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QUALITATIVE ANALYSIS AND TRAVELLING WAVE SOLUTIONS FOR THE SI MODEL WITH VERTICAL TRANSMISSION

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Abstract. In this note we analyze a spatially structured SI epidemic model with vertical transmission, a logistic effect on vital dynamics and a density dependent incidence. For a bounded spatial domain we show global stability of the endemic state when it is feasible. Then we look at the existence of travelling wave solutions connecting the endemic and the disease free states.

1. Introduction.

We consider a generic S-I epidemic model for the spread of an epidemic disease within a spatially structured theoretical population of density $P(x,t)$, $x \in \mathbb{R}^n$ and $t > 0$, split into susceptible, $S(x,t)$, and infective individuals, $I(x,t)$, that is $P = S + I$. Assume both subpopulation fluxes follow a Fick's law, $-d_S \nabla S$ for susceptibles and $-d_I \nabla I$ for infectives, with constant diffusivities, $d_S > 0$ and $d_I > 0$. Let $b(P)$, $b_I(P)$ and $m(p)$ be the respective birth rate for susceptibles, birth rate for infectives, and mortality rate. Let θ, $0 \le \theta \le 1$, be the proportion of offspring born from an infective individual that is susceptible at birth, and let $1/\alpha > 0$ be the average time spent in the infectious class. Last let $\sigma(S,I)$ be the incidence function, that is the recruitment of infectives into the susceptible class by direct contact (horizontal transmission). A generic Reaction-Diffusion (RD) system modelling the spatial spread of the epidemic disease at hand reads,

$$\begin{aligned} &\partial_t S - d_S \triangle S = -\sigma(S,I) + b(P)S + \theta b_I(P)I - m(P)S, \;\; x \in \mathbb{R}^n, \; t > 0, \\ &\partial_t I - d_I \triangle I = +\sigma(S,I) - \alpha I + (1-\theta) b_I(P)I - m(P)I, \; x \in \mathbb{R}^n, \; t > 0, \end{aligned} \tag{1}$$

supplemented by a set of nonnegative and bounded initial conditions,

$$S(x,0) = S_0(x) \ge 0, \;\; I(x,0) = I_0(x) \ge 0, \; x \in \mathbb{R}^n. \tag{2}$$

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When the spatial domain is a bounded domain $\Omega \subset \mathbb{R}^n$ instead of $\mathbb{R}^n$ no-flux boundary conditions are to be supplemented along the boundary $\partial\Omega$ of Ω. We assume that the population is isolated and we impose no-flux boundary conditions,

$$d_S \nabla S(x,t) \cdot \eta(x) = d_I \nabla I(x,t) \cdot \eta(x) = 0, \;\; x \in \partial\Omega, \; t > 0,$$

where $\eta(x)$ is the unit outer normal vector at $x \in \partial\Omega$.
In this study the incidence function will take a density dependent form,

$$\sigma(S,I) = \sigma SI, \text{ for some } \sigma > 0.$$

Here we refer to [1, 3, 4, 5, 7, 20] for a nice overview on epidemic models. We also refer to the book of [16], and the recent survey papers [17, 18] for a nice survey on travelling waves in this context.

In Section 2 we collect basic results for the underlying Ordinary Differential Equation (ODE) system, including the emergence of a boundary endemic equilibrium that becomes stable, eventually. In Section 3 we analyze the RD system posed on a bounded open domain Ω of $\mathbb{R}^n$ and prove the Global Asymptotic Stability (GAS) of the endemic state when it exists; the case without vertical transmission was studied in [9]. In Section 4 we look for travelling wave solutions connecting the endemic and Disease Free Equilibrium (DFE). Technical results are supplied in an Appendix.

2. **The underlying ODE model with logistic effect.** Let us assume,

Assumption 1. Coefficients b, b_I, m, k, α and σ are nonnegative constants and satisfy $0 \le b_I \le b$, $0 < m < b$, $0 \le \theta \le 1$, $k > 0$ and $\sigma > 0$.

One wants to handle the ODE system,

$$\begin{aligned} S' &= -\sigma SI + bS + \theta b_I I - (m + kP)S, \\ I' &= +\sigma SI - \alpha I + (1-\theta) b_I I - (m + kP)I, \end{aligned} \tag{3}$$

with initial conditions, $S(0) = S_0 > 0, \; I(0) = I_0 > 0$. The existence and uniqueness of a componentwise positive and globally bounded solution is granted.
Qualitative properties are summarized in the following three statements. Set,

$$T_0^{dd} = \frac{\sigma K}{b + \alpha - (1-\theta) b_I}, \;\; K = \frac{b-m}{k}.$$

Lemma 2.1. *When $T_0^{dd} < 1$ the DFE, $(K, 0)$, is GAS with respect to initial data $(S_0, I_0) \in (0, +\infty)^2$. It is unstable when $T_0^{dd} > 1$.*

Lemma 2.2. *Assume either $\theta > 0$ or $\theta = 0$ and $b_I - m - \alpha \le 0$. When $T_0^{dd} > 1$ there exists a unique endemic state, (S^*, I^*) with $0 < S^*, I^* < P^* = S^* + I^* < K$, that is GAS with respect to initial data $(S_0, I_0) \in (0, +\infty)^2$.*

Lemma 2.3. *Assume $\theta = 0$ and $b_I - m - \alpha > 0$. Set $K^{**} = \frac{b_I - m - \alpha}{k} \in (0, K)$. There is a unique "boundary" endemic state $(0, I^{**})$, with $I^{**} = K^{**}$. When $T_0^{dd} > 1$ and $\sigma K^{**} < b + \alpha - b_I$ there exists a unique endemic state, (S^*, I^*) with $0 < S^*, I^* < P^* < K$, that is GAS for initial data $(S_0, I_0) \in (0, +\infty)^2$. When $T_0^{dd} > 1$ and $\sigma K^{**} > b + \alpha - b_I$ $(0, I^{**})$ is GAS with respect to initial data $(S_0, I_0) \in (0, +\infty)^2$.*

Proof. Existence and uniqueness proofs for admissible (S^*, I^*) and $(0, I^{**})$ are obtained through a detailed graphical analysis, see the Appendix. GAS analysis are similar to that for the RD model supplied below. □

Note that $T_0^{dd} < 1$ yields $\sigma K^{**} < \sigma K < b + \alpha - b_I$, while $(0, I^{**})$ is GAS for boundary initial data $(S_0, I_0) \in \{0\} \times (0, +\infty)$ as long as $b_I - m - \alpha > 0$.

3. **The RD system with logistic effect in a bounded domain.** Let us assume,

Assumption 3.1. *Ω is a bounded open subset of $\mathbb{R}^n$ with a smooth (at least $C^{2+\gamma}$ for some $\gamma \in (0,1)$) boundary $\partial\Omega$ so that locally Ω lies on one side of $\partial\Omega$; η is a normal vector to Ω along $\partial\Omega$.*

Diffusivities, d_S and d_I, are positive.

