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Integrated semigroups and parabolic equations. Part I: linear perburbation of almost sectorial operators

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Abstract. The paper deals with linear abstract Cauchy problem with non-densely defined and almost sectorial operators, whenever the part of this operator in the closure of its domain is sectorial. This kind of problem naturally arises for parabolic equations with non-homogeneous boundary conditions. Using the integrated semigroup theory, we prove an existence and uniqueness result for integrated solutions. Moreover, we study the linear perturbation problem.

1. Introduction

In this article, we study abstract linear parabolic equations of the form

$$\frac{dv(t)}{dt} = Av(t) + f(t),\ t > 0, \quad v(0) = x \in \overline{D(A)}, \tag{1.1}$$

wherein $A : D(A) \subset X \to X$ is a linear operator on a Banach space X. When dealing with parabolic equations, it is usually assumed that the operator A is a sectorial elliptic operator. This operator property usually holds true when considering elliptic operators in Lebesgue spaces or Hölder spaces and together with homogeneous boundary conditions. As pointed out, for instance, by Lunardi [15], this property does no longer hold true when dealing with these operators in some more regular spaces. As observed in Prevost [22] in the context of parabolic with non-homogeneous boundary conditions, it turns to be natural to impose a weaker conditions than sectoriality. Motivated by these examples, in the sequel we will make the following assumption.

ASSUMPTION 1.1. We assume that

(a) A_0, the part of a A in $\overline{D(A)}$, is a sectorial operator.
(b) A is almost sectorial.

Using functional calculus and $1 - \alpha$ growth semigroups, this kind of problem has been considered by Periago and Straub [21]. In [21], they use functional calculus to define the fractional power of $\lambda I - A$ for some $\lambda > 0$ large enough. Here, we will also consider this question, but using integrated semigroups and an approach similar to the

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one used by Pazy [20]. We also refer to DeLaubenfels [8], and Haase [12] for more update results on functional calculus, and to Da Prato [7] for pioneer work on $1-\alpha$ growth semigroups. More recently, the case of non-autonomous Cauchy problems has also been studied by Carvalho et al. [5]. In these works, based on the existence of $1-\alpha$ growth semigroups, a notion of solution has been developed.

Here, we intend to reconsider the Cauchy problem (1.1) using integrated semigroup theory. Our goal is to study the existence of integrated solutions for the Cauchy problem (1.1). Let us recall the following definition:

DEFINITION 1.2. Let us assume that $f \in L^1(0,\tau;X)$ for some $\tau > 0$. A map $v \in C([0,\tau], X)$ is called **an integrated solution** of the Cauchy problem (1.1) on $[0,\tau]$ if the two following conditions are satisfied:

$$\int_0^t v(s)\mathrm{d}s \in D(A), \forall t \in [0,\tau],$$

$$v(t) = x + A\int_0^t v(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s.$$

Under Assumption 1.1, the linear operator A is not (in general) a Hille-Yosida operator. In the context of integrated semigroup, a class of non-Hille-Yosida operators has been recently studied by Magal and Ruan [16] and Thieme [27]. The aim of this paper is to apply the theory developed [16,27], in the context of linear operator with a sectorial part. In Remark 3.10, we will also give a brief comparison between the integrated semigroup approach and the approach used by Periago and Straub [21].

As presented by Magal and Ruan [17,18], integrated semigroup theory allows to construct a bifurcation theory for semi-linear Cauchy problems of the form

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)),\ t > 0,\quad v(0) = x \in \overline{D(A)},$$

where the map F acts smoothly from $\overline{D(A)}$ into X. So the goal of this paper is to present a linear theory, which will be used in a companion paper [9] for semi-linear systems.

In order to motivate our theoretical framework, we consider the following parabolic problem with non-autonomous boundary condition:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + g(t,x), & t>0,\quad x>0 \\ -\dfrac{\partial u(t,0)}{\partial x} = h(t) \\ u(0,.) = u_0 \in L^p((0,+\infty),\mathbb{R}), \end{cases} \tag{1.2}$$

where $g \in L^1((0,\tau), L^p((0,+\infty),\mathbb{R}))$, and $h \in L^q((0,\tau),\mathbb{R})$.

As in [18], we consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi'(0) \\ \varphi'' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty),\mathbb{R}\right).$$

One may observe that A_0, the part of A in $\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p\left((0,+\infty),\mathbb{R}\right)$, is the linear operator defined by

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty),\mathbb{R}\right) : \varphi'(0) = 0 \right\}.$$

In particular, it is well known that A_0 is the infinitesimal generator of an analytic semigroup on $\overline{D(A)}$. But the resolvent of A is defined by the formula

$$\begin{aligned}(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} &= \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(x) = \frac{e^{-\sqrt{\lambda}x}}{-\sqrt{\lambda}}\alpha + \frac{e^{-\sqrt{\lambda}x}}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-\sqrt{\lambda}s}\psi(s)ds \\ &\quad + \frac{1}{2\sqrt{\lambda}} \int_0^{+\infty} e^{-\sqrt{\lambda}|x-s|}\psi(s)\mathrm{d}s\end{aligned}$$

for $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$.

Due to the boundary condition, we obtain the following inequalities

$$0 < \liminf_{\lambda\to+\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < \limsup_{\lambda\to+\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty,$$

where

$$p^* := \frac{2p}{1+p}.$$

It follows that A is not a Hille-Yosida operator when $p \in (1, +\infty)$. Set

$$f(t) := \begin{pmatrix} h(t) \\ g(t) \end{pmatrix}.$$

By identifying $u(t,.)$ to $v(t) = \binom{0}{u(t,.)}$, the PDE problem (1.2) can be re-written as the following abstract Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = x \in \overline{D(A)}. \tag{1.3}$$

In this article we will prove (see Theorem 3.11) that for each $\widehat{p} > p^*$, and each $f \in L^{\widehat{p}}(0,\tau;X)$ (with $\tau > 0$), the Cauchy problem (1.3) has a unique integrated solution, and there exists a constant $M_{\tau,\widehat{p}} > 0$ such that

$$\|u(t)\| \leq M_{\tau,\widehat{p}} \left(\int_0^t \|f(s)\|^{\widehat{p}} \mathrm{d}s \right)^{1/\widehat{p}}, \forall t \in [0,\tau].$$

For parabolic problems in dimension n, the same difficulty arises, and we refer to Tanabe [23, Section 3.8, p.82], Agranovich [1], and Volpert and Volpert [28] for general estimates for the resolvent of elliptic operators in the n dimensional case.

The paper is organized as follows. Section 2 recalls some results about integrated semigroups. Section 3 deals with almost sectorial operators as well as the associated linear Cauchy problem. This section will also provide an alternative construction of functional calculus for almost sectorial operators using integrated semigroups. Finally, in Sect. 4 we give some perturbation results of almost sectorial operators.

2. Integrated semigroup

In this section, we first recall some results about integrated semigroups. We refer to Arendt [2,3], Neubrander [19], Kellermann and Hieber [14], Thieme [25–27], and Arendt et al. [4], Xiao and Liang [29], and Magal and Ruan [16–18] for more detailed results on the subject. We consider a Banach space $(X, \|.\|_X)$ and the non-homogeneous Cauchy problem

$$\frac{du}{dt} = Au + f(t), \text{ for all } t \geq 0, \text{ and } u(0) = x \in \overline{D(A)}, \tag{2.4}$$

wherein $A : D(A) \subset X \to X$ is a linear operator satisfying $\overline{D(A)} \neq X$. We also denote by $\mathcal{L}(X)$ the space of bounded linear operators from X into X, $\rho(A)$ the resolvent set of A, X_0 the closure of $D(A)$ into X, and A_0 the part of A in X_0. Note that $A_0 : D(A_0) \subset X_0 \to X_0$ is a linear operator defined by

$$A_0 x = Ax, \forall x \in D(A_0) = \{y \in D(A) : Ay \in X_0\}.$$

Assume that there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, then it is easy to check that for each $\lambda > \omega$,

$$D(A_0) = (\lambda I - A)^{-1} X_0 \text{ and } (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} |_{X_0}.$$

Moreover, we have the following result, which is proved in Magal and Ruan [16].

