Travelling wave solutions for an infection-age structured model with diffusion

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Abstract

In this article, we study the existence of travelling waves for a class of epidemic model structured in space and with respect to the age of infection. We obtain a necessary and sufficient condition for the existence of travelling waves for such a class of problems. As a consequence of our main result, we also derive the existence of travelling waves of class of functional partial derivative equation.

1 Introduction

This work is devoted to the existence of travelling wave solutions for a Kermack and McKendrick's model where both infectivity and recovery can depend on the duration of infection and where individuals can diffuse in space. As in the pioneer work of Kermack and McKendrick [9] (see also Anderson [1] for a nice survey on Kermack-McKendrick models), we consider a population which is divided into the three classes, the susceptible, the infected, and the recovered. Here we assume that the total population is homogeneous in space and constant in time. This means that the model do not take into account the vital dynamics of the population, that is neither natural birth rate nor natural death rate. Here the main novelty with respect to the existing literature on the subject is that we introduce the age of infection. The age of infection was used previously in epidemic model to describe the period of latency which is necessary for an infected individuals to become infectious (see D'Agata et al. [3], and Thieme and Chavez [18][19] and references therein for a nice survey). In particular the age of infection allows us to follow the history of infected individuals. The model is following

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_s \Delta_x S - S(t,x) \displaystyle\int_0^{a_\dagger} \beta(a) i(t,a,x) da, \ x \in \mathbb{R}, \ t \geq 0, \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} = d_i \Delta_x i - (\mu_R(a) + \mu_M(a)) \, i(t,a,x), \ a \in (0, a_\dagger) \,, \ x \in \mathbb{R}, \ t \geq 0, \\ i(t,0,x) = S(t,x) \displaystyle\int_0^{a_\dagger} \beta(a) i(t,a,x) da, \ x \in \mathbb{R}, \ t \geq 0 \\ \dfrac{\partial R}{\partial t} = d_r \Delta_x R + \displaystyle\int_0^{a_\dagger} \mu_R(a) i(t,a,x) da, \\ S(0,x) = S_0(x), \ \ i(0,a,x) = i_0(a,x) \text{ and } R(0,x) = R_0(x). \end{cases} \tag{1.1}$$

where a is the time since the infection, $a_\dagger \in (0, +\infty]$ is the maximum attainable age of infection. Here $S(t,x)$, $i(t,a,x)$ and $R(t,x)$ denotes respectively the density of susceptible,infected and recovered at time t and located at $x \in \mathbb{R}$ and the age a for the density of infected. Parameter $d_s > 0$ (respectively $d_i > 0$, $d_r \geq 0$) is the diffusion coefficient of susceptible (respectively infected, recovered) individuals. The function $a \to \beta(a)$ denotes the transmission rate coefficient which is assumed to depend explicitly on the duration of infection. The function $a \to \mu(a) := \mu_R(a) + \mu_M(a)$ is the sum of the recovery rate $\mu_R(a)$ and the death rate $\mu_M(a)$ in the class of infected individuals. If the disease does not induce mortality the class R denotes the class of individuals who have recovered and are immune to reinfection. Since the function R is known as soon as the functions S and i are known, from here on we only focus on the partial differential equations

$$\begin{aligned}
&\frac{\partial S}{\partial t}=d_s\Delta_x S-S(t,x)\int_0^{a_\dagger}\beta(a)i(t,a,x)da,\ x\in\mathbb{R},\ t\geq 0,\\
&\frac{\partial i}{\partial t}+\frac{\partial i}{\partial a}=d_i\Delta_x i-\mu(a)i(t,a,x),\ a\in(0,a_\dagger),\ x\in\mathbb{R},\ t\geq 0,\\
&i(t,0,x)=S(t,x)\int_0^{a_\dagger}\beta(a)i(t,a,x)da,\ x\in\mathbb{R},\ t\geq 0\\
&S(0,x)=S_0(x),\ \text{and}\ i(0,a,x)=i_0(a,x).
\end{aligned}\tag{1.2}$$

This system without structuration in space was considered by Kermack and McKendrick in [9]. They prove the existence of an epidemic threshold parameter

$$R_0=\int_0^{a_\dagger}\beta(a)e^{-\int_0^a\mu(s)ds}da,$$

such that the infectious epidemic can spread in the population if $R_0>1$ while the infection die out when $R_0<1$.

In order to understand the role of the infectiouness function $\beta(a)$, one may first observe that the solution of the system satisfies a Volterra integral equation. So the role of the function $\beta(a)$ is to describe the intensity of the disease, the incubation period of the disease. When the incubation is exactly equal to $\tau>0$, then the function β takes the following form

$$\beta(a)=\widehat{\beta}1_{[\tau,+\infty)}(a),\forall a\geq 0.$$

Moreover if we assume in addition that $a_\dagger=+\infty$, and that the function $\mu(a)$ is constant and equal to $\widehat{\mu}$, then the system (1.2) can be rewritten as the following partial differential equation with delay (see section 2 for more details)

$$\begin{cases}
\frac{dS}{dt}=d_s\Delta_Z S(t)-\widehat{\beta}e^{-\widehat{\mu}\tau}S(t)T_{d_i\Delta_Y}(\tau)I(t-\tau),\\
\frac{dI}{dt}=d_i\Delta_Y I(t)-\widehat{\mu}I(t)+\widehat{\beta}e^{-\widehat{\mu}\tau}S(t)T_{d_i\Delta_Y}(\tau)I(t-\tau),\\
S(\theta)=S_0(\theta),I(\theta)=I_0(\theta),\ \forall\theta\in[-\tau,0].
\end{cases}\tag{1.3}$$

where

$$I(t)=\int_0^{+\infty}i(t,a)da,$$

and

$$T_{d_i\Delta_Y}(\tau)(\varphi)(x) = \frac{1}{\sqrt{4\pi d_i\tau}}\int_{\mathbb{R}}\varphi(x-y)e^{-|y|^2/4d_i\tau}dy.$$

So we obtain an integro-differential partial derivative equation with delay. Recently such a spatially structured epidemic system with delay has been extensively studied in the litterature (see Wang Li and Ruan [21] for a nice survey). But as far as we know, the above system has not been considered.

System (1.2) was also extensively studied without infection age structure (with corresponds to the case $\tau = 0$ in system (1.3)). A particular interest has been given to the study of long time behaviour of the system and to travelling wave solutions. These questions have been partially solved by Kallen [7], and Kallen et al. [8] who prove the existence of travelling wave solutions when succeptible individuals cannot spread in space. This model has been used to study rabies epizootic. The travelling wave solutions have been investigated for the model with both diffusion of succeptibles and infected by Hosono and Ilyas in [5].

In this work we focus on travelling wave solutions for problem (1.2). The mathematical arguments of phase plane analysis used by Hosono et al. [5] cannot be applied for system (1.2). Moreover the system does not have any comparison principle and monotonic properties. As a consequence the classical methods to study travelling fronts solutions cannot be applied (see for instance [4, 6, 10, 11, 20, 21, 23, 24, 25] and references therein). Nevertheless, each equation of the system admits separately some monotonic properties. Following the idea proposed in [2], we will take into account this particular form in order to construct some invariant convex set for some suitable operator. The problem will then become a fixed point problem on finite intervals. The main difficulty is to obtain some a priori estimations which are independent of the length of the interval in order to apply a limit procedure (see section 4 for more precision).

The plan of the paper is the following. In section 2, we describe the evolution problem associated to system (1.2). In particular, we establish the existence and uniqueness of mild solutions for this system. We also describe the relationship between system (1.2) and the PDE systems with delay (1.3). In section 3, we present the main result of this paper, and also derive a corollary for a class of PDE with delay. More specifically we show that the system of partial differential equation has positive travelling wave solutions if and only if $R_0 > 1$.

2 Preliminary

In this section we consider system (1.2) as an evolution problem. We will make the following assumption.

Assumption 2.1: $d_s > 0$, $d_i > 0$, $\beta \in L_+^\infty\left((0, a_\dagger), \mathbb{R}\right)$, and $\mu \in L_{Loc,+}^1\left([0, a_\dagger), \mathbb{R}\right)$.