We look at the RD model posed on a bounded domain Ω,

$$\begin{aligned} \partial_t S - d_S \triangle S &= -\sigma SI + bS + \theta b_I I - (m + kP)S, \quad x \in \Omega,\ t > 0 \\ \partial_t I - d_I \triangle I &= +\sigma SI - \alpha I + (1-\theta) b_I I - (m + kP)I, \quad x \in \Omega,\ t > 0 \end{aligned} \tag{4}$$

wherein $P = S + I$, supplemented by no-flux boundary conditions along $\partial\Omega$,

$$d_S \nabla S(x,t) \cdot \eta(x) = d_I \nabla I(x,t) \cdot \eta(x) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{5}$$

and a set of nonnegative and bounded initial conditions at $t = 0$,

$$S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq 0, x \in \Omega. \tag{6}$$

Lemma 3.2. *Let Assumptions 1 and 2 be satisfied. Assume (S_0, I_0) are nonnegative elements of $C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$. There exists a unique componentwise nonnegative and globally bounded classical solution to (4)-(5)-(6) whose semi-orbits $\{(S(\cdot,t), I(\cdot,t)),\ t > 0)\}$ are relatively compact in $[C^0(\bar{\Omega})]^2$.*

Proof. This follows from a general methodology, cf. [9, 10, 11, 15, 19]. □

3.1. **Stability analysis: disease persistence.** Let us assume,

Assumption 3.3. *Concerning the parameter set one shall assume one of the following three conditions holds: either $\theta > 0$, or $\theta = 0$ and $b_I - m - \alpha \leq 0$, or $\theta = 0$ and $b_I - m - \alpha > 0$ and $\sigma K^{**} < b + \alpha - b_I$, see Lemma 2.3 for K^{**}.*

Theorem 3.4. *Let Assumptions 1, 2, and 3 be satisfied. When $T_0^{dd} > 1$ the unique endemic state, (S^*, I^*) with $0 < S^*, I^*, P^* = S^* + I^* < K$, of the ODE system (3) is GAS for (4)-(5)-(6) with respect to initial conditions (S_0, I_0) that are nonnegative elements of $C^\gamma(\overline{\Omega})$, $I_0 \neq 0$ and $S_0 \neq 0$, for some $\gamma \in (0,1)$.*

Proof. From the minimum principle applied to the parabolic equation for I in (4), I_0 nonnegative and $I_0 \neq 0$ it follows $I(x,t) > 0$ for $x \in \Omega$ and $t > 0$. When $S_0 \neq 0$ from the parabolic equation for S in (4) one also gets $S(x,t) > 0$, $x \in \Omega$ and $t > 0$. For any positive elements u and v in $C^0(\bar{\Omega})$ and positive numbers ν_S and ν_I set,

$$\mathcal{L}(u,v) = \nu_S \int_\Omega \Big(u(x) - S^* - S^* \ln \frac{u(x)}{S^*}\Big) dx + \nu_I \int_\Omega \Big(v(x) - I^* - I^* \ln \frac{v(x)}{I^*}\Big) dx, \tag{7}$$

(S^*, I^*) being the componentwise positive endemic state of the ODE system (3), see [12]. Rather straightforward calculations yield,

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(S(\cdot,t),I(\cdot,t)) = &-\nu_S d_S \int_\Omega \frac{|\nabla S|^2}{S^2}(x,t)dx - \nu_I d_I \int_\Omega \frac{|\nabla I|^2}{I^2}(x,t)dx \\
&-\nu_S \theta b_I \int_\Omega I(x,t)\frac{(S(x,t)-S^*)^2}{S^* S(x,t)}dx \\
&-\nu_S k \int_\Omega (S(x,t)-S^*)^2 dx - \nu_I k \int_\Omega (I(x,t)-I^*)^2 dx \\
&+\varpi \int_\Omega (S(x,t)-S^*)(I(x,t)-I^*)dx,
\end{aligned} \tag{8}
$$

wherein,

$$\varpi = -\nu_S \frac{\sigma S^* + kS^* - \theta b_I}{S^*} + \nu_I(\sigma - k). \tag{9}$$

Using the ODE for S in (3) at equilibrium one has,

$$\sigma S^* + kS^* - \theta b_I = (b - [m + kS^*])\frac{S^*}{I^*} > 0 \text{ because } 0 < S^* < K.$$

As a first consequence, when $\sigma > k$ there exists a couple of positive parameters (ν_S, ν_I) so that $\varpi = 0$ that implies $\frac{d}{dt}\mathcal{L}(S(\cdot,t),I(\cdot,t)) \le 0$.

The quadratic form on $[C^0(\bar\Omega)]^2$ featuring in the last two lines of (8) is nonnegative provided a matrix $M(\nu_S, \nu_I)$ be positive, with,

$$M(\nu_S,\nu_I) = \begin{pmatrix} 2\nu_S k & -\varpi \\ -\varpi & 2\nu_I k \end{pmatrix}.$$

We follow a methodology from [4, 5] and split $M(\nu_S,\nu_I) = WA + A^\top W$, wherein,

$$W = \begin{pmatrix} 2\nu_S & 0 \\ 0 & 2\nu_I \end{pmatrix}, \quad A = \begin{pmatrix} k & k-\sigma \\ k+\sigma-\frac{\theta b_I}{S^*} & k \end{pmatrix}.$$

When $\sigma \le k$ matrix $M(\nu_S,\nu_I)$ is positive for some set of positive parameters (ν_S,ν_I) because matrix A has positive diagonal elements and a positive determinant,

$$\det(A) = \sigma^2 + (k-\sigma)\frac{\theta b_I}{S^*} > 0, \text{ cf. Lemma 5.1.} \tag{10}$$

As a consequence, when $\sigma \le k$ there exists a couple of positive parameters (ν_S,ν_I) so that $\frac{d}{dt}\mathcal{L}(S(\cdot,t),I(\cdot,t)) \le 0$.

Hence given any positive σ and k there exists a couple of positive parameters (ν_S,ν_I) so that $\mathcal{L}$ is a Lyapunov functional. From the LaSalle invariance principle, cf. [13], $\mathcal{L}$ is constant on the largest invariant subset of the ω-limit set of (4)-(5)-(6) in $[C^0(\bar\Omega)]^2$and this ω-limit set reduces to (S^*,I^*). □

3.2. **Stability analysis: disease extinction.**

3.2.1. *Case $\sigma \le k$.* This is the simple case.