LEMMA 2.1. *Let $(X, \|.\|_X)$ be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. Assume that there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and*

$$\limsup_{\lambda \to +\infty} \lambda \left\| (\lambda I - A_0)^{-1} \right\| = \limsup_{\lambda \to +\infty} \lambda \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X_0)} < +\infty. \tag{2.5}$$

Then the following assertions are equivalent:

(i) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X,$
(ii) $\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \ \forall x \in X_0,$
(iii) $X_0 = \overline{D(A_0)}.$

Recall that A is a **Hille-Yosida operator** if there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k}, \ \forall \lambda > \omega, \ \forall k \geq 1.$$

In the following, we assume that A satisfies the following weaker conditions.

ASSUMPTION 2.2. Let $(X, \|.\|_X)$ be a Banach space and $A : D(A) \subset X \to X$ be a linear operator. Assume that

(a) There exist $M_A \geq 1$ and $\omega_A \in \mathbb{R}$, such that $(\omega_A, +\infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A_0)^{-k}\right\|_{\mathcal{L}(X_0)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \ \forall \lambda > \omega_A, \ \forall k \geq 1.$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X.$

Using Lemma 2.1 and the Hille-Yosida's theorem (see [20], theorem 5.3, p.20), one obtains the following lemma.

LEMMA 2.3. *Assumption 2.2 is satisfied if and only if there exist two constants* $M \geq 1$ *and* $\omega_A \in \mathbb{R}$ *such that* $(\omega_A, +\infty) \subset \rho(A)$ *and* A_0 *the part of* A *in* $X_0 = \overline{D(A)}$ *is the infinitesimal generator of a* C_0*-semigroup* $\left\{T_{A_0}(t)\right\}_{t \geq 0}$ *on* X_0 *which satisfies* $\|T_{A_0}(t)\|_{\mathcal{L}(X_0)} \leq M_A e^{\omega_A t}, \ \forall t \geq 0.$

Now, we define the integrated semigroup generated by A. The notion of the generator of an integrated semigroup is taken from Thieme [25].

DEFINITION 2.4. Let $(X, \|.\|_X)$ be a Banach space. A family of bounded linear operator $\{S(t)\}_{t \geq 0}$ on X is called an **integrated semigroup** if

(i) $S(0) = 0$,
(ii) The map $t \to S(t)$ is strongly continuous (i.e. $t \to S(t)x$ is continuous from $[0, +\infty)$ into X for any $x \in X$),
(iii) $S(t)$ satisfies

$$S(r)S(t) = \int_0^r (S(l + t) - S(l))\mathrm{d}l, \ \forall r, t \geq 0. \tag{2.6}$$

From this definition, let us remark that for each $r, t \geq 0$

$$\int_0^r (S(l + t) - S(l))\mathrm{d}l = \int_0^{r+t} S(l)\mathrm{d}l - \int_0^r S(l)\mathrm{d}l - \int_0^t S(l)\mathrm{d}l,$$

so that

$$S(r)S(t) = S(t)S(r), \ \forall r, t \geq 0. \tag{2.7}$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ on X is said to be **non-degenerate** if

$$S(t)x = 0, \forall t \geq 0 \Rightarrow x = 0.$$

We will say that a linear operator $A : D(A) \subset X \to X$ is the **generator** of a non-degenerate integrated semigroup $\{S(t)\}_{t \geq 0}$ on X if and only if

$$x \in D(A),\ y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)y\mathrm{d}s,\ \forall t \geq 0. \tag{2.8}$$

From Lemma 2.5 in Thieme [25], we know that if A generates $\{S(t)\}_{t \geq 0}$, then for each $x \in X$ and $t \geq 0$ one has

$$\int_0^t S(s)x\mathrm{d}s \in D(A) \text{ and } S(t)x = A\int_0^t S(s)x\mathrm{d}s + tx,\ \forall x \in X. \tag{2.9}$$

An integrated semigroup $\{S(t)\}_{t \geq 0}$ is said to be **exponentially bounded** if there exist two constants $\hat{M} \geq 1$ and $\hat{\omega} > 0$, such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq \hat{M}e^{\hat{\omega}t},\ \forall t \geq 0.$$

Combining Theorem 3.1 in Arendt [4] together with Proposition 3.10 in Thieme [25], one has the following result:

THEOREM 2.5. *Let $\{S(t)\}_{t \geq 0}$ be a strongly continuous exponentially bounded family of bounded linear operator on a Banach space $(X, \|.\|_X)$ and $A : D(A) \subset X \to X$ be a linear operator. Then, $\{S(t)\}_{t \geq 0}$ is a non-degenerate integrated semigroup, and A its generator, if and only if there exist $\hat{\omega} > 0$ such that $(\hat{\omega}, +\infty) \subset \rho(A)$ and*

$$(\lambda I - A)^{-1}x = \lambda\int_0^{+\infty} e^{-\lambda t}S(t)x\mathrm{d}t,\ \forall \lambda > \hat{\omega}. \tag{2.10}$$

The following proposition summarizes some properties of integrated semigroup.

PROPOSITION 2.6. *Let Assumption 2.2 be satisfied. Then A generates a non-degenerate integrated semigroup $\{S_A(t)\}_{t \geq 0}$ and for each $x \in X$, each $t \geq 0$, and each $\mu > \omega$, $S(t)x$ is given by*

$$S_A(t)x = (\mu I - A)\int_0^t T_{A_0}(s)\mathrm{d}s\,(\mu I - A)^{-1}x, \tag{2.11}$$

or equivalently

$$S_A(t)x = \mu\int_0^t T_{A_0}(s)\,(\mu I - A)^{-1}x\mathrm{d}s + \left[I - T_{A_0}(t)\right](\mu I - A)^{-1}x. \tag{2.12}$$

Moreover, we have the following properties:

(i) *The map $t \to S_A(t)x$ is continuously differentiable on $[0, \infty)$ if and only if $x \in X_0$ and*

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x, \forall t \geq 0,\ \forall x \in X_0. \tag{2.13}$$

(ii) *$S_A(t)$ commutes with $(\lambda I - A)^{-1}$ and*

$$S_A(t)x = \int_0^t T_{A_0}(s)x\mathrm{d}s, \forall t \geq 0, \ \forall x \in X_0. \tag{2.14}$$

We set for each $f \in L^1((0,T), X)$,

$$(S_A * f)(t) := \int_0^t S_A(t-s)f(s)\mathrm{d}s, \ \forall t \in [0,T],$$

and whenever $t \to (S_A * f)(t)$ is continuously differentiable, we set

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t), \ \forall t \in [0,T].$$

Let us finally recall the following theorem which has been proved in Magal and Ruan [16, Theorem 4.3].

THEOREM 2.7. *Let Assumption 2.2 be satisfied. Let $B : D(A) \to Y$ be a bounded linear operator from $D(A)$ into a Banach space $(Y, \|.\|)$ and $\chi : (0,+\infty) \to \mathbb{R}$ a non-negative measurable function such that*

$$\inf\left\{\delta > 0 : e^{-\delta.}\chi(.) \in L^1((0,+\infty), \mathbb{R})\right\} < +\infty.$$

Then the following assertions are equivalent.