We denote by $BUC\left(\mathbb{R}\right)$ the space of bounded and uniformly continuous map from $\mathbb{R}$ into itself, endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{R}} |\varphi(x)| .$$

We denote by

$$BUC_+\left(\mathbb{R}\right) = \left\{\varphi \in BUC\left(\mathbb{R}\right) : \varphi\left(s\right) \geq 0, \forall s \in \mathbb{R}\right\}.$$

We consider the Laplacian operator Δ as a linear operator from

$$D(\Delta) = \left\{\varphi \in BUC\left(\mathbb{R}\right) \cap C^2\left(\mathbb{R}, \mathbb{R}\right) : \varphi'' \in BUC\left(\mathbb{R}\right)\right\},$$

into $BUC\left(\mathbb{R}\right)$. It is readily checked that $(0, \infty) \subset \rho\left(\Delta\right)$ the resolvent set of Δ, and

$$\left(\lambda - \Delta\right)^{-1}\left(\varphi\right)\left(x\right) = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|s|} \varphi(s + x) ds, \forall \lambda > 0.$$

It is well known that Δ is the infinitesimal generator of $\left\{T_\Delta(t)\right\}_{t \geq 0}$ a positive analytic semigroup of contraction on $BUC\left(\mathbb{R}\right)$, and for $t > 0$,

$$T_\Delta(t)\left(\varphi\right)\left(x\right) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \varphi(x - y) e^{-|y|^2/4t} dy. \tag{2.1}$$

It follows that for each $d > 0$, $d\Delta$ is the infinitesimal generator of $\left\{T_{d\Delta}(t)\right\}_{t \geq 0}$, with $T_{d\Delta}(t) = T_\Delta(dt), \forall t \geq 0$. We set

$$Z := \left\{\varphi \in C\left(\mathbb{R}, \mathbb{R}\right) : \lim_{x \to +\infty} \varphi(x) \text{ and } \lim_{x \to -\infty} \varphi(x) \text{ exist}\right\},\ Z_+ := BUC_+\left(\mathbb{R}\right) \cap Z,$$

and

$$Y := \left\{\varphi \in C\left(\mathbb{R}, \mathbb{R}\right) : \lim_{x \to +\infty} \varphi(x) = \lim_{x \to -\infty} \varphi(x) = 0\right\},\ \text{and } Y_+ := BUC_+\left(\mathbb{R}\right) \cap Y.$$

The subspace Z and Y are two closed subspaces of $BUC(\mathbb{R})$. So $(Z, \|.\|_\infty)$ and $(Y, \|.\|_\infty)$ are two Banach spaces. We denote by Δ_Y (resp. Δ_Z) the part of Δ in Y (resp. Z) the linear operators defined for $V = Y$ or Z by

$$\Delta_V \varphi = \Delta\varphi, \forall \varphi \in D(\Delta_V) = \{\varphi \in D(\Delta) \cap V : \Delta\varphi \in V\}.$$

We observe that

$$(\lambda - \Delta)^{-1} V \subset V, \forall \lambda > 0, \forall V = Y, Z.$$

So for each $\lambda > 0$, and $V = Y, Z$,

$$(0, \infty) \subset \rho(\Delta_V), \ D(\Delta_V) = (\lambda - \Delta)^{-1} V, \text{ and } (\lambda - \Delta_V)^{-1} = (\lambda - \Delta)^{-1} |_V .$$

It follows that for each $d > 0$, and $V = Y, Z$, $d\Delta_V$ is the infinitesimal generator of an analytic semigroup $\{T_{d\Delta_V}(t)\}_{t \geq 0}$ on V, with $T_{d\Delta_V}(t) = T_{d\Delta}(t) |_V$, $\forall t \geq 0$.

In the sequel, we consider $S(t, .)$ the first component of system (1.2) as an element of Z, and $i(t, a, .)$ the second component of system (1.2) as an element of Y. In order to express the second component of system (1.2) as an abstract evolution equation we use the approach of Thieme [15] (see also [16, 17, 13] and references therein for more detailed description of the problem). We consider the Banach space

$$W := Y \times L^1((0, \infty), Y),$$

endowed with the usual product norm. We consider

$$W_+ = Y_+ \times L^1((0, \infty), Y_+), \ W_0 := \{0_Y\} \times L^1((0, \infty), Y),$$
$$\text{and } W_{0+} := W_0 \cap W_+.$$

Then the family of bounded linear operators $\{R_\lambda\}_{\lambda > 0}$ on W, defined by

$$R_\lambda \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$
$$\Leftrightarrow \varphi(a) = e^{-\int_0^a \mu(l) + \lambda dl} T_{d_i\Delta_Y}(a)\alpha + \int_0^a e^{-\int_s^a \mu(l) + \lambda dl} T_{d_i\Delta_Y}(a - s)\psi(s) ds.$$

One may observe that $\{R_\lambda\}_{\lambda > 0}$ is a pseudo-resolvent on W, that is to say that

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \forall \lambda, \mu > 0.$$

Moreover we have

$$R_\lambda x = 0, \text{ and } x \in W \Rightarrow x \in W_0,$$

and

$$\lim_{\lambda \to +\infty} \lambda R_\lambda x = x, \forall x \in W_0.$$

By using similar arguments as in Pazy [14, p. 36-37], we deduce that there exists a unique closed linear $\mathcal{B} : D(\mathcal{B}) \subset W \to W$, with $\overline{D(\mathcal{B})} = W_0$, and $R_\lambda = (\lambda I - \mathcal{B})^{-1}, \forall \lambda > 0$. We set

$$\begin{aligned} X &: = Z \times W, \ X_0 := Z \times W_0, \\ X_+ &: = Z_+ \times W_+, \text{ and } X_{0+} = X_+ \cap X_0, \end{aligned}$$

and we consider $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ the linear operator defined by

$$\mathcal{A} \left(\begin{array}{c} \varphi \\ \left(\begin{array}{c} 0 \\ \psi \end{array} \right) \end{array} \right) = \left(\begin{array}{c} d_s \Delta_Z \varphi \\ \mathcal{B} \left(\begin{array}{c} 0 \\ \psi \end{array} \right) \end{array} \right) \text{ with } D(\mathcal{A}) = D(\Delta_Z) \times D(\mathcal{B}).$$

We also define $F : X_0 \to X$

$$F \left(\begin{array}{c} \varphi \\ \left(\begin{array}{c} 0 \\ \psi \end{array} \right) \end{array} \right) = \left(\begin{array}{c} -\eta \varphi \mathcal{F}_\beta(\psi) \\ \left(\begin{array}{c} \eta \varphi \mathcal{F}_\beta(\psi) \\ 0 \end{array} \right) \end{array} \right),$$

with

$$\mathcal{F}_\gamma(\varphi)(x) := \int_0^{a_\dagger} \gamma(a)\varphi(a)da, \forall \gamma \in L^\infty((0, a_\dagger), \mathbb{R}), \forall \varphi \in L^1((0, a_\dagger), Y).$$

Then the system can be re-written as the following abstract Cauchy problem

$$\frac{du}{dt} = \mathcal{A}u(t) + F(u(t)), t \geq 0, \text{ with } u(0) = x \in X_{0+}. \tag{2.2}$$

We note that F is Lipschitz on bounded sets of X_0, and for each $M > 0$, there exists $\lambda > 0$, such that

$$F(x) + \lambda x \in X_+, \forall x \in X_{0+} \cap B_X(0, M).$$

By using the fact that $\mathcal{A}$ is a Hille-Yosida operator, and by using integrated semigroup theory (see [15, 12] and references therein). We deduce the following results.

Theorem 2.1 *Let Assumption 2.1 be satisfied. There exist* $\{U(t)\}_{t\geq 0}$ *a* C_0*-semigroup of continuous nonlinear operator on* X_{0+}*, such that for each* $x \in X_{0+}$*, the map* $t \to U(t)x$ *is the unique mild solution (2.2), that is to say that satisfied*

$$\int_0^t U(s)x ds \in D(\mathcal{A}), \quad \forall t \geq 0,$$

and

$$U(t)x = x + \mathcal{A}\int_0^t U(s)xds + \int_0^t F(U(s)x)ds, \ \forall t \geq 0.$$

Volterra Formulation:

But using Laplace's transformed arguments, one may establish that the mild solutions of (2.2) take the following form

$$U(s)x = \begin{pmatrix} S(t,.) \\ 0_Y \\ i(t,.,.) \end{pmatrix},$$

where $S(t)$ and $i(t)$ satisfy the following Volterra formulation of the problem (1.2)

$$\begin{cases} S(t) = T_{d_s\Delta_Z}(t)S_0 + \int_0^t T_{d_s\Delta_Z}(t-s)\left[-S(s)\int_0^{a_\dagger}\beta(a)i(s,a,.)da\right]ds \\ i(t,a) = \begin{cases} e^{-\int_{a-t}^a \mu(l)dl}T_{d_i\Delta_Y}(t)i_0(a-t), \text{ if } a \geq t, \\ e^{-\int_0^a \mu(l)dl}T_{d_i\Delta_Y}(a)B(t-a), \text{ if } t \geq a, \end{cases} \end{cases} \tag{2.3}$$

and the map $B(.) \in C([0,+\infty), Y)$ is the unique solution of the following Volterra integral equation

$$B(t) = S(t)\left[\begin{array}{c} \int_{\min(t,a_\dagger)}^{a_\dagger}\beta(a)e^{-\int_{a-t}^a \mu(l)dl}T_{d_i\Delta_Y}(t)i_0(a-t)da \\ +\int_0^{\min(t,a_\dagger)}\beta(a)e^{-\int_0^a\mu(l)dl}T_{d_i\Delta_Y}(a)B(t-a)da \end{array}\right]. \tag{2.4}$$

In order to derive the PDE with delay, we will make the following assumption.