Theorem 3.5. *Let Assumptions 1 and 2 be satisfied. Assume $\sigma \le k$. When $T_0^{dd} < 1$ the DFE, $(K,0)$, is GAS for (4)-(5)-(6) with respect to initial conditions (S_0, I_0) that are nonnegative elements of $C^\gamma(\overline{\Omega})$, $I_0 \ne 0$ and $S_0 \ne 0$, for some $\gamma \in (0,1)$.*

Proof. We build a Lyapunov functionnal similar to (7), cf. [12]. For any positive elements u and v in $C^0(\bar{\Omega})$ and any positive ν_S and ν_I set,

$$\mathcal{L}(u,v) = \nu_S \int_\Omega \Big(u(x) - K - K \ln \frac{u(x)}{K} \Big) dx + \nu_I \int_\Omega v(x) dx.$$

Rather straightforward calculations yield,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(S(\cdot,t), I(\cdot,t)) = & -\nu_S d_S \int_\Omega \frac{|\nabla S|^2}{S^2}(x,t)dx - \nu_S \theta b_I \int_\Omega I(x,t) \frac{(S(x,t)-K)^2}{KS(x,t)} dx \\ & - \nu_I \frac{1 - T_0^{dd}}{b + \alpha - (1-\theta) b_I} \int_\Omega I(x,t) dx \\ & - \nu_S k \int_\Omega (S(x,t) - K)^2 dx - \nu_I k \int_\Omega I^2(x,t) dx \\ & + \varpi \int_\Omega (S(x,t) - K)\, I(x,t) dx,\end{aligned}$$

wherein,

$$\varpi = -\nu_S \frac{\sigma K + kK - \theta b_I}{K} + \nu_I (\sigma - k). \tag{11}$$

Proceeding as in the second part of the proof of Theorem 3.4 one gets that there exists a couple of positive (ν_S, ν_I) so that $\frac{d}{dt}\mathcal{L}(S(\cdot,t), I(\cdot,t)) \leq 0$ provided,

$$\sigma^2 + (k - \sigma)\frac{\theta b_I}{K} > 0, \text{ compare to (10)}. \tag{12}$$

This completes the proof of Theorem 3.5 because $\sigma \leq k$. □

3.2.2. *Case $\sigma > k$.* This is less simple than the previous one. When $T_0^{dd} < 1$ the DFE $(K, 0)$ is linearly stable for (4)-(5)-(6) for any positive (σ, k).

Remark 1. When $\sigma > k$ the functional used in the proof of Theorem 3.5 remains a Lyapunov functional, yielding the DFE $(K, 0)$ is GAS, as long as either $\sigma K + kK - \theta b_I = \sigma K + b - m - \theta b_I > 0$ (see (11)) or inequality (12) still hold. This requires additional constraints on $(b, b_I, \theta, m, k, \sigma)$, not on α.

For identical diffusivities a new Lyapunov function is built for infectives, relying on an a priori estimate for the total population not known for distinct diffusivities.

Theorem 3.6. *Let Assumptions 1 and 2 be satisfied. Assume $\sigma > k$ and identical diffusivities, $d_S = d_I = d > 0$. When $T_0^{dd} < 1$ the DFE, $(K, 0)$, is GAS for (4)-(5)-(6) with respect to initial conditions (S_0, I_0) that are nonnegative elements of $C^\gamma(\overline{\Omega})$, $I_0 \neq 0$ and $S_0 \neq 0$, for some $\gamma \in (0,1)$.*

Proof. Identical diffusivities yield the total population, $P = S + I$, is a solution to,

$$\partial_t P - d \triangle P = \big(b - m - kP\big)P - \big(\alpha + b - b_I\big) I \leq \big(b - m - kP\big)P,$$

that together with (5)-(6) and a comparison result for parabolic equations give,

$$0 \leq P(x,t) \leq y(t),\ x \in \Omega,\ t < 0,$$

y being a solution to the ODE $y' = (b - m - ky)y$ with $y(0) = \|P_0\|_{\infty,\Omega}$. It follows,

$$\exists \varphi \in L^1(0, +\infty), \varphi(t) \geq 0,\ 0 \leq P(t) \leq K + \varphi(t),\ t \geq 0.$$

It is unclear whether this estimate holds true for distinct diffusivities.

As a consequence one gets, $(\sigma-k)SI \leq (\sigma-k)\left[K+\varphi(t)\right]I$, and

$$\begin{aligned}\partial_t I - d\triangle I &\leq \left[\sigma K - kK - (m+\alpha-(1-\theta)b_I)\right]I - kI^2 + (\sigma-k)\varphi(t)I,\\ &\leq -\frac{1-T_0^{dd}}{b+\alpha-(1-\theta)b_I}I - kI^2 + (\sigma-k)\varphi(t)I.\end{aligned}$$

because $kK = b-m$. Integrating over Ω one finds $I \in L^p(\Omega\times(0,\infty))$, $p=1,2$. Now, (S,I) being uniformly bounded over $\Omega\times(0,\infty)$ and $I\in L^1(\Omega\times(0,\infty))$ a further integration of the equation for I in (4) supplies the existence of a $I^*\geq 0$ so that $\int_\Omega I(x,t)dx \to I^*$ as $t\to+\infty$. One may conclude $I^*=0$. Hence, the ω-limit set in $[C^0(\bar\Omega)]^2$ of the nonnegative semi-orbits $\{(S(\cdot,t),I(\cdot,t)),\ t>0)\}$ is $(K,0)$. □

3.3. **Stability analysis: boundary equilibrium.** We go back to the general case of distinct diffusivities.

Theorem 3.7. *Let Assumptions 1 and 2 be satisfied. Assume $\theta=0$ and $b_I-m-\alpha>0$. When $\sigma K^{**} > b+\alpha-b_I$ the boundary equilibrium from Lemma 2.3, $(0,I^{**})$, is GAS for (4)-(5)-(6) with respect to initial conditions (S_0,I_0) that are nonnegative elements of $C^\gamma(\overline{\Omega})$, $I_0\neq 0$ and $S_0\neq 0$, for some $\gamma\in(0,1)$.*

Proof. We build a further Lyapunov functionnal similar to (7). For any positive elements u and v in $C^0(\bar\Omega)$ and any positive ν_S and ν_I set,

$$\mathcal{L}(u,v) = \nu_S\int_\Omega u(x)dx + \nu_I\int_\Omega\Big(v(x) - I^{**} - I^{**}\ln\frac{v(x)}{I^{**}}\Big)dx.$$

The proof follows the lines of the proofs from Theorem 3.4 and Theorem 3.5 with the simplification $\theta=0$. □

4. Travelling waves solutions to the RD system.

Let us assume,

Assumption 4.1. . *We assume $\theta b_I>0$ and $T_0^{dd}>1$.*

In this section we come back to (1) posed on the whole space $\mathbb{R}^n$. The dynamical properties of such reaction-diffusion system posed on unbounded domains generally leads to more complicated dynamics than model posed on some bounded domain. Indeed, since the pioneering works of Fisher [8] and Kolmogov, Petrovskii and Piskunov in [14], it is well known that reaction-diffusion problems posed on the whole space may admit some special complete (in time) solution, the so-called travelling wave solutions. In the context of epidemic models, such particular solutions show the ability of reaction-diffusion problems to propagate the infection. We shall deal with (1) together with logistic growth as well as Assumption 1.