(i) $\|B(S_A \diamond f)(t)\| \leq \int_0^t \chi(t-s)\|f(s)\|ds, \forall t \geq 0, \forall f \in C^1((0,+\infty), X)$.

(ii) $\left\|B(\lambda I - A)^{-n}\right\|_{L(X,Y)} \leq \frac{1}{(n-1)!}\int_0^{+\infty} s^{n-1}e^{-\lambda s}\chi(s)ds, \forall n > 1$.

(iii) $\|B(S_A(t+h) - S_A(t)\|_{L(X,Y)} \leq \int_t^{t+h}\chi(s)ds, \forall t, h \geq 0$.

Moreover, if one of the above three conditions is satisfied, then $\chi \in L^q_{loc}([0,+\infty), \mathbb{R})$ for some $q \in [1,+\infty]$ and $p \in [1,+\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for each $\tau > 0$ and each $f \in L^p((0,\tau), X)$, the map $t \to B(S_A * f)(t)$ is continuously differentiable and

$$\left\|\frac{d}{dt}B(S_A * f)(t)\right\| \leq \int_0^t \chi(t-s)\|f(s)\|\mathrm{d}s, \forall t \in [0,\tau].$$

3. Almost sectorial operators and linear Cauchy problem

The aim of this section is to recall some results about almost sectorial operators and to consider some associated linear Cauchy problem.

3.1. Almost sectorial operators

We first recall some definitions.

DEFINITION 3.1. Let $L : D(L) \subset X \to X$ a linear operator on a Banach space X. L is said to be a **sectorial** operator if there are constants $\widehat{\omega} \in \mathbb{R}, \theta \in]\pi/2, \pi[$, and $\widehat{M} > 0$ such that

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$,

(ii) $\left\|(\lambda I - L)^{-1}\right\| \leq \dfrac{\widehat{M}}{|\lambda - \widehat{\omega}|}, \forall \lambda \in S_{\theta,\widehat{\omega}}$.

We refer, for instance, to Friedmann [11], Tanabe [23], Henry [13], Pazy [20], Temam [24], Lunardi [15], Cholewa and Dlotko [6] and Engel and Nagel [10] for more details on the subject. In particular, when L is a sectorial operator and densely defined, then L is the infinitesimal generator of a strongly continuous analytic semigroup $T(t)$ given by

$$T_L(t) = \frac{1}{2\pi i}\int_{\hat{\omega}+\gamma_{r,\eta}} (\lambda I - L)^{-1} e^{\lambda t} \mathrm{d}\lambda, \ t > 0, \text{ and } T_L(0)x = x, \ \forall x \in X,$$

where $r > 0, \eta \in (\pi/2, \theta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta, |\lambda| = r\}$, oriented counterclockwise (see Proposition 2.1.1. p.35 in [15]). We recall that a family of bounded linear operator $T(t)$ which satisfies the semigroup property is said to be an analytic semigroup (following, for instance, [15]) if the function $t \to T(t)$ is analytic in $(0, +\infty[$ with values in $\mathcal{L}(X)$ (i.e. $T(t) = \sum_{n=0}^{+\infty} (t - t_0)^n L_n$ for $|t - t_0|$ small enough).

Let us now introduce the notion of almost sectorial operators:

DEFINITION 3.2. Let $L : D(L) \subset X \to X$ a linear operator on a Banach space X and $\alpha \in (0, 1]$ be given. L is said to be a α—**almost sectorial** operator if there are constants $\widehat{\omega} \in \mathbb{R}, \theta \in (\pi/2, \pi)$, and $\widehat{M} > 0$ such that

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$,

(ii) $\left\|(\lambda I - L)^{-1}\right\| \leq \frac{\widehat{M}}{|\lambda - \widehat{\omega}|^{\alpha}}, \quad \forall \lambda \in S_{\theta,\widehat{\omega}}$.

This class of operators has been used by Periago et al. [21] as well as by Carvalho et al. [5]. In these works the authors construct functional calculus for such operators and define a notion of solution for the corresponding linear abstract Cauchy problem.

In this work we shall focus on linear almost sectorial operators with a sectorial part over the closure of its domain. In order to use this notion, we first derive some characterization for this class of operators.

PROPOSITION 3.3. *Let $A : D(A) \subset X \to X$ be a linear operator and A_0 be its part in $X_0 = \overline{D(A)}$. Then the following statements are equivalent:*

(i) *The operator A_0 is sectorial in X_0 and A is $\frac{1}{p^*}$—almost sectorial for some $p^* \in [1, +\infty)$.*

(ii) *There exist two constants $\omega_A \in \mathbb{R}$, and $M_A > 0$ such that the following properties are satisfied:*

(iia) $\{\lambda \in \mathbb{C} : \mathcal{Re}(\lambda) > \omega_A\} \subset \rho(A_0)$ *and*

$$\left\|(\lambda I - A_0)^{-1}\right\|_{\mathcal{L}(X_0)} \leq \frac{M_A}{|\lambda - \omega_A|}, \ \forall \lambda \in \mathbb{C} \textit{ with } \mathcal{Re}(\lambda) > \omega_A,$$

(iib) $(\omega_A, +\infty) \subset \rho(A)$, *and*

$$\overline{\lim_{\lambda \to +\infty}} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty.$$

Proof. From definitions 3.1 and 3.2 it directly follows that (i) implies (ii).
Let us now assume that (ii) holds. Using Proposition 2.1.11, p.43 in Lunardi [15], we know that (iia) ensures that A_0 is a sectorial operator on X_0. Now since $\rho(A)$ and $\rho(A_0)$ are non-empty we have $\rho(A) = \rho(A_0)$ (see Lemma 2.1 in [17]). Since A_0 is sectorial on X_0, there exist $\omega_A \in R$ and $\theta \in (\pi/2, \pi)$ such that $S_{\theta,\omega_A} \subset \rho(A)$. Without loss of generality (by replacing ω_A by $\omega_A + \varepsilon$ for some $\varepsilon > 0$ large enough), one may assume that there exist some positive constant $C > 0$ and $\theta \in (\pi/2, \pi)$ such that

$$\left\| (\lambda - \omega_A)(\lambda I - A_0)^{-1} \right\| \leq C, \quad \forall \lambda \in S_{\theta,\omega_A}$$
$$(\mu - \omega_A)^{1/p^*} \left\| (\mu I - A)^{-1} \right\|_{\mathcal{L}(X)} \leq C, \ \forall \mu \in (\omega_A, +\infty).$$

Now for each $\lambda \in \rho(A_0)$ and each $\mu \in (\omega_A, +\infty)$, one has

$$(\lambda I - A)^{-1} = (\mu - \lambda)(\lambda I - A_0)^{-1} (\mu I - A)^{-1} + (\mu I - A)^{-1}. \tag{3.15}$$

Therefore for each $\lambda \in S_{\theta,\omega_A}$, we choose $\mu = \omega_A + |\lambda - \omega_A|$ and one has

$$\begin{aligned}\left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} &= \left\| (\mu - \lambda)(\lambda I - A_0)^{-1} (\mu I - A)^{-1} + (\mu I - A)^{-1} \right\| \\ &\leq C \frac{|\mu - \lambda|}{|\lambda - \omega_A|} \frac{C}{(\mu - \omega_A)^{1/p^*}} + \frac{C}{(\mu - \omega_A)^{1/p^*}}.\end{aligned}$$

From the definition of μ one has

$$|\lambda - \omega_A| = |\mu - \omega_A|.$$

Thus, $|\lambda - \mu| \leq 2|\lambda - \omega_A|$ and this implies that

$$\left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} \leq C \frac{(2C+1)}{|\lambda - \omega_A|^{1/p^*}} = \frac{\tilde{M}}{|\lambda - \omega_A|^{1/p^*}} \quad \forall \lambda \in S_{\theta,\omega_A}.$$

This completes the proof of the result. □

From now on, we only use the following assumption.