Assumption 2.2: *We assume that* $a_\dagger = +\infty$*, and there exist* $\widehat{\beta} \geq 0, \tau > 0$*, and* $\widehat{\mu} > 0$*, such that*

$$\beta(a) = \widehat{\beta}1_{[\tau,+\infty)}(a), \text{ and } \mu(a) = \widehat{\mu}, \forall a \geq 0.$$

From here on we set

$$I(t) := \int_0^{+\infty} i(t,a)da.$$

Let Assumption 2.2 be satisfied. Then the system reduces to

$$\begin{aligned} S(t) &= T_{d_s\Delta_Z}(t)S_0 + \int_0^t T_{d_s\Delta_Z}(t-s)\left[-S(s)\int_0^{+\infty}\beta(a)i(s,a,.)da\right]ds, \\ I(t) &= T_{d_i\Delta_Y-\mu I}(t)I(0) + \int_0^t T_{d_i\Delta_Y-\mu I}(t-s)S(s)\int_0^{+\infty}\beta(a)i(s,a,.)da, \end{aligned}$$

and $t \to B(t)$ satisfies

$$B(t) = S(t)\left[T_{d_i\Delta_Y-\mu I}(t)\int_t^{+\infty}\beta(a)i_0(a-t)da + \int_0^t \beta(a)T_{d_i\Delta_Y-\mu I}(a)B(t-a)da\right].$$

It follows that for $t \geq \tau$,

$$B(t) = S(t)T_{d_i\Delta_Y-\mu I}(\tau)I(t-\tau).$$

So for $t \geq \tau$, $S(t)$ and $I(t)$ is a mild solution of the PDE with delay (see Wu [22] for a nice survey on the subject)

$$\begin{cases} \dfrac{dS}{dt} = d_s\Delta_Z S(t) - \widehat{\beta}e^{-\widehat{\mu}\tau}S(t)T_{d_i\Delta_Y}(\tau)I(t-\tau), \\ \dfrac{dI}{dt} = d_i\Delta_Y I(t) - \widehat{\mu}I(t) + \widehat{\beta}e^{-\widehat{\mu}\tau}S(t)T_{d_i\Delta_Y}(\tau)I(t-\tau), \\ S(\theta) = S_0(\theta), I(\theta) = I_0(\theta), \ \forall\theta \in [-\tau, 0]. \end{cases} \tag{2.5}$$

where $T_{d_i\Delta_Y}(\tau) = T_\Delta(d_i\tau)$ and $T_\Delta(t)$ is given by formula (2.1).

3 Main results

In order to investigate the travelling wave of system (1.2), it is sufficient to consider the following system

$$\begin{cases} \dfrac{\partial S}{\partial t} = d\Delta_x S - S(t)\mathcal{F}_\gamma(i(t)), \\ \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} = \Delta_x i(t,a,x), \ a \in (0, a_\dagger), \\ i(t,0,.) = S(t)\mathcal{F}_\gamma(i(t)), \\ S(0) = S_0 \in Z, \text{ and } i(0,.,.) = i_0 \in L^1((0,a_\dagger), Y), \end{cases} \tag{3.1}$$

where $d = d_s/d_i$ and

$$\gamma(a) = \beta(a)e^{-\int_0^a \mu(s)ds} \text{ for almost every } a \in (0, a_\dagger). \tag{3.2}$$

We are looking for travelling wave solutions for the system (3.1), which are positive solutions of the form

$$S(t,x) = S(x - ct), \quad i(t,a,x) = i(a, x - ct). \tag{3.3}$$

Such solutions satisfies the following system

$$dS'' + cS' - S\mathcal{F}_\gamma(i) = 0, \tag{3.4}$$

$$\partial_a i = \partial_x^2 i + c\partial_x i, \tag{3.5}$$

$$i(0,x) = S(x)\mathcal{F}_\gamma(i)(x). \tag{3.6}$$

This system is posed on the whole real line $x \in \mathbb{R}$, and is supplemented with the following conditions at infinity

$$S(-\infty) = S^+, \quad S(+\infty) = 1,$$

$$i(a, -\infty) = 0, \quad i(a, +\infty) = 0.$$

Here $c > 0$ and $S^+ \in [0,1)$ are unknown numbers that should be found together with the unknown functions i and S. The parameter c is the wave speed while S^+ describes the severity of the epidemic. It is the density of susceptible individuals after the epidemic.

In the following we suppose that the following assumption holds:

Assumption 3.1: $d > 0$, $a_\dagger \in (0, +\infty]$, $\gamma \in L^1_+([0, a_\dagger), \mathbb{R}) \cap L^\infty_+([0, a_\dagger), \mathbb{R})$.

From now on, we set

$$R_0 := \int_0^{a_\dagger} \gamma(s)ds. \tag{3.7}$$

The main result of this paper is the following theorem.

Theorem 3.1 *Let Assumption 3.1 be satisfied. Then the system (3.1) has a positive travelling wave if and only if* $R_0 > 1$.

Remark 3.2 *When $R_0 > 1$ we can easily prove by integrating equation (3.5) over $\mathbb{R}$ that the limit of the travelling wave $S^+ = S(-\infty)$ satisfies the inequality*

$$0 \leq S^+ \leq \frac{1}{R_0}. \tag{3.8}$$

We now turn to the existence of travelling wave for the system (2.5). We look for travelling wave solutions of the following form

$$S(t,x) = S(x-ct), \quad I(t,x) = I(x-ct).$$

with the following conditions at infinity

$$S(x) \in [0,1], \forall x \in \mathbb{R}, \ S(-\infty) = S^+ \geq 0, \ S(+\infty) = 1, \ I(-\infty) = I(+\infty) = 0.$$

Combining the transformations of section 2 to derive the PDE with delay, and Theorem 3.1 we obtain the existence of travelling wave for the PDE with delay (2.5).

Corollary 3.3 *Let Assumption 2.2 be satisfied. Then the system (2.5) has a positive travelling wave if and only if*

$$R_0 = \frac{\widehat{\beta} e^{-\widehat{\mu}\tau}}{\widehat{\mu}} > 1.$$

Note that when $R_0 > 1$ estimate 3.8 holds.

4 Proof of Theorem 3.1

It is well known that $\Delta + c\partial_x : D(\Delta) \subset BUC(\mathbb{R}) \to BUC(\mathbb{R})$ generates a positive analytic semigroup of contraction on $BUC(\mathbb{R})$. Moreover combined with (2.1), we have the following explicit formula

$$T_{\Delta + c\partial_x}(t)(\varphi)(x) = T_{c\partial_x}(t)(T_\Delta(t)(\varphi))(x) = T_\Delta(t)(\varphi)(x+ct).$$

As before we denote by $(\Delta + c\partial_x)_Y$ the part of $\Delta + c\partial_x$ in Y, which generates a positive analytic semigroup of contraction $\left\{T_{(\Delta+c\partial_x)_Y}(t)\right\}_{t\geq 0}$ and $T_{(\Delta+c\partial_x)_Y}(t) = T_{(\Delta+c\partial_x)}(t)\mid_Y, \forall t \geq 0$.

In order to prove Theorem 3.1, it is sufficient to investigate the following system

$$dS''(x) + cS'(x) - S(x)J(x) = 0, \tag{4.1}$$

$$B(x) = S(x)J(x), \tag{4.2}$$

with

$$J(x) := \int_0^{a_\dagger} \gamma(a) T_{(\Delta + c\partial_x)_Y}(a)\,(B)\,(x) da, \tag{4.3}$$

and with the following constraint

$$\begin{aligned} &S(x) \in [0,1], \forall x \in \mathbb{R},\ S(-\infty) = S^+,\ S(+\infty) = 1, \\ &B(-\infty) = B(+\infty) = 0. \end{aligned} \tag{4.4}$$

Before starting the proof, we give some explanations on the different steps of the proof. The fact that $R_0 > 1$ is a necessary condition for the existence of travelling waves is relatively easy to prove. The main difficulty here is to prove that this condition is sufficient. To do so, we will use the following procedure. We first construct some suitable sub and super-solutions for problem (3.4)-(3.6) together with the corresponding limit behavior at infinity. Then we consider a similar problem posed on a bounded domain. The boundedness of the domain ensures the compactness for some operators and allows us to use some classical fixed point arguments. Finally we let the length of the bounded domain tending to infinity. This limit procedure requires to obtain some estimates of the solutions that are independent of the length of the bounded domain. Finally the sub and super-solutions allow us to avoid some possible degeneracy during the limit procedure.