Note that, in view of Lemma 2.2, Assumption 3 ensures the existence of a unique endemic steady state $(S^*,I^*)\in(0,\infty)^2$ with $S^*+I^*<K$ of the ordinary differential equations (3). We now consider the reaction-diffusion system

$$\begin{aligned}&\frac{\partial S}{\partial t} - d_S\Delta S = -\sigma SI + bS + \theta b_I I - (m+kP)S,\ t>0,\ \ x\in\mathbb{R}^n\\ &\frac{\partial I}{\partial t} - d_I\Delta I = \sigma SI - \alpha I + (1-\theta)b_I I - (m+kP)I,\ t>0,\ \ x\in\mathbb{R}^n\end{aligned}\tag{13}$$

In order to give some hints about the dynamics of this problem, we shall look for one-dimensional travelling wave solutions, that are particular solutions of system (13) of the form

$$S(t,x)=u(e.x-ct),\ \ I(t,x)=v(e.x-ct),$$

where the unit vector e belongs to $\mathbb{R}^n$. According to Assumption 3, we shall supplement this travelling wave problem together with the following conditions

$$\begin{aligned} &u(z) > 0, \quad v(z) > 0 \, \forall z \in \mathbb{R} \\ &\lim_{z\to-\infty} \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = \begin{pmatrix} S^* \\ I^* \end{pmatrix}, \quad \lim_{z\to\infty} \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix}. \end{aligned} \tag{14}$$

Now this travelling wave problem re-writes in the moving frame coordinates as the following problem

$$\begin{aligned} &d_S u''(z) + cu'(z) + F(u(z), v(z)) = 0, \quad z \in \mathbb{R}, \\ &d_I v''(z) + cv'(z) + G(u(z), v(z)) = 0, \end{aligned} \tag{15}$$

wherein we have set $F : \mathbb{R}^2 \to \mathbb{R}$ and $G : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} &F(S, I) = bS + \theta b_I I - \sigma SI - (m + kS + kI)S \\ &G(S, I) = (1-\theta)b_I I + \sigma SI - \alpha I - (m + kS + kI)I. \end{aligned} \tag{16}$$

Recall that Problem (15) is supplemented together with (14). In this problem, the parameter $c > 0$ denotes the wave speed. It is an unknown real parameter that should be found together with the unknown functions u and v.

The main results of this section are the following:

Proposition 4.2. *Let Assumptions 1 and 4 be satisfied. Let $d_S > 0$ and $d_I > 0$ be given. If system (14)-(15) has solution (u, v) for some value $c > 0$ then*

$$c \geq c^* := 2\sqrt{d_I}\sqrt{(1-\theta)b_I - m - \alpha + (\sigma - k)K}. \tag{17}$$

Note that the positivity of the quantity $(1-\theta)b_I - m - \alpha + (\sigma - k)K$ is ensured by Assumption 4, namely $T_0^{dd} > 1$.

Theorem 4.3. *Let Assumptions 1 and 4 be satisfied. Let $d_S > 0$ and $d_I > 0$ be given. Then the following holds true*

- *(i) If $\sigma - k < 0$ then for each $c \geq c^*$ where c^* is defined in (17), system (14)-(15) has a solution (u, v) such that u is increasing and v is decreasing.*
- *(ii) If $\sigma - k \geq 0$ and $d_S = d_I$ then for each $c > c^*$ system (14)-(15) has a positive solution (u, v).*

Remark 2. Proposition 4.2 provides a lower bound for the wave speed. On the other hand, Theorem 4.3 shows that the value c^* is actually the minimal wave speed for Problem (13).

Let us now prove these two results.

Proof of Proposition 4.2. Let us consider a sequence of real number $\{x_n\}_{n\geq 0}$ such that $x_n \to \infty$ when $n \to \infty$ and consider the map $v_n(x) = v(x + x_n)$. It satisfies the equation

$$d_I v_n'' + cv_n' + a(x + x_n)v_n = 0, \; x \in \mathbb{R},$$

wherein we have set

$$a(x) = (1-\theta)b_I + \sigma u(x) - \alpha - (m + ku(x) + kv(x)).$$

Due to Harnack inequality, for each $r > 0$ there exists some constant $M_r > 0$ such that

$$v(x + x_n) \leq M_r v(x_n), \quad \forall x \in [-r, r], \quad \forall n \geq 0.$$

As a consequence, we deduce that $v(x_n) \neq 0$ for each $n \geq 0$ and the map $w_n(x) = \frac{v(x+x_n)}{v(x_n)}$ satisfies the equation

$$\begin{aligned} & d_I w_n'' + c w_n' + a(x+x_n) w_n = 0 \\ & w_n(0) = 1, \;\; 0 < w_n(x) \leq M_r, \;\; \forall x \in [-r, r], \;\; \forall r > 0. \end{aligned}$$

As a consequence, by considering eventually a subsequence, one may assume that the sequence $\{w_n\}$ converges locally uniformly towards some function w that satisfies

$$\begin{aligned} & d_I w'' + c w' + ((1-\theta) b_I + (\sigma - k) K - \alpha - m) \, w = 0 \\ & w(x) \geq 0, \;\; w(0) = 1. \end{aligned}$$

As a consequence, there exists $\lambda \in \mathbb{R}$ such that

$$d_I \lambda^2 + c\lambda + ((1-\theta) b_I + (\sigma - k) K - \alpha - m) = 0,$$

and the result follows. □

It remains to Prove Theorem 4.3. The proof of this result is split into several steps. We first prove (i). To do that we first need the following lemma

Lemma 4.4. *Let Assumptions 1 and 4 be satisfied. Then we have*

$$S^* > \frac{\theta b_I}{\sigma + k}.$$

Proof. Let us first show that $S^* \neq \frac{\theta b_I}{\sigma + k}$. To do so we shall argue by contradiction by assuming that these two quantities equal. Recall that

$$bS^* + \theta b_I I^* - \sigma S^* I^* - (m + kP^*) S^* = 0$$

Then we obtain that

$$b \frac{\theta b_I}{\sigma + k} + \theta b_I I^* - \sigma I^* \frac{\theta b_I}{\sigma + k} - (m + kP^*) \frac{\theta b_I}{\sigma + k} = 0$$

Since $\theta b_T > 0$, this leads us to

$$b + kI^* - (m + kP^*) = 0$$

thus $S^* = K$ a contradiction.

The proof of Lemma 4.4 will follow a continuation argument with respect to the parameter θ. To do so we explicitely write down the dependence of the different quantities with respect to $\theta \in [0,1]$. Let us recall that

$$T_0^{dd}(\theta) = \frac{\sigma K}{b + \alpha - (1-\theta) b_I},$$

so that the map $\theta \to T_0^{dd}(\theta)$ is decreasing. Let us assume that $T_0^{dd}(\theta_0) > 1$ for some $\theta_0 \in (0,1)$. Then we obtain $T_0^{dd}(\theta) > 1$ for all $\theta \in [0, \theta_0]$. Next we consider a continuous branch of endemic equilibria $\{(S^*(\theta), I^*(\theta)), \; \theta \in [0, \theta_0]\}$, that $I^*(\theta) > 0$ for all $\theta \in [0, \theta_0]$. We introduce the map $f(\theta) = S^*(\theta) - \frac{\theta b_I}{\sigma + k}$ then this map is continuous on $[0, \theta_0]$. Moreover it does not vanish on $(0, \theta_0]$. We now split split the argument into two parts according to $S^*(0) > 0$ or $S^*(0) = 0$. If $S^*(0) > 0$ then $f(0) = S^*(0) > 0$. As a consequence we obtain that $S^*(\theta) > \frac{\theta b_I}{\sigma + k}$ for each $\theta \in [0, \theta_0]$ and the result follows. If $S^*(0) = 0$ then it is easily checked that

$$S^*(\theta) \sim \frac{\theta b_I I^*(0)}{m - b + (k + \sigma) I^*(0)}.$$

Since $S^*(\theta) > 0$ for all $\theta > 0$ we obtain that

$$\frac{I^*(0)}{m - b + (k+\sigma)I^*(0)} > 0.$$

Thus we obtain that

$$\lim_{\theta \to 0^+} \frac{f(\theta)}{\theta b_I} = \frac{I^*(0)}{m - b + (k+\sigma)I^*(0)} - \frac{1}{\sigma + k}.$$