ASSUMPTION 3.4. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. We assume that there exist two constants $\omega_A \in \mathbb{R}$, $M_A > 0$, such that
(a) $\rho(A_0) \supset \{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) > \omega_A\}$ and

$$\left\| (\lambda - \omega_A)(\lambda I - A_0)^{-1} \right\|_{\mathcal{L}(X_0)} \leq M_A, \ \forall \lambda \in \mathbb{C} : \mathcal{R}e(\lambda) > \omega_A,$$

(b) $(\omega_A, +\infty) \subset \rho(A)$, and there exists $p^* \geq 1$ such that

$$\limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)} < +\infty.$$

Note that under Assumption 3.4-(a), the operator $A_0 : D(A_0) \subset X_0 \to X_0$ is the infinitesimal generator of a strongly continuous and analytic semigroup on X_0. It will be denoted by $\{T_{A_0}(t)\}_{t \geq 0}$ in the sequel. We also remark that if $p^* = 1$ in Assumption 3.4-(b), it is clear that A is a Hille-Yosida operator. It is also clear that if Assumption 3.4-(b) is satisfied, then for each $\delta \in \mathbb{R}$, we have

$$\limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - (A + \delta I))^{-1} \right\|_{\mathcal{L}(X)} = \limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathcal{L}(X)}.$$

3.2. Semigroup estimates and fractional powers

The aim of this section is to give some estimates and differentiability properties for the integrated semigroup $\{S_A(t)\}_{t \geq 0}$. Moreover these estimates will allow us to give an alternative construction for the fractional powers of the operator-A using a semigroup approach. The definition of fractional powers as well as functional calculus has been well developed for almost sectorial operators (see, for instance, [21]). These constructions essentially use the resolvent operator. Our construction will follow the one given by Pazy [20] using integrated semigroup theory.

LEMMA 3.5. *Let Assumption 2.2-(a) and Assumption 3.4-(b) be satisfied. Then for each* $\delta \in (-\infty, -\omega_A)$, *(i.e.* $\omega_A + \delta \leq 0$ *) there exist* $\tilde{M}_1 = \tilde{M}_1(\delta) > 0$ *and* $\tilde{M}_2 = \tilde{M}_2(\delta) > 0$ *such that*

$$\left\| S_{(A+\delta I)}(t) \right\|_{\mathcal{L}(X)} \leq \tilde{M}_1, \ \forall t \geq 0, \tag{3.16}$$

and

$$\left\| S_{(A+\delta I)}(t) \right\|_{\mathcal{L}(X)} \leq \tilde{M}_2 t^{1/p^*}, \ \forall t \in [0, 1]. \tag{3.17}$$

Proof. Let $\delta \in (-\infty, -\omega_A)$ be fixed. By replacing A by $(A + \delta I)$ in (2.11), we have for each $\mu > \omega_A + \delta$, each $t \geq 0$, and $x \in X$,

$$\begin{aligned} S_{(A+\delta I)}(t)x = \mu \int_0^t & T_{(A_0+\delta I)}(s) \left(\mu I - (A + \delta I)\right)^{-1} x \mathrm{d}s \\ & + \left[I - T_{(A_0+\delta I)}(t) \right] \left(\mu I - (A + \delta I)\right)^{-1} x. \end{aligned}$$

So if we fix $\mu > 0$, we obtain

$$\begin{aligned} \left\| S_{(A+\delta I)}(t)x \right\|_X \leq \mu & \left\| (\mu I - (A + \delta I))^{-1} x \right\| \int_0^t \left\| T_{(A_0+\delta I)}(s) \right\| \mathrm{d}s \\ & + \left\| (\mu I - (A + \delta I))^{-1} x \right\| \left[1 + \left\| T_{(A_0+\delta I)}(t) \right\| \right]. \end{aligned}$$

But $\left\| T_{(A_0+\delta I)}(s) \right\| \leq M e^{(\omega_A+\delta)t}$, thus

$$\left\| S_{(A+\delta I)}(t)x \right\|_X \leq \mu \left\| (\mu I - (A+\delta I))^{-1} x \right\| \int_0^t M e^{(\omega_A+\delta)s} \mathrm{d}s \tag{3.18}$$

$$+ \left\| (\mu I - (A+\delta I))^{-1} x \right\| \left[1 + M e^{(\omega_A+\delta)t} \right] \tag{3.19}$$

and since $\omega_A + \delta < 0$, one gets

$$\begin{aligned} \left\| S_{(A+\delta I)}(t)x \right\|_X &\leq \mu \left\| (\mu I - (A+\delta I))^{-1} x \right\| \int_0^\infty M e^{(\omega_A+\delta)s} \mathrm{d}s \\ &\quad + \left\| (\mu I - (A+\delta I))^{-1} x \right\| [1+M] \end{aligned}$$

and (3.16) follows.

Let $\tilde{C} > 0$ and $\lambda_0 > 0$ be fixed such that $(\lambda_0, +\infty) \subset \rho(A)$ and,

$$\lambda^{1/p^*} \left\| (\lambda I - (A+\delta I))^{-1} \right\|_{\mathcal{L}(X)} \leq \tilde{C}, \ \forall \lambda \in [\lambda_0, +\infty). \tag{3.20}$$

For each $t \in (0,1]$ we replace μ by $\frac{\lambda_0}{t} \in [\lambda_0, \infty)$ in (3.18). Since $\omega_A + \delta < 0$ one has $e^{(\omega_A+\delta)t} \leq 1$ while $\int_0^t e^{(\omega_A+\delta)s} \mathrm{d}s \leq t$ for each $t \in [0,1]$. This yields to

$$\begin{aligned} \left\| S_{(A+\delta I)}(t)x \right\| &\leq \lambda_0 M \left\| \left(\frac{\lambda_0}{t} I - (A+\delta I) \right)^{-1} x \right\| \\ &\quad + \left\| \left(\frac{\lambda_0}{t} I - (A+\delta I) \right)^{-1} x \right\| [1+M]. \end{aligned} \tag{3.21}$$

On the other hand, from (3.20) with $\lambda = \frac{\lambda_0}{t}$ we have

$$\left\| \left(\frac{\lambda_0}{t} I - (A+\delta I) \right)^{-1} x \right\| \leq \frac{\tilde{C}}{\left(\frac{\lambda_0}{t} \right)^{1/p^*}} \|x\|. \tag{3.22}$$

Finally combining (3.21) together with (3.22) one obtains

$$\begin{aligned} \left\| S_{(A+\delta I)}(t)x \right\| &\leq \frac{\lambda_0 M \tilde{C} t^{1/p^*}}{\lambda_0^{1/p^*}} \|x\| + \frac{\tilde{C} t^{1/p^*}}{\lambda_0^{1/p^*}} [1+M] \|x\| \\ &\leq \frac{\tilde{C}}{\lambda_0^{1/p^*}} [(\lambda_0+1)M+1] t^{1/p^*} \|x\|, \end{aligned}$$

and (3.17) follows. □

We shall now assume that $\omega_A < 0$ and let $\delta \in (-\infty, -\omega_A)$ be given and fixed. Then (see [20], p.70) one has for each $\gamma > 0$ and each $x \in X_0$,

$$(\delta I - A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} T_{A_0-\delta I}(t) x \mathrm{d}t, \tag{7.2.17}$$

and

$$(\delta I - A_0)^0 = I.$$

We first derive some estimates for the fractional powers of operator A_0.