4.1 Non existence results: $R_0 \leq 1$

This section concerns the nonexistence results claimed in Theorem 3.1 and in Corollary 3.3.

Proposition 4.1 *Let Assumption 3.1 be satisfied. Assume that $R_0 \leq 1$. Then for any $c \geq 0$, the trivial solution $(S \equiv 1,\ i \equiv 0)$ is the unique positive solution of system (3.4)-(3.6).*

Proof. Consider the equation

$$B(x) = S(x) \int_0^{a_\dagger} \gamma(a) T_{(\Delta + c\partial_x)_Y}(a)\,(B)\,(x) da.$$

Since $B(x)$ is a bounded function tending to zero at infinity, we can consider $x_0 \in \mathbb{R}$ such that

$$B(x_0) = \sup_{x\in\mathbb{R}} |B(x)|,$$

and since $\left\{T_{\Delta + c\frac{\partial}{\partial x}}(t)\right\}_{t\geq 0}$ is a contraction semigroup, we obtain

$$B(x_0) = S(x_0) \int_0^{a_\dagger} \gamma(a) T_{(\Delta + c\partial_x)_Y}(a)\,(B)\,(x_0)da \leq S(x_0) R_0 ||B||_\infty,$$

so

$$||B||_\infty \leq S(x_0) R_0 ||B||_\infty. \tag{4.5}$$

Assume that $B(x_0) > 0$. Since $S(x)$ is a bounded solution of (4.1), we have

$$S'(x) = \frac{1}{d} \int_{-\infty}^{x} e^{-\frac{c}{d}(x-l)} B(l) dl,$$

and by integrating this formula between x and y, we obtain

$$S(x) - S(y) = \frac{1}{d} \int_0^{+\infty} e^{-\frac{c}{d}r} \int_{y-r}^{x-r} B(l) dl dr,$$

and it follows that S strictly increasing over $[x_0, +\infty)$. So $S(x_0) < 1$. Now by using (4.5) we obtain $||B||_\infty = 0$. ∎

We can also prove a similar non existence result for system 2.5. More precisely we have

Proposition 4.2 *Let Assumption 2.2 be satisfied. Assume that* $R_0 = \frac{\hat{\beta}}{\hat{\mu}} e^{-\tau \hat{\mu}} \leq 1$. *Then for any* $c \geq 0$, *the trivial solution* $(S \equiv 1,\ i \equiv 0)$ *is the unique positive travelling wave solution of system 2.5.*

Proof. This proof uses similar arguments as in the proof of Proposition 4.1. ∎

4.2 Building of sub and super-solutions

In this section we construct suitable sub and super-solutions that will be essential to prove the sufficient condition in Theorem 3.1. Trought this section we suppose that $R_0 > 1$, with R_0 defined in (3.7). Then since $R_0 > 1$ and $\gamma \in L^1_+(0, a_\dagger)$, we can find $\alpha^* > 0$ satisfying of the following integral equation

$$\int_0^{a_\dagger} \gamma(a) e^{-\alpha^* a} da = 1. \tag{4.6}$$

Then we have the following lemma.

Lemma 4.3 *Let Assumption 3.1 be satisfied, and assume that* $R_0 > 1$. *For each* $c > 2\sqrt{\alpha^*}$, *we set*

$$\lambda^* = \frac{c - \sqrt{c^2 - 4\alpha^*}}{2} \in \left(0, \frac{c}{2}\right),$$

and

$$\bar{j}^+(a, x) := e^{-\lambda^* x} e^{-\alpha^* a} = e^{-\lambda^* x} e^{\left(\lambda^{*2} - c\lambda^*\right) a}. \tag{4.7}$$

Then $\bar{j}^+$ *satisfies the following equation*

$$\partial_a j = \partial_x^2 j + c\partial_x j, \quad j(0, x) = \int_0^{a_\dagger} \gamma(a) j(a, x) da. \tag{4.8}$$

Proof. The proof is trivial. ■

Next we have the following lemma.

Lemma 4.4 *Under the same assumptions and notations of the Lemma 4.3. For each* $\gamma^* > 0$ *sufficiently small, and* $\beta > 1$ *a large enough, the map* $\underline{s}^+$ *defined by*

$$\underline{s}^+(x) := 1 - \beta e^{-\gamma^* x}, \tag{4.9}$$

satisfies the following differential inequality

$$ds'' + cs' - e^{-\lambda^* x} s \geq 0, \quad s(+\infty) = 1. \tag{4.10}$$

Proof. The inequality (4.10) is equivalent to

$$c\gamma^* \beta \geq d\beta\gamma^{*2} + e^{-\lambda^* x}(e^{\gamma^* x} - \beta).$$

When $\gamma^* \leq \lambda^*$ the function $h(x) = e^{-\lambda^* x}(e^{\gamma^* x} - \beta)$ is non-increasing. So for $x \geq 0$, this inequality will be satisfied if

$$0 < \gamma^* \leq \lambda^* \text{ and } c\gamma^* \beta \geq d\beta\gamma^{*2} + (1 - \beta),$$

so it is sufficient to verify

$$0 < \gamma^* < \min(\lambda^*, \frac{c}{d}), \text{ and } \beta \geq 1/(c\gamma^* - d\gamma^{*2}).$$

When $x < 0$, then the inequality (4.10) holds if

$$\beta e^{-\lambda^* x} + c\gamma^*\beta \geq d\beta\gamma^{*2} + e^{-(\lambda^*-\gamma^*)x}$$

This last inequality holds true if $\beta \geq 1$ and $\gamma^* < \min(\lambda^*, \frac{c}{d})$. ■

Finally we have the following.

Lemma 4.5 *Under the same assumptions and notations of the Lemmas 4.3 and 4.4. For each $\eta > 0$ sufficiently small and each $k > 1$ sufficiently large the function $\underline{j}^+$ defined by*

$$\underline{j}^+(a,x) := e^{-\lambda^* x} e^{\left(\lambda^{*2} - c\lambda^*\right)a} - k e^{-(\lambda^*+\eta)x} e^{((\lambda^*+\eta)^2 - c(\lambda^*+\eta))a}, \tag{4.11}$$

satisfies the following differential inequality

$$\frac{\partial j}{\partial a} = j'' + cj', \quad j(0,x) \leq (1 - \beta e^{-\gamma^* x})^+ \int_0^{a_\dagger} \gamma(a) j(a,x) da. \tag{4.12}$$

Proof. Let us first note that the PDE in (4.12) is satisfied for any η and k. So it remains to verify the inequality in (4.12). Since $(\lambda^{*2} - c\lambda^*) = -\alpha^*$, this inequality will be satisfied if and only if

$$e^{-\lambda^* x} - k e^{-(\lambda^*+\eta)x} \leq (1 - \beta e^{-\gamma^* x})^+ \left[e^{-\lambda^* x} - k e^{-(\lambda^*+\eta)x} \alpha(\eta) \right]$$

where

$$\alpha(\eta) = \int_0^{a_\dagger} \gamma(a) e^{((\lambda^*+\eta)^2 - c(\lambda^*+\eta))a} da.$$

So we must verify that

$$1 - k e^{-\eta x} \leq (1 - \beta e^{-\gamma^* x})^+ \left[1 - k e^{-\eta x} \alpha(\eta) \right]. \tag{4.13}$$

We note that

$$\alpha(0) = 1,$$

and

$$\alpha'(\eta) = (2(\lambda^* + \eta) - c) \int_0^{a_\dagger} a\gamma(a) e^{((\lambda^*+\eta)^2 - c(\lambda^*+\eta))a} da < 0,$$

whenever

$$(\lambda^* + \eta) < \frac{c}{2}.$$

Since $\lambda^* < \frac{c}{2}$, this last inequality is satisfied for each $\eta > 0$ sufficiently small. So for each $\eta > 0$ small enough we have

$$0 < \alpha(\eta) < 1 \text{ and } \eta < \gamma^*.$$

Let $x_0 \in \mathbb{R}$ be fixed such that

$$1 - \beta^{-\gamma^* x_0} = 0.$$

Let us first consider the case $x \leq x_0$. Then the inequality (4.13) is equivalent to

$$1 - ke^{-\eta x} \leq 0,$$

which is true for each k sufficiently large. Next consider the case $x \geq x_0$. Then the inequality (4.13) is equivalent to

$$1 - ke^{-\eta x} \leq (1 - \beta e^{-\gamma^* x}) \left[1 - ke^{-\eta x}\alpha(\eta)\right]$$

which is equivalent to

$$k\left(\alpha(\eta) - 1\right) \leq \beta e^{-\gamma^* x}\left[k\alpha(\eta) - e^{\eta x}\right] \tag{4.14}$$