Now since $m - b < 0$ we obtain that this last quantity is nonnegative and the result follows. This completes the proof of the result. $\square$

Proof of Theorem 4.3 (i). We re-write system (14)-(15) with the independent variables $(u, J = I^* - v)$ and we get

$$\begin{aligned} &d_S u'' + cu' + F_1(u, J) = 0, \\ &d_I J'' + cJ' + F_2(u, J) = 0, \\ &\lim_{x \to -\infty} (u, J)(x) = (S^*, 0), \quad \lim_{x \to \infty} (u, I)(x) = (K, I^*). \end{aligned}$$

Here we have set

$$F_1(u, J) = bu + \theta b_I(I^* - J) - (m + ku + kI^* - kJ)u - \sigma u(I^* - J)$$
$$F_2(u, J) = -(1-\theta)b_I(I^* - J) - \sigma u(I^* - J) + \alpha(I^* - J) + (m + ku + kI^* - kJ)(I^* - J).$$

Next due to Lemma 4.4, one has

$$(\sigma + k)u - \theta b_I > 0, \quad \forall u \geq S^*.$$

Then one has

$$\begin{aligned} &\frac{\partial F_1(u, J)}{\partial J} = (\sigma + k)u - \theta b_I > 0 \;\; \forall (u, J) \in (S^*, K) \times (0, I^*) \\ &\frac{\partial F_2(u, J)}{\partial u} = (k - \sigma)(I^* - J) > 0. \end{aligned}$$

Thus this case enters the monotone framework that has been extensively studied in the litterature and we get the existence of monotone travelling wave (see for instance the monograph of Volpert et al. [21] and the references cited therein). This completes the proof of the result. $\square$

We shall now prove Theorem 4.3 (ii). Let us first notice that since $d_S = d_I$, up to change z by $\sqrt{d_S} z$, without loss of generality, we shall assume that $d_S = d_I = 1$. Therefore, in the sequel, we shall consider the following system

$$\begin{aligned} &u''(z) + cu'(z) + F(u(z), v(z)) = 0, \quad z \in \mathbb{R}, \\ &v''(z) + cv'(z) + G(u(z), v(z)) = 0, \\ &(u, v)(-\infty) = (S^*, I^*), \quad (u, v)(\infty) = (K, 0). \end{aligned} \tag{18}$$

We first deals with some preliminary lemmas

Lemma 4.5. *Let $c > c^*$ be given and fixed. Let $\lambda > 0$ be defined by*

$$\lambda = \frac{c - \sqrt{c^2 - c^{*2}}}{2}.$$

Then the following properties hold true

(i) The map $\overline{u}(x) = e^{-\lambda x}$ satisfies the equation

$$u'' + cu' + ((1-\theta)b_I - m - \alpha + K\sigma - kK)\, u = 0.$$

(ii) Let $\gamma > 0$ be defined by

$$(b - b_I + \alpha) = \gamma \frac{c^{*2}}{4}.$$

Then the map $\underline{v}(x) = K - \gamma e^{-\lambda x}$ satisfies

$$(b - b_I + \alpha)\, e^{-\lambda x} \leq v'' + cv'.$$

(iii) There exists $\varepsilon \in (0, \lambda)$ such that

$$(\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + (1 - \theta) b_I - \alpha - m + \sigma K > 0$$

$$(\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + \frac{c^{*2}}{4} < 0.$$

For such a value of $\varepsilon > 0$ there exists $\kappa > 0$ sufficiently large such that the map $w(x) = e^{-\lambda x} - \kappa e^{-(\lambda+\varepsilon)x}$ satisfies the differential inequality

$$w'' + cw' + ((1 - \theta) b_I - \alpha - m + \sigma K)\, w \leq k \left(K - \gamma e^{-\lambda x}\right)^+ w.$$

Proof. The proof of (i) is obvious.

The proof of (ii). The differential inequality is equivalent to

$$(b - b_I + \alpha) \leq \gamma(c\lambda - \lambda^2).$$

Due to the definition of λ we obtain that

$$(b - b_I + \alpha) \leq \gamma \frac{c^{*2}}{4}.$$

Proof of (iii). Let us notice that λ is the smallest root of the equation

$$\lambda^2 + c\lambda + \frac{c^{*2}}{4} = 0.$$

Therefore for each $x \in (\lambda, \lambda^+)$, where λ^+ is defined by

$$\lambda^+ = \frac{c + \sqrt{c^2 - c^{*2}}}{2},$$

we have $x^2 + cx + \frac{c^{*2}}{4} < 0$. Thus for each $\varepsilon \in (0, \lambda^+ - \lambda)$ we obtain that

$$(\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + \frac{c^{*2}}{4} < 0.$$

Finally let us notice that

$$\begin{aligned} &\lambda^2 - c\lambda + (1 - \theta) b_I - \alpha - m + \sigma K \\ &\quad = -(1 - \theta) b_I + m + \alpha - (\sigma - k)K + (1 - \theta) b_I - \alpha - m + \sigma K = kK > 0. \end{aligned}$$

Thus when $\varepsilon > 0$ is small enough we get that

$$(\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + (1 - \theta) b_I - \alpha - m + \sigma K > 0.$$

Next let us set $\mu = (1 - \theta) b_I - \alpha - m + \sigma K$. Then the differential inequality in (iii) may be re-written for any $x \in \mathbb{R}$

$$\begin{aligned} &\left(\lambda^2 - c\lambda + \mu\right) - \kappa e^{-\varepsilon x} \left((\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + \mu\right) \\ &\leq k \left(K - \gamma e^{-\lambda x}\right)^+ \left(1 - \kappa e^{-\varepsilon x}\right). \end{aligned}$$

Due to the definition of λ we obtain

$$\begin{aligned} &kK - \kappa e^{-\varepsilon x} \left((\lambda + \varepsilon)^2 - c(\lambda + \varepsilon) + \mu\right) \\ &\leq k \left(K - \gamma e^{-\lambda x}\right)^+ \left(1 - \kappa e^{-\varepsilon x}\right), \quad x \in \mathbb{R}. \end{aligned} \tag{19}$$

We split this inequality into two parts, $x \leq x_0$ and $x \geq x_0$ where x_0 is the solution of

$$K - \gamma e^{-\lambda x_0} = 0.$$

We set $\Phi(\varepsilon) = (\lambda+\varepsilon)^2 - c(\lambda+\varepsilon) + \mu$ and (19) re-writes as

$$\begin{cases} kK \leq \kappa e^{-\varepsilon x}\Phi(\varepsilon), \quad \forall x \leq x_0 \\ -\kappa\Phi(\varepsilon) \leq -kK\kappa - k\gamma e^{(\varepsilon-\lambda)x} + k\gamma\kappa e^{-\lambda x}, \quad \forall x \geq x_0. \end{cases}$$

This re-writes as

$$\begin{cases} kK \leq \kappa e^{-\varepsilon x_0}\Phi(\varepsilon) \\ -\kappa\left(\Phi(\varepsilon) - kK\right) \leq k\gamma\left(\kappa e^{-\lambda x} - e^{(\varepsilon-\lambda)x}\right), \quad x \geq x_0. \end{cases}$$

