Integrated Semigroups and Parabolic Equations. Part II: Semilinear Problems

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October 3, 2018

Abstract

In this note we study of a class of non-autonomous semilinear abstract Cauchy problems involving non-densely defined almost sectorial operator. The nonlinearity may contain unbounded terms and acts on suitable fractional power spaces associated with the almost sectorial operator. We use the framework of the so-called integrated semigroups to investigate the well posedness of the problems. This note is a continuation of a previous work [9] dealing with linear equations. Here, using a suitable notion of mild solutions, we first study the existence of a maximal and strongly continuous evolution semiflow for semilinear equations under rather mild assumptions. Under additional conditions we prove that the semiflow is Frechet differentiable and state some consequences about the linear stability of equilibria. In addition we prove that the solutions become immediately smooth so that the mild solutions turn out to be classical. We complete this work with an application of the results presented in this note to a reaction-diffusion equation with nonlinear and nonlocal boundary conditions arising, in particular, in mathematical biology.

Key words. Integrated semigroups, almost sectorial operators, semilinear parabolic equations.

MSC: 47D06, 47D62, 35K90.

1 Introduction

In this note we consider the following class of non-autonomous abstract Cauchy problems

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + F(t, v(t)), \text{ for } t > s. \tag{1.1}$$

Herein $s \geq 0$ is given and $A : D(A) \subset X \to X$ is a non-densely defined linear operator on a Banach space $(X, \|.\|)$ that is assumed to be an almost sectorial operator. The nonlinear function F is defined for all time $t \geq 0$ and may contain unbounded terms so that it is defined on a space smaller than $\overline{D(A)}$, the closure of $D(A)$, involving suitable fractional power spaces of the linear operator A. The precise definition for an almost section operator is given below (see Assumption 1.1 and Definition 1.2 below) while the assumptions on the function F are also precisely stated below. The above problem is supplemented with a suitable initial condition $v(s) = x$ that will also be discussed below.

Let us precise some of the main assumptions we shall need in this work. Throughout this article, we will make the following assumption on the linear operator A.

Assumption 1.1 *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|\cdot\|)$. We assume that*

(a) the operator A_0, the part of A in $\overline{D(A)}$, is the infinitesimal generator of an analytic semigroup of bounded linear operators on $\overline{D(A)}$ that is denoted by $\{T_{A_0}(t)\}_{t\geq 0}$.

(b) There exist $\omega \in \mathbb{R}$ and $p^* \in [1, +\infty)$ such that $(\omega, +\infty) \subset \rho(A)$, the resolvent set of A, and

$$\limsup_{\lambda\to+\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty. \tag{1.2}$$

Using the results proved by Ducrot et al. in [9] (see Proposition 3.3 in that paper), this above set of assumptions can be reformulated using the notion of almost sectorial operators and, this re-writes as follows.
Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|\cdot\|)$. Then Assumption 1.1 is satisfied if and only if the two following conditions are satisfied:

(a) A_0 is a sectorial operator.

(b) A is a $\frac{1}{p^*}-$almost sectorial operator.

Here let us recall $(\alpha-)$almost sectorial operator is defined as follows.

Definition 1.2 (Almost sectorial operator) *Let $L : D(L) \subset X \to X$ be a linear operator on the Banach space X and let $\alpha \in (0, 1]$ be given. Then L is said to be a $\alpha-$**almost sectorial** operator if there are constants $\widehat{\omega} \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $\widehat{M} > 0$ such that*

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$,

(ii) $\left\| (\lambda I - L)^{-1} \right\|_{\mathcal{L}(X)} \leq \dfrac{\widehat{M}}{|\lambda - \widehat{\omega}|^{\alpha}}, \ \forall \lambda \in S_{\theta,\widehat{\omega}}.$

Moreover L is called ***sectorial operator*** *if L is 1-almost sectorial.*

Since A_0, the part of A in $\overline{D(A)}$, is assumed to be a sectorial operator, the fractional power $(\mu I - A_0)^\gamma$ is well defined for each $\gamma \geq 0$, and for each μ large enough. This assumption on the linear operator A allows us to roughly speak about the nonlinear term $F = F(t,v)$ and the initial data. In this note we assume that the function $F = F(t,v)$ is defined from $[0,\infty) \times D\left((\mu I - A_0)^\alpha\right)$ with values in X, for some given value $\alpha \geq 0$ and some constant $\mu > 0$ large enough. Further smoothness assumptions will be detailed below (see Section 3 and 4). And, the initial data $v(s) = x$ is assumed to live in some other fractional space $D\left((\mu I - A_0)^\beta\right)$ for some $\beta \geq 0$.

Observe that when the domain of A is dense, namely $\overline{D(A)} = X$, then the operators A and A_0 coincide, and Problem (1.1) enters the classical framework. We refer for instance to Friedmann [12], Tanabe [25], Henry [14], Pazy [23], Temam [26], Lunardi [17], Cholewa and Dlotko [6], Engel and Nagel [11] and Yagi [29] for more details on the subject.

In this note we are mainly concerned with the non-densely defined case, that is $\overline{D(A)} \neq X$, and when the linear operator A is almost sectorial.
When dealing with parabolic equations (densely defined or not), it is usually assumed that the operator A is a sectorial elliptic operator. This operator property usually holds true when considering elliptic operators in Lebesgue spaces or Hölder spaces and together with homogeneous boundary conditions. As pointed out by Lunardi in [17], this property does no longer hold true when dealing with such operators in some more regular spaces. Typical examples of non-sectorial but almost sectorial parabolic problems may also arise when dealing with parabolic equations with nonlinear boundary conditions. This point will be discussed below on a particular motivating example and in the last section of this work.

Almost sectorial operators have been studied in the literature, by using functional calculus and the so-called growth semigroups. In [24] the authors use functional calculus to define the fractional powers of $\lambda I - A$ for some $\lambda > 0$ large enough. We also refer to DeLaubenfels [10], and Haase [13] for more update results on functional calculi, and to Da Prato [7] for pioneer work on growth semigroups. More recently the case of non-autonomous Cauchy problems has also been studied by Carvalho et al. in [5] by using a notion of solution based on growth semigroups. We also refer to the recent work of Matsumoto and Tanaka [22] who deal with semilinear problems with growth semigroup and Volterra integral equations techniques.

In the companion paper [9] an integrated semigroup approach has been developed to handle linear equations involving almost sectorial operators. So the goal of this article is to study some properties of the nonlinear semiflow generated by (1.1) by using integrated semigroups. In addition to the existence of a maximal nonlinear semiflow for (1.1) using a suitable notion of mild solutions, we also investigate differentiability property of this semiflow. First we investigate the Frechet differentiability with respect to the state variable and derive stability results. Second we investigate the differentiability of the solutions with respect to time and show that they immediately become smooth, in the sense that the solutions belong to the domain $D(A)$ as soon as the time is positive and mild solutions are somehow classical solutions.

One may note that when $\alpha = \beta = 0$ (that is when the nonlinearity F is defined on $\mathbb{R} \times \overline{D(A)}$ and the initial data belongs to $\overline{D(A)}$), the results obtained in Magal and Ruan in [20] – combined with the results in [9] – apply and allow us to study the abstract Cauchy problem (1.1). But as far as we know the problem with $\alpha > 0$ and\or $\beta > 0$ has not been considered in the literature by using an integrated semigroup approach.

The motivation for using integrated semigroup theory here comes from the fact that it has been successfully used to develop a bifurcation theory for abstract non-densely defined Cauchy problems. The results in [9] can be combined with those in Magal and Ruan [19] to obtain some results on the existence and smoothness of a center manifold. These results can also be combined with the ones in Liu, Magal and Ruan [15] to obtain a Hopf bifurcation theorem, and with the results in Liu, Magal, Ruan and Wu [16] to derive an abstract normal form theory. Here let us emphasis that these earlier results can only be applied in the case $\alpha = \beta = 0$. And, the results presented in this note can also be viewed as a preparation for a center manifold and bifurcation theory for almost sectorial abstract Cauchy problems (with $\alpha > 0$ and\or $\beta > 0$). This point will be investigated in a forthcoming work.

We now discuss a motivating example that enters the framework of this note. To that aim we consider a model introduced by Armstrong, Painter and Sherratt in [4] to describe the motion of cells. This model takes the following form:

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x) - \partial_x \left[u(t,x) L(u(t,.))(x)\right], \ x \in (0,1), \ t > 0 \\ \partial_x u(t,x) - u(t,x) L(u(t,.))(x) = 0 \text{ for } x \in \{0,1\} \text{ and } t > 0. \\ u(0,.) = u_0(.) \in L^p\left((0,1), \mathbb{R}\right), \end{cases} \tag{1.3}$$

for some $p \in (1, +\infty)$ while

$$L(u(t,.))(x) = \eta(x) \int_0^1 g(x-y) h(u(t,y)) \mathrm{d}y,$$

for suitable functions η and g, and where the nonlinear function $h : \mathbb{R} \to \mathbb{R}$ typically reads, for some constant $M > 0$, as

$$h(x) := \begin{cases} x\left(1 - \dfrac{x}{M}\right), & \text{if } x \in [0, M], \\ 0, & \text{else.} \end{cases}$$

In [4] the above problem is posed on the whole space so that the nonlinear and nonlocal boundary conditions are not needed. However when posed on the interval $(0,1)$ the above boundary conditions correspond to no-flux at the boundary and ensure that the total mass (that is the total number of cells in that context) $U(t) = \int_0^1 u(t,x)\mathrm{d}x$ is preserved in time. To see this one may observe that the quantities $[\partial_x u(t,1) - L(u(t,.))(1)u(t,1)]$ and $[\partial_x u(t,0) - L(u(t,.))(0)u(t,0)]$ correspond to the flux at $x = 1$ and $x = 0$ respectively.

Next in order to re-write (1.3) in the framework of this note and make use of integrated semigroup theory, we extend the state space in order to incorporate the boundary conditions into the state variable. We thus define the Banach space

$$X = \mathbb{R}^2 \times L^p(0,1),$$

and we consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(1) \\ \varphi'(0) \\ \varphi'' \end{pmatrix} \text{ with } D(A) = \{0_{\mathbb{R}}\}^2 \times W^{2,p}(0,1).$$

This linear operator A turns out to be non-densely defined since $\overline{D(A)} = \{0_{\mathbb{R}}\}^2 \times L^p(0,1) \neq X$.

Then we defined the nonlinear function $F : D(F) \subset \overline{D(A)} \to X$ as follows

$$F\begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi(1)L(\varphi)(1) \\ -\varphi(0)L(\varphi)(0) \\ -(\varphi L(\varphi))' \end{pmatrix}.$$

One may note that this function is not well defined on $\overline{D(A)}$. Now identifying $u(t,.)$ with $v(t) = \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ u(t,.) \end{pmatrix}$, System (1.3) re-writes as the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + F(v(t)), \text{ for } t > 0, \quad v(0) \in \overline{D(A)}.$$

As it will be seen latter (see Section 5), the linear operator A satisfies Assumption 1.1 with

$$p^* = \frac{2p}{1+p}.$$

This example will be further discussed in Section 6 where we shall investigate a more general multi-dimensional equations. In the above example, the boundary conditions are both nonlinear and nonlocal. Here we refer the readers to the work of Amann in [2] for a theory dealing with quasi-linear parabolic equations (with local nonlinear boundary conditions). As far as we know the case of nonlocal boundary conditions case has been scarcely treated while it naturally arises in the context of population dynamics.

This paper is organized as follows. In Section 2, we recall some preliminary materials on linear equations mostly taken from [9]. In Section 3, we study the existence of the nonlinear and non-autonomous semiflow generated by (1.1).

Then in Section 4 we turn to the linearized equations and prove a local stability result for equilibria . In Section 5 we investigate the smoothness of the solutions and roughly prove that they become smooth as soon as the time is positive. Finally Section 6 deals with an example of application that consists in a generalisation of System (1.3) discussed above.

2 Analytic Integrated Semigroup

In this section we present some materials on linear equations and recall some important results that will be used in the sequel. Let X and Z be two Banach spaces. We denote by $\mathcal{L}(X,Z)$ the space of bounded linear operators from X into Z and by $\mathcal{L}(X)$ the space $\mathcal{L}(X,X)$. Let $A : D(A) \subset X \to X$ be a linear operator. We set

$$X_0 := \overline{D(A)},$$

and we denote by A_0, the part of A in X_0, the linear operator on X_0 defined by

$$A_0 x = Ax,\ \forall x \in D(A_0) := \{y \in D(A) : Ay \in X_0\}.$$

Throughout this section we assume that A satisfies Assumption 1.1 for some $p^* \in [1,\infty)$ and $\omega \in \mathbb{R}$. Note that it is easy to check that for each $\lambda > \omega$ one has

$$D(A_0) = (\lambda I - A)^{-1} X_0 \text{ and } (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} |_{X_0}.$$

From here on, we define $q^* \in (1,+\infty]$ by

$$q^* := \frac{p^*}{p^*-1} \ \Leftrightarrow \frac{1}{q^*} + \frac{1}{p^*} = 1, \tag{2.4}$$

wherein $p^* \geq 1$ is defined in Assumption 1.1.

In order to prepare our semilinear theory, we firstly recall some results for the non-homogeneous Cauchy problems

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), t \geq 0,\ u(0) = x \in \overline{D(A)}. \tag{2.5}$$

To that aim let us recall the following definition.

Definition 2.1 (Integrated solution) *Let $f \in L^1(0,\tau;X)$ be a given function for some given $\tau > 0$. A map $v \in C([0,\tau],X)$ is said to be **an integrated solution** of the Cauchy problem (2.5) on $[0,\tau]$ if the two following conditions are satisfied:*

$$\int_0^t v(s)\mathrm{d}s \in D(A), \forall t \in [0,\tau],$$

and

$$v(t) = x + A\int_0^t v(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s,\ \forall t \in [0,\tau].$$

In order to go further recall that $\omega_0(A_0)$ the growth rate of the semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ is defined by

$$\omega_0(A_0) := \lim_{t\to+\infty} \frac{\ln\left(\|T_{A_0}(t)\|_{\mathcal{L}(X_0)}\right)}{t}.$$

Since $p^* \neq +\infty$, one has $\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \to 0$ as $\lambda \to +\infty$ and by using the Lemma 2.1 in Magal and Ruan [18], we deduce that

$$\overline{D(A)} = \overline{D(A_0)}.$$

Since by assumption $\rho(A) \neq \varnothing$, it is follows that (see Magal and Ruan [19, Lemma 2.1])

$$\rho(A) = \rho(A_0).$$

This in particular yields

$$(\omega_0(A_0), +\infty) \subset \rho(A).$$

Next the integrated semigroup $\{S_A(t)\}_{t\geq 0}$ generated by A is the family of bounded linear operator on X defined by

$$S_A(t) = (\lambda I - A_0)\int_0^t T_{A_0}(s)\mathrm{d}s\,(\lambda I - A)^{-1}, \tag{2.6}$$

for all $\lambda \in (\omega_0(A_0), +\infty)$.