LEMMA 3.6. *Let Assumption 3.4-(a) be satisfied and assume that* $\omega_A < 0$. *Let* $\gamma \in (0, 1)$ *be given. Then there exists some constant* $\check{M} > 0$ *such that*

$$\left\| (\lambda I - A_0)^{-\gamma} \right\| \leq \frac{\check{M}}{|\lambda|^{\gamma}}, \ \forall \lambda \in \mathbb{C} : \mathcal{R}e\,(\lambda) > 0. \tag{3.23}$$

Proof. Under Assumption 3.4-(a) we know that A_0 generates an analytical semigroup on X_0. Hence using the formula (6.4) p. 69 in Pazy [20], we have

$$(\lambda I - A_0)^{-\gamma} = \frac{\sin \pi\gamma}{\pi} \int_0^{+\infty} t^{-\gamma} \left((t+\lambda) I - A_0 \right)^{-1} \mathrm{d}t.$$

Then,

$$\begin{aligned}
\left\| (\lambda I - A_0)^{-\gamma} \right\| &\leq M \frac{\sin \pi\gamma}{\pi} \int_0^{+\infty} \frac{1}{t^{\gamma}} \frac{1}{|t+\lambda|} \mathrm{d}t, \\
&= M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^{\gamma}} \int_0^{+\infty} \frac{1}{l^{\gamma}} \frac{1}{\left| l + e^{-i \arg(\lambda)} \right|} \mathrm{d}l, \\
&= M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^{\gamma}} \left[\int_0^{1/2} \frac{1}{l^{\gamma}} \frac{1}{\left| l + e^{i \arg(\lambda)} \right|} \mathrm{d}l \right. \\
&\quad \left. + \int_{1/2}^{+\infty} \frac{1}{l^{\gamma}} \frac{1}{\left| l + e^{i \arg(\lambda)} \right|} \mathrm{d}l \right], \\
&\leq M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^{\gamma}} \left[\int_0^{1/2} \frac{1}{l^{\gamma}} \frac{1}{1-l} \mathrm{d}l + \int_{1/2}^{+\infty} \frac{1}{l^{\gamma}} \frac{1}{l} \mathrm{d}l \right], \\
&\leq \frac{\check{M}}{|\lambda|^{\gamma}}.
\end{aligned}$$

This completes the proof of Lemma 3.6. □

Note now that we have for each $x \in X_0$, each $\delta \in (0, -\omega_A)$ and each $\gamma > 0$,

$$(-A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} e^{-\delta t} T_{(A_0+\delta I)}(t) x \mathrm{d}t.$$

By integrating by parts, we obtain

$$(-A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \left[1 - \gamma + \delta t \right] t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x \mathrm{d}t. \tag{3.24}$$

Note that (3.24) is well defined for each $x \in X_0$, and each $\gamma > 0$ because $\left\| S_{(A+\delta I)}(t) \right\| \leq \int_0^t \left\| T_{(A_0+\delta I)}(s) \right\| ds \leq Mt$, for each $t \geq 0$.

Hence (3.24) leads us to the following definition of fractional power of the resolvent of A.

LEMMA 3.7. *Let Assumption 3.4 be satisfied and assume that $\omega_A < 0$. Then for each $\gamma > 1 - 1/p^*$, and each $\delta \in (0, -\omega_A)$, the operator $(-A)^{-\gamma}$ is well defined by*

$$(-A)^{-\gamma}\, x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \left[1 - \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x \mathrm{d}t,\ \forall x \in X. \quad (3.25)$$

Moreover, we have the following properties:

(i) $(\mu I - A_0)^{-1} (-A)^{-\gamma} = (-A_0)^{-\gamma} (\mu I - A)^{-1}, \forall \mu > \omega_A$.

(ii) $(-A_0)^{-\gamma}\, x = (-A)^{-\gamma}\, x, \forall x \in X_0$.

(iii) *When $\gamma = 1$, $(-A)^{-1}$ defined by (3.25) is the inverse of $-A$.*

(iv) *For each $\gamma \geq 0$, $\beta > 1 - 1/p^*$,*

$$(-A_0)^{-\gamma} (-A)^{-\beta} = (-A)^{-(\gamma+\beta)}.$$

Proof. Let $x \in X$ be given and the function $H : [0, +\infty) \to \mathcal{L}(X)$ defined by

$$t \to H(t) = \begin{cases} \left[1 - \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) \text{ if } t > 0, \\ 0 \text{ if } t = 0. \end{cases}$$

The map $t \to S_A(t)$ is Hölder continuous since

$$S_A(t + r) - S_A(r) = T_{A_0}(r) S_A(t)$$

so there exists $C > 0$, such that

$$\| S_A(t + r) - S_A(r) \| \leq C t^{1/p^*}, \forall t \in [0, 1], \forall r \geq 0.$$

It follows that H is continuous on $(0, +\infty)$, with respect to the operator norm topology and therefore is a Bochner's measurable map. Moreover, for each $\gamma > 1 - 1/p^*$, one has

$$\begin{aligned} \int_0^{+\infty} \|H(t)\| \,\mathrm{d}t &= \int_0^{+\infty} \left\| \left[1 - \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x \right\| \mathrm{d}t, \\ &\leq \int_0^{+\infty} \left[1 - \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| \mathrm{d}t, \\ &\leq \int_0^{1} \left[1 - \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| \mathrm{d}t, \\ &\quad + \int_1^{+\infty} \left[1 + \gamma + \delta t\right]\, t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| \mathrm{d}t, \\ &\leq \left[1 + \gamma + \delta\right] \left(\int_0^1 t^{\gamma-2} t^{1/p^*} \tilde{M}_2 \|x\| \,\mathrm{d}t \right. \\ &\quad \left. + \int_1^{+\infty} t^{\gamma-1} e^{-\delta t} \tilde{M}_1 \|x\| \,\mathrm{d}t \right), \end{aligned}$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are the constants introduced in Lemma 3.5. So, we have

$$\begin{aligned} \int_0^{+\infty} \|H(t)\| \,\mathrm{d}t \leq \left[1 + \gamma + \delta\right] \|x\| &\left(\tilde{M}_2 \int_0^1 t^{(\gamma-2)+1/p^*} \mathrm{d}t \right. \\ &\left. + \tilde{M}_1 \int_1^{+\infty} t^{\gamma-1} e^{-\delta t} \mathrm{d}t \right), \end{aligned} \quad (3.26)$$

and since

$$(\gamma - 2) + 1/p^* > -1 - 1/p^* + 1/p^* = -1,$$

function H is Bochner's integrable.

On the other hand one has

$$\begin{aligned}\left\|(-A)^{-\gamma}x\right\| &= \left\|\frac{1}{\Gamma(\gamma)}\int_0^{+\infty}\left[1-\gamma+\delta t\right]t^{\gamma-2}e^{-\delta t}S_{(A+\delta I)}(t)x\mathrm{d}t\right\| \\ &= \frac{1}{\Gamma(\gamma)}\left\|\int_0^{+\infty}H(t)\mathrm{d}t\right\| \\ &\leq \frac{1}{\Gamma(\gamma)}\int_0^{+\infty}\|H(t)\|\,\mathrm{d}t.\end{aligned}$$

We conclude from (3.26) that

$$\left\|(-A)^{-\gamma}\right\| \leq \frac{[1+\gamma+\delta]}{\Gamma(\gamma)}.\left(\tilde{M}_2\int_0^1 t^{(\gamma-2)+1/p^*}\mathrm{d}t + \tilde{M}_1\int_1^{+\infty}t^{\gamma-1}e^{-\delta t}\mathrm{d}t\right),$$

and $(-A)^{-\gamma}$ is well defined by (3.25) for each $x \in X$, each $\gamma > 1 - 1/p^*$ and each $\delta \in (0, -\omega_A)$.
Assertions (i)–(iii) are direct consequences from definition (3.25). It remains to prove (iv).