Due to the definition of ε we have $\Phi(\varepsilon) > 0$ while $\Phi(\varepsilon) - kK < 0$. Since the map $x \to \frac{k\gamma e^{-\lambda x}}{kK - \Phi(\varepsilon) + k\gamma e^{(\eta-\lambda)x}}$ is continuous and bounded on the interval $[x_0, \infty)$ it is sufficient to choose $\kappa > 0$ large enough such that

$$\begin{cases} \kappa \geq \dfrac{kK}{\Phi(\varepsilon)} e^{\varepsilon x_0} \\ \kappa \geq \sup_{x\in[x_0,\infty)} \dfrac{k\gamma e^{-\lambda x}}{kK - \Phi(\varepsilon) + k\gamma e^{(\varepsilon-\lambda)x}}. \end{cases}$$

This completets the proof of the lemma. □

Our second preliminary result is the following

Lemma 4.6. *Let $c > c^*$ be given and fixed. Let $a > 0$, $\eta_1 > 0$, $\eta_2 > 0$ be given such that*

$$\begin{aligned} &\eta_2 \leq \eta_1 \\ &I^* \leq \overline{u}(-a), \quad \eta_2 \leq \overline{u}(-a) \\ &I^* \geq w(-a), \quad \eta_2 \geq w(a) \\ &P^* \geq v(-a), \quad K - \eta_1 + \eta_2 \geq v(a). \end{aligned}$$

Then there exists a solution (u, v) of the elliptic boundary value problem:

$$\begin{aligned} &u'' + cu' + F(u, v) = 0, \quad x \in (-a, a) \\ &v'' + cv' + G(u, v) = 0, \quad x \in (-a, a) \\ &(u, v)(-a) = (S^*, I^*), \quad (u, v)(a) = (K - \eta_1, \eta_2). \end{aligned}$$

Moreover (u, v) satisfies

$$u \geq 0, \quad v \geq 0, \quad \max(0, w) \leq v \leq \overline{u}, \quad \max(0, \underline{v}) \leq u + v \leq K.$$

Proof. Let $\lambda_0 > b$ be given and fixed such that for all $(S_0, I_0, P_0) \in [0, K]^3$

$$\begin{aligned} &\lambda_0 - (-b + \sigma I_0 + m + kP_0) > 0 \\ &\lambda_0 + (1-\theta)b_I + \sigma S_0 - (m + kP_0) - \alpha > 0 \\ &\lambda_0 + b - m - 2kK \geq 0. \end{aligned}$$

Next let us consider the Banach space $E = C([-a, a]) \times C([-a, a))$, the closed convex set

$$\mathcal{C} = \{(P, S) \in E : \quad S \geq 0, \ \max(0, v) \leq P \leq K, \max(0, w) \leq P - S \leq u(x)\},$$

as well as the map $\Phi : \mathcal{C} \to E$ defined by

$$\Phi(S_0, P_0) = (S, P),$$

where (S, P) is the solution of the linear boundary value problem

$$\begin{aligned} & -S'' - cS' + \lambda_0 S = \lambda_0 S_0 + F(S_0, P_0 - S_0) \\ & -P'' - cP' + \lambda_0 P = (\lambda_0 + b - m - kP_0)P_0 - (\alpha + b - b_I)(P_0 - S_0) \\ & S(-a) = S^*, \;\; S(a) = K - \eta_1 \\ & P(-a) = P^* := S^* + I^*, \;\; P(a) = K - \eta_1 + \eta_2. \end{aligned} \tag{20}$$

We claim that

Claim 4.7. $\Phi(\mathcal{C}) \subset \mathcal{C}$.

Before proving this claim, we complete the proof of the lemma by applying Schauder fixed point theorem. Indeed, due to elliptic regularity, the map Φ is completely continuous. It remains to complete the proof of Claim 4.7. Let $(P_0, I_0) \in \mathcal{C}$ be given and $(P, S) = \Phi(P_0, S_0)$. Set also $I = P - S$, that satisfies the equation

$$\begin{aligned} & -I'' - cI' + \lambda_0 I = (\lambda_0 + (1-\theta)b_I + \sigma S_0 - (m + kP_0) - \alpha)\, I_0 \\ & I(-a) = I^*, \;\; I(a) = \eta_2. \end{aligned}$$

Next due to the maximum principle we have $S \geq 0$, $I \geq 0$.

If P has a maximum in $x_0 \in (-a, a)$ then $P''(x_0) \leq 0$ and $P'(x_0) = 0$. Thus

$$\lambda P(x_0) \leq (\lambda + b - m - kP_0(x_0))P_0(x_0) \leq \lambda K$$

because the map $s \to (\lambda + b - m - ks)s$ is increasing on $[0, K]$. As a consequence we get $P(x) \leq \max(K, P(a)) \leq K$. Next, let us show that $I \leq u$. Because $\sigma - k \geq 0$, one has

$$\begin{aligned} -I'' - cI' + \lambda_0 I &= (\lambda_0 + (1-\theta)b_I + (\sigma - k)S_0 - m - \alpha - kI_0)\, I_0 \\ &\leq (\lambda_0 + (1-\theta)b_I + (\sigma - k)K - m - \alpha)\, u \\ &\leq -u'' - cu' + \lambda_0 u \end{aligned}$$

Since $I(-a) = I^* \leq u(-a)$ and $I(a) = \eta_2 \leq u(-a)$ we obtain that $I(x) \leq u(x)$ for each $x \in [-a, a]$.

Next let us show that $P \geq v$. Indeed we have

$$\begin{aligned} -P'' - cP' + \lambda_0 P &= (\lambda_0 + b - m - kP_0)P_0 - (\alpha + b - b_I)I_0 \\ &\geq \lambda_0 P_0 - (\alpha + b - b_I)u \\ &\geq \lambda_0 v - (\alpha + b - b_I)u \\ &\geq -v'' - cv' + \lambda_0 v. \end{aligned}$$

Since $P(-a) = P^* \geq v(-a)$ and $P(a) = K - \eta_1 + \eta_2 \geq v(a)$ we obtain that $P(x) \geq v(x)$ for all $x \in [-a, a]$.

It remains to show that $I \geq w$. To do so let us notice that

$$\begin{aligned} -I'' - cI' + \lambda_0 I &= (\lambda_0 + (1-\theta)b_I + \sigma S_0 - m - \alpha - kP_0)\, I_0 \\ &\leq (\lambda_0 + (1-\theta)b_I + \sigma K - m - \alpha)\, w - kw \max(0, v) \\ & -w'' - cw' + \lambda_0 w. \end{aligned}$$

Since $I(-a) = I^* \geq w(-a)$ and $I(a) = \eta_2 \geq w(a)$ we obtain that $I(x) \geq w(x)$ for all $x \in [-a, a]$. This completes the proof of Claim 4.7. □

We now prove Theorem 4.3 (ii).