The relationship between the integrated semigroups $\{S_A(t)\}_{t\geq 0}$, and the semigroup, used in paticular by Lunardi in [17], comes from the fact that the map $t \to S_A(t)$ is continuously differentiable from $(0, +\infty)$ into $\mathcal{L}(X)$, and that the family

$$T(t) := \frac{\mathrm{d}S_A(t)}{\mathrm{d}t} = (\lambda I - A_0)\, T_{A_0}(t)\,(\lambda I - A)^{-1}, \text{ for } t > 0, \text{ and } T(0) = I, \tag{2.7}$$

defines a semigroup of bounded linear operators on X. However it has to be noted that when A is not densely defined then the family $\{T(t)\}_{t\geq 0}$ of bounded linear operator on X is not strongly continuous at $t = 0$.

For completeness, we also recall that the analyticity of $t \to S_A(t)$ and $t \to T(t)$, follows from the formula

$$S_A(t) = (\mu I - A_0)\int_0^t T_{A_0}(l)\mathrm{d}l\,(\mu I - A)^{-1}, \text{ and } T(t) = \int_\Gamma e^{\lambda t}(\lambda - A)^{-1}\mathrm{d}\lambda,$$

where $\mu > \omega_0(A_0)$, and Γ is the path $\omega + \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta, |\lambda| = r\}$, oriented counterclockwise for some $r > 0$, $\eta \in \left(\frac{\pi}{2}, \pi\right)$.

In the context of Assumption 1.1, recall also that the fractional powers $(\lambda I - A_0)^{-\alpha}$ are well defined, for any $\lambda > \omega_0(A_0)$, by

$$(\lambda I - A_0)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T_{(A_0 - \lambda I)}(t) \mathrm{d}t, \text{ for } \alpha > 0, \text{ and } (\lambda I - A_0)^0 = I.$$

Now since A is only assumed to be almost sectorial, the fraction powers of $(\lambda I - A)^{-\alpha}$ are not defined for any $\alpha > 0$ but for α large enough. More precisely, we have following result (see [24] or [9, Lemma 3.7]).

Lemma 2.2 *Let Assumption 1.1 be satisfied. The fractional power* $(\lambda I - A)^{-\alpha} \in \mathcal{L}(X)$ *is well defined for each* $\alpha \in \left(\frac{1}{q^*}, +\infty\right)$ *and* $\lambda > \omega_0(A_0)$*. Moreover one has*

$$(\lambda I - A)^{-\alpha}(X) \subset \overline{D(A)},$$

and the following properties are satisfied:

(i) $(\mu I - A_0)^{-1} (\lambda I - A)^{-\alpha} = (\lambda I - A_0)^{-\alpha} (\mu I - A)^{-1}, \forall \mu > \omega_0(A_0)$.

(ii) $(\lambda I - A_0)^{-\alpha} x = (\lambda I - A)^{-\alpha} x, \ \forall x \in \overline{D(A)} = X_0$.

(iii) For each $\alpha \geq 0$, $\beta > \frac{1}{q^*}$,

$$(\lambda I - A_0)^{-\alpha} (\lambda I - A)^{-\beta} = (\lambda I - A)^{-(\alpha+\beta)}.$$

Now observe that since $(\lambda I - A)^{-\alpha}$ and $(\mu I - A)^{-1}$ commute, it follows that $(\lambda I - A)^{-\alpha}$ commutes with $S_A(t)$ and $T_{A_0}(t)$. This in particular yields

$$S_A(t) = (\lambda I - A_0)^{\alpha} \int_0^t T_{A_0}(s) ds \, (\lambda I - A)^{-\alpha}$$

for any $\alpha \in \left(\frac{1}{q^*}, +\infty\right)$ and for each $\lambda > \omega_0(A_0)$.

Let us also observe that for $\alpha \in \left(\frac{1}{q^*}, 1\right]$,

$$(\lambda I - A)^{-1} = (\lambda I - A_0)^{-(1-\alpha)} (\lambda I - A)^{-\alpha}.$$

Hence, due to (2.7), for each $t > 0$ we get

$$\begin{aligned} \frac{\mathrm{d}S_A(t)}{\mathrm{d}t} &= (\lambda I - A_0) T_{A_0}(t) (\lambda I - A)^{-1} \\ &= (\lambda I - A_0) T_{A_0}(t) (\lambda I - A_0)^{-(1-\alpha)} (\lambda I - A)^{-\alpha} \end{aligned}$$

and, since $T_{A_0}(t)$ and $(\lambda I - A_0)^{-(1-\alpha)}$ commute, we also obtain the following expression for the derivative of S_A:

$$\frac{\mathrm{d}S_A(t)}{\mathrm{d}t} = (\lambda I - A_0)^{\alpha} T_{A_0}(t) (\lambda I - A)^{-\alpha}, \forall t > 0, \forall \alpha \in \left(\frac{1}{q^*}, 1\right]. \tag{2.8}$$

Now the main tool to deal with integrated solutions for the Cauchy problem relies on the constant variation formula. Hence before coming back to the non-homogeneous Problem (2.5) let us recall the following result.

Theorem 2.3 *Let Assumption 1.1 be satisfied. Let $f \in L^p(0,\tau;X)$ with $p > p^*$. Then the map $t \to (S_A * f)(t) := \int_0^t S_A(t-s)f(s)\mathrm{d}s$ is continuously differentiable, $(S_A * f)(t) \in D(A)$, $\forall t \in [0,\tau]$, and if we denote by*

$$(S_A \diamond f)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S_A(t-s)f(s)\mathrm{d}s, \tag{2.9}$$

then

$$(S_A \diamond f)(t) = A\int_0^t (S_A \diamond f)(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s,\ \forall t \in [0,\tau].$$

Moreover for each $\beta \in \left(\frac{1}{q^*}, \frac{1}{q}\right)$ (with $\frac{1}{q} + \frac{1}{p} = 1$), each $\lambda > \omega_0(A_0)$, and each $t \in [0,\tau]$, the following holds true

$$(S_A \diamond f)(t) = \int_0^t (\lambda I - A_0)^{\beta} T_{A_0}(t-s)(\lambda I - A)^{-\beta} f(s)\mathrm{d}s, \tag{2.10}$$

and, the following estimate also holds true

$$\|(S_A \diamond f)(t)\| \le M_\beta \left\|(\lambda I - A)^{-\beta}\right\|_{\mathcal{L}(X)} \int_0^t (t-s)^{-\beta} e^{\omega_A(t-s)} \|f(s)\|\mathrm{d}s, \tag{2.11}$$

wherein M_β denotes some positive constant, and $\omega_A > \omega_0(A_0)$.

By using integrated semigroups, or formula (2.10), we derive the extended variation of constant formula:

$$(S_A \diamond f)(t) = T_{A_0}(t-s)(S_A \diamond f)(s) + (S_A \diamond f(s+.))(t-s), \forall t \ge s \ge 0. \tag{2.12}$$

By using the above theorem, and the usual uniqueness result of Thieme [27, Theorem 3.7], one derive the following result.

Corollary 2.4 *Let Assumption 1.1 be satisfied. Let $p \in (p^*, +\infty)$ be given. Then for each $f \in L^p(0,\tau;X)$ and for each $x \in X_0$ the Cauchy problem (2.5) has a unique integrated solution $u \in C([0,\tau], X_0)$ that is given by*

$$u(t) := T_{A_0}(t)x + (S_A \diamond f)(t),\ \forall t \in [0,\tau]. \tag{2.13}$$

Moreover, as an immediate consequence of Theorem 2.3 we have the following regularity lemma.

Lemma 2.5 *Let $\beta > 0$, $\alpha > \frac{1}{q^*}$, and $p \in (p^*, +\infty)$ be three real numbers. Let $f \in L^p(0, \tau; X)$ be given and assume that*

$$\alpha + \beta < \frac{1}{q} := 1 - \frac{1}{p}.$$

Then for each $\lambda > \omega_0(A_0)$ one has

$$(S_A \diamond f)(t) \in D\left((\lambda I - A_0)^\beta\right), \forall t \in [0, \tau].$$

The map $t \to (\lambda I - A_0)^\beta (S_A \diamond f)(t)$ is continuous from $[0, \tau]$ into X_0 and the following estimate holds true for each $t \in [0, \tau]$,

$$\left\|(\lambda I - A_0)^\beta (S_A \diamond f)(t)\right\| \leq M_\beta \left\|(\lambda I - A)^{-\alpha}\right\|_{\mathcal{L}(X)} \int_0^t (t-s)^{-(\beta+\alpha)} e^{\omega_A (t-s)} \|f(s)\| \, \mathrm{d}s,$$

wherein M_β is some positive constant, and $\omega_A > \omega_0(A_0)$.

Proof. By using (2.10) we have

$$\begin{aligned} (S_A \diamond f)(t) &= \int_0^t (\lambda I - A_0)^{-\beta} (\lambda I - A_0)^{\alpha+\beta} T_{A_0}(t-s) (\lambda I - A)^{-\alpha} f(s) \mathrm{d}s \\ &= (\lambda I - A_0)^{-\beta} \int_0^t (\lambda I - A_0)^{\alpha+\beta} T_{A_0}(t-s) (\lambda I - A)^{-\alpha} f(s) \mathrm{d}s. \end{aligned}$$

Note that the last integral is well defined since $q(\alpha + \beta) < 1$ and the result follows. ■

We conclude this section by recalling some results about linear perturbation of A. To that aim we shall make use of the following assumption.

Assumption 2.6 *Let $B : D(B) \subset X_0 \to Y$ be a linear operator from $D(B)$ into a Banach space $Y \subset X$. We assume that there exists $\alpha \in (0, 1)$ such that the operator B is $(\lambda I - A_0)^\alpha -$ bounded for some $\lambda > \omega_0(A_0)$ in the sense that $B(\lambda I - A_0)^{-\alpha}$ is a bounded linear operator.*

Using the above assumption we obtain various perturbation results depending on the choice of the space Y.

When $Y = X$ the following result holds true.

Theorem 2.7 *[9, Theorem 4.2] Let Assumptions 1.1 and 2.6 be satisfied with $Y = X$. We assume in addition that*

$$\alpha < \frac{1}{p^*}.$$

Then $A + B : D(A) \cap D(B) \subset X \to X$ satisfies the Assumption 1.1.

If we now assume that the range of B is included in $\overline{D(A)}$ one obtains the following result.

Theorem 2.8 *[9, Theorem 4.6] Let Assumptions 1.1 and 2.6 be satisfied and assume that* $Y = X_0$. *Then* $A+B : D(A)\cap D(B) \subset X \to X$ *satisfies Assumption 1.1.*

Moreover we finally recall that the semigroup the generated by the part of $(A+B)_0$ in $\overline{D(A)}$ is the unique solution of a Cauchy problem coupled with a suitable integral equation.

Theorem 2.9 *[9, Theorem 4.8] Let Assumptions 1.1 and 2.6 be satisfied and assume that* $Y = X$. *Assume in addition that* $\omega_0(A_0) < 0$. *If there exists* $\hat{p} \in [1, +\infty)$ *such that*

$$p^* < \hat{p} < \frac{1}{\alpha}. \tag{2.14}$$

Then, $\left\{T_{(A+B)_0}(t)\right\}_{t\geq 0}$ *the* C_0*-semigroup generated by* $(A+B)_0$ *is the unique solution of the fixed point problem*

$$T_{(A+B)_0}(t) = T_{A_0}(t) + (S_A \diamond V)(t), \tag{2.15}$$

where $V(.)x \in L^{\widehat{p}}_{\omega^*}(0, +\infty; X)$ *(for some* $\omega^* > 0$ *large enough) is the solution of*

$$V(t)x = BT_{A_0}(t)x + B\left(S_A \diamond V(.)x\right)(t), \text{ for } t > 0. \tag{2.16}$$

Herein $L^{\widehat{p}}_{\omega^*}(0, +\infty; X)$ *denotes the space of the maps* $f : (0, +\infty) \to X$ *Bochner's measurable and such that*

$$\|f\|_{L^{\widehat{p}}_{\omega^*}} := \left(\int_0^{+\infty} \left\|e^{-\omega^* t} f(t)\right\|^{\widehat{p}} \mathrm{d}t\right)^{1/\widehat{p}} < +\infty.$$

3 Semilinear Cauchy problems

Throughout this section $A : D(A) \subset X \to X$ denotes a linear operator satisfying Assumption 1.1. From here on we fix

$$\mu > \omega_0(A_0).$$

For each $\alpha \in [0, 1)$, the linear operator $(\mu I - A_0)^\alpha : D((\mu I - A_0)^\alpha) \to X_0$ is closed (see Pazy [23]). Moreover we have for each $x \in D((\mu I - A_0)^\alpha)$,

$$\|(\mu I - A_0)^\alpha x\| \leq \|x\| + \|(\mu I - A_0)^\alpha x\| \leq \left[\|(\mu I - A_0)^{-\alpha}\|_{\mathcal{L}(X_0)} + 1\right] \|(\mu I - A_0)^\alpha x\|.$$

It follows that

$$X_0^\alpha := D((\mu I - A_0)^\alpha)) \subset X_0$$

is Banach space endowed with the norm $\| \|_\alpha$ defined by

$$\|x\|_\alpha := \|(\mu I - A_0)^\alpha x\|, \ \forall x \in X_0^\alpha. \tag{3.17}$$

In the case $\alpha = 0$, we have

$$(\mu I - A_0)^\alpha = (\mu I - A_0)^{-\alpha} = I_{X_0}$$

so that $X_0^0 = X_0$ and $\|.\|_0 = \|.\|$ on X_0.

In this section, we extend some results of Henry [14] and Lunardi [17] about the exitence of a maximal semiflow in the context of almost sectorial operators. More specifically we consider the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + F(t, v(t)), \text{ for } t > s \geq 0, \;\; v(s) = x \in X_0^\beta, \tag{3.18}$$

where F maps $[0, +\infty) \times X_0^\alpha$ into X for given parameters α and β.

Our goal is to prove the existence of a maximal non-autonomous semiflow generated by (3.18) on the space Banach X_0^β. In order to do so we shall make use of the following assumption.

Assumption 3.1 *Let $\alpha \in (0, 1)$, $\beta \in [0, 1)$ and $p \in [1, +\infty)$ be given such that*

$$0 \leq \beta \leq \alpha < \beta + \frac{1}{p} < \frac{1}{p^*}. \tag{3.19}$$

We assume that there exists a non-decreasing (with respect to both arguments) function $K : [0, +\infty)^2 \to [0, +\infty)$ such that $F : [0, \infty) \times X_0^\alpha \to X$ satisfies the following conditions:

(i) For each $\tau > 0$ and each $M > 0$ one has

$$\|F(t, x) - F(t, y)\|_X \leq K(\tau, M) \left[(\|x\|_\alpha + \|y\|_\alpha + 1) \|x - y\|_\beta + \|x - y\|_\alpha \right],$$

whenever $t \in [0, \tau]$, $x, y \in X_0^\alpha$, and $\max(\|x\|_\beta, \|y\|_\beta) \leq M$.