On the other hand function $g(x) := e^{-\gamma^* x}(\alpha(\eta)k - e^{\eta x})$ achieves its maximum at a point x_k on $\mathbb{R}$ and x_k satisfies the following equation:

$$\frac{\alpha(\eta)k\gamma^*}{\gamma^* - \eta} = e^{\eta x_k}.$$

We obtain that $x_k \to +\infty$ as $k \to +\infty$. So let us choose k large enough to have $x_k > x_0$, and we obtain that for any $x \geq x_0$, $g(x) \geq \min(0, g(x_0))$. The function g is increasing from $-\infty$ to $g(x_k)$ on $(-\infty, x_k)$, and decreasing from $g(x_k)$ to 0 on $[x_k, +\infty)$. Finally, since the right hand side of inequality (4.14) is negative, it is sufficient to find k large enough and satisfying $g(x_0) \geq 0$. That can be re-written as

$$\alpha(\eta)k - e^{\eta x_0} > 0,$$

this last inequality holds true for k sufficiently large. This completes the proof. ■

4.3 A similar problem on a bounded domain

In the sequel we suppose that $R_0 > 1$ and we fix $c > 2\sqrt{\alpha^*}$ with α^* defined in (4.6). We consider the following functions

$$\begin{aligned} &\bar{j}(a,x) = \bar{j}^+(a,x), \;\; \underline{j}(a,x) = \max(0, \underline{j}^+)(a,x), \\ &\bar{S}(x) = 1, \;\; \underline{S}(x) = \max(0, s^+(x)), \end{aligned}$$

where the functions $\bar{j}^+$, $\underline{j}^+$ and s^+ are defined in Lemmas 4.3, 4.5 and 4.4.

Let $X > 0$ be given and consider the following problem posed in the domain $(-X, X)$:

$$\begin{aligned} &\frac{\partial i}{\partial a} = i'' + ci', \\ &i(a, \pm X) = \underline{j}(a, \pm X), \;\; i(0,x) = S(x)J(x), \end{aligned} \tag{4.15}$$

$$dS'' + cS' = SJ(x), \;\; S(\pm X) = \underline{S}(\pm X). \tag{4.16}$$

where we have set $J(x) = \int_0^{a_\dagger} \gamma(a)i(a,x)da$. In equations (4.15)-(4.16) prime denotes the derivative with respect to x.

First note that when X is sufficiently large $\underline{j}(a, -X) \equiv 0$ and $\underline{S}(-X) = 0$. Therefore we introduce $X_0 > 0$ such that for any $X \geq X_0$, $\underline{j}(a, -X) \equiv 0$ and $\underline{S}(-X) = 0$. Then we will prove the following result:

Proposition 4.6 *Let Assumptions 3.1 be satisfied. Assume in addition that $R_0 > 1$. Then for any $X > X_0$, problem (4.15)-(4.16) has a classical solution (i, S) satisfying*

$$\begin{aligned} &\underline{j}(a,x) \leq i(a,x) \leq \bar{j}(a,x), \;\; \forall (a,x) \in [0, a_\dagger) \times (-X, X), \\ &\underline{S}(x) \leq S(x) \leq 1, \;\; \forall x \in (-X, X). \end{aligned} \tag{4.17}$$

Moreover function S is increasing.

Proof. We start by investigating the existence of the solution. We first re-formulate problem (4.15)-(4.16) as a fixed point problem. For that purpose let us introduce the following parabolic initial data problem

$$\frac{\partial i}{\partial a} = i'' + ci', \;\; i(a, \pm X) = \underline{j}(a, \pm X), \;\; i(0,x) = i_0. \tag{4.18}$$

Let us denote by i the solution of the above linear problem. Next we consider S a solution of the linear elliptic problem:

$$dS'' + cS' = SJ(x), \quad S(\pm X) = \underline{S}(\pm X), \tag{4.19}$$

where $J(x) = \int_0^{a_\dagger} \gamma(a)i(a,x)da$. Finally problem (4.15)-(4.16) is equivalent to the following one:

$$i_0(x) = S(x) \int_0^{a_\dagger} \gamma(a)i(a,x)da.$$

In order to solve this problem we introduce the following closed and convex subset E of the continuous functions on the compact set $[-X, X]$:

$$E = \{i_0 \in C([-X,X]), \quad \underline{j}(0,x) \leq i_0(x) \leq \bar{j}(0,x)\}.$$

Next we now consider this fixed point problem and we consider the operator

$$\Phi : i_0 \in E \to S \int_0^{a_\dagger} \gamma(a)i(a,.)da,$$

where i is the solution of

$$\frac{\partial i}{\partial a} = i'' + ci', \quad i(a, \pm X) = \underline{j}(a, \pm X), \quad i(0,x) = i_0(x)$$

and then S is the solution of

$$dS'' + cS' = S \int_0^{a_\dagger} \gamma(a)i(a,.)da, \quad S(\pm X) = \underline{S}(\pm X).$$

Let us first show that operator Φ is a compact operator from E into $C([-X,X])$. We first note function i can be re-written as

$$i(a,x) = T_{\Delta + c\partial_x}(a)\,(i_0) + \widehat{i}(a)(\underline{j}(.,\pm X))$$

where $\widehat{i}(a,x)$ satisfies

$$\frac{\partial i}{\partial a} = i'' + ci', \quad i(a, \pm X) = \underline{j}(a, \pm X), \quad i(0,x) = 0,$$

and the last semigroup $\{T_{\Delta + c\partial_x}(t)\}_{t\geq 0}$ is generated $\Delta + c\partial_x$ with Dirichlet boundary conditions. It follows

$$\int_0^{a_\dagger} \gamma(a)i(a,.)da = \lim_{\varepsilon \to 0} \int_\varepsilon^{a_\dagger} \gamma(a)T_{\Delta + c\partial_x}(a)\,(i_0)\,da + \int_0^{a_\dagger} \gamma(a)\widehat{i}(a,.)da,$$

and since $\gamma \in L^\infty$ the limit converges uniformly with respect to i_0 in bounded sets. But the linear operator

$$\int_\varepsilon^{a_\dagger} \gamma(a) T_{\Delta + c\partial_x}(a)\,(i_0)\,da = T_{\Delta + c\partial_x}(\varepsilon) \int_\varepsilon^{a_\dagger} \gamma(a) T_{\Delta + c\partial_x}(a-\varepsilon)\,(i_0)\,da$$

is compact (since $T_{\Delta + c\partial_x}(\varepsilon)$ is compact). It follows that the operator $i_0 \to \int_0^{a_\dagger} \gamma(a) i(a,.)da$ is compact from $C([-X,X])$ into itself. Moreover, from standard elliptic estimates, we also obtain that the operator $J \in E \to S$ is bounded from $C([-X,X])$ into $C^1([-X,X])$. Finally, we obtain that Φ is completely continuous from E into $C([-X,X])$.

Next let us prove that $\Phi(E) \subset E$. This result follows from successive applications of the comparison principle. Indeed let $i_0 \in E$ be given. From $i_0 \geq 0$ and $i(a, \pm X) \geq 0$ we obtain from the comparison principle that $i(a,x) \geq 0$ for any (a,x). Therefore $J(x) \geq 0$ and the maximum principle applies to equation (4.19). Since $0 \leq S(\pm X) \leq 1$, we obtain that $0 \leq S(x) \leq 1$ for any $x \in [-X,X]$.

Then, since functions $\bar{j}^+$ satisfy equation (4.8), $i_0(x) = i(0,x) \leq \bar{j}^+(0,x)$ and $i(a,\pm X) \leq \bar{j}^+(a, \pm X)$ we obtain from the comparison principle that

$$i(a,x) \leq \bar{j}(a,x), \quad \text{for any } (a,x) \in [0,a_\dagger) \times [-X,X]. \tag{4.20}$$

On the other hand, since function $\underline{j}^+$ satisfies (4.12), $i_0(x) = i(0,x) \geq \underline{j}^+(0,x)$ and $i(a,\pm X) \geq \underline{j}^+(a,\pm X)$, we obtain that

$$i(a,x) \geq \underline{j}(a,x), \quad \text{for any } (a,x) \in [0,a_\dagger) \times [-X,X]. \tag{4.21}$$

From inequality (4.21), we obtain that

$$\begin{aligned} J(x) &= \int_0^{a_\dagger} \gamma(a) i(a,x) da \\ &\leq \int_0^{a_\dagger} \gamma(a) \bar{j}^+(a,x) da \leq e^{-\lambda^* x}. \end{aligned}$$

As a consequence, function S satisfies the following differential inequality

$$dS'' + cS' - e^{-\lambda^* x} S \leq 0, \quad S(\pm X) = \underline{S}(\pm X),$$

Next recalling that function s^+ defined in Lemma 4.4 satisfies inequality (4.10), we obtain the following equation for $w = S - s^+$:

$$\begin{aligned} &dw'' + cw' - e^{-\lambda^* x} w \leq 0, \\ &w(-X) = S(-X) - s^+(-X) > 0, \;\; w(X) = S(X) - s^+(X) = 0. \end{aligned}$$

We conclude from the maximum principle that $w \geq 0$, that is $S \geq s^+$. Since $S \geq 0$, we obtain that $S \geq \underline{S}$.