Proof of Theorem 4.3 (ii). Let $c > c^*$ be given. Let $\{a_n\}_{n\geq 0}$ be a sequence of positive numbers tending to infinity. Then there exists $n_0 \geq 0$ such that for each $n \geq n_0$

$$I^* \leq \overline{u}(-a_n), \quad I^* \geq w(-a_n),$$
$$P^* \geq v(-a_n), \quad K \geq v(a_n).$$

Then according to Lemma 4.6 for each $n \geq n_0$ we choose $\eta_1 = \eta_2 = w(a_n) > 0$ and we obtain a solution (u_n, v_n) of the problem

$$u_n'' + cu_n' + F(u_n, v_n) = 0, \quad x \in (-a_n, a_n)$$
$$v_n'' + cv_n' + G(u_n, v_n) = 0, \quad x \in (-a_n, a_n),$$
$$(u_n, v_n)(-a_n) = (S^*, I^*), \quad (u_n, v_n)(a_n) = (K - w(a_n), w(a_n)).$$

and such that for each $x \in [-a_n, a_n]$

$$u_n(x) \geq 0, \quad \max(0, w(x)) \leq v_n(x) \leq \overline{u}(x), \quad \max(0, \underline{v}(x)) \leq u_n(x) + v_n(x) \leq K.$$

Next due to elliptic regularity, up to a subsequence, one may assume that the sequences $\{u_n\}$ and $\{v_n\}$ converge locally uniformly towards some functions (u, v) solution of the problem

$$\begin{aligned} u'' + cu' + F(u, v) = 0, \quad x \in \mathbb{R} \\ v'' + cv' + G(u, v) = 0, \quad x \in \mathbb{R}, \end{aligned} \tag{21}$$

and such that for each $x \in \mathbb{R}$

$$u(x) \geq 0, \quad \max(0, w(x)) \leq v(x) \leq \overline{u}(x), \quad \max(0, \underline{v}(x)) \leq u(x) + v(x) \leq K.$$

Due to this bound, we know that (u, v) does not indentically equal to $(K, 0)$ and that

$$\lim_{x\to\infty} (u(x), v(x)) = (K, 0).$$

It remains to show that the expected behaviour at $x = -\infty$ holds. To do so, let us re-write (21) under the following form:

$$\begin{pmatrix} u \\ v \end{pmatrix}'' + c \begin{pmatrix} u \\ v \end{pmatrix}' + \begin{pmatrix} c_{11}(x) & \theta b_I \\ c_{21}(x) & c_{22}(x) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \quad x \in \mathbb{R}, \tag{22}$$

wherein we have set

$$c_{11}(x) = b - \sigma v(x) - (m + ku(x) + kv(x))$$
$$c_{22}(x) = (1 - \theta)b_I - \alpha - (m + ku(x) + kv(x))$$
$$c_{21}(x) = \sigma v(x).$$

Let us first notice that since v is positive and not indentically zero, Harnack inequality for the equation for v implies that $v(x) > 0$ for all $x \in \mathbb{R}$. Thus $c_{21}(x) > 0$ for all $x \in \mathbb{R}$. Since these maps are bounded and since $\theta b_I > 0$ one can apply Harnack inequality for system (22) (see [6, 2]) to get that there exists some constant $M > 0$ such that for any $x \in \mathbb{R}$

$$\max\left(\max_{(x-1,x+1)} u, \max_{(x-1,x+1)} v\right) \leq M \min\left(\min_{(x-1,x+1)} u, \min_{(x-1,x+1)} v\right).$$

As a consequence, we deduce that $u > 0$ and that there exists some constant $\widehat{M} > 0$ such that

$$\left|\frac{u'(x)}{u(x)}\right| + \left|\frac{v'(x)}{v(x)}\right| \leq \widehat{M}, \quad \forall x \in \mathbb{R}. \tag{23}$$

We complete the proof of the result by introducing the map

$$\mathcal{V}(x) = c\mathcal{V}_1(x) + \mathcal{V}_2(x).$$

where we have set

$$\mathcal{V}_1(x) = \nu_S g\left(\frac{u(x)}{S^*}\right) + \nu_I g\left(\frac{v(x)}{I^*}\right)$$
$$\mathcal{V}_2(x) = \nu_S g'\left(\frac{u(x)}{S^*}\right)\frac{u'(x)}{S^*} + \nu_I g'\left(\frac{v(x)}{I^*}\right)\frac{v'(x)}{I^*},$$

and wherein

$$g(s) = s - 1 - \ln s.$$

Here we choose $\nu_S > 0$ and $\nu_I > 0$ such that

$$-\nu_S(m + \alpha + \theta b_I) + I^*\nu_I\sigma = 0.$$

With such a choice we obtain that

$$\begin{aligned}\frac{d\mathcal{V}(x)}{dx} &= \nu_S g'\left(\frac{u(x)}{S^*}\right)\frac{u''(x) + cu'(x)}{S^*} + \nu_S g''\left(\frac{u(x)}{S^*}\right)\left(\frac{u'(x)}{S^*}\right)^2 \\ &+ \nu_I g'\left(\frac{v(x)}{I^*}\right)\frac{v''(x) + cv'(x)}{I^*} + \nu_I g''\left(\frac{v(x)}{I^*}\right)\left(\frac{v'(x)}{I^*}\right)^2 \\ &= \nu_S\left(\frac{S^*}{u(x)} - 1\right)F(u,v) + \nu_I\left(\frac{I^*}{v(x)} - 1\right)G(u,v) + \nu_S\left(\frac{u'(x)}{u(x)}\right)^2 + \nu_I\left(\frac{v'(x)}{v(x)}\right)^2.\end{aligned}$$

On the other hand, one has

$$\begin{aligned}&\nu_S\left(\frac{S^*}{u(x)} - 1\right)F(u,v) + \nu_I\left(\frac{I^*}{v(x)} - 1\right)G(u,v) \\ &= \nu_S\theta b_I v(x)\frac{(u(x) - S^*)^2}{u(x)} + \omega(v(x) - I^*)(u(x) - S^*),\end{aligned}$$

where

$$\omega = \nu_S(\theta b_I - \sigma S^*) + I^*\nu_I\sigma.$$

Due to the choice of ν_S and ν_I, one has $\omega = 0$ and therefore

$$\frac{d\mathcal{V}(x)}{dx} = \nu_S\theta b_I v(x)\frac{(u(x) - S^*)^2}{u(x)} + \nu_S\left(\frac{u'(x)}{u(x)}\right)^2 + \nu_I\left(\frac{v'(x)}{v(x)}\right)^2.$$

As a consequence, the map $x \to V(x)$ is increasing. Let us also notice that due to (23), the map $x \to \mathcal{V}_2(x)$ is bounded. This yields,

$$0 < \mathcal{V}_1(x) \le \mathcal{V}(0) + \sup_{s\in\mathbb{R}}\mathcal{V}_2(s), \quad \forall x \le 0.$$

Therefore we get

$$\liminf_{x\to-\infty} u(x) > 0, \quad \liminf_{x\to-\infty} v(x) > 0. \tag{24}$$