(ii) For each $x \in X_0^\alpha$, the map $t \to F(t, x)$ belongs to $L^p_{\text{loc}}([0, +\infty); X)$.

Remark 3.2 *Using (3.19), first note that, since $\beta \leq \alpha$, we have*

$$D(A_0) \subset X_0^\alpha \subset X_0^\beta \subset X_0 = \overline{D(A)} \subset X.$$

Moreover for each $\alpha \in \left(0, 1 - \frac{1}{q^*}\right)$ we have

$$D(A) = (\mu I - A)^{-1} X = (\mu I - A_0)^{-\alpha} (\mu I - A)^{-(1-\alpha)} X,$$

thus

$$D(A_0) \subset D(A) \subset X_0^\alpha \subset X_0, \forall \alpha \in \left(0, 1 - \frac{1}{q^*}\right).$$

One may also observe that for each $x \in X_0^\alpha$,

$$\|x\|_\beta = \|(\mu I - A_0)^\beta x\| = \|(\mu I - A_0)^{-(\alpha-\beta)} (\mu I - A_0)^\alpha x\|$$

thus the embedding from X_0^α into X_0^β is continuous, and the following comparison estimate holds true

$$\|x\|_\beta \leq \|(\mu I - A_0)^{-(\alpha-\beta)}\|_{\mathcal{L}(X_0)} \|x\|_\alpha, \forall x \in X_0^\alpha.$$

Example 3.3 *Let* $(Y, \|.\|_Y)$ *and* $(Z, \|.\|_Z)$ *be two Banach spaces. As a prototype example for function* F*, we consider* $F : X_0^\alpha \to X$ *the map defined by*

$$F(x) = L(\psi_1(x), Bx) + \psi_2(x)$$

where $L : Y \times Z \to X$ *is a bounded bilinear map,* $B : X_0^\alpha \to Z$ *is a bounded linear operator and, the maps* $\psi_1 : X_0^\beta \to Y$ *and* $\psi_2 : X_0^\beta \to X$ *are Lipschitz continuous on the bounded sets of* X_0^β*. This function satisfies the above set of assumptions (and more precisely* (i) *since it is independent of time). To see this observe that for each* $M > 0$*, and each* $x, y \in X_\alpha$ *with* $\max(\|x\|_\beta, \|y\|_\beta) \leq M$*, one has*

$$\begin{aligned}\|F(x) - F(y)\| &\leq \|L(\psi_1(x), Bx) - L(\psi_1(y), By)\| + \|\psi_2(x) - \psi_2(y)\| \\ &\leq \|L(\psi_1(x) - \psi_1(y), By)\| + \|L(\psi_1(x), B(x-y))\| + \|\psi_2(x) - \psi_2(y)\| \\ &\leq \|L\|_{\mathcal{L}(Y\times Z,X)} \|\psi_1\|_{\text{Lip},B_{X_0^\beta}(0,M)} \|x-y\|_\beta \|By\| \\ &\quad + \|L\|_{\mathcal{L}(Y\times Z,X)} \left[\|\psi_1\|_{\text{Lip},B_{X_0^\beta}(0,M)} \|x\|_\beta + \|\psi_1(0)\|\right] \|B(x-y)\| \\ &\quad + \|\psi_2\|_{\text{Lip},B_{X_0^\beta}(0,M)} \|x-y\|_\beta .\end{aligned}$$

Thus for each $x, y \in X_0^\alpha$ *with* $\max(\|x\|_\beta, \|y\|_\beta) \leq M$*, this yields*

$$\|F(x) - F(y)\| \leq K(M)\left[(\|x\|_\alpha + \|y\|_\alpha)\|x-y\|_\beta + \|x-y\|_\alpha + \|x-y\|_\beta\right],$$

with

$$K(M) = \left[\|L\| \|B\| \left(\|\psi_1\|_{\text{Lip},B_{X_0^\beta}(0,M)} + \|\psi_1(0)\|\right)(1+M) + \|\psi_2\|_{\text{Lip},B_{X_0^\beta}(0,M)}\right].$$

Before going further let us observe that the inequality in (3.19) implies that

$$p > p^* \Leftrightarrow \frac{1}{q^*} < \frac{1}{q}, \text{ with } q := \frac{p}{p-1}.$$

Hence Theorem 2.3 applies and ensures $(S_A \diamond f)(t)$ is well defined for $t \in [0, \tau]$ whenever $f \in L^p(0, \tau; X)$.

Now we turn to the study of the Cauchy problem (3.18). To handle this problem, for each $\tau > 0$ we consider the Banach space Z_τ defined by

$$Z_\tau := C\left([0,\tau], X_0^\beta\right) \cap L^p(0, \tau; X_0^\alpha), \tag{3.20}$$

endowed with the usual norm

$$\|u\|_{Z_\tau} = \sup_{t\in[0,\tau]} \|u(t)\|_\beta + \|u\|_{L^p\left(0,\tau;X_0^\alpha\right)}, \quad \forall u \in Z_\tau.$$

With this notation, the next two lemmas provide crucial estimates to handle (3.18).

Lemma 3.4 *Let Assumptions 1.1 and 3.1 be satisfied. Let $\tau > 0$, $M > 0$, and $u \in Z_\tau$ be given. Then, for any $s \geq 0$, the map $t \to F(s+t, u(t))$ belongs to $L^p(0, \tau; X)$, and satisfies for any $s \geq 0$*

$$\begin{aligned}\|F(s+.,u(.))\|_{L^p(0,\tau;X)} \leq& (M+1)K(s+\tau, M)\|u\|_{L^p(0,\tau;X_0^\alpha)} \\ &+ \tau^{\frac{1}{p}} K(s+\tau, M)M + \|F(s+.,0)\|_{L^p(0,\tau;X)},\end{aligned}$$

whenever $\sup_{t\in[0,\tau]} \|u(t)\|_\beta \leq M$.

Proof. The proof of this result is split into two steps. We first prove that $F(s+.,u(.))$ is Bochner mesurable and then we derive the estimate stated above.

First step: Bochner's Measurability. For notational simplicity here we assume that $s = 0$. Let $u \in Z_\tau$ be fixed. Let $\rho_n : \mathbb{R} \to \mathbb{R}$ for $n \geq 0$ be a sequence of mollifier. We define for each $n \geq 0$,

$$u^n(t) := \int_0^\tau \rho_n(t-s)u(s)\mathrm{d}s,$$

where the last integral takes place in X_0^α. Then, for a subsequence denoted here with the same indexes, we have

$$u^n(t) \to u(t), \text{ in } X_0^\alpha, \ \forall t \in [0,\tau] \setminus N, \tag{3.21}$$

where N is a Lebesgue's negligible set. To prove this one may for example proceed as in the proof of Proposition 1.2.2 b) in Arendt et al. [3]. Moreover since $u \in C([0,\tau], X_0^\beta)$, and using the continuous embedding $X_0^\alpha \hookrightarrow X_0^\beta$, we deduce that there exists $M \geq 0$ such that

$$\|u\|_{C([0,\tau],X_0^\beta)} \leq M, \text{ and } \|u^n\|_{C([0,\tau],X_0^\beta)} \leq M, \forall n \geq 0, \tag{3.22}$$

and

$$u^n(t) \to u(t), \text{ in } X_0^\beta, \ \forall t \in [0,\tau] \setminus N.$$

By using (3.21), (3.22) and Assumption 3.1-(i), we deduce that

$$F(t, u^n(t)) \to F(t, u(t)), \text{ in } X, \ \forall t \in [0,\tau] \setminus N.$$

Hence using this approximation argument, it remains to prove that $t \to F(t, u(t))$ is Bochner's measurable, whenever $u \in C([0,\tau], X_0^\beta) \cap C([0,\tau], X_0^\alpha)$. To that aim we set for each $n \geq 1$, and each $k = 1, ..., n$,

$$v^n(t) := u\left(\frac{k\tau}{n}\right), \text{ if } t \in \left[\frac{(k-1)\tau}{n}, \frac{k\tau}{n}\right).$$

Since u is uniformly continuous on $[0,\tau]$, it follows that

$$v^n(t) \to u(t), \text{ in } X_0^\alpha, \ \forall t \in [0,\tau].$$

Now due to Assumption 3.1-(ii), and the fact that v^n is a step function, and we deduce that $t \to F(t, v^n(t))$ is Bochner's measurable. By using Assumption 3.1 (i), and the fact that

$$\|u\|_{C([0,\tau],X_0^\beta)} \leq M,$$

for some constant $M > 0$, it follows that

$$F(t, v^n(t)) \to F(t, u(t)), \text{ in } X, \ \forall t \in [0, \tau] \, .$$

Hence the Bochner's measurability of $t \to F(t, u(t))$ follows (see Arendt et al. [3, Corollary 1.1.2-d)]).

Second step: Integrability estimate. Let $s \geq 0$ be given. Let $u \in Z_\tau$ be given. Define M by

$$M = \sup_{t \in [0,\tau]} \|u\,(t)\|_\beta \, .$$

Then using Assumption 3.1 we get for almost every $t \in [0, \tau]$:

$$\begin{aligned} \|F(s+t, u(t))\| &\leq \|F(s+t, u(t)) - F(s+t, 0)\| + \|F(s+t, 0)\| \\ &\leq K\,(s+\tau, M)\,[\|u(t)\|_\alpha \|u(t)\|_\beta + \|u(t)\|_\alpha + \|u(t)\|_\beta] + \|F(s+t, 0)\| \\ &\leq (M+1)K\,(s+\tau, M)\,\|u(t)\|_\alpha + K\,(s+\tau, M)\,M + \|F(s+t, 0)\| \, . \end{aligned}$$

Using the first step, this yields

$$\begin{aligned} \left(\int_0^\tau \|F(s+t, u(t))\|^p \, \mathrm{d}t \right)^{1/p} \leq & (M+1)K\,(s+\tau, M)\,\|u\|_{L^p(0,\tau;X_0^\alpha)} \\ & + \tau^{\frac{1}{p}} K\,(s+\tau, M)\,M + \|F(s+.,0)\|_{L^p(0,\tau;X)} \, , \end{aligned}$$

and the proof is completed. ■

The second main ingredient to deal with the Cauchy problem (3.18) is the following lemma.

Lemma 3.5 *Let Assumptions 1.1 and 3.1 be satisfied. Then there exist two continuous and non decreasing maps* $\delta : [0, \infty) \to [0, +\infty)$ *with* $\lim_{t \to 0^+} \delta(t) = 0$, *and* $m : [0, \infty) \to [0, +\infty)$ *such that:*

(i) For each $\tau > 0$ *and for any* $f \in L^p\,(0, \tau; X)$ *the map* $t \to (S_A * f)\,(t)$ *is of the class* C^1 *from* $[0, \tau]$ *into* X_0^β *and from* $(0, \tau]$ *into* X_0^α. *Moreover* $(S_A \diamond f) \in Z_\tau$ *and*

$$\|(S_A \diamond f)\|_{Z_\tau} \leq \delta(\tau) \|f\|_{L^p(0,\tau;X)}. \tag{3.23}$$

(ii) For each $\tau > 0$ *and for any* $x \in X_0^\beta$ *the map* $t \to T_{A_0}(t)x$ *belongs to* Z_τ *and satisfies the following estimates:*

$$\|T_{A_0}(.)x\|_{Z_\tau} \leq m(\tau) \|x\|_\beta. \tag{3.24}$$

Proof. One may first observe that (3.19) implies that

$$\beta + \frac{1}{p} < \frac{1}{p^*} \Leftrightarrow \frac{1}{q^*} + \beta < \frac{1}{q}.$$

Let γ be given such that

$$\frac{1}{q^*} < \gamma, \text{ and } \gamma + \beta < \frac{1}{q}.$$

Then we have

$$\gamma + \alpha < \gamma + \beta + \frac{1}{p} < 1.$$

Since (see (2.8))

$$\frac{\mathrm{d}S_A(t)}{\mathrm{d}t} = (\mu I - A_0)^{\gamma} T_{A_0}(t) (\mu I - A)^{-\gamma},$$

we have, for each $\sigma \in \{\alpha, \beta\}$,

$$(\mu I - A_0)^{\sigma} \frac{\mathrm{d}S_A(t)}{\mathrm{d}t} = (\mu I - A_0)^{\sigma+\gamma} T_{A_0}(t) (\mu I - A)^{-\gamma}.$$

Next recall that there exist constants $M > 0$ and $\omega > \omega_0(A_0)$ such that

$$\left\| \frac{\mathrm{d}S_A(t)}{\mathrm{d}t} \right\|_{\mathcal{L}(X, X_0^{\beta})} \leq M t^{-(\beta+\gamma)} e^{\omega t}, \quad \left\| \frac{\mathrm{d}S_A(t)}{\mathrm{d}t} \right\|_{\mathcal{L}(X, X_0^{\alpha})} \leq M t^{-(\alpha+\gamma)} e^{\omega t}, \quad \forall t > 0.$$

Now let $f \in L^p(0, \tau; X)$ be given. Then one has

$$(S_A \diamond f)(t) = \int_0^t S_A'(s) f(t-s) \mathrm{d}s,$$

so that

$$\|(S_A \diamond f)\|_{C([0,\tau], X_0^{\beta})} \leq \left\| \frac{\mathrm{d}S_A(.)}{\mathrm{d}t} \right\|_{L^q(0,\tau;\mathcal{L}(X, X_0^{\beta}))} \|f\|_{L^p(0,\tau;X)},$$

and

$$\|(S_A \diamond f)\|_{L^p([0,\tau], X_0^{\alpha})} \leq \left\| \frac{\mathrm{d}S_A(.)}{\mathrm{d}t} \right\|_{L^1(0,\tau;\mathcal{L}(X, X_0^{\alpha}))} \|f\|_{L^p(0,\tau;X)}.$$

Hence this yields

$$\|(S_A \diamond f)\|_{Z_\tau} \leq \delta(\tau) \|f\|_{L^p(0,\tau;X)},$$

wherein the function δ is defined by

$$\delta(\tau) := M \left(\int_0^{\tau} t^{-q(\gamma+\beta)} e^{q\omega t} \mathrm{d}t \right)^{\frac{1}{q}} + M \left(\int_0^{\tau} t^{-(\alpha+\gamma)} e^{\omega t} \mathrm{d}t \right).$$

But since $q(\gamma + \beta) < 1$ and $(\alpha + \gamma) < 1$ this proves (i) .