Now let $\gamma \geq 0$, $\beta > 1 - 1/p^*$, $x \in X$ and $\mu > \omega_A$ be fixed. Since $(\mu I - A)^{-1}$ commutes with $(-A_0)^{-\gamma}$ and $(-A)^{-\beta}$, we have

$$(\mu I - A)^{-1}(-A_0)^{-\gamma}(-A)^{-\beta}x = (-A_0)^{-\gamma}(-A)^{-\beta}(\mu I - A)^{-1}x.$$

Since $(\mu I - A)^{-1} \in X_0$, we obtain

$$(\mu I - A)^{-1}(-A_0)^{-\gamma}(-A)^{-\beta}x = (-A_0)^{-\gamma}(-A_0)^{-\beta}(\mu I - A)^{-1}x.$$

Recalling that (see [20, Lemma 6.2, p.70]) for each $\gamma, \beta \geq 0$, the following relation holds

$$(-A_0)^{-\gamma}(-A_0)^{-\beta} = (-A_0)^{-(\gamma+\beta)},$$

one obtains that

$$\begin{aligned}(\mu I - A)^{-1}(-A_0)^{-\gamma}(-A)^{-\beta}x &= (-A_0)^{-(\gamma+\beta)}(\mu I - A)^{-1}x, \\ &= (-A)^{-(\gamma+\beta)}(\mu I - A)^{-1}x, \\ &= (\mu I - A)^{-1}(-A)^{-(\gamma+\beta)}x.\end{aligned}$$

Finally (iv) follows because $(\mu I - A)^{-1}$ is a one-to-one operator. This completes the proof of the result. □

Let us recall that, since $\{T_{A_0}(t)\}_{t\geq 0}$ is an analytic semigroup, we can define for each $\beta \geq 0$ the operator $(-A_0)^\beta : D\left((-A_0)^\beta\right) \subset X_0 \to X_0$ as the inverse of $(-A_0)^{-\beta}$ (i.e. $(-A_0)^\beta = \left((-A_0)^{-\beta}\right)^{-1}$). Moreover (see [20, Theorem 6.8, p. 72]), we know that $(-A_0)^\beta$ is a closed operator, $D\left((-A_0)^\gamma\right) \subset D\left((-A_0)^\beta\right)$ for all $\gamma \geq \beta \geq 0$, $\overline{D\left((-A_0)^\beta\right)} = X_0$, and for each $\gamma, \beta \in \mathbb{R}$

$$(-A_0)^\gamma (-A_0)^\beta = (-A_0)^{\gamma+\beta}.$$

We also know that (see [20, see Theorem 6.13, p. 74]) for each $t > 0$, $(-A_0)^\beta T_{A_0}(t)$ is a bounded operator, and

$$\left\|(-A_0)^\beta T_{A_0}(t)\right\| \leq M_\beta t^{-\beta} e^{\omega_A t}, \ \forall t > 0. \tag{3.27}$$

As a consequence of these above results, we have the following lemma.

LEMMA 3.8. *Let Assumption 3.4 be satisfied and assume that $\omega_A < 0$. Then for each $q^* \in \left[1, \frac{1}{1-1/p^*}\right)$ and each $\tau > 0$ we have*

$$S_A(.)_{|(0,\tau)} \in W^{1,q^*}\left((0,\tau), \mathcal{L}(X)\right). \tag{3.28}$$

Proof. Taking $\mu = 0$ in (2.12) leads to

$$S_A(t)x = (-A)^{-1} x - T_{A_0}(t) (-A)^{-1} x, \ \forall t \geq 0, \ \forall x \in X.$$

Since $\{T_{A_0}(t)\}_{t\geq 0}$ is an analytic semigroup, the map $t \to T_{A_0}(t)$ is operator norm continuously differentiable on $(0, +\infty)$, so $t \to S_A(t)$ is continuously differentiable on $(0, +\infty)$, and

$$\frac{dS_A(t)x}{dt} = -A_0 T_{A_0}(t) (-A)^{-1} x, \forall t > 0, \ \forall x \in X.$$

Due to Lemma 3.7, we have for each $t > 0$, each $\beta > 1 - 1/p^*$ and each $x \in X$,

$$\begin{aligned}\frac{dS_A(t)x}{dt} &= -A_0 T_{A_0}(t) (-A_0)^{-(1-\beta)} (-A)^{-\beta} x,\\ &= -A_0 (-A_0)^{-(1-\beta)} T_{A_0}(t) (-A)^{-\beta} x,\\ &= (-A_0)^\beta T_{A_0}(t) (-A)^{-\beta} x,\end{aligned}$$

so the map $t \to S_A(t)$ is continuously differentiable on $(0, +\infty)$, and

$$\frac{dS_A(t)x}{dt} = (-A_0)^\beta T_{A_0}(t) (-A)^{-\beta} x, \ \forall t > 0, \ \forall x \in X, \ \forall \beta > 1 - 1/p^*. \tag{3.29}$$

Since $\{S_A(t)\}_{t\geq 0}$ is exponentially bounded, we have for each $\tau > 0$ and each $\hat{q} \geq 1$ that $S_A(.)_{|(0,\tau)} \in L^{\hat{q}}\left((0,\tau), \mathcal{L}(X)\right)$.

Let $\beta \geq 1$ be fixed such that $\beta > 1 - 1/p^*$ and let $x \in D\left((-A_0)^\beta\right)$. It is well known that

$$T_{A_0}(t)x = T_{A_0}(s)x + A_0 \int_s^t T_{A_0}(l)x\mathrm{d}l,$$

so

$$(-A_0)^{\beta} T_{A_0}(t)x = (-A_0)^{\beta} T_{A_0}(s)x - \int_s^t (-A_0)^{\beta+1} T_{A_0}(l)x\mathrm{d}l.$$

We deduce that

$$(-A_0)^{\beta} \left[T_{A_0}(t)x - T_{A_0}(s)x\right] = \int_s^t (-A_0)^{\beta+1} T_{A_0}(l)x\mathrm{d}l.$$

It follows that

$$\left\|(-A_0)^{\beta} \left[T_{A_0}(t)x - T_{A_0}(s)x\right]\right\| \leq M_{\beta+1} \int_s^t \frac{1}{l^{\beta+1}}\mathrm{d}l \,\|x\|,$$

and the map $t \to (-A_0)^{\beta} T_{A_0}(t)$ is continuous from $(0, +\infty)$ into $\mathcal{L}(X_0)$. Since $(-A)^{-\beta}$ is a bounded operator, the map $H : t \to (-A_0)^{\beta} T_{A_0}(t) (-A)^{-\beta}$ is continuous from $(0, +\infty)$ into $\mathcal{L}(X)$ and thus Bochner's measurable. Now due to (3.27) one obtains that

$$\begin{aligned}\left\|\frac{dS_A(t)x}{dt}\right\| &= \left\|(-A_0)^{\beta} T_{A_0}(t) (-A)^{-\beta} x\right\|, \\ &\leq M_{\beta} t^{-\beta} e^{\omega_A t} \left\|(-A)^{-\beta} x\right\|, \\ &\leq M_{\beta} t^{-\beta} e^{\omega_A t} \left\|(-A)^{-\beta}\right\| \|x\|,\end{aligned}$$

so

$$\int_0^{\tau} \left\|\frac{\mathrm{d}S_A(t)}{\mathrm{d}t}\right\|_{\mathcal{L}(X)}^{q^*} \mathrm{d}t \leq \left(M_{\beta} \left\|(-A)^{-\beta}\right\|\right)^{q^*} \int_0^{\tau} t^{-q^*\beta} e^{\omega_A t}\mathrm{d}t.$$

Since $q^* \in \left[1, \frac{1}{1-1/p^*}\right)$ then $q^*\beta < 1$ and the map $t \to t^{-q^*\beta}$ is integrable on $(0, \tau)$ and the result follows. □

As a direct corollary we get the following result.