Now we can easily conclude that operator Φ maps E into E. Indeed, from (4.21) and $S \leq 1$, we obtain that

$$\Phi(i_0)(x) = S(x) \int_0^{a_\dagger} \gamma(a) i(a,x) da \leq \int_0^{a_\dagger} \gamma(a) \bar{\imath}(a,x) da = \bar{\jmath}(0,x).$$

Next, from inequality (4.21) and $S \geq \underline{S}$ we obtain that

$$\Phi(i_0)(x) = S(x) \int_0^{a_\dagger} \gamma(a) i(a,x) da \geq \underline{S}(x) \int_0^{a_\dagger} \gamma(a) \underline{j}(a,x) da \geq \underline{j}(0,x).$$

This concludes the proof of $\Phi(E) \subset E$. Now from Schauder theorem we obtain the existence of a fixed point for the map Φ, that is $i_0 \in E$ satisfying $i_0 = \Phi(i_0)$.

In order to obtain the regularity of the solution we use bootstrapping arguments. From parabolic and elliptic regularity we obtain that functions (i, S) defined by the resolution of (4.18) and (4.19) satisfy problem (4.15)-(4.16). Let $\varepsilon \in (0, a_\dagger)$ be fixed. Since $i_0 \in C([-X, X])$, we obtain that $i \in L^{2-\eta}((0,\varepsilon), W^{1,p}(-X,X)) \cap L^\infty((\varepsilon, a_\dagger), W^{1,p}(-X,X))$ for any $p \in (1, +\infty)$ and any $\eta \in (0,1)$. Since $\gamma \in L^1 \cap L^\infty$, and

$$J = \int_\varepsilon^{a_\dagger} \gamma(a) i(a,x) da + \int_0^\varepsilon \gamma(a) i(a,x) da,$$

we obtain $J \in W^{1,p}(-X,X) \hookrightarrow C^\alpha([-X,X])$ for some $\alpha \in (0,1)$ if p is sufficiently large. From elliptic equation (4.19) we obtain that $S \in C^{2+\alpha}([-X,X])$. Since $i_0 = SJ$ we deduce that $i_0 \in C^\alpha([-X,X])$ and the parabolic regularity shows that $i \in L^{2-\eta}((0,\varepsilon), C^{1+\alpha}([-X,X])) \cap L^\infty((\varepsilon, a_\dagger), C^{1+\alpha}([-X,X]))$ for any $\eta \in (0,1)$. This proves that $i_0 \in C^{1+\alpha}([-X,X])$. Using the same argument as above we can show that $i_0 \in C^{2+\alpha}([-X,X])$ and therefore $i \in C^{1+\alpha/2, 2+\alpha}([0,a_\dagger) \times [-X,X])$ that proves that (i,S) is a classical solution of problem (4.15)-(4.16).

It remains to prove that function S is increasing. From $J \geq 0$ and $S \geq 0$, we obtain that

$$dS'' + cS' = SJ(x) \geq 0.$$

Therefore function S satisfies $(S'(x)e^{\frac{c}{d}x})' \geq 0$ and integrating this inequality from $-X$ to x we obtain

$$S'(x)e^{\frac{c}{d}x} \geq S'(-X)e^{-\frac{c}{d}X}.$$

Finally, recalling that $X > X_0$ we have $S(-X) = 0$. Then since $S \geq 0$ we obtain that $S'(-X) \geq 0$ and $S'(x) \geq 0$ for any $x \in [-X, X]$. This proof is completed. ■

4.4 Limit procedure $X \to +\infty$

In this section we complete the proof of Theorem 2.1. More precisely, for any $X > X_0$ we consider a solution (i, S) of problem (4.15)-(4.16) and we let $X \to +\infty$ in order to obtain a solution of problem (3.4)-(3.6) together with the associated limit behavior. Let $(X_n)_{n\geq 0}$ be a given sequence of positive number and tending to $+\infty$ as n grows. We denote by (i_n, S_n) a solution of problem (4.15)-(4.16) provided by Proposition 4.6 with $X = X_n$. Recall that $c > 2\sqrt{\alpha^*}$ is fixed. So (i_n, S_n) satisfies the following problem

$$\begin{aligned} &\frac{\partial i_n}{\partial a} = i_n'' + c i_n', \\ &i_n(a, -X_n) = 0, \;\; i_n(a, X_n) = \underline{j}(a, X_n), \\ &i_n(0, x) = S_n(x)\mathcal{F}_\gamma(i_n)(x), \end{aligned} \tag{4.22}$$

$$\begin{aligned} &dS_n'' + cS_n' = S_n \mathcal{F}_\gamma(i_n), \\ &S_n(-X_n) = 0, \;\; S_n(X_n) = \underline{S}(X_n), \end{aligned} \tag{4.23}$$

with $\mathcal{F}_\gamma(i_n)(x) = \displaystyle\int_0^{a_\dagger} \gamma(a) i_n(a, x) da$.

We also introduce the following notations

$$\omega_n = (-X_n, X_n) \;\text{ and }\; \Omega_n = (0, a_\dagger) \times (-X_n, X_n).$$

Before passing to the limit $n \to +\infty$ we will obtain some a priori estimates independent of n for the solution (i_n, S_n). For convenience in the sequel we denote by M a certain constant (which may change) but which is independent of n.

Lemma 4.7 *Let Assumption 3.1 be satisfied, and assume that $R_0 > 1$. There exist $n_0 \geq 0$ sufficiently large and some constant $M > 0$ such that for any $n \geq n_0$ we have*

$$\frac{\partial i_n}{\partial x}(a, -X_n) \geq 0, \;\; S_n'(-X_n) \geq 0,$$

and

$$\int_{\omega_n} i_n(a,x)dx + \int_0^{a_\dagger} \frac{\partial i_n}{\partial x}(a', -X_n)da' + dS_n'(-X_n) \leq M. \tag{4.24}$$

Moreover we have for any $x \in \omega_n$

$$\int_0^{a_\dagger} i_n(a,x)da \leq M. \tag{4.25}$$

Proof. Let us introduce the quantity $J_n(a,x) = \int_0^a i_n(a',x)da'$. Then J_n satisfies the equation

$$\partial_x^2 J_n(a,x) + c\partial_x J_n(a,x) + S_n(x)\mathcal{F}_\gamma(i_n)(x) = i_n(a,x).$$

On the other hand function S_n satisfies $dS_n'' + cS_n' - S_n\mathcal{F}_\gamma(i_n) = 0$ therefore we deduce that

$$\partial_x^2 J_n + c\partial_x J_n + dS_n'' + cS_n' = i_n(a,x). \tag{4.26}$$

Integrating this equality over ω_n provides that

$$\begin{aligned} &\partial_x J_n(a,X_n) + cJ_n(a,X_n) + dS_n'(X_n) + cS_n(X_n) \\ &= \int_{\omega_n} i_n(a,x)dx + \partial_x J_n(a,-X_n) + dS_n'(-X_n). \end{aligned}$$

If n is sufficiently large we have $J_n(a,-X_n) = 0$ and $S_n(-X_n) = 0$, and since $i_n \geq 0$ and $i_n(a,-X_n) = 0$ we obtain $\partial_x i_n(a,-X_n) \geq 0$. In addition we have $S_n \geq 0$ and $S_n(-X_n) = 0$ that allows us to conclude that

$$\partial_x i_n(a,-X_n) \geq 0, \;\; S_n'(-X_n) \geq 0.$$

Now since $S_n(X_n) = \underline{S}(X_n)$ and $S_n \geq \underline{S}$ we obtain that $S_n'(X_n) \leq \underline{S}'(X_n)$. In the same way $i_n(a,X_n) = \underline{j}(a,X_n)$ and $i_n \geq \underline{j}$ therefore we have for n sufficiently large

$$\partial_x i_n(a,X_n) \leq \partial_x \underline{j}(a,X_n) \leq 0 \;\; S_n'(X_n) \leq \underline{S}'(X_n).$$