In order to prove (ii), recall that there exist constants $M>0$ and $\omega>\omega_0(A_0)$ such that

$$\|T_{A_0}(t)\|_{\mathcal{L}\left(X_0^\beta\right)}\le Me^{\omega t},\ \ \forall t\ge 0\ \ \|T_{A_0}(t)\|_{\mathcal{L}\left(X_0^\beta,X_0^\alpha\right)}\le Mt^{(\beta-\alpha)}e^{\omega t},\ \ \forall t>0.$$

Now since $p\,(\beta-\alpha)>-1$, the result follows defining the function m by

$$m(\tau)=M\sup_{0\le s\le\tau}e^{\omega s}+M\left(\int_0^\tau t^{p(\beta-\alpha)}e^{p\omega t}\mathrm{d}t\right)^{1/p}.$$

■

Motivated by the above lemmas, namely Lemma 3.4 and 3.5, one can state the following definition for the mild solutions of Problem (3.18).

Definition 3.6 *Let Assumptions 1.1 and 3.1 be satisfied. Let $x\in X_0^\beta$, $s\ge 0$, and $\tau>0$ be given. Then a map $u:[s,\tau+s]\to X_0^\beta$ is said to be* ***a mild solution of*** (3.18) ***on*** $[s,\tau+s]$ *if the two following conditions are satisfied:*

(a) $u\in C([s,\tau+s];X_0^\beta)\cap L^p(s,\tau+s;X_0^\alpha)$;

(b) the function u satisfies

$$u(t)=T_{A_0}(t-s)x+\left(S_A\diamond F\left(.+s,u(.+s)\right)\right)(t-s),\ \ \forall t\in[s,\tau+s],\tag{3.25}$$

or equivalently

$$\int_s^t u(l)\mathrm{d}l\in D(A),\ \ \forall t\in[s,\tau+s],$$
and (3.26)
$$u(t)=x+A\int_s^t u(l)\mathrm{d}l+\int_s^t F\left(l,u(l)\right)\mathrm{d}l,\ \ \forall t\in[s,\tau+s].$$

In order to deal with (3.18) we also recall the notion of maximal semiflow.

Definition 3.7 *Consider two maps $\chi:[0,\infty]\times X_0^\beta\to(0,+\infty]$ and $U:D_\chi\to X_0^\beta$, where*

$$D_\chi:=\left\{(t,s,x)\in[0,+\infty)^2\times X_0^\beta:s\le t<s+\chi\left(s,x\right)\right\}.$$

We say that U (and more precisely (U,χ)) is ***a maximal non-autonomous semiflow on*** X_0^β *if U and χ satisfy the following properties:*

(i) $\chi\left(r,U(r,s)x\right)+r=\chi\left(s,x\right)+s,\ \forall s\ge 0,\forall x\in X_\beta,\ \forall r\in\left[s,s+\chi\left(s,x\right)\right).$

(ii) $U(s,s)x=x,\ \forall s\ge 0,\ \forall x\in X_0^\beta.$

(iii) $U(t,r)U(r,s)x=U(t,s)x,\ \forall s\ge 0,\forall x\in X_0^\beta,\ \forall t,r\in\left[s,s+\chi\left(s,x\right)\right)$ *with* $t\ge r$.

(iv) *If $\chi(s,x) < +\infty$, then*

$$\lim_{t \to (s+\chi(s,x))^-} \|U(t,s)x\|_\beta = +\infty.$$

Next set

$$D := \left\{ (t,s,x) \in [0,+\infty)^2 \times X_\beta : t \geq s \right\}.$$

In addition in order to state our main result we shall need some continuity property for the function $F = F(t,u)$ with respect to t that reads as follows.

Assumption 3.8 (Continuity) *For each $\tau > 0$ and for $s \geq 0$ one has*

$$\lim_{\sigma \to s} \|F(. + \sigma, v(.)) - F(. + s, v(.))\|_{L^p(0,\tau;X)} = 0, \ \forall v \in Z_\tau.$$

Using all the above definitions, we shall prove the following result for Problem (3.18).

Theorem 3.9 *Let Assumptions 1.1 and 3.1 be satisfied. Then there exists a maximal non-autonomous semiflow (U,χ), (with $\chi : [0,+\infty) \times X_0^\beta \to (0,+\infty]$ and $U : D_\chi \to X_0^\beta$) such that for each $x \in X_0$ and each $s \geq 0$, $U(.,s)x \in C\left([s, s+\chi(s,x)), X_0^\beta\right) \cap L^p_{\text{loc}}([s, s+\chi(s,x)), X_0^\alpha)$ is the unique maximal solution of* (3.18) *(or equivalently the unique maximal solution of* (3.25)*). Moreover if Assumption 3.8 is furthermore satisfied then D_χ is an open set in D and the map $(t,s,x) \to U(t,s)x$ is continuous from D_χ into X_0^β.*

The rest of this section is devoted to the proof of the above theorem. We shall first prove the uniqueness of the mild solutions.Then we prove the local existence of the solution by using a suitable contraction fixed point argument. Finally we shall derive some properties of the semiflow and the proof will be completed by showing the continuity of the semiflow with respect to (t,s,x) in D_χ under the additional Assumption 3.8. Here we closely follow some of the arguments presented by Magal and Ruan in [20] (see also the references therein).

Lemma 3.10 ***(Uniqueness)*** *Let Assumptions 1.1 and 3.1 be satisfied. Then for each $x \in X_0^\beta$, each $s \geq 0$, and each $\tau > 0$, Problem* (3.18) *has at most one mild solution $u \in C\left([s, \tau+s], X_0^\beta\right) \cap L^p([s,\tau+s], X_0^\alpha)$.*

Proof. Assume that (3.18) has two mild solutions on $[s, \tau+s]$, denoted by $u_1, u_2 \in C([s,\tau+s], X_\beta) \cap L^p([s,\tau+s], X_\alpha)$, such that $u_1(s) = u_2(s) = x$. Then let us consider the number $t_0 \geq s$ defined by

$$t_0 = \sup\{t \in [s,\tau+s] : u_1(l) = u_2(l), \forall l \in [s,t]\}.$$

In order to prove our uniqueness result we argue by contradiction by assuming that

$$t_0 < \tau + s. \tag{3.27}$$

Then let us recall that we have, for any $i=1,2$,

$$u_i(t)=T_{A_0}(t-s)x+\left(S_A\diamond F\left(.+s,u_i(.+s)\right)\right)(t-s),\ \ \forall t\in[s,\tau+s],$$

or equivalently

$$\begin{aligned} u_i(t) &= x+A\int_s^t u_i(l)\mathrm{d}l+\int_s^t F\left(l,u_i(l)\right)\mathrm{d}l,\\ &= u_i(t_0)+A\int_{t_0}^t u_i(l)\mathrm{d}l+\int_{t_0}^t F\left(l,u_i(l)\right)\mathrm{d}l. \end{aligned}$$

Due to the definition of t_0 we also get

$$u_i(t)=T_{A_0}(t-t_0)u_i(t_0)+\left(S_A\diamond F\left(.+t_0,u_i(.+t_0)\right)\right)(t-t_0),\ \ \forall t\in[t_0,\tau+s],\forall i=1,2,$$

so that, since $u_1(t_0)=u_2(t_0)$, we infer that for each $t\in[t_0,\tau+s]$,

$$u_1(t)-u_2(t)=\left(S_A\diamond\left[F\left(.+t_0,u_1\left(.+t_0\right)\right)-F\left(.+t_0,u_2(.+t_0)\right)\right]\right)(t-t_0).$$

Let us consider $t=t_0+\varepsilon$ for some $\varepsilon>0$ small enough such that $t\in(t_0,\tau)$ and

$$\delta(\varepsilon)K\left(\tau+s,M\right)\left(2M+1+\tau^{\frac{1}{p}}\right)<1, \tag{3.28}$$

wherein we have set $M:=\max_{i=1,2}\|u_i(s+.)\|_{Z_\tau}$ while δ is defined in Lemma 3.5.

Next Lemma 3.5 applies and yields, using Assumption 3.1, the following estimate

$$\begin{aligned} &\|(u_1-u_2)(t_0+.)\|_{Z_{t-t_0}}\\ &\le\delta(t-t_0)\|\left(F\left(.+t_0,u_1(.+t_0)\right)-F\left(.+t_0,u_2(.+t_0)\right)\right\|_{L^p(0,t-t_0;X)}\\ &\le\delta(\varepsilon)K\left(\tau+s,M\right)\left(2M+1+\tau^{\frac{1}{p}}\right)\|(u_1-u_2)(t_0+.)\|_{Z_{t-t_0}}. \end{aligned}$$

Due to (3.28) one concludes that $u_1(l)=u_2(l)$ for any $l\in[s,t_0+\varepsilon]$, a contradiction with the definition of t_0. Thus $t_0=\tau+s$ and this completes the proof of the lemma. ■

We now turn to the proof of the existence of a local semiflow as stated in our next lemma.

Lemma 3.11 ***(Local Existence)*** *Let Assumptions 1.1 and 3.1 be satisfied. Then for each $\xi>0$ and $\sigma>0$ there exist some constant $M(\xi,\sigma)>0$ and $\tau(\xi,\sigma)>0$ such that for each $x\in X_0^\beta$ with $\|x\|_\beta\le\xi$ and each $s\in[0,\sigma]$, Problem* (3.18) *has a unique mild solution $U(.,s)x\in C\left([s,s+\tau(\xi,\sigma)],X_0^\beta\right)\cap L^p(s,s+\tau(\xi,\sigma);X_0^\alpha)$ that satisfies*

$$\|U(s+.,s)x\|_{Z_{\tau(\xi,\sigma)}}\le M(\xi,\sigma).$$

Proof. Let $\xi > 0$ and $\sigma > 0$ be given and fixed. Consider two constants $\tau(\xi, \sigma) = \tau > 0$ and $M(\xi, \sigma) = M > 0$ such that for all $s \in [0, \sigma]$

$$\xi m(\tau) + \delta(\tau) \left[\left(M + 1 + \tau^{\frac{1}{p}} \right) MK(s + \tau, M) + \|F(.,0)\|_{L^p(s,s+\tau;X)} \right] \leq M, \tag{3.29}$$

and

$$\delta(\tau)(2M + 1 + \tau^{\frac{1}{p}})K(\tau + s, M) < 1. \tag{3.30}$$

Next consider the Banach space $Z = C\left([s, s+\tau]; X_0^\beta\right) \cap L^p(s, s+\tau; X_0^\alpha)$ endowed with the norm

$$\|u\|_Z = \sup_{t \in [s,s+\tau]} \|u(t)\|_\beta + \|u\|_{L^p\left(s,s+\tau;X_0^\alpha\right)}, \quad \forall u \in Z,$$

and consider the set

$$C = \{u \in Z : \ \|u\|_Z \leq M\}.$$

Next let $x \in X_\beta$ be given such that $\|x\|_\beta \leq \xi$ and consider the map $\Psi : C \to Z$ defined for $u \in C$ by

$$\Psi(u)(t) = T_{A_0}(t-s)x + (S_A \diamond F(.+s, u(s+.)) (t-s), \ t \in [s, s+\tau]. \tag{3.31}$$

Let us first check that $\Psi(C) \subset C$. Indeed Lemma 3.5 yields, for any $u \in C$,

$$\|\Psi(u)\|_Z \leq \|x\|_\beta m(\tau) + \delta(\tau)\|F(.+s, u(s+.))\|_{L^p(0,\tau;X)}.$$

Next, due to Lemma 3.4, one obtains

$$\|\Psi(u)\|_Z \leq \xi m(\tau) + \delta(\tau) \left[\left(M + 1 + \tau^{\frac{1}{p}} \right) MK(s+\tau, M) + \|F(.,0)\|_{L^p(s,s+\tau;X)} \right].$$

Thus (3.29) yields $\Psi(C) \subset C$.

Next let $u \in C$ and $v \in C$ be given. Then we infer from Lemma 3.5 and Assumption 3.1 that

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_Z &\leq \delta(\tau)\|F(.+s, u(s+.) - F(.+s, v(s+.)\|_{L^p(0,\tau;X)} \\ &\leq \delta(\tau)(2M + 1 + \tau^{\frac{1}{p}})K(\tau + s, M)\|u - v\|_Z. \end{aligned}$$

Finally due to (3.30) one concludes that the map Ψ has a unique fixed point in C and this completes the proof of the result. ∎

Now for each $x \in X_0^\beta$ and $s \geq 0$, we define the maximal existence time

$$\chi(s, x) = \sup\left\{t \geq 0 : U(s+., s)x \in Z_t \text{ is a solution of (3.18))}\right\}.$$

From Lemma 3.11, we already know $\chi(s, x) > 0$ for all $s \geq 0$ and $x \in X_0^\beta$. More precisely, for each $\xi > 0$ and $\sigma > 0$ there exists $\tau = \tau(\xi, \sigma) > 0$ such that

$$\chi(s, x) \geq \tau > 0, \quad \text{for all } x \in X_0^\beta \text{ with } \|x\|_\beta \leq \xi \text{ and } s \in [0, \sigma].$$

The two above lemmas allow us to uniquely define $U : D_\chi \to X_0^\beta$ such that for each $x \in X_0$ and each $s \geq 0$,

$$U(.,s)x \in C\left([s, s+\chi(s,x)), X_0^\beta\right) \cap L^p_{\text{loc}}([s, s+\chi(s,x)), X_0^\alpha)$$

is the unique weak solution of (3.18).

Our next lemma proves that the pair (U, χ) turns out to be maximal non-autonomous semiflow according to Definition 3.7.

Lemma 3.12 (Maximality of (U, χ)) *Let Assumptions 1.1 and 3.1 be satisfied. Then (U, χ) is a maximal non-autonomous semiflow associated to* (3.18) *and according to Definition 3.7.*

Proof. In order to prove this lemma we shall check that Assertions $(i) - (iv)$ in Definition 3.7 hold true. First note that $(i) - (iii)$ hold true because of the definition of mild solutions and the uniqueness result provided by Lemma 3.10. It remains to prove (iv). To that aim, let $x \in X_0^\beta$ and $s \geq 0$ be given and fixed. In order to prove this assertion we argue by contradiction by assuming that $\chi(s,x) < +\infty$ and that there exist a sequence $(t_n)_{n\geq 0}$ and a constant $\xi > 0$ such that

$$\begin{array}{l} t_n < \chi(s,x), \ \ \forall n \geq 0 \text{ and } t_n \to \chi(s,x) \text{ as } n \to \infty \\ \|U(t_n+s,s)x\|_\beta \leq \xi, \ \ \forall n \geq 0 \text{ and } \chi(s,x) \leq \xi. \end{array}$$

Let $\tau_\xi = \tau(\xi, \sigma) > 0$ be the time provided by Lemma 3.11 with $\sigma = \xi$. Let $n \geq 0$ be fixed large enough such that

$$t_n + \tau_\xi > \chi(s,x). \tag{3.32}$$

Then let us consider the map $V \in Z_{\tau_\xi}$, provided by Lemma 3.11, solution of the equation

$$V(t) = T_{A_0}(t)\left(U(t_n+s,s)x\right) + (S_A \diamond F(.+t_n, V(.))(t) \text{ for } t \in [0, \tau_\xi].$$

Next the function $W : [0, t_n+\tau_\xi] \to X_0^\beta$ defined by

$$W(t) = \begin{cases} U(t+s,s)x \text{ if } t \in [0,t_n], \\ V(t-t_n) \text{ if } t \in [t_n, t_n+\tau_\xi], \end{cases}$$

belongs to $Z_{t_n+\tau_\xi}$ and the function $\tilde{W}(t) = W(t-s)$ satisfies (3.18) on $[s, s+t_n+\tau_\xi]$. Thus (3.32) contradicts the definition of $\chi(s,x)$ and, this completes the proof of the lemma. ∎

We now complete the proof of Theorem 3.9 by proving the – semiflow – continuity properties. To do so we furthermore assume that Assumption 3.8 holds true.