COROLLARY 3.9. *Let Assumption 3.4 be satisfied. Then the family of bounded linear operator* $\{T(t) = \frac{dS_A(t)}{dt}\}_{t>0}$ *satisfies* $T(s+t) = T(s)T(t)$ *for all* $s, t > 0$ *and for each* $\beta > 1 - \frac{1}{p^*}$ *there exists some constant* $M_{\beta} > 0$ *such that*

$$\|t^{\beta} T(t)\| \leq M_{\beta} e^{\omega_A t}, \quad \forall t > 0.$$

REMARK 3.10. The above results are related to work of Periago and Straub [21] using the following formulas for the integrated semigroups, and its derivative with respect to $t > 0$,

$$S(t) = \int_{\Gamma} \lambda^{-1} \left(e^{\lambda t} - 1\right) (\lambda - A)^{-1}\mathrm{d}\lambda, \quad T(t) = \int_{\Gamma} e^{\lambda t}(\lambda - A)^{-1}\mathrm{d}\lambda, \quad (3.30)$$

where Γ is an angle

$$\{\lambda \in \mathbb{C} : |\arg(\lambda - \omega_A)| = \eta, |\lambda - \omega_A| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega_A)| \leq \eta, |\lambda - \omega_A| = r\}$$

oriented counterclockwise where $r > 0$, $\eta \in (\pi/2, \theta)$. To prove (3.30) simply observe that for $x \in X$,

$$\begin{aligned} T(t)x &= \lim_{\mu \to \infty} \mu (\mu I - A_0)^{-1} T(t)x = \lim_{\mu \to \infty} T(t)\mu (\mu I - A_0)^{-1} x \\ &= \lim_{\mu \to \infty} \int_{\Gamma} e^{\lambda t} (\lambda - A_0)^{-1} \mathrm{d}\lambda \mu (\mu I - A)^{-1} x \\ &= \lim_{\mu \to \infty} \mu (\mu I - A_0)^{-1} \int_{\Gamma} e^{\lambda t} (\lambda - A)^{-1} x \mathrm{d}\lambda, \end{aligned}$$

and second formula in (3.30) follows. By taking the time derivative in the first formula of (3.30), we also deduce that the first equality in (3.30) holds true. Now using Theorem 3.9 in Periago and Straub [21], we obtain that $t \to T(t)$ and $t \to S(t)$ are analytic functions on $(0, +\infty)$.

3.3. The linear Cauchy problem

In this section we investigate the existence and uniqueness for integrated solution for the linear Cauchy problem (2.4). The following theorem is the main result of this section

THEOREM 3.11. *Let Assumption 3.4 be satisfied. Let $\lambda > \omega_A$ and $\hat{p} \in (p^*, +\infty)$ be fixed. Then for each $f \in L^{\hat{p}}((0, \tau), X)$, the map $t \to (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0, \tau]$, and if we denote by $u(t) = (S_A \diamond f)(t) = \frac{d}{dt}(S_A * f)(t)$, then*

$$u(t) = A \int_0^t u(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s, \ \forall t \in [0, \tau].$$

Moreover, for each $\beta \in (1 - \frac{1}{p^*}, 1 - \frac{1}{\hat{p}})$, and each $t \in [0, \tau]$, the following holds true

$$(S_A \diamond f)(t) = \int_0^t (\lambda I - A_0)^{\beta} T_{A_0}(t - s)(\lambda I - A)^{-\beta} f(s)\mathrm{d}s, \tag{3.31}$$

as well as the estimate:

$$\|(S_A \diamond f)(t)\| \leq M_\beta \left\|(\lambda I - A)^{-\beta}\right\|_{\mathcal{L}(X)} \int_0^t (t - s)^{-\beta} e^{\omega_A (t-s)} \|f(s)\| \mathrm{d}s, \tag{3.32}$$

wherein M_β is some positive constant.

Proof. Without loss of generality, one may assume that $\omega_A < 0$. Let $\hat{p} \in (p^*, +\infty)$ be fixed. Let $f \in C_c^1((0, \tau), X)$, then the map $t \to (S_A * f)(t)$ is continuously differentiable on $[0, \tau]$ and

$$(S_A \diamond f)(t) = \int_0^t S_A(t - s) f'(s)\mathrm{d}s, \ \forall t \in [0, \tau].$$

Using Fubini 's Theorem one obtains that for each $t \in [0, \tau]$,

$$\int_0^t S_A(t-s) f'(s) \mathrm{d}s = \int_0^t \int_0^{t-s} \frac{\mathrm{d}S_A(r)}{\mathrm{d}r} f'(s) \mathrm{d}r \mathrm{d}s = \int_0^t \int_0^{t-r} \frac{\mathrm{d}S_A(r)}{\mathrm{d}r} f'(s) \mathrm{d}s \mathrm{d}r = \int_0^t \frac{\mathrm{d}S_A(r)}{\mathrm{d}r} f(t-r) \mathrm{d}r.$$

Since $1 \leq \frac{1}{1-1/\hat{p}} < \frac{1}{1-1/p^*}$, we infer from Lemma 3.8 that $S_A(.) \in W^{1,\frac{1}{1-1/\hat{p}}}((0,\tau), \mathcal{L}(X))$. Thus Hölder inequality provides, for each $t \in [0, \tau]$,

$$\|(S_A \diamond f)(t)\| \leq \left\| \frac{d}{dt} S_A(.) \right\|_{L^{\frac{1}{1-1/\hat{p}}}((0,\tau),\mathcal{L}(X))} \cdot \|f\|_{L^{\hat{p}}((0,\tau),X)}.$$

The first part of the theorem follows from the density of $C_c^1((0,\tau), X)$ into $L^{\hat{p}}((0,\tau), X)$ and Theorem 2.11 in Magal and Ruan [16]. Moreover for each $\beta \in \left(1 - \frac{1}{p^*}, 1 - \frac{1}{\hat{p}}\right)$ and each $\lambda > \omega_A$ one has

$$\frac{dS_A(t)}{dt} = (\lambda I - A_0)^\beta T_{A_0}(t) (\lambda I - A)^{-\beta}, \ \forall t > 0$$

hence

$$\|(S_A \diamond f)(t)\| \leq M_\beta \left\|(\lambda I - A)^{-\beta}\right\|_{\mathcal{L}(X)} \int_0^t (t-s)^{-\beta} e^{\omega_A(t-s)} \|f(s)\| \mathrm{d}s.$$

This completes the proof of the result. □

As a consequence of this result, one can derive some estimate for the resolvent of operator A. Indeed, estimate (3.32) re-writes for each $\beta > 1 - \frac{1}{p^*}$:

$$\|(S_A \diamond f)(t)\| \leq \int_0^t \chi(t-s) \|f(s)\| \mathrm{d}s, \forall t \geq 0, \forall f \in C^1((0,+\infty), X).$$

with $\chi(s) = M_\beta s^{-\beta} e^{\omega_A s}$ wherein $M_\beta > 0$ is some constant. Therefore, Theorem 2.7 applies (with $B = Id$, $\chi(s) = M_\beta s^{-\beta} e^{\omega_A s}$) and provides that for any $n \geq 1$ and each $\beta > 1 - \frac{1}{p^*}$

$$\left\|(\lambda I - A)^{-n}\right\|_{\mathcal{L}(X)} \leq \frac{M_\beta}{(n-1)!} \int_0^{+\infty} s^{n-1-\beta} e^{-(\lambda-\omega_A)s} \mathrm{d}s = \frac{M_\beta}{(n-1)!(\lambda-\omega_A)^{(n-\beta)}} \int_0^{+\infty} l^{(n-\beta)-1} e^{-l} \mathrm{d}l = \frac{M_\beta \Gamma(n-\beta)}{\Gamma(n)} \frac{1}{(\lambda-\omega_A)^{(n-\beta)}}.$$

Conversely if the above inequality is satisfied then Theorem 2.7 and Theorem 2.11 in Magal and Ruan [16] imply that for each $f \in L^{\hat{p}}(0, \tau, X)$ with $1 - \frac{1}{\hat{p}} < \beta$, the map $t \to (S_A * f)(t)$ is continuously differentiable and

$$\left\| \frac{d}{dt}(S_A * f) \right\| \leq C \int_0^t (t-s)^{-\beta} e^{-\omega_A (t-s)} \|f(s)\| \, \mathrm{d}s \quad \forall t \in [0, \tau].$$

But this condition is not easy to verify in practice compared to Assumption 3.4-(b).

