline{w}_\nu)(x)(t) = -\left[\partial_\nu\partial_w f(\nu,\overline{w}_\nu)+\partial_w^2 f(\nu,\overline{w}_\nu)\partial_\nu\overline{w}_\nu\right]\int_0^\infty \chi(l)x(t-\omega l)dl, \tag{4.6}$$

and by Lemma 4.2

$$\partial_\omega\partial_x F(\omega,\nu,\overline{w}_\nu)(x)(t) = -\partial_w f(\nu,\overline{w}_\nu)\int_0^\infty x(t-\omega l)\mu_\tau(dl). \tag{4.7}$$

State space decomposition: Note that by Assumption 3.7-(i) we have

$$1=\partial_w f(\nu_0,\overline{w}_{\nu_0})\int_0^\infty \chi(l)e^{\pm i\,l}dl,$$

which re-writes as

$$1=\partial_w f(\nu_0,\overline{w}_{\nu_0})\int_0^\infty \chi(l)\cos l dl \text{ and } 0=\partial_w f(\nu_0,\overline{w}_{\nu_0})\int_0^\infty \chi(l)\sin l dl, \tag{4.8}$$

and thus the kernel $N(\partial_x F(1,\nu_0,\overline{w}_{\nu_0}))$ contains the space X_1 given by

$$X_1 := \operatorname{span}\{\cos,\sin\}\subset C_{2\pi}(\mathbb{R}).$$

Now define the closed space

$$X_2 := \left\{z\in C_{2\pi}(\mathbb{R}) : \int_0^{2\pi} z(t)\cos(t)dt=\int_0^{2\pi} z(t)\sin(t)dt=0\right\}.$$

This space turns out to be a complement of X_1 as stated in the next lemma.

Lemma 4.3. *We have the following state space decomposition*

$$C_{2\pi}(\mathbb{R})=X_1\oplus X_2.$$

Proof. This property is directly inherited from the decomposition of $L^2((0,2\pi),\mathbb{R})$ as

$$L^2((0,2\pi),\mathbb{R})=X_1\oplus X_1^{\perp},$$

with

$$X_1^{\perp}=\left\{z\in L^2((0,2\pi),\mathbb{R}) : \int_0^{2\pi} z(t)e^{i\,t}dt=0\right\},$$

Now if $z\in C_{2\pi}(\mathbb{R})$ then $z\in L^2((0,2\pi),\mathbb{R})$ and the above $L^2((0,2\pi),\mathbb{R})-$decomposition ensures that there exist unique $z_1\in X_1$ and $z_2\in X_1^{\perp}$ such that

$$z=z_1+z_2.$$

Now since $z_1=c_1\cos+c_2\sin$ (for some constants c_1 and c_2) this ensures that $z_2=z-z_1$ is also continuous and $z_2\in X_2$. The state space decomposition follows. $\square$

Now let us define the map $h:\mathbb{R}^3\times X_2\to C_{2\pi}(\mathbb{R})$ by

$$h(s,\omega,\nu,z)=\begin{cases} s^{-1}F(\omega,\nu,\overline{w}_\nu+s(u_1+z)), & \text{if } s\neq 0,\\ \partial_x F(\omega,\nu,\overline{w}_\nu)(u_1+z), & \text{if } s=0,\end{cases}$$

where

$$u_1(t)=\cos(t),\ \forall t\in\mathbb{R}.$$

Now let us observe that since $f=f(\nu,x)$ is of class C^2, h is of class C^1. One also has $h(0,1,\nu_0,0)=0$ while the derivative with respect to (ω,ν,z) is given, for all $(\tilde{\omega},\tilde{\nu},\tilde{z})\in\mathbb{R}\times\mathbb{R}\times C_{2\pi}(\mathbb{R})$, by