Lemma 3.13 (Continuity of the semiflow) *Let Assumptions 1.1, 3.1 and 3.8 be satisfied. Then the following properties hold* *(i) The map $(s,x) \to \chi(s,x)$ is lower semi-continuous on $[0,\infty) \times X_0^\beta$.*

(ii) The set D_χ is an open subset of D.

(iii) The map $(t,s,x) \to U(t,s)x$ is continuous from D_χ into X_0^β.

Proof. Let $x \in X_0^\beta$ and $s \geq 0$ be given and fixed. Consider a sequence $(s_n, x_n)_{n\geq 0} \subset [0,\infty) \times X_0^\beta$ such that $(s_n, x_n) \to (s,x)$. In order to prove the lemma let us fix $\tau \in (0, \chi(s,x))$ and let us define

$$\xi = \frac{1}{2}\left[\|U(s+.,s)x\|_{Z_\tau} + 1\right] > 0.$$

Define also the sequence $\{\tau_n\}$ by

$$\tau_n = \sup\left\{t \in [0, \chi(s_n,x_n)) : \ \|U(.+s_n,s_n)x_n\|_{Z_t} \leq 2\xi\right\}. \tag{3.33}$$

Then we claim that

Claim 3.14 *The following limit holds true:*

$$\lim_{n\to\infty} \|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\min(\tau_n,\tau)}} = 0. \tag{3.34}$$

Proof of Claim 3.14. Let $\varepsilon > 0$ small enough be given such that

$$\Theta(\varepsilon) := \delta(\varepsilon)K\left(\widehat{s}+\tau,\xi\right)\left(2\xi+1+\tau^{\frac{1}{p}}\right) < 1 \text{ with } \widehat{s} := \sup_{n\geq 0} s_n. \tag{3.35}$$

Along this proof, for notational simplicity, we write χ and χ_n respectively instead of $\chi(s,x)$ and $\chi(s_n,x_n)$. Then observe that due to the semiflow property one has, for all $0 \leq r \leq l < \chi$,

$$\begin{aligned}
&U(l+s,s)x = U(l+s,r+s)U(r+s,s)x \\
&= T_{A_0}(l-r)U(r+s,s)x + \left(S_A \diamond F\left(.+r+s, U(.+r+s,s)x\right)\right)(l-r).
\end{aligned}$$

Using the same equality with $s = s_n$ and $x = x_n$ one gets, for each $n \geq 0$, for each $0 \leq r \leq l < \chi_n$,

$$\begin{aligned}
U(l+s_n,s_n)x_n =& T_{A_0}(l-r)U(r+s_n,s_n)x_n \\
&+ \left(S_A \diamond F(.+r+s_n, U(.+r+s_n,s_n)x_n)\right)(l-r).
\end{aligned}$$

Hence, for any $n \geq 0$ and for any $0 \leq r \leq l < \min\left(\chi,\chi_n\right)$, one has

$$\begin{aligned}
&U(l+s,s)x - U(l+s_n,s_n)x_n \\
&= T_{A_0}(l-r)\left[U(r+s,s)x - U(r+s_n,s_n)x_n\right] \\
&+ \left[S_A \diamond \left(F(.+r+s, U(.+r+s,s)x) - F(.+r+s_n, U(.+r+s_n,s_n)x_n)\right)\right](l-r).
\end{aligned}$$

Next, setting $\varepsilon_n := \min(\varepsilon, \tau_n, \tau)$, we infer from Lemma 3.4 and 3.5 that, for all $n \geq 0$ and each $r \in [0, \min\left(\tau,\tau_n\right) - \varepsilon_n]$, one has

$$\begin{aligned}
&\|U(.+r+s_n,s_n)x_n - U(.+r+s,s)x\|_{Z_{\varepsilon_n}} \\
&\leq m(\varepsilon)\left\|U(r+s_n,s_n)x_n - U(r+s,s)x\right\|_\beta \\
&+ \delta(\varepsilon)\|F(.+r+s_n, U(.+r+s_n,s_n)x_n) - F(.+r+s, U(.+r+s,s)x)\|_{L^p(0,\varepsilon_n;X)}.
\end{aligned}$$

On the other hand, by setting, for any $n \geq 0$ and $r \geq 0$ with $r \leq \min(\chi, \chi_n) - \varepsilon_n$,

$$u_n = U(. + r + s_n, s_n)x_n \text{ and } u = U(. + r + s, s)x,$$

let us observe that, for any $n \geq 0$,

$$\begin{aligned}&\|F(. + r + s_n, u_n(.)) - F(. + r + s, u(.))\|_{L^p(0,\varepsilon_n;X)} \\ &\quad \leq \|F(. + r + s_n, u_n(.)) - F(. + r + s_n, u(.))\|_{L^p(0,\varepsilon_n;X)} + \gamma_n^r,\end{aligned}$$

wherein we have set $\gamma_n^r = \|F(. + r + s_n, u(.)) - F(. + r + s, u(.))\|_{L^p(0,\varepsilon_n;X)}$.
Next Assumption 3.1 yields

$$\begin{aligned}&\|F(. + r + s_n, u_n(.)) - F(. + r + s_n, u(.))\|_{L^p(0,\varepsilon_n;X)} \\ &\quad \leq K(\widehat{s} + \tau, \xi)(4\xi + 1 + \tau^{\frac{1}{p}})\|U(r + . + s_n, s_n)x_n - U(r + . + s, s)x\|_{Z_{\varepsilon_n}}.\end{aligned}$$

As a consequence, one obtains, for any $n \geq 0$ and $r \geq 0$ with $r \leq \min(\tau, \tau_n) - \varepsilon_n$,

$$\begin{aligned}&\|U(r + . + s_n, s_n)x_n - U(r + . + s, s)x\|_{Z_{\varepsilon_n}} \\ &\quad \leq m(\varepsilon)\|U(r + s_n, s_n)x_n - U(r + s, s)x\|_\beta + \delta(\varepsilon)\gamma_n^r \\ &\quad + \Theta(\varepsilon)\|U(r + . + s_n, s_n)x_n - U(r + . + s, s)x\|_{Z_{\varepsilon_n}},\end{aligned}$$

that is, since $\Theta(\varepsilon) < 1$, for any $n \geq 0$ and $r \geq 0$ with $r < \min(\tau, \tau_n) - \varepsilon_n$,

$$\begin{aligned}&\|U(r + . + s_n, s_n)x_n - U(r + . + s, s)x\|_{Z_{\varepsilon_n}} \\ &\quad \leq \frac{m(\varepsilon)}{1 - \Theta(\varepsilon)}\|U(r + s_n, s_n)x_n - U(r + s, s)x\|_\beta + \frac{\delta(\varepsilon)}{1 - \Theta(\varepsilon)}\gamma_n^r.\end{aligned}$$

Now choosing $r = 0$ yields

$$\|U(. + s_n, s_n)x_n - U(. + s, s)x\|_{Z_{\varepsilon_n}} \leq \frac{m(\varepsilon)}{1 - \Theta(\varepsilon)}\|x_n - x\|_\beta + \frac{\delta(\varepsilon)}{1 - \Theta(\varepsilon)}\gamma_n^0. \quad (3.36)$$

Hence, recalling that Assumption 3.8 ensures that $\gamma_n^0 \to 0$ as $n \to \infty$, one obtains that

$$\lim_{n \to \infty}\|U(. + s_n, s_n)x_n - U(. + s, s)x\|_{Z_{\varepsilon_n}} = 0.$$

This already completes the proof of Claim 3.14 if $\min(\tau, \tau_n) \leq \varepsilon$ for all n large enough. In the general case, we proceed by induction and we consider, for any $n \geq 0$, the integer $k_n \geq 0$ such that

$$k_n\varepsilon < \min(\tau, \tau_n) \leq (k_n + 1)\varepsilon.$$

First, for any n such that $k_n = 0$, one has

$$\|U(. + s_n, s_n)x_n - U(. + s, s)x\|_{Z_{\min(\tau,\tau_n)}} \leq \frac{m(\varepsilon)}{1 - \Theta(\varepsilon)}\|x_n - x\|_\beta + \frac{\delta(\varepsilon)}{1 - \Theta(\varepsilon)}\gamma_n^0.$$

Next for any n such that $k_n = 1$, one has $\varepsilon_n = \varepsilon$ and choosing $r_n = \min(\tau_n, \tau) - \varepsilon \in (0, \varepsilon]$,

$$\begin{aligned}
&\|U(.+r_n+s_n,s_n)x_n - U(.+r_n+s,s)x\|_{Z_\varepsilon} \\
&\quad\le \frac{m(\varepsilon)}{1-\Theta(\varepsilon)}\|U(r_n+s_n,s_n)x_n - U(r_n+s,s)x\|_\beta + \frac{\delta(\varepsilon)}{1-\Theta(\varepsilon)}\gamma_n^{r_n}.
\end{aligned}$$

And, adding-up with (3.36) and recalling that $r_n = \min(\tau_n, \tau) - \varepsilon \le \varepsilon$ yield

$$\begin{aligned}
&\|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\min(\tau,\tau_n)}} \\
&\quad\le \frac{m(\varepsilon)}{1-\Theta(\varepsilon)}\left[\|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\varepsilon_n}} + \|x_n - x\|_\beta\right] \\
&\quad+ \frac{\delta(\varepsilon)}{1-\Theta(\varepsilon)}\left(\gamma_n^0 + \gamma_n^{r_n}\right) \\
&\quad\le \frac{m(\varepsilon)}{1-\Theta(\varepsilon)}\left[\left(\frac{m(\varepsilon)}{1-\Theta(\varepsilon)}+1\right)\|x_n - x\|_\beta + \frac{\delta(\varepsilon)}{1-\Theta(\varepsilon)}\gamma_n^0\right] \\
&\quad+ \frac{\delta(\varepsilon)}{1-\Theta(\varepsilon)}\left(\gamma_n^0 + \gamma_n^{r_n}\right)
\end{aligned}$$

One may continue this process and, since (k_n) is bounded, there exists some constant $K > 0$ such that, for any $n \ge 0$, one has

$$\|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\min(\tau,\tau_n)}} \le K\|x-x_n\|_\beta + K\left[\sum_{k=0}^{(k_n-1)^+}\gamma_n^{k\varepsilon} + \gamma_n^{r_n^+}\right].$$

In the above formula, the superscript $+$ denotes the positive part. Now let us observe that for each n one has, for any $k = 0, .., (k_n-1)^+$,

$$\gamma_n^{k\varepsilon} \le \Gamma_n := \|F(.+s_n, U(.+s,s)x) - F(.+s, U(.+s,s)x)\|_{L^p(0,\tau;X)},$$

and

$$\gamma_n^{r_n^+} \le \Gamma_n.$$

As a consequence, since (k_n) is bounded, there exists some constant $\widehat{K} > 0$ such that for any n one has

$$\|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\min(\tau,\tau_n)}} \le K\|x-x_n\|_\beta + \widehat{K}\Gamma_n.$$

Finally Assumption 3.8 ensures that $\Gamma_n \to 0$ as $n \to \infty$. So that we get

$$\lim_{n\to\infty}\|U(.+s_n,s_n)x_n - U(.+s,s)x\|_{Z_{\min(\tau,\tau_n)}} = 0,$$

that completes the proof of Claim 3.14.

■

Equipped with the above claim we complete the proof of the lemma. For that purpose note that to prove (i) and (ii) it is sufficient to show that

$$\liminf_{n\to\infty}\chi(s_n,x_n) \ge \chi(s,x). \tag{3.37}$$

In order to prove the above inequality, recall that $\tau \in (0, \chi(s,x))$ is fixed but arbitrary. And, since $\chi(s_n, x_n) \geq \tau_n$, it is sufficient to show that

$$\liminf_{n \to +\infty} \tau_n \geq \tau.$$

To prove this we argue by contradiction by assuming that

$$\liminf_{n \to +\infty} \tau_n < \tau.$$

Then we can find a subsequence still denoted with the same indexes such that

$$\tau_n < \tau, \ \forall n \geq 0 \text{ and } \tau_n \to \tilde{\tau} < \tau.$$

Next one has, for each $n \geq 0$,

$$\|U(. + s_n, s_n)x_n\|_{Z_{\tau_n}} \leq \|U(. + s_n, s_n)x_n - U(. + s, s)x\|_{Z_{\tau_n}} + \|U(. + s, s)x\|_{Z_{\tau_n}} .$$

Since $\tau_n < \tau$, $\forall n \geq 0$, one gets

$$\|U(. + s_n, s_n)x_n\|_{Z_{\tau_n}} \leq \|U(. + s_n, s_n)x_n - U(. + s, s)x\|_{Z_{\min(\tau_n,\tau)}} + \|U(. + s, s)x\|_{Z_{\tau}} .$$

Next, from the definition of τ_n in (3.33) and using (3.34), one obtains

$$2\xi \leq \|U(. + s, s)x\|_{Z_\tau} .$$

This contradicts the definition of ξ and (3.37) follows. Finally one may also observe that (iii) directly follows from (3.34) and this completes the proof of the lemma. ∎

4 Stability and instability of equilibrium

The aim of this section is to consider the linear stability of equilibrium points of the autonomous Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)), \ \ t > 0, \ \ u(0) = x \in X_0. \tag{4.38}$$

Throughout this section we assume that Assumptions 1.1 holds true. In addition in order to deal with linear stability of equilibrium points we slightly strength Assumption 3.1 as follows.

Assumption 4.1 *Let $\alpha \in (0,1)$ and $p \in [1, +\infty)$ be given such that*

$$0 < \alpha < \frac{1}{p} < \frac{1}{p^*}.$$

We assume that there exists a non-decreasing function $K : [0,+\infty) \to [0,+\infty)$ such that $F : X_0^\alpha \to X$ satisfies, for each $M > 0$,

$$\|F(x) - F(y)\|_X \leq K(M)\|x - y\|_\alpha,$$

whenever $x, y \in X_0^\alpha$, and $\max(\|x\|_0, \|y\|_0) \leq M$. Here recall that $X_0^0 = X_0$ so that $\|.\|_0 = \|.\|_{X_0}$.