We conclude that we have the following estimate

$$\int_{\omega_n} i_n(a,x)dx + \partial_x J_n(a,-X_n) + dS_n'(-X_n) \leq c\int_0^a \underline{j}(a',X_n)da' + d\underline{S}'(X_n) + c\underline{S}(X_n).$$

Finally we conclude that there exists some constant $M > 0$ such that for each $n \geq 0$ large enough, and for each $a \in (0, a_\dagger)$,

$$\int_{\omega_n} i_n(a,x)dx + \int_0^{a_\dagger} \partial_x i_n(a', X_n)da' + dS_n'(-X_n) \leq M.$$

It remains to prove estimate (4.25). We first note that such an estimate is obvious for $x \geq 0$ (because of the inequality $i_n \leq \bar{j}$). By integrating (4.26) over $(-X_n, x)$ for some $x \leq 0$, we obtain

$$\begin{aligned}\partial_x J_n(a,x) + cJ_n(a,x) = &- dS_n'(x) - cS_n(x) + \int_{-X_n}^x i_n(a,x')dx' \\ &+ \partial_x J_n(a, -X_n) + dS_n'(-X_n).\end{aligned}$$

Since S_n is increasing and positive, we obtain using (4.24) that for any $x \leq 0$,

$$\partial_x J_n(a,x) + cJ_n(a,x) \leq M.$$

Integrating this differential inequality provides that

$$J_n(a,x) \leq M,$$

that completes the proof of estimate (4.25). ∎

Lemma 4.8 *Let Assumption 3.1 be satisfied, and assume that $R_0 > 1$. Then there exist an integer $n_1 \geq 0$ and some constant $M > 0$ such that for any $n \geq n_1$ we have*

$$i_n(a,x) \leq ||\gamma||_\infty M, \quad \text{for any } (a,x) \in [0, a_\dagger) \times (-X_n, X_n). \tag{4.27}$$

Moreover we have the following estimates

$$\mathcal{F}_\gamma(i_n)(x) \leq M, \ \forall x \in \omega_n, \quad \int_{\omega_n} \mathcal{F}_\gamma(i_n)(x)dx \leq M. \tag{4.28}$$

Proof. From (4.25) we have for any $x \in (-X_n, X_n)$,

$$S_n(x) \int_0^{a_\dagger} \gamma(a) i_n(a,x)dx \leq ||\gamma||_\infty M.$$

Since $\underline{j}(a, X_n)$ converges towards 0 when n tends to infinity uniformly with respect to $a \in [0, a_\dagger)$, we can find $n_1 \geq 0$ such that $\underline{j}(a, X_n) \leq ||\gamma||_\infty$ for any

$a \in [0, a_\dagger)$ and $n \geq n_1$. Therefore from the comparison principle applied to equation (4.15), and we obtain that for each $n \geq n_1$, and each $(a, x) \in \Omega_n$,

$$i_n(a, x) \leq ||\gamma||_\infty M,$$

which completes the proof of (4.27). The first estimate in (4.28) easily follows from (4.27), and from the fact that $\gamma \in L^1(0, a_\dagger)$. Next due to (4.24), we obtain that

$$\int_{\omega_n} \mathcal{F}_\gamma(i_n)(x)dx = \iint_{\Omega_n} \gamma(a) i_n(a, x) da dx \leq M \int_0^{a_\dagger} \gamma(a) da = R_0 M.$$

This completes the proof of Lemma 4.8. ■

Next we prove the following estimate.

Lemma 4.9 *Let Assumption 3.1 be satisfied, and assume that $R_0 > 1$. Then there exists some constant $M > 0$ such that for any $n \geq n_2 := \max(n_0, n_1)$, we have*

$$0 \leq S_n'(x) \leq M, \quad \int_0^{a_\dagger} \partial_x i_n(a, X_n) da \geq -M. \tag{4.29}$$

Furthermore we have $|S_n''(x)| \leq M$ for any $x \in (-X_n, X_n)$.

Proof. The estimates for the derivatives of function S_n easily follows from equation (4.16) together with the uniform bound (4.28). ■

Lemma 4.10 *Let Assumption 3.1 be satisfied, and assume that $R_0 > 1$. There exists some constant $M > 0$ such that for any $n \geq n_2$ we have*

$$\int_{\omega_n} i_n(a, x)^2 + (\partial_x i_n)^2(a, x) dx \leq M, \ \forall a \in (0, a_\dagger), \tag{4.30}$$

$$\iint_{\Omega_n} (\partial_x i_n)^2(a, x) + (\partial_a i_n)^2(a, x) da dx \leq M, \tag{4.31}$$

$$\iint_{\Omega_n} (\partial_x^2 i_n)^2(a, x) da dx \leq M. \tag{4.32}$$

Proof. These estimates follow from classical energy estimates. We multiply equation (4.22) by i_n and integrate over ω_n. We obtain

$$\frac{1}{2} \frac{d}{da} \int_{\omega_n} i_n(a, x)^2 dx + \int_{\omega_n} (\partial_x i_n)^2(a, x) dx \leq \frac{1}{2} \partial_x(\underline{j}^2)(a, X_n) + \frac{c}{2} (\underline{j}^2)(a, X_n). \tag{4.33}$$

Moreover due to (4.28), and $S_n \in [0,1]$, we have

$$\begin{aligned}\int_{\omega_n} i_n(0,x)^2 dx &= \int_{\omega_n} \Big(S_n(x)\mathcal{F}_\gamma(i_n)(x)\Big)^2 dx \\ &\le M \int_{\omega_n} \mathcal{F}_\gamma(i_n)(x)dx \\ &\le M.\end{aligned}$$

Therefore integrating (4.33) over $(0,a)$ provides that

$$\frac{1}{2}\int_{\omega_n} i_n(a,x)^2 dx + \int_0^a \int_{\omega_n} (\partial_x i_n)^2(a',x)dxda' \le M. \tag{4.34}$$

Next we multiply (4.22) by $\partial_a i_n$ and integrate over $(0,a)\times\omega_n$. We obtain that

$$\begin{aligned}&\iint_{(0,a)\times\omega_n} (\partial_a i_n)^2(a',x)da'dx + \frac{1}{2}\int_{\omega_n} (\partial_x i_n)^2(a,x)dx = \frac{1}{2}\int_{\omega_n} (\partial_x i_n)^2(0,x)dx \\ &+ \int_0^a \partial_a \underline{j}(a',X_n)\partial_a i_n(a',X_n)da' + c\iint_{(0,a)\times\omega_n} \partial_a i_n(a',x)\partial_x i_n(a',x)da'dx.\end{aligned} \tag{4.35}$$

On the one hand we have

$$\begin{aligned}\int_0^a \partial_a \underline{j}(a',X_n)\partial_a i_n(a',X_n)da' &= \int_0^a (\partial_a \underline{j}(a',X_n))^2 da' \\ &\le \int_0^{a_\dagger} (\partial_a \underline{j}(a',X_n))^2 da'.\end{aligned}$$

Therefore there exists some constant $M>0$ such that for any n we have

$$\int_0^a \partial_a \underline{j}(a',X_n)\partial_a i_n(a',X_n)da' \le M.$$

On the other hand we have

$$\partial_x i_n(0,x) = S_n'(x)\mathcal{F}_\gamma(i_n)(x) + S_n(x)\mathcal{F}_\gamma(\partial_x i_n).$$

Therefore using (4.29), we obtain

$$||\partial_x i_n(0,.)||_2 \le M||\mathcal{F}_\gamma(i_n)||_2 + \int_0^{a_\dagger} \gamma(a)||\partial_x i_n(a,.)||_2 da. \tag{4.36}$$

The first term in the right hand side in (4.36) can be estimated using (4.28)

$$||\mathcal{F}_\gamma(i_n)||_2 \leq ||\mathcal{F}_\gamma(i_n)||_\infty ||\mathcal{F}_\gamma(i_n)||_1.$$

The second term in the right hand side of (4.36) can be majorized by using the Cauchy-Scharwz's inequality

$$\int_0^{a_\dagger} \gamma(a)||\partial_x i_n(a,.)||_2 da \leq ||\gamma||_{L^2(0,a_\dagger)}||\partial_x i_n||_{L^2(\Omega_n)}.$$

As a consequence, due to (4.34) there exists some constant $M > 0$ that is independent of n such that

$$||\partial_x i_n(0,.)||_{L^2(0,a_\dagger)} \leq M. \tag{4.37}$$

Now by using (4.35), Cauchy-Scharwz's inequality, (4.34), and (4.37), we obtain that there exists some constant $M > 0$ such that

$$\iint_{(0,a)\times\omega_n} (\partial_a i_n)^2(a',x)da'dx + \int_{\omega_n} (\partial_x i_n)^2(a,x)dx \leq M. \tag{4.38}$$