$$D_{(\omega,\nu,z)}h(0,1,\nu_0,0)(\tilde{\omega},\tilde{\nu},\tilde{z})=\partial_x F(1,\nu_0,\overline{w}_{\nu_0})\tilde{z}+\tilde{\nu}\partial_\nu\partial_x F(1,\nu_0,\overline{w}_{\nu_0})u_1+\tilde{\omega}\partial_\omega\partial_x F(1,\nu_0,\overline{w}_{\nu_0})u_1,$$

hence by using (4.5)-(4.7) we obtain

$$\begin{aligned} D_{(\omega,\nu,z)}h(0,1,\nu_0,0)(\tilde{\omega},\tilde{\nu},\tilde{z}) = \; & \tilde{z}(t)-\partial_w f(\nu_0,\overline{w}_{\nu_0})\textstyle\int_0^\infty \chi(l)\tilde{z}(t-l)dl \\ & -\tilde{\nu}\left[\partial_\nu\partial_w f(\nu_0,\overline{w}_{\nu_0})+\partial_w^2 f(\nu_0,\overline{w}_{\nu_0})\partial_\nu\overline{w}_{\nu_0}\right]\textstyle\int_0^\infty \chi(l)u_1(t-l)dl \\ & -\tilde{\omega}\partial_w f(\nu_0,\overline{w}_{\nu_0})\textstyle\int_0^\infty u_1(t-l)\mu_\tau(dl).\end{aligned}$$

The second main step of the proof is the following lemma.

Lemma 4.4. *The bounded linear operator*

$$D_{(\omega,\nu,z)}h(0,1,\nu_0,0)\in\mathcal{L}(\mathbb{R}^2\times X_2, C_{2\pi}(\mathbb{R}))$$

is invertible.

Proof. Let us first define the linear bounded operator $K : X_2 \to C_{2\pi}(\mathbb{R})$ by

$$K\phi := \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l)\phi(\cdot - l)dl.$$

Step 1: Let us prove that $K(X_2) \subset X_2$. Indeed, by using Fubini's theorem, for all $\phi \in C_{2\pi}(\mathbb{R})$ one has

$$\int_0^{2\pi} \int_0^\infty \chi(l)\phi(t-l)dl e^{i\,t}dt = \int_0^\infty \int_0^{2\pi} \phi(t-l)e^{i\,(t-l)}dt e^{i\,l}\chi(l)dl$$

and since $t \to \phi(t)e^{i\,t}$ is $2\pi-$periodic we deduce that

$$\int_0^{2\pi} \phi(t-l)e^{i\,(t-l)}dt = 0, \forall l \geq 0.$$

This completes the first step.

Step 2: Let us now prove that $N(I-K) = \{0\}$ whenever $I-K \in \mathcal{L}(X_2)$. In order to compute the kernel of $I-K$ in X_2, consider $g \in N(I-K)$, that is $g \in X_2$ such that

$$g(t) - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l)g(t-l)dl = 0, \ \forall t \in \mathbb{R}.$$

Multiplying the above equality by e^{-int}, for some $n \in \mathbb{Z}$ and integrating between 0 and 2π we obtain

$$\left[1 - \partial_w f(\nu_0, \overline{w}_{\nu_0})\widehat{\chi}(n)\right] \widehat{g}(n) = 0,$$

wherein we have set

$$\widehat{g}(n) := \int_0^{2\pi} g(l)e^{-inl}dl,$$

and

$$\widehat{\chi}(n) := \int_0^\infty \chi(l)e^{-inl}dl.$$

Now for $n \neq \pm 1$ we deduce by Assumption 3.7-(ii) that

$$\widehat{g}(n) = \left[1 - \partial_w f(\nu_0, \overline{w}_{\nu_0})\widehat{\chi}(n)\right]^{-1} 0 = 0.$$

Since $g \in X_2$, it follows that $g = 0$ and $N(I-K) = \{0\}$.

Step 3: Let us prove that $I - K \in \mathcal{L}(X_2)$ is invertible. Next note that it follows from the continuity of the translation in L^1 that K is a compact operator. Thus one has $R(I-K) = X_2$ by Fredholm Alternative (see [15, Lemma 4.3.17]), where $R(I-K)$ denotes the range of $I-K$. Hence we have that $I-K$ is invertible and that the inverse is continuous by bounded inverse theorem.