In addition to the above set of hypothesis we also make use of the following assumption related to steady state and local behaviour of the nonlinearity F in its neighbourhood.

Assumption 4.2 *We assume that there exists a point $x^* \in D(A)$ such that*

$$Ax^* + F(x^*) = 0.$$

And, there exist a linear operator $L \in \mathcal{L}(X_0^\alpha, X)$ and a map $G : X_0^\alpha \to X$ such that

(i) $F(x) = F(x^*) + L(x - x^*) + G(x)$ for all $x \in X_0^\alpha$,

(ii) For each $\eta > 0$ there exists $\delta > 0$ such that

$$\|G(x)\| \leq \eta \|x - x^*\|_\alpha, \quad \forall x \in B_0(x^*, \delta) \cap X_0^\alpha.$$

Using the above set of assumptions our next result proves that the semiflow U generated by (4.38) in X_0, and provided by Theorem 3.9, is differentiable in X_0 at the equilibrium point $x = x^*$. In order to state our result, let us first observe that Theorem 2.9 and Lemma 3.5 ensure that the operator $(A + L)_0$, the part of $(A + L)$ in X_0, generates a strongly continuous semigroup on X_0 denoted by $\{V(t)\}_{t \geq 0}$. Let us also recall that for all $x \in X_0$

$$V(t)x \in X_0^\alpha, \ \forall t > 0 \text{ and } V(.)x \in L^p_{\text{loc}}([0, \infty); X_0^\alpha).$$

With this notation, our precise result reads as follows.

Theorem 4.3 *Under the above set of assumptions, Problem* (4.38) *defines a maximal semiflow (U, χ) on X_0. For each $\tau > 0$ there exists $\varepsilon_\tau > 0$ such that*

$$\tau < \chi(x), \ \forall x \in B_0(x^*, \varepsilon_\tau),$$

and, for each $t \in [0, \tau]$, the map $x \to U(t)x$ acting from $B_0(x^*, \varepsilon_\tau)$ into X_0 is Frechet differentiable at $x = x^*$ and its derivative, denoted by $D_{x^*}U(t)$, is given by

$$D_{x^*}U(t)h = V(t)h, \ \forall t \in [0, \tau], \ h \in X_0.$$

Proof. The existence of a maximal strongly continuous semiflow follows from Theorem 3.9. Note that since (4.38) is autonomous, Assumption 3.8 is also satisfied so that the maximum existence time χ is lower semi-continuous in X_0. Since $\chi(x^*) = \infty > 0$, the lower semi-continuity of χ implies that $\chi(x) \to \infty$ as $x \to x^*$. Let $\tau > 0$ be given. Then there exists $\varepsilon = \varepsilon_\tau > 0$ such that

$$\tau < \inf_{x \in B_0(x^*, \varepsilon)} \chi(x).$$

Now let $h \in X_0$ be given such that $\|h\|_0 < \varepsilon$. Define

$$W(t) = U(t)(x^* + h) - U(t)x^* - V(t)h, \quad t \in [0, \tau],$$

and put

$$G_1(t) = F\left(U(t)(x^* + h)\right) - F\left(U(t)x^* + V(t)h\right),$$
$$G_2(t) = F\left(x^* + V(t)h\right) - F(x^*) - LV(t)h, \quad t \in [0, \tau].$$

Observe that, since $U(t)x^* = x^*$ for any $t \geq 0$, one has

$$W(t) = (S_A \diamond G_1)(t) + (S_A \diamond G_2)(t), \quad \forall t \in [0, \tau].$$

To estimate the function W we shall first derive suitable estimates for G_1 and G_2.

Estimate for G_1: This estimate follows from the Lipschitz property of the function F as stated in Assumption 4.1. Define $M > 0$ by

$$M = \sup_{h \in B_0(0,\varepsilon),\ t \in [0,\tau]} \left\{ \|U(.)(x^* + h)\|_0 + \|x^* + V(.)h\|_0 \right\}.$$

Hence due to Assumption 4.1 one gets for all $t \in (0, \tau]$:

$$\|G_1(t)\| \leq K(M)\|W(t)\|_\alpha. \tag{4.39}$$

Estimate for G_2: Due to Assumption 4.2 for each $\eta > 0$ there exists $\delta > 0$ such that

$$\|F(x^* + \zeta) - F(x^*) - L\zeta\|_X \leq \eta\|\zeta\|_\alpha, \quad \forall \zeta \in X_0^\alpha \cap B_0(0, \delta).$$

Let $\eta > 0$ be given and $\delta > 0$ be the corresponding value satisfying the above property. Up to reduce the value of ε if necessary one may assume that

$$\|V(t)\zeta\|_0 < \delta, \quad \forall t \in [0, \tau], \quad \forall \zeta \in B_0(0, \varepsilon).$$

From this one derives that

$$\|G_2(t)\|_X \leq \eta\|V(t)h\|_\alpha, \quad \forall t \in (0, \tau],\ \forall h \in B_0(0, \varepsilon). \tag{4.40}$$

To complete our proof we make use of the above estimates, namely (4.39) and (4.40), coupled together with Lemma 2.5. Doing so, for each $\gamma > 0$ such that $1 - \frac{1}{p^*} < \gamma$ and $\gamma + \alpha < 1 - \frac{1}{p}$, there exists some constant $M_{\gamma,\tau} > 0$ such that

$$\|(S_A \diamond G_1)(t)\|_\alpha \leq M_{\gamma,\tau} \int_0^t (t - s)^{-(\alpha+\gamma)} \|W(s)\|_\alpha \mathrm{d}s,$$
$$\|(S_A \diamond G_2)(t)\|_\alpha \leq M_{\gamma,\tau} \int_0^t (t - s)^{-(\alpha+\gamma)} \eta \|V(s)h\|_\alpha \mathrm{d}s.$$

On the other hand, using Holder inequality one has

$$\|(S_A \diamond G_2)(t)\|_\alpha \leq \eta M_{\gamma,\tau} \left(\int_0^t s^{-q(\alpha+\gamma)} \mathrm{d}s \right)^{1/q} \|V(.)h\|_{L^p(0,\tau;X_0^\alpha)}.$$

Moreover, since there exists some constant $C_\tau > 0$ such that

$$\|V(.)h\|_{L^p(0,\tau;X_0^\alpha)} \leq C_\tau \|h\|_0,$$

this yields

$$\|(S_A \diamond G_2)(t)\|_\alpha \leq \eta M_{\gamma,\tau} C_\tau t^{1-\frac{1}{p}-\alpha-\gamma} \|h\|_0.$$

Finally coupling all the above estimates leads us to the existence of some constant $\widehat{M} > 0$ such that

$$\|W(t)\|_\alpha \leq \widehat{M} \int_0^t (t-s)^{-(\alpha+\gamma)} \|W(s)\|_\alpha \mathrm{d}s + \widehat{M}\eta \|h\|_0, \ \ \forall t \in (0,\tau], \ h \in B_0(0,\varepsilon).$$

Thus from Gronwall inequality (see Henry [14] Lemma 7.1.1) we get that there exists a continuous function $E : [0,\infty) \to [0,\infty)$ with $E(0) = 0$ and such that

$$\|W(t)\|_\alpha \leq \eta \|h\|_0 E(t), \ \ \forall t \in (0,\tau].$$

Since $X_0^\alpha \hookrightarrow X_0$, we have proved the following properties: For any given $\tau > 0$ there exists some constant $\widetilde{M} > 0$ such that for each $\eta > 0$ small enough there exists $\tilde{\varepsilon} > 0$ such that

$$\begin{aligned} &\chi(x^* + h) > \tau, \ \ \forall h \in B_0(0,\tilde{\varepsilon}), \\ &\|U(t)(x^*+h) - U(t)x^* - V(t)h\|_0 \leq \widetilde{M}\eta\|h\|_0, \ \ \forall t \in (0,\tau], \ \ \forall h \in B_0(0,\tilde{\varepsilon}). \end{aligned}$$

This proves the differentiability of the nonlinear semigroup $U(t)$ at $x = x^*$ and this completes the proof of the result. ∎

As a consequence of the above theorem one may directly apply the results of Desch and Schappacher in [8] to obtain the following linear stability result.

Corollary 4.4 *Let Assumptions 1.1, 4.1 and 4.2 be satisfied. Then the following properties hold true:*

(i) (Stability) Assume that the zero equilibrium of $\{V(t)\}_{t\geq 0}$ *is exponentially asymptotically stable, that is there exists* $M > 0$ *and* $\omega > 0$ *such that*

$$\|V(t)\|_{\mathcal{L}(X_0)} \leq M e^{-\omega t}, \forall t \geq 0. \tag{4.41}$$

Then there exists $\varepsilon > 0$ *such that*

$$\chi(x) = \infty, \ \forall x \in B_0(x^*,\varepsilon), \tag{4.42}$$

and the equilibrium x^* *of* $\{U(t)\}_{t\geq 0}$ *is locally exponentially asymptotically stable, in the sense that there exist constants* $K > 0$, $\mu > 0$ *and* $0 < \delta < \varepsilon$ *such that*

$$\|U(t)x - x^*\|_0 \leq K e^{-\mu t} \|x - x^*\|_0, \ \ \forall t \geq 0, \ \ \forall x \in B_0(x^*,\delta).$$

(ii) (Instability) Assume that X_0 can be split as $X_0 = X_1 \oplus X_2$ where X_i are both closed $V-$invariant subspaces such that X_1 is finite dimensional while

$$\inf\{|\lambda| : \quad \lambda \in \sigma\left(V(t)|_{X_1}\right)\} > e^{\omega t},$$

with

$$\omega := \lim_{s\to\infty} \frac{1}{s} \log \|V(s)|_{X_2}\|_{\mathcal{L}(X_2)}.$$

Then x^* is instable with respect to the semiflow $\{U(t)\}_{t\geq 0}$, in the sense of the following alternative:

(a) There exists $\{x_n\}_{n\geq 0} \subset X_0$, such that

$$\lim_{n\to\infty} \|x_n - x^*\|_0 = 0,$$

and

$$\chi(x_n) < \infty, \forall n \geq 0,$$

that is to say that every solution starting from x_n is blowing up in finite time.

(b) There exists $\varepsilon > 0$ such that

$$\chi(x) = \infty, \ \forall x \in B_0(x^*, \varepsilon),$$

and there exist $\delta > 0$, $\{x_n\}_{n\geq 0} \subset X_0$ and $\{t_n\}_{n\geq 0} \subset (0, \infty)$ with $t_n \to \infty$, such that

$$\lim_{n\to\infty} \|x_n - x^*\|_0 = 0, \quad \|U(t_n)x_n - x^*\|_0 \geq \delta, \quad \forall n \geq 0.$$

Proof. Let us first observe that (ii) directly follows from Theorem 4.3 and the results from Desch and Schappacher in [8]. Now in order to prove (i), note that it is sufficient to prove the global existence property (4.42). Indeed, here again the differentiability property derived in Theorem 4.3 together with the stability results of Desch and Schappacher in [8] ensure that (i) holds true.

In order to prove (4.42), recall that we assume that (4.41) is satisfied. Let $\gamma > 0$ be given and fixed such that $\gamma + \alpha < 1$. Fix $\eta > 0$ small enough such that

$$\eta M \int_0^\infty e^{-\omega s} s^{-\gamma-\alpha} \mathrm{d}s < 1. \tag{4.43}$$

Let $\delta > 0$ be given (see Assumption 4.2 (ii)) such that

$$\|G(x)\| \leq \eta \|x - x^*\|_\alpha, \quad \forall x \in B_0\left(x^*, \delta\right) \cap X_0^\alpha.$$

Next define, for each $x \in B_0\left(x^*, \delta\right)$,

$$\tau(x) := \sup\{t \in (0, \chi(x)) : \|U(s)x - x^*\|_0 \leq \delta, \forall s \in [0, t]\}$$

Then we have, for each $x \in B_0(x^*, \delta/2)$ and each $t \in [0, \tau(x))$,

$$U(t)x = T_{(A+L)_0}(t)x + (S_{A+L} \diamond (F(x^*) - Lx^* + G(U(.)x)))(t)$$

and, since $G(x^*) = 0$,

$$x^* = U(t)x^* = T_{(A+L)_0}(t)x^* + (S_{A+L} \diamond F(x^*) - Lx^*)(t).$$

Therefore for each $t \in [0, \tau(x))$ one has

$$\begin{aligned} \|U(t)x - x^*\|_0 &\leq Me^{-\omega t}\|x - x^*\|_0 \\ &+M \int_0^t e^{-\omega(t-s)}(t-s)^{-\gamma}\eta\|U(s)x - x^*\|_\alpha \mathrm{d}s, \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \|U(t)x - x^*\|_\alpha &\leq Me^{-\omega t}t^{-\alpha}\|x - x^*\|_0 \\ &+M \int_0^t e^{-\omega(t-s)}(t-s)^{-(\gamma+\alpha)}\eta\|U(s)x - x^*\|_\alpha \mathrm{d}s. \end{aligned} \tag{4.45}$$

Next set

$$Y(t) := \|U(t)x - x^*\|_\alpha.$$

Then, using the Young's inequality to the convolution in (4.45) yields

$$\|Y(.)\|_{L^p(0,t)} \leq K\|x - x^*\|_0, \ \forall t \in [0, \tau(x)),$$

wherein the constant $K > 0$ is defined by (see (4.43) above for the property of η)

$$K = \frac{M\left(\int_0^\infty e^{-\omega ps}s^{-\alpha p}\mathrm{d}s\right)^{1/p}}{1 - \eta M\int_0^\infty e^{-\omega s}s^{-\gamma-\alpha}\mathrm{d}s}.$$

Finally, by plugging this inequality into (4.44), we obtain for each $t \in [0, \tau(x))$

$$\|U(t)x - x^*\|_0 \quad \leq \|x - x^*\|_0 \left[Me^{-\omega t} + M\eta\left(\int_0^\infty e^{-\omega qs}s^{-\gamma q}\mathrm{d}s\right)^{1/q} K\right],$$

and as a consequence, by choosing $\|x - x^*\|_0$ small enough, it follows that

$$\tau(x) = \chi(x) = \infty.$$

This completes the proof of (4.42) and thus the proof of the corollary. ■

5 Differentiability with respect to time

In this section we deal with the differentiability in time of the semiflow provided in Section 3. For the simplicity of the exposition, we consider an autonomous problem of the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)), \ t > 0 \text{ and } u(0) = x \in X_0^\alpha. \tag{5.46}$$

Here we assume that $A : D(A) \subset X \to X$ satisfies Assumption 1.1 and that $F : X_0^\alpha \to X$ is of the class C^1 for some given $\alpha < \frac{1}{p^*}$.