Combining (4.35) and (4.38) provides (4.30) and (4.31). Finally (4.32) follows from (4.22) together with (4.30) and (4.31). This completes the proof of Lemma 4.10. ■

We can now pass to the limit $n \to +\infty$. From Lemmas 4.7-4.10 there exists some constant $M > 0$ such that for any n sufficiently large we have

$$\int_{\omega_n} i_n(a,x)dx \leq M, \quad \int_0^{a_\dagger} i_n(a,x)da \leq M, \quad i_n(a,x) \leq M, \tag{4.39}$$

$$||i_n||_{H^1((\Omega_n)} + ||i_n||_{L^2((0,a_\dagger),H^2(\omega_n))} \leq M, \tag{4.40}$$

$$||S_n||_{W^{2,\infty}(\omega_n)} \leq M. \tag{4.41}$$

We set $\Omega = (0,a_\dagger) \times \mathbb{R}$ and we can extract from the sequence (i_n, S_n) a subsequence, still denoted (i_n, S_n), tending towards a function (i, S) for the following topologies

$$\begin{aligned} & i_n \to i \text{ in } L^2_{loc}(\Omega), \text{ almost everywhere,} \\ & \quad \text{in } H^1_{loc}(\Omega) \text{ weakly and in } L^2_{loc}((0,a_\dagger),H^2_{loc}(\mathbb{R})) \text{ weakly} \\ & S_n \to S \text{ in } C^1_{loc}(\mathbb{R}). \end{aligned} \tag{4.42}$$

Moreover from estimates (4.39) and Fatou's Lemma function i satisfies

$$\int_{\mathbb{R}} i(a,x)dx \leq M, \quad \int_0^{a_\dagger} i(a,x)da \leq M, \quad i(a,x) \leq M, \quad \text{a.e..} \tag{4.43}$$

Next from the weak convergence and estimates (4.40) we obtain that

$$i \in H^1(\Omega) \cap L^2((0,a_\dagger), H^2(\mathbb{R})). \tag{4.44}$$

while function S satisfies

$$0 \leq S \leq 1, \quad S' \in L^\infty(\mathbb{R}). \tag{4.45}$$

First of all, since $\gamma \in L^1(0,a_\dagger)$ and i_n is uniformly bounded, from Lebesgue's dominated convergence theorem, we have

$$\mathcal{F}_\gamma(i_n)(x) = \int_0^{a_\dagger} \gamma(a) i_n(a,x)da \to \mathcal{F}_\gamma(i)(x) \text{ for } x \in \mathbb{R} \text{ a.e.}$$

Then function S satisfies the equation

$$dS'' + cS' - S\mathcal{F}_\gamma(i) = 0.$$

Finally due to (4.43) and (4.45) we obtain that $S'' \in L^\infty(\mathbb{R})$.

Now let $\phi \in \mathcal{D}([0,a_\dagger) \times \mathbb{R})$ be given. Then for n sufficiently large such that *supp* $(\phi) \subset [0,a_\dagger) \times (-X_n, X_n)$ function i_n satisfies the equality

$$\iint_\Omega \phi(0,x)\gamma(a)S_n(x)i_n(a,x)dadx = \iint_\Omega i_n(a,x)\partial_a\phi(a,x)dadx \\ - \iint_\Omega \partial_x i_n \partial_x \phi dadx + c\iint_\Omega \partial_x i_n \phi dadx.$$

Therefore due to (4.42) we obtain that function i satisfies the equation

$$\iint_\Omega \phi(0,x)\gamma(a)S(x)i(a,x)dadx = \iint_\Omega i(a,x)\partial_a\phi(a,x)dadx \\ - \iint_\Omega \partial_x i \partial_x \phi dadx + c\iint_\Omega \partial_x i \phi dadx,$$

for any $\phi \in \mathcal{D}([0,a_\dagger) \times \mathbb{R})$. Then we conclude that i satisfies the following equation

$$\partial_a i = \partial_x^2 i + c\partial_x i \text{ in } \mathcal{D}'((0,a_\dagger) \times \mathbb{R}).$$

Note that due to (4.44) each term in this equality belongs to $L^2(\Omega)$. From (4.44) we obtain that $i(0,x) = S(x)\mathcal{F}_\gamma(i)(x)$ almost everywhere. Moreover from (4.43) this equality holds in $L^p(\mathbb{R})$ for any $p \geq 1$. Therefore we can write the function i under the following integral formulation

$$i(a) = T_{\Delta+c\partial_x}(a)(B), \ \text{ with } B(x) = S(x)\mathcal{F}_\gamma(i)(x).$$

Here $T_{\Delta+c\partial_x}(a)$ is the C_0 analytic semigroup generated by the operator $\Delta + c\partial_x$ in $L^2(\mathbb{R})$. ¿From $S \in W^{2,\infty}(\mathbb{R})$, and (4.44) we see that $B \in W^{2,2}(\mathbb{R})$. Therefore using a similar bootstrap argument as those used in Proposition 4.1 we easily show that (i, S) is a classical solution of problem (4.15)-(4.16) and that i belongs to $C^1([0, a_\dagger) \times \mathbb{R})$.

Next from the estimates $\underline{j} \leq i_n \leq \bar{j}$ and $\underline{S} \leq S \leq 1$ we conclude that

$$\underline{j} \leq i \leq \bar{j}, \quad \underline{S} \leq S \leq 1.$$

This estimate first shows that i is a nonzeros function. Then this implies that

$$\lim_{x\to+\infty} S(x) = 1, \quad \lim_{x\to+\infty} i(a,x) = 0, \text{ uniformly with respect to } a \in [0, a_\dagger).$$

It remains to study the limit when $x \to -\infty$.

Since S_n is an increasing and bounded function for any $n \geq 0$ we conclude that function S is also increasing and bounded. Thus there exists some constant $S^+ \in [0,1]$ such that $S(-\infty) = S^+$. It remains to prove that $i(.,x)$ tends to zeros as $x \to -\infty$ in the topology of $C^0_{loc}([0, a_\dagger))$. For that purpose we set $\omega = (-1, 0)$ and we consider a sequence (t_n) tending to $-\infty$. We consider the sequence $j_n(a,x) = i(a, x + t_n)$. From i of the class C^1 on $[0, a_\dagger) \times \mathbb{R}$ we obtain that the sequence j_n is bounded in $C^1([0, a_\dagger) \times \omega)$. Therefore from the sequence (t_n) we can extract a subsequence still denoted by (t_n) such that (t_n) is decreasing, for any n we have $t_n - t_{n+1} > 1$ and j_n converges towards a function j for the topology of $C^0_{loc}([0, a_\dagger) \times \omega)$. Finally we have for any $a \in [0, a_\dagger)$

$$\sum_{n=0}^{+\infty} \int_{(t_{n+1}, t_n)} i(a,x)dx \leq \int_{\mathbb{R}} i(a,x)dx. \tag{4.46}$$

Finally since $t_n - t_{n+1} > 1$ we obtain that

$$\int_{(t_{n+1}, t_n)} i(a,x)dx \geq \int_{\omega} j_n(a,x)dx. \tag{4.47}$$

From (4.43), the serie in the left hand side in (4.46) is convergent we obtain from (4.47) that

$$\int_{\omega} j_n(a,x)dx \to 0, \text{ for any } a \in [0, a_\dagger) \text{ when } n \to +\infty.$$

On the other hand we have

$$\int_{\omega} j_n(a,x)dx \to \int_{\omega} j(a,x)dx, \text{ for any } a \in [0, a_\dagger) \text{ when } n \to +\infty.$$

Therefore since function j is continuous and positive we conclude that $j \equiv 0$ and we have that function i tends to zeros when $x \to -\infty$ for the topology of $C^0_{loc}([0, a_\dagger))$.

Finally let us notice that the limit $S^+ = S(-\infty)$ belongs to $[0, 1)$. Indeed if $S^+ = 1$ then, since function S is increasing, we obtain $S \equiv 1$. Therefore function i is an integrable and positive solution of the equation

$$\partial_a i = \partial_x^2 i + c\partial_x i, \quad i(0,x) = \int_0^{a_\dagger} \gamma(a)i(a,x)da.$$

Since $R_0 = \int_0^{a_\dagger} \gamma(a)da > 1$ we obtain $i \equiv 0$, in contradiction with the inequality $i \geq \underline{j}$. This completes the proof of Theorem 3.1.

Remark 4.11 *We can notice that we have proved that when $R_0 > 1$, system (4.15)-(4.16) has a solution for any wave speed $c > 2\sqrt{\alpha^*}$ with α^* defined in (4.6). We expect that $c^* = 2\sqrt{\alpha^*}$ corresponds to the minimal wave speed but this problem remains for the moment an open question.*

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