Step 4: Let us prove that $D_{(\omega,\nu,z)}h(0,1,\nu_0,0)$ is invertible. To prove this, let $y \in C_{2\pi}(\mathbb{R})$ be given. Set $L := D_{(\omega,\nu,z)}h(0,1,\nu_0,0)$ and let us consider the equation

$$(\tilde{\omega}, \tilde{\nu}, \tilde{z}) \in \mathbb{R} \times \mathbb{R} \times X_2, \ \ L(\tilde{\omega}, \tilde{\nu}, \tilde{z}) = y. \tag{4.9}$$

Define the projectors $P_1 : C_{2\pi}(\mathbb{R}) \to X_1$ and $P_2 : C_{2\pi}(\mathbb{R}) \to X_2$ associated to the state space decomposition of Lemma 4.3 and set $y_1 := P_1 y$ and $y_2 := P_2 y$. Next projecting (4.9) along P_1 and P_2 yields the following system, for all $t \in \mathbb{R}$,

$$\begin{aligned} y_1(t) = \ & -\tilde{\nu}\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \textstyle\int_0^\infty \chi(l)u_1(t-l)dl \\ & -\tilde{\omega}\partial_w f(\nu_0, \overline{w}_{\nu_0}) \textstyle\int_0^\infty u_1(t-l)\mu_\tau(dl), \end{aligned} \tag{4.10}$$

and

$$y_2(t) = \tilde{z}_2(t) - \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty \chi(l)\tilde{z}_2(t-l)dl.$$

Observe that $I - K$ is invertible in X_2, thus $\tilde{z}_2$ can be solved by

$$\tilde{z}_2 = (I-K)^{-1} y_2.$$

Let us focus on the resolution of (4.10). To that aim, recall that $u_1 = \cos(\cdot)$ and (4.8) so that (4.10) rewrites

$$\begin{aligned} y_1(t) = & -\tilde{\nu}\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty \chi(l) \cos l dl \cos t \\ & -\tilde{\omega}\partial_w f(\nu_0, \overline{w}_{\nu_0}) \left[\int_0^\infty \cos l \mu_\tau(dl) \cos t + \int_0^\infty \sin l \mu_\tau(dl) \sin t\right], \end{aligned}$$

Furthermore by applying the integration by parts formula (3.1), we obtain

$$\begin{aligned} & \int_0^\infty \cos l \mu_\tau(dl) = \lim_{M\to\infty} \int_0^M \cos l \mu_\tau(dl) \\ & = \lim_{M\to\infty} \left\{ [\cos M\tau(M) - \cos 0 \tau(0)] + \int_0^M \tau(l) \sin l dl \right\}, \end{aligned}$$

hence

$$\int_0^\infty \cos(l)\, \mu_\tau(dl) = \int_0^\infty l\chi(l)\sin(l)dl.$$

Similarly, one can obtain

$$\int_0^\infty \sin l \mu_\tau(dl) = -\int_0^\infty l\chi(l)\cos l dl.$$

On the other hand, since $y_1 \in X_1$, there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $y_1 = c_1 \cos(\cdot) + c_2 \sin(\cdot)$, while the y_1-equation can be rewritten as the following system, for all $t \in \mathbb{R}$,

$$\begin{aligned} & c_1 \cos t + c_2 \sin t \\ = & -\tilde{\nu}\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty \chi(l) \cos l dl \cos t \\ & -\tilde{\omega}\partial_w f(\nu_0, \overline{w}_{\nu_0}) \left[\int_0^\infty l\chi(l) \sin l dl \cos t - \int_0^\infty l\chi(l)\cos l dl \sin t\right], \end{aligned}$$

and identifying the coefficients of $\cos(\cdot)$ and $\sin(\cdot)$ we end-up with the resolution of the following two-dimensional linear system

$$\begin{pmatrix} -\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty \chi(l) \cos l dl & -\partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty l\chi(l) \sin l dl \\ 0 & \partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty l\chi(l)\cos l dl \end{pmatrix} \begin{pmatrix} \tilde{\nu} \\ \tilde{\omega} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