Using the results in Section 3 (with $\beta = \alpha$ and p large enough), the above problem generates a maximal semiflow in X_0^α, denoted by (U_α, χ_α) and since for $\alpha = 0$ we have $X_0^\alpha = X_0$ we define (U_0, χ_0) the maximal semiflow in X_0.

Hence (5.46) has a solution $u = u(t) \in C([0, \tau_M), X_0^\alpha)$ wherein we have set $\tau_M = \chi_\alpha(x)$. In this section we investigate the time differentiability of the function u and our result reads as follows.

Theorem 5.1 *There exists $r > 1$ such that the function u satisfies:*

$$u \in W^{1,r}_{\text{loc}}((0, \tau_M); X_0^\alpha) \text{ and } u(t) \in D(A) \text{ a.e. } t \in (0, \tau_M).$$

Furthermore $u = u(t)$ satisfies

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)), \text{ a.e. } t \in (0, \tau_M),$$

with $\frac{\mathrm{d}u(.)}{\mathrm{d}t} \in L^r(t_1, t_2; X_0^\alpha)$ and $F(u(.)) \in C([t_1, t_2]; X)$ for each $t_1 < t_2$ such that $[t_1, t_2] \subset (0, \tau_M)$.

To prove this result, let us first observe that the function $t \mapsto F(u(t))$ belongs to $C([0, \tau_M); X)$. Hence, according to Lemma 2.5, the function $t \mapsto (S_A \diamond F(u))(t)$ is continuous from $[0, \tau_M)$ into X_0^β for all $0 < \beta < \frac{1}{p^*}$. Furthermore the function $t \mapsto T_{A_0}(t)x$ is continuous from $(0, \infty)$ into X_0^β for any $\beta \in (0, 1)$.

Now we fix $\beta > 0$ such that

$$\beta > \alpha \text{ and } \beta < \frac{1}{p^*},$$

so that the function $u = u(t)$ is continuous from $(0, \infty)$ into X_0^β.

In order to prove the above theorem we fix $\gamma > 0$ such that

$$\gamma > \frac{1}{q^*} = 1 - \frac{1}{p^*} \text{ and } \alpha + \gamma < 1.$$

Next we fix $r \in (1, \infty)$ such that

$$r(1 - \beta + \alpha) < 1 \text{ and } r(\alpha + \gamma) < 1.$$

Now to prove the above theorem, note that if we formally set $v(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$ then (u, v) satisfies the following system of equations:

$$\begin{cases} u(t) = T_{A_0}(t)x + \int_0^t T_A(s)F(u(t-s))\mathrm{d}s, \\ v(t) = A_0 T_{A_0}(t)x + T_A(t)F(x) + \int_0^t T_A(s)DF(u(t-s))v(t-s)\mathrm{d}s. \end{cases} \tag{5.47}$$

Here we have set $T_A(t) = \frac{\mathrm{d}S_A(t)}{\mathrm{d}t}$.

Next we shall investigate (5.47) and, to that aim we shall prove the following key lemma.

Lemma 5.2 *Let $C > 0$ be given. Then there exists $\tau = \tau(C) > 0$ such that for any $z \in X_0^\beta$ with $\|z\|_\beta \le C$, there exists $(u,v) = (u(.;z), v(.;z)) \in C([0,\tau]; X_0^\alpha) \cap W^{1,r}(0,\tau; X_0^\alpha) \times L^r(0,\tau; X_0^\alpha)$ such that for any $t \in [0,\tau]$*

$$\begin{cases} u(t) = T_{A_0}(t)z + \int_0^t T_A(s)F(u(t-s))\mathrm{d}s, \\ v(t) = A_0 T_{A_0}(t)z + T_A(t)F(z) + \int_0^t T_A(s)DF(u(t-s))v(t-s)\mathrm{d}s, \\ u(t) = z + \int_0^t v(s)\mathrm{d}s. \end{cases} \tag{5.48}$$

Before proving this lemma let us complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $0 < t_1 < t_2 < \tau_M$ be given. Recall that the function $u = u(t)$, the solution of (5.46), is continuous from $[0,\tau_M)$ into X_0^α and from $(0,\tau_M)$ into X_0^β. Hence set $C = \max\{\|u(t)\|_\beta,\ t \in [t_1,t_2]\}$. Let $\tau = \tau(C)$ be the constant provided by Lemma 5.2. Next the uniqueness of the solution shows that

$$u(t_1 + t) = u(t; u(t_1)),\ \forall t \in [0, \min(t_2 - t_1, \tau)].$$

Hence $u_{[t_1, \min(t_2, t_1+\tau)]}$ belongs to $W^{1,r}$ and Theorem 5.1 follows if $t_2 \le t_1 + \tau$. If $t_2 > t_1 + \tau$, then since $\|u(t_1+\tau)\|_\beta \le C$ one may reproduce the same argument on the interval $[t_1+\tau, \min(t_2, t_1+2\tau)]$. Hence if $t_2 \le t_1 + 2\tau$ the result follows and if $t_2 > t_1 + 2\tau$ one continues the argument. Since there exists $n \in \mathbb{N}$ such that $t_2 \le t_1 + n\tau$, the result follows. This proves that for any $0 < t_1 < t_2 < \tau_M$, one has $u_{|[t_1,t_2]} \in W^{1,r}(t_1,t_2; X_0^\alpha)$ and the result follows. ■

Now it remains to prove Lemma 5.2. Before going further, let us recall that for each $\tau^* > 0$ there exists some constant $M(\tau^*) > 1$ such that

$$\|T_A(t)\|_{\mathcal{L}(X, X_0^\alpha)} \le \frac{M(\tau^*)}{t^{\alpha+\gamma}},\ \forall t \in (0,\tau^*]. \tag{5.49}$$

Next to prove Lemma 5.2 we shall make use of a suitable fixed point argument based on the so-called fibre contraction theorem proved by Vanderbauwhede in [28]. To that aim, fix $z \in X_0^\beta \subset X_0^\alpha$ and note that due to the above estimate, one has

$$\|A_0 T_{A_0}(t)z\|_\alpha = O\left(\frac{1}{t^{1+\alpha-\beta}}\right) \text{ and } \|T_A(t)F(z)\|_\alpha = O\left(\frac{1}{t^{\alpha+\gamma}}\right) \text{ as } t \to 0^+.$$

Hence,

$$A_0 T_{A_0}(.)z + T_A(.)F(z) \in L^r_{\mathrm{loc}}([0,\infty); X_0^\alpha).$$

Furthermore it readily follows that, for each $C > 0$ there exists some constant $K = K(C)$ such that

$$\|A_0 T_{A_0}(.)z + T_A(.)F(z)\|_{L^r(0,1;X_0^\alpha)} \le K(C), \text{ for any } z \in X_0^\beta \text{ with } \|z\|_\beta \le C.$$

Now we fix $C > 0$ as in the statement of Lemma 5.2. And, for each $\tau \in (0,1]$ and $z \in X_0^\beta$ with $\|z\|_\beta \le C$, we consider the complete metric spaces

$$M_1^{z,\tau} = \{\varphi_1 \in C([0,\tau]; X_0^\alpha) :\ \varphi_1(0) = z \text{ and } \|\varphi_1(t) - T_{A_0}(t)z\|_\alpha \le 1,\ \forall t \in [0,\tau]\},$$
$$M_2^{z,\tau} = \left\{\varphi_2 \in L^r(0,\tau; X_0^\alpha) :\ \|\varphi_2(.) - A_0 T_{A_0}(.)z - T_A(.)F(z)\|_{L^r(0,\tau;X_0^\alpha)} \le 1\right\}.$$

Now consider the map $\Psi^z : M_1^{z,\tau} \times M_2^{z,\tau} \to C([0,\tau]; X_0^\alpha) \times L^r(0,\tau; X_0^\alpha)$ defined by

$$\Psi^z(u,v) = (\Psi_1^z(u), \Psi_2^z(u,v)) = (\widehat{u}, \widehat{v}),$$

wherein each component is defined for $t \in [0,\tau]$ and for $t \in (0,\tau]$ respectively by

$$\widehat{u}(t) = T_{A_0}(t)z + \int_0^t T_A(s)F(u(t-s))\mathrm{d}s,$$
$$\widehat{v}(t) = A_0 T_{A_0}(t)z + T_A(t)F(z) + \int_0^t T_A(s)DF(u(t-s))v(t-s)\mathrm{d}s.$$

Our next lemma collects suitable estimates for the map Ψ^z.

Lemma 5.3 *There exists some constant $K = K(C) > 0$ such that for any $\tau \in (0,1]$ and $z \in X_0^\beta$ with $\|z\|_\beta \le C$ the following estimates hold true:*

(i) For any $(u,v) \in M_1^{z,\tau} \times M_2^{z,\tau}$, the function $(\widehat{u},\widehat{v}) = \Psi^z(u,v)$ satisfies

$$\|\widehat{u}(t) - T_{A_0}(t)z\|_\alpha \le K t^{1-\alpha-\gamma}, \ \forall t \in [0,\tau].$$

and

$$\|\widehat{v}(.) - (A_0 T_{A_0}(.)z + T_A(.)F(z))\|_{L^r(0,\tau;X_0^\alpha)} \le K\tau^{1-\alpha-\gamma}.$$

(ii) For $u, \tilde{u} \in M_1^{z,\tau}$ one has:

$$\|\Psi_1^z(u) - \Psi_1^z(\tilde{u})\|_{C([0,\tau];X_0^\alpha)} \le K\tau^{1-\alpha-\gamma} \|u - \tilde{u}\|_{C([0,\tau];X_0^\alpha)},$$

and, for any $u \in M_1^{z,\tau}$ and any $v, \tilde{v} \in M_2^{z,\tau}$ one gets

$$\|\Psi_2^z(u,v) - \Psi_2^z(u,\tilde{v})\|_{L^r(0,\tau;X_0^\alpha)} \le K\tau^{1-\alpha-\gamma} \|v - \tilde{v}\|_{L^r(0,\tau;X_0^\alpha)}.$$

Proof. Set $B_\beta(C) = \{z \in X_0^\beta, \ \|z\|_\beta \le C\}$ and let us introduce the quantity $K_1 > 0$ and $K_2 > 0$ defined by

$$K_1 = \sup\left\{\|F(u)\| : \ u \in X_0^\alpha, \ \|u\|_\alpha \le 1 + \sup_{t\in[0,1], \ z\in B_\beta(C)} \|T_{A_0}(t)z\|_\alpha\right\},$$

and

$$K_2 = \sup\left\{\|DF(u)\|_{\mathcal{L}(X_0^\alpha,X)} : \ u \in X_0^\alpha, \ \|u\|_\alpha \le 1 + \sup_{t\in[0,1], \ z\in B_\beta(C)} \|T_{A_0}(t)z\|_\alpha\right\}.$$

Next, to prove (i), let $z \in B_\beta(C)$ and $\tau \in (0,1]$ be given. Let $(u,v) \in M_1^{z,\tau} \times M_2^{z,\tau}$ be given. Then, recalling (5.49), one has

$$\|\widehat{u}(t) - T_{A_0}(t)z\|_\alpha \le \int_0^t M K_1 s^{-\alpha-\gamma}\mathrm{d}s,$$

and the first estimate follows with $K = MK_1/(1-\alpha-\gamma)$. On the other hand one also has

$$\begin{aligned} &\|\widehat{v}(.) - (A_0 T_{A_0}(.)z + T_A(.)F(z))\|_{L^r(0,\tau;X_0^\alpha)} \\ &\leq \left\| MK_2 \int_0^. s^{-\alpha-\gamma}\|v(.-s)\|_\alpha \mathrm{d}s \right\|_{L^r(0,\tau)} \leq K\tau^{1-\alpha-\gamma}, \end{aligned}$$

with

$$K = \frac{MK_2}{1-\alpha-\gamma} \sup_{z\in B_\beta(C)} \left\{1 + \|A_0 T_{A_0}(.)z + T_A(.)F(z)\|_{L^r(0,1;X_0^\alpha)}\right\}.$$

Hence (i) follows while (ii) follows from the same arguments. ∎

Now we fix $\tau = \tau(C) \in (0,1]$ small enough such that $K(C)\tau^{1-\alpha-\gamma} < 1$ where $K = K(C)$ is the constant provided by Lemma 5.3. With such a choice, one gets

$$\Psi^z\left(M_1^{z,\tau} \times M_2^{z,\tau}\right) \subset M_1^{z,\tau} \times M_2^{z,\tau},\ \forall z \in B_\beta(C),\ \tau \in (0,\tau(C)).$$

Finally we apply the fibre contraction theorem to complete the proof of Lemma 5.2.
For the sake of completeness, we recall the fibre contraction theorem, we shall use and we refer the reader to Vanderbauwhede [28, Theorem 3.5] or to Magal and Ruan [21, Lemma 6.7] for a proof of the result.

Theorem 5.4 (Fibre contraction theorem) *Let M_1, M_2 be two complete metric spaces and $\Psi : M_1 \times M_2 \to M_1 \times M_2$ be a map of the form*

$$\Psi(x,y) = (\Psi_1(x), \Psi_2(x,y)),$$

satisfying the following set of assumptions:

(i) The map Ψ_1 has a fixed point $\overline{x} \in M_1$ and

$$\lim_{n\to\infty} \Psi_1^{(n)}(x) = \overline{x},\ \forall x \in M_1.$$

Here $\Psi_1^{(n)} = \Psi_1 \circ \cdots \circ \Psi_1$ denotes the $n-$fold composition of Ψ_1;

(ii) There exists $k \in [0,1)$ such that for all $x \in M_1$ the map $y \mapsto \Psi_2(x,y)$ is $k-$Lipschitz continuous on M_2;

(iii) The map $x \mapsto \Psi_2(x,\overline{y})$ is continuous where $\overline{y} \in M_2$ denotes the unique fixed point of $\overline{y} = \Psi_2(\overline{x},\overline{y})$.

Then, for each $(x,y) \in M_1 \times M_2$, one has

$$\lim_{n\to\infty} \Psi^{(n)}(x,y) = (\overline{x},\overline{y}).$$

We are now able to complete the proof of Lemma 5.2 and thus the one of Theorem 5.1.