Let us now consider a nonincreasing sequence of real numbers $\{x_n\}_{n\ge0}$ such that $x_n \to -\infty$ when $n \to \infty$ and consider the sequences $\{u_n(x) = u(x + x_n)\}_{n\ge0}$ and $\{v_n(x) = v(x + x_n)\}_{n\ge0}$. Next, due to elliptic regularity, up to a subsequence, one may assume that u_n and v_n converges towards some nonnegative functions u_∞ and v_∞, solution of (18). Next due to (24) we know that u_∞ and v_∞ are positive functions and due to the monotonicity of the map $x \to \mathcal{V}(x)$, these maps satisfy for any $x \in \mathbb{R}$

$$\nu_S \theta b_I v_\infty(x) \frac{(u_\infty(x) - S^*)^2}{u_\infty(x)} + \nu_S \left(\frac{u'_\infty(x)}{u_\infty(x)} \right)^2 + \nu_I \left(\frac{v'_\infty(x)}{v_\infty(x)} \right)^2 = 0.$$

We deduce that $u_\infty(x) \equiv S^*$, $u'_\infty(x) \equiv 0$ and $v'_\infty(x) \equiv 0$. Due to (24), v_∞ is a positive constant. Since $u_\infty \equiv S^*$ and v_∞ are also solution of system (18), we conclude that $u_\infty \equiv S^*$ and $v_\infty \equiv I^*$. Finally, since the sequence $\{x_n\}$ is arbitrary, this leads us to

$$\lim_{x \to -\infty} (u(x), v(x)) = (S^*, I^*).$$

This completes the proof of the Theorem 4.3 (ii). □

5. Appendix.

5.1. Endemic states for (3).

Using (P, I) as state variables system (3) reads,

$$P' = \big(b - m - kP\big)P - \big(\alpha + b - b_I\big)I,$$
$$I' = (\sigma - k)PI - \big(\sigma I + m + \alpha - (1 - \theta)b_I\big)I.$$

Looking for admissible stationary states, $0 < S^*,\ I^* < P^* = S^* + I^* < K$, one finds $I^* = \varphi(P^*) = \psi(P^*)$ wherein,

$$\varphi(p) = \frac{(b - m - kp)p}{\alpha + b - b_I}, \quad \psi(p) = \frac{(\sigma - k)p - \big(m + \alpha - (1 - \theta)b_I\big)}{\sigma}, \ p \geq 0.$$

Note that $\varphi(p)$ is a concave and positive function in the range $0 < p < K$ vanishing at $p = 0$ and $p = K$, while ψ is affine with,

$$\psi(0) = \frac{(1 - \theta)b_I - m - \alpha}{\sigma}, \quad \psi(K) = (T_0^{dd} - 1)\frac{b + \alpha - (1 - \theta)b_I}{\sigma}.$$

5.1.1. *The weak vertical transmission case,* $(1 - \theta)b_I - m - \alpha \leq 0$. Thus, $\psi(0) \leq 0$. When $T_0^{dd} < 1$ then ψ remains nonpositive in the range $0 < p < K$ and there is no admissible stationary solutions.

When $T_0^{dd} > 1$ then ψ is increasing in the range $0 < p < K$ with a slope $0 < \frac{\sigma - k}{\sigma} < 1$; it follows there is a unique admissible stationary solution $(0 < S^*, I^* < P^* < K)$.

5.1.2. *The strong vertical transmission case,* $(1 - \theta)b_I - m - \alpha > 0$. Thus $\psi(0) > 0$. Function φ crosses the first bissectrix $\chi(p) = p$ at $p = p_\varphi$ while function ψ crosses the first bissectrix at $p = K^{**}$ with $0 < p_\psi \leq K^{**} < K$, wherein,

$$p_\psi = \frac{(1 - \theta)b_I - m - \alpha}{k}; \quad K^{**} = \frac{b_I - m - \alpha}{k} \text{ (see Lemma 2.3)}.$$

Case $\theta > 0$. This implies $0 < p_\psi < K^{**} < K$.

When $T_0^{dd} > 1$ a concavity argument yields two solutions $P^* \in (0, K)$ to $\varphi(P^*) = \psi(P^*)$. First one has $\psi(0) > \varphi(0) = 0$ and $\varphi(p_\psi) > p_\psi = \psi(p_\psi)$ supplying a non admissible solution $P_1^* \in (0, p_\psi)$ because $I_1^* = \varphi(P_1^*) = \psi(P_1^*) > P_1^*$. Next $K^{**} = \varphi(K^{**}) > \psi(p_\varphi)$ and $\psi(K) > \varphi(K) = 0$ supply an admissible solution $P_2^* \in (K^{**}, K)$ because $I_2^* = \varphi(P_2^*) = \psi(P_2^*) < P_2^*$.

Likewise when $T_0^{dd} < 1$ a concavity argument yields a unique but non admissible solution $P_1^* \in (0, K)$ to $\varphi(P^*) = \psi(P^*)$ with $P_1^* \in (0, p_\psi)$ and $I_1^* > P_1^*$.

Case $\theta = 0$. This implies $0 < p_\psi = K^{**} < K$.

$(0, K^{**})$ is a boundary equilibrium, see Lemma 2.3. One has,

$$\varphi(p) - \psi(p) = \frac{k}{\sigma}(p - K^{**})\Big(1 - \frac{\sigma p}{\alpha + b - b_I}\Big).$$

When $T_0^{dd} < 1$ equation $\varphi(p) = \psi(p)$ has a unique root in $(0, K)$, $P^* = K^{**}$.

When $T_0^{dd} > 1$ equation $\varphi(p) = \psi(p)$ has two roots in $(0, K)$, $P^{**} = K^{**}$ and $P^* = \frac{\alpha+b-b_I}{\sigma}$; one may check that P^* is admissible, that is $0 < I^* = \varphi(P^*) = \psi(P^*) < P^*$, if and only if $\sigma K^{**} < \alpha + b - b_I$.

5.2. **An auxiliary lemma for subsection 3.1.**

Lemma 5.1. *Let* $W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, *for some* $w_1 > 0$, $w_2 > 0$, $a_{11} > 0$, $a_{22} > 0$ *and assume* $\det(A) > 0$. *Set* $M(w_1, w_2) = WA + A^\top W$.

Then there exists a set of positive parameters (w_1, w_2) *such that* $M(w_1, w_2)$ *is symmetric positive definite.*

Proof. $M(w_1, w_2)$ is symmetric. Its characteristic polynomial reads,

$$\det\big(\lambda I_d - M(w_1, w_2)\big) = \lambda^2 - 2(a_{11}w_1 + a_{22}w_2)\lambda + \det(M(w_1, w_2)).$$

Suppose $a_{12} \neq 0$ and set $w_1 = \rho\, w_2$ for some positive ρ. One finds,

$$\frac{1}{w_1^2}\det(M(w_1, w_2)) = 4a_{11}a_{22}\rho - (a_{12}\rho + a_{21})^2 = \Phi(\rho).$$

Then Φ is a concave and quadratic function, achieving its maximal value at $\rho^* > 0$, with $a_{12}^2\,\rho^* = a_{11}a_{22} + \det(A)$. Elementary algebraic manipulations yield $\Phi(\rho^*) > 0$, $a_{12}^2\,\Phi(\rho^*) = 4a_{11}a_{22}\det(A)$.

As a consequence, for ρ close to $\rho^* > 0$ the determinant of $M(w_1, w_2)$ is positive. Because its trace is also positive, we may conclude that the real roots $M(w_1, w_2)$ are positive. □

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