To solve this linear system, let us show that our assumptions for the characteristic equation, namely Assumption 3.7, ensures that the determinant of the above matrix is non-zero, that reads as

$$-\partial_w f(\nu_0, \overline{w}_{\nu_0}) \left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty \chi(l)\cos l dl \int_0^\infty l\chi(l)\cos l dl \neq 0. \tag{4.11}$$

To check this property, recalling (4.8), it is sufficient to check that

$$\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty l\chi(l)\cos(l)dl \neq 0.$$

Next set for $\theta \in \mathbb{R}$,

$$\widehat{\chi}(\theta) = \int_0^\infty \chi(l) e^{-i\theta l} dl,$$

and observe that the above condition (or equivalently (4.11)) is equivalent to

$$\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \operatorname{Im} \widehat{\chi}'(1) \neq 0 \tag{4.12}$$

On the other hand, by differentiating the characteristic equation (3.2) with respect to ν at $\nu = \nu_0$ (and recalling that $\omega_0 = 1$), we have

$$\frac{d\lambda(\nu_0)}{d\nu}\partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^{\infty} l\chi(l)e^{-i\,l}dl = \left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \widehat{\chi}(1),$$

which implies

$$\frac{d\lambda(\nu_0)}{d\nu} = \frac{\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \int_0^\infty \chi(l)e^{-il}dl}{\partial_w f(\nu_0, \overline{w}_{\nu_0}) \int_0^\infty l\chi(l)e^{-il}dl}.$$

Thus the transversality condition, that is Assumption 3.7-(iv), becomes

$$\begin{aligned}
\mathrm{Re}\frac{d\lambda(\nu_0)}{d\nu} &= \mathrm{Re}\left\{\frac{\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \widehat{\chi}(1)}{i\partial_w f(\nu_0, \overline{w}_{\nu_0})\widehat{\chi}'(1)}\right\} \\
&= \mathrm{Re}\left\{\left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \underbrace{\widehat{\chi}(1)\partial_w f(\nu_0, \overline{w}_{\nu_0})}_{=1} \overline{i\widehat{\chi}'(1)}\right\} \\
&= \left[\partial_\nu \partial_w f(\nu_0, \overline{w}_{\nu_0}) + \partial_w^2 f(\nu_0, \overline{w}_{\nu_0})\partial_\nu \overline{w}_{\nu_0}\right] \mathrm{Im}\,(\widehat{\chi}'(1)) \neq 0,
\end{aligned}$$

therefore (4.12) holds true and we can find a unique $(\tilde{\nu}, \tilde{\omega}) \in \mathbb{R}^2$ solving the above two-dimensional linear system. This completes the proof of the lemma.

□

Last part of the proof of Theorem 4.1: To conclude the proof of the Hopf bifurcation Theorem 4.1, we apply the implicit function theorem (see Deimling [5, Theorem 15.2]) to the function $h : \mathbb{R}^3 \times X_2 \to C_{2\pi}(\mathbb{R})$ and we deduce that there exists a C^1-mapping $(\omega, \nu, z) : (-\delta, \delta) \to \mathbb{R}^2 \times X_2$, for some $\delta > 0$ small enough, such that

$$h(s, \omega(s), \nu(s), z(s)) = 0, \quad \forall s \in (-\delta, \delta).$$

By the definition of h, this is equivalent to say that

$$F(\omega(s), \nu(s), \overline{w}_{\nu(s)} + s(u_1 + z(s))) = 0,$$

when $s \neq 0$ with $(\omega(0), \nu(0), z(0)) = (1, \nu_0, \overline{w}_{\nu_0})$. We see that $(\omega(s), \nu(s), \overline{w}_{\nu(s)} + s(u_1 + z(s)))$ is the desired curve of solutions of $F = 0$. Thus the proof is complete.

□

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