Proof of Lemma 5.2. Since $M_1^{z,\tau}$ and $M_2^{z,\tau}$ are complete metric spaces when respectively endowed with the distance associated to the norms of $C([0,\tau];X_0^\alpha)$ and $L^r(0,\tau;X_0^\alpha)$, and due to the choice of $\tau=\tau(C)$, it readily follows from the fibre contraction theorem recalled above that, for each $z\in B_\beta(C)$, the map Ψ^z has a unique fixed point $(u(.;z),v(.;z))$ in $M_1^{z,\tau}\times M_2^{z,\tau}$ that attracts any points in $M_1^{z,\tau}\times M_2^{z,\tau}$ under the action of Ψ^z. To complete the proof of Lemma 5.2, consider the closed set $\mathcal{E}^z\subset M_1^{z,\tau}\times M_2^{z,\tau}$ defined by

$$\mathcal{E}=\left\{(\varphi_1,\varphi_2)\in M_1^{z,\tau}\times M_2^{z,\tau}:\ \varphi_1(.)=x+\int_0^. \varphi_2(s)\mathrm{d}s\right\}.$$

Note that it is invariant under the action of Ψ^z, namely $\Psi^z(\mathcal{E}^z)\subset\mathcal{E}^z$. Hence, because of the attractiveness of the unique fix point $(u(.;z),v(.;z))$ of Ψ^z, one obtains that $(u(.;z),v(.;z))\in\mathcal{E}^z$, for any $z\in B_\beta(C)$, and this completes the proof of Lemma 5.2. ■

We end this section by a direct corollary of Theorem 5.1, that is concerned with the solution of (5.46) with less smooth initial data that only belongs to X_0. To state our result we consider the problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}=Au(t)+F(u(t)),\ t>0 \text{ and } u(0)=x\in X_0. \tag{5.50}$$

Here again, we assume that $A:D(A)\subset X\to X$ satisfies Assumption 1.1 while $F:X_0^\alpha\to X$ is of the class C^1 for some given $\alpha<\frac{1}{p^*}$. As already mentioned at the beginning of this section, the above problem generates a maximal semiflow in X_0, denoted by (U_0,χ_0). We denote by $u=u(t)=U_0(t)x$ the solution of (5.50) that is defined and continuous on $[0,\chi_0(x))$. Furthermore, one has $u(t)\in X_0^\alpha$ for all $t\in(0,\chi_0(x))$. And, this function enjoys the following regularity properties.

Theorem 5.5 *Let $x\in X_0$ be given. Consider the solution $u=u(t):=U_0(t)x$ defined for $t\in[0,\chi_0(x))$. Then it enjoys the following properties:*

(i) One has $u(t)\in X_0^\alpha$ for all $t\in(0,\chi_0(x))$ and for each $t\in(0,\chi_0(x))$

$$\lim_{h\to 0,\,h>0}\|u(t+h)-u(t)\|_\alpha=0.$$

(ii) If we consider the set $\mathcal{D}$ of – left – discontinuity points of u defined by

$$\mathcal{D}=\left\{t\in(0,\chi_0(x)):\ \limsup_{h\to 0^+}\|u(t-h)-u(t)\|_\alpha>0\right\}.$$

Then $(0,\chi_0(x))\setminus\mathcal{D}\neq\emptyset$ and for each $t\in\mathcal{D}$ there exists $\varepsilon_t>0$ such that $(t,t+\varepsilon_t)\subset(0,\chi_0(x))\setminus\mathcal{D}$ and, for all $t\in\mathcal{D}$, one has $\|u(s)\|_\alpha\to\infty$ as $s\to t$ and $s<t$.

(iii) *For any* $t_1 < t_2$ *such that* $[t_1, t_2] \subset (0, \chi_0(x)) \setminus \mathcal{D}$ *then* $u \in W^{1,r}(t_1, t_2; X_0^\alpha)$, $u(t) \in D(A)$ *a.e. for* $t \in [t_1, t_2]$ *and* $u = u(t)$ *satisfies*

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)), \text{ a.e. } t \in [t_1, t_2],$$

with $\frac{\mathrm{d}u(.)}{\mathrm{d}t} \in L^r(t_1, t_2; X_0^\alpha)$ *and* $F(u(.)) \in C([t_1, t_2]; X)$.

Proof. Recall that (U_α, χ_α) denotes the maximal semiflow associated to (5.50) in X_0^α. Let $t_0 \in (0, \chi_0(x))$ be given. Then, because of the uniqueness of the solution (see Lemma 3.10), one has

$$u(t_0 + t) = U_\alpha(t)u(t_0) \; t \in [0, \chi_\alpha(u(t_0))).$$

Hence $u(t_0 + t)$ is right continuous at $t = 0$ in X_0^α and (i) follows. Moreover we have also proved that $(0, \chi_0(x)) \setminus \mathcal{D} \neq \emptyset$ and for each $t_0 \in \mathcal{D}$ there exists $\varepsilon_{t_0} > 0$ such that $(t_0, t_0 + \varepsilon_{t_0}) \subset (0, \chi_0(x)) \setminus \mathcal{D}$. The proof for the – left – blow-up in X_0^α at the points of $\mathcal{D}$ directly follows from a continuation argument similar to the one used in the proof of Lemma 3.12. This proves (ii). Finally (iii) follows from Theorem 5.1 above. This completes the proof of the result. ∎

6 Applications

As an application of the above results and more particularly Theorem 3.9 we investigate the existence of solutions for a reaction-diffusion equation with nonlinear and nonlocal boundary conditions.

Let $q \in (1, \infty)$ be given. Let $\Omega \subset \mathbb{R}^N$ be a given bounded and smooth domain. We consider the following reaction-diffusion equation posed in Ω

$$\begin{cases} \partial_t u = \Delta u + \operatorname{div}(u\mathbf{v}), \; t > 0, \; x \in \Omega, \\ \partial_\nu u + u \; (\mathbf{v} \cdot \nu) = 0 \; t > 0, \; x \in \Gamma := \partial\Omega, \\ u(0, .) = u_0 \in L^q(\Omega). \end{cases} \tag{6.51}$$

In the above problem $\nu = \nu(x)$ denotes the unit outward normal vector at $x \in \Gamma$. The vector $\mathbf{v} = \mathbf{v}(t, x)$ denotes a velocity field that is assumed to depend on the density function u and that takes the form

$$\mathbf{v}(t, x) = [\mathbf{L}h\,(u(t, .))]\,(x), \; x \in \Omega, \tag{6.52}$$

wherein $\mathbf{L}$ is a – smoothing – bounded linear operator from $L^q(\Omega)$ into $\left(W^{1,r}(\Omega) \cap L^\infty(\Omega)\right)^N$ for some integer $r \in [1, \infty]$ and $h : \mathbb{R} \to \mathbb{R}$ denotes a continuous function. Here to handle the example presented in the introduction we assume for simplicity that the function h satisfies the following set of assumptions:

$$h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}), \; h' \in L^\infty(\mathbb{R}),$$
$$|h(u)| = O(|u|) \text{ as } u \to \pm\infty.$$

Because of this assumption, one may observe that the operator $u \mapsto h(u)$ maps $L^q(\Omega)$ into itself and it is globally Lipschitz continuous on $L^q(\Omega)$.

Remark 6.1 *Note that this set of assumptions for the function h allows us to consider the case where $h(u) = u$ but also the case presented in the introduction where $h(u) = u \max\left(0, 1 - \frac{u}{M}\right)$ for some constant $M > 0$. One may also observe that when the function $\eta = \eta(x)$ and $g = g(x)$, arising in (1.3) are smooth enough, then the assumption presented above are satisfied.*

To handle this problem we consider the Banach space

$$X = W^{1-\frac{1}{q},q}(\Gamma) \times L^q(\Omega),$$

as well as the linear operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \{0\} \times W^{2,q}(\Omega) \text{ and } A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_\nu \varphi \\ \Delta\varphi \end{pmatrix}, \ \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A).$$

Here one has $X_0 := \overline{D(A)} = \{0\} \times L^q(\Omega)$.
In the above definition the set $W^{1-\frac{1}{q},q}(\Gamma)$ represents the trace space that is defined by $\gamma_\Gamma\left(W^{1,q}(\Omega)\right)$ where γ_Γ denotes the trace operator. This boundary space becomes a Banach space when it is endowed with the trace norm defined by

$$\|\varphi\|_{W^{1-\frac{1}{q},q}(\Gamma)} = \inf\left\{\|\psi\|_{W^{1,q}(\Omega)} : \ \gamma_\Gamma\psi = \varphi\right\}.$$

Here recall also that $\gamma_\Gamma \in \mathcal{L}\left(W^{1,p}(\Omega), W^{1-\frac{1}{q},q}(\Gamma)\right)$.

Now according to the results obtained by Agranovich et al. in [1] (see also the references cited therein for a nice review on elliptic boundary estimates), the operator A satisfies Assumption 1.1 with $p^* = \frac{2q}{1+q}$.

We shall now discuss the nonlinear part associated to Problem (6.51)-(6.52). To that aim we set $Y_0 := L^q(\Omega)$ and consider $\Delta : D(\Delta) \subset Y_0 \to Y_0$ the Laplace operator on $L^q(\Omega)$ supplemented with the homogeneous Neumann boundary conditions on $\Gamma = \partial\Omega$. We this notation one has

$$D(A_0) = \{0\} \times D(\Delta) \text{ and } A_0 = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}.$$

Using this set of notations as well as the fractional spaces associated to Δ, denoted by Y_0^α for $\alpha \in [0, 1]$, the following lemma holds true.

Lemma 6.2 *Let us assume that $r > q$ and $r > \frac{N}{2}$ then, by setting $\alpha_r = \max\left(\frac{1}{2}, \frac{N}{2r}\right) \in (0, 1)$, for each $\alpha \in (\alpha_r, 1]$ the bilinear map B defined by*

$$B(\varphi, \psi) = \varphi\left[\mathbf{L}\psi\right],$$

is bounded from $Y_0^\alpha \times L^q(\Omega)$ into $\left(W^{1,q}(\Omega)\right)^N$.

Proof. First note that since $\mathbf{L} \in \mathcal{L}\left(L^q(\Omega), L^\infty(\Omega)^N\right)$ one already obtains

$$\|B(\varphi, \psi)\|_{L^q(\Omega)} \leq \|\mathbf{L}\|_{\mathcal{L}(L^q(\Omega), L^\infty(\Omega)^N)}\|\psi\|_{L^q(\Omega)}\|\varphi\|_{L^q(\Omega)}, \ \forall \varphi, \psi \in L^q(\Omega).$$

Next let $i = 1, .., N$ be given and set $D_i = \partial_{x_i}$. Then for any smooth functions φ and ψ, let's say $C^\infty(\overline{\Omega})$, one has

$$D_i B(\varphi, \psi) = D_i \varphi \, \mathbf{L}\psi + \varphi \, D_i \mathbf{L}\psi.$$

Next observe that due to Hölder inequality and since $r > q$ one has

$$\begin{aligned} \|\varphi \, D_i \mathbf{L}\psi\|_{L^q(\Omega)} &\leq \|\varphi\|_{L^{\frac{qr}{(r-q)}}(\Omega)} \|D_i \mathbf{L}\psi\|_{L^r(\Omega)} \\ &\leq \|\mathbf{L}\|_{\mathcal{L}(L^q(\Omega), W^{1,r}(\Omega)^N)} \|\varphi\|_{L^{\frac{qr}{(r-q)}}(\Omega)} \|\psi\|_{L^q(\Omega)}. \end{aligned}$$

Next recall that the continuous embeddings, proved in Theorem 1.6.1 of the monograph of Henry [14], state that for $\alpha \in [0, 1]$ one has

$$Y_0^\alpha \hookrightarrow W^{k,l}(\Omega) \text{ if } k - \frac{N}{l} < 2\alpha - \frac{N}{q}, \; l \geq q. \tag{6.53}$$

As a consequence, since $r > \frac{N}{2}$, one obtains that $Y_0^\alpha \hookrightarrow L^{\frac{qr}{r-q}}(\Omega)$ for all $\alpha \in \left(\frac{N}{2r}, 1\right]$. Next let us observe that

$$\|D_i \varphi \, \mathbf{L}\psi\|_{L^q(\Omega)} \leq \|\varphi\|_{W^{1,q}(\Omega)} \|\mathbf{L}\psi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{W^{1,q}(\Omega)} \|\mathbf{L}\|_{\mathcal{L}(L^q(\Omega), L^\infty(\Omega)^N)} \|\psi\|_{L^q(\Omega)}.$$

Now recall that due to (6.53), if $\alpha \in \left(\frac{1}{2}, 1\right]$ then $Y_0^\alpha \hookrightarrow W^{1,q}(\Omega)$.

Finally we infer from the above estimates that, for each $\alpha \in (\alpha_r, 1]$, there exists $M_\alpha > 0$ such that for all smooth functions φ and ψ one has

$$\|B(\varphi, \psi)\|_{W^{1,q}(\Omega)} \leq M_\alpha \|\varphi\|_{Y_0^\alpha} \|\psi\|_{L^q(\Omega)}.$$

This completes the proof of the lemma using a usual density argument. ■

In the rest of this section we assume that

$$r > q \text{ and } r > \frac{N}{2}, \tag{6.54}$$

and we fix $\alpha \in (\alpha_r, 1]$. Now consider the function $F : X_0^\alpha = \{0\} \times Y_0^\alpha \to X$ defined by

$$F \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \gamma_\Gamma \left(B\left(\varphi, h(\varphi)\right) \cdot \nu \right) \\ \operatorname{div} B\left(\varphi, h(\varphi)\right) \end{pmatrix}, \; \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_0^\alpha,$$

and observe that, due to Lemma 6.2 and recalling that $u \mapsto h(u)$ is globally Lipschitz continuous on $L^q(\Omega)$, this function is well defined and satisfies the following Lipschitz property.

Lemma 6.3 *There exists a constant $M > 0$ such that for all $\varphi_1, \varphi_2 \in Y_0^\alpha$ one has*

$$\left\| F \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} - F \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \right\|_X \leq M \left[\|\varphi_1\|_{Y_0^\alpha} \|\varphi_1 - \varphi_2\|_{L^q} + \|\varphi_2\|_{L^q} \|\varphi_1 - \varphi_2\|_{Y_0^\alpha} \right].$$

We are now able to come back to Problem (6.51)-(6.52). Using all the above notations and identifying the function $u = u(t, x)$ with $v(t) = \begin{pmatrix} 0 \\ u(t, .) \end{pmatrix}$, Problem (6.51)-(6.52) re-writes as the following abstract Cauchy problem

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + F(v(t)),\ t > 0 \text{ and } v(0) = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in X_0.$$

To conclude this section, we collect all the above information and we show that with a suitable choice of q, r the above abstract Cauchy problem satisfies both Assumptions 1.1 and 2.6 with $\beta = 0$ and with a suitable choice of the parameter α. Here recall that the linear operator A satisfies Assumption 1.1 with $p^* = \frac{2q}{1+q}$. Equipped with Lemma 6.3, if r and q satisfies (6.54) and

$$\alpha_r < \frac{1+q}{2q},$$

then, due to Lemma 6.3, Assumption 2.6 holds true for any pair of parameters $(\alpha, p) \in (0, 1) \times (1, \infty)$ such that $\alpha_r < \alpha < \frac{1}{p} < \frac{1+q}{2q}$.
As a consequence of Theorem 3.9 one obtains the following result:

Theorem 6.4 *Let q and r be given such that*

$$r > q > 1,\ r > \frac{N}{2} \text{ and } \frac{N}{r} < \frac{1+q}{q},$$

then Problem (6.51) *generates a maximal strongly continuous semiflow in $L^q(\Omega)$.*

Acknowledgement

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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