COROLLARY 3.12. *Let Assumption 3.4 be satisfied. Let $\hat{p} \in (p^*, +\infty)$ be fixed. Then for each $f \in L^{\hat{p}}((0, \tau), X)$ and for each $x \in X_0$, the Cauchy problem (2.4) has a unique integrated solution $u \in C([0, \tau], X_0)$ given by*

$$u(t) = T_{A_0}(t)x + (S_A \diamond f)(t), \ \forall t \in [0, \tau]. \tag{3.33}$$

4. Perturbation results

In this section, we investigate the properties of $A + B : D(A) \cap D(B) \subset X \to X$, where $B : D(B) \subset X \to X$ is a linear operator. Inspired by Pazy [20], we will make the following assumption.

ASSUMPTION 4.1. Recall that $X_0 = \overline{D(A)}$ and let $B : D(B) \subset X_0 \to Y$ be a linear operator from $D(B)$ into a Banach space $Y \subset X$. We assume that there exists $\alpha \in (0, 1)$, such that the operator B *is* $(\lambda I - A_0)^\alpha$ − bounded for some $\lambda > \omega_A$ (that means $B(\lambda I - A_0)^{-\alpha}$ is a bounded linear operator).

Let us first notice that when Assumption 4.1 holds true then

$$D((\lambda I - A_0)^\alpha) \subset D(B),$$

and thus

$$D((\lambda I - A_0)^\alpha) \subset D(A) \cap D(B).$$

The main result of this section is the following theorem.

THEOREM 4.2. *Let Assumptions 3.4 and 4.1 be satisfied and assume that $Y = X$. We assume in addition that*

$$\alpha < 1/p^*.$$

Then $A + B : D(A) \cap D(B) \subset X \to X$ satisfies the Assumption 3.4. More precisely, there exist two constants $\widehat{\omega} > 0$ and $\widetilde{M} > 1$ such that:

(i) $(\widehat{\omega}, +\infty) \subset \rho(A + B)$ *and the resolvent of $A + B$ is given by*

$$(\lambda I - (A + B))^{-1} = (\lambda I - A)^{-1} + (\lambda I - A)^{-1} \sum_{k=0}^{\infty} \left[B(\lambda I - A)^{-1} \right]^k B(\lambda I - A)^{-1},$$

whenever $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) > \widehat{\omega}$.

(ii) $(A+B)_0$ *the part of* $A+B$ *in* X_0 *is the infinitesimal generator of an analytic semigroup* $\left\{T_{(A+B)_0}(t)\right\}_{t\geq 0}$ *on* X_0 *and*

$$\left\|(\lambda I-(A+B))^{-1}\right\| \leq \frac{\widetilde{M}}{(\lambda-\widehat{\omega})^{1/p^*}}, \forall\lambda>\widehat{\omega}.$$

By taking $p^*=1$ we also have the following immediate corollary.

COROLLARY 4.3. *If* A *is a Hille-Yosida operator and* A_0 *is the infinitesimal generator of an analytic semigroup such that* $B(\lambda I-A_0)^{-\alpha}$ *is a bounded operator for some* $\alpha\in(0,1)$ *and some* $\lambda>\omega_A$ *then* $(A+B)$ *is a Hille-Yosida operator and* $(A+B)_0$ *is the infinitesimal generator of an analytic semigroup.*

Using Proposition 3.3 we have the following immediate corollary:

COROLLARY 4.4. *Let* $A: D(A)\subset X\to X$ *be a* $\frac{1}{p^*}$*−almost sectorial operator for some* $p^*\geq 1$ *and with sectorial part* A_0 *on* X_0*. Let* $B: D(B)\subset X_0\to X$ *be a linear closed operator such that there exists* $\alpha<\frac{1}{p^*}$ *and* $D\left((\lambda I-A_0)^{\alpha}\right)\subset D(B)$ *for some* $\lambda>\omega_A$*. Then* $A+B$ *is a* $\frac{1}{p^*}$*−almost sectorial operator with a sectorial part* $(A+B)_0$ *in* X_0*.*

In order to prove Theorem 4.2, we first need to prove the following lemma:

LEMMA 4.5. *Let Assumptions 3.4 and 4.1 be satisfied and assume that* $Y=X$*. Let* $\gamma\in(\alpha,1)$ *be given and fixed. Then there exists some constant* $C>0$ *such that*

$$\left\|B(\delta I-A_0)^{-\gamma}\right\| \leq \frac{C}{(\delta-\omega_A)^{\gamma-\alpha}}, \text{ for all } \delta>\omega_A, \tag{4.34}$$

and

$$\left\|B(\lambda I-A_0)^{-1}\right\| \leq \frac{C}{|\lambda-\omega_A|^{1-\alpha}}, \text{ for all } \lambda\in\mathbb{C} \text{ with } \mathcal{R}e(\lambda)>\omega_A. \tag{4.35}$$

Moreover, if

$$\alpha<1/p^*,$$

then for $\widetilde{\omega}>0$ *large enough*

$$\left\|B(\lambda I-A)^{-1}\right\| \leq 1/2, \text{ for all } \lambda\in\mathbb{C} \text{ with } \mathcal{R}e(\lambda)>\omega_A+\widetilde{\omega}. \tag{4.36}$$

Proof. Let $\lambda>\omega_A$ be fixed. We have that for $\delta>\omega_A$

$$\begin{aligned} B(\delta I-A_0)^{-\gamma} &= B(\lambda I-A_0)^{-\alpha}(\lambda I-A_0)^{\alpha}(\delta I-A_0)^{-\gamma}, \\ &= B(\lambda I-A_0)^{-\alpha}(\lambda I-A_0)^{\alpha}\left[\frac{1}{\Gamma(\gamma)}\int_0^{+\infty} t^{\gamma-1}e^{-\delta t}T_{A_0}(t)\mathrm{d}t\right]. \end{aligned}$$

Since $(\lambda I - A_0)^{-\alpha}$ is bounded and $(\lambda I - A_0)^{\alpha}$ is closed, we have

$$B(\delta I - A_0)^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} e^{-\delta t} B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t) \mathrm{d}t,$$

so that

$$\begin{aligned}\left\| B(\delta I - A_0)^{-\gamma} \right\| &\leq \frac{\tilde{C}}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-\alpha-1} e^{-\delta t} e^{\omega_A t} \mathrm{d}t, \\ &\leq \frac{\tilde{C}}{\Gamma(\gamma)} \int_0^{+\infty} \left(\frac{l}{\delta - \omega_A} \right)^{\gamma-\alpha-1} e^{-l} \frac{1}{\delta - \omega_A} \mathrm{d}l, \\ &\leq \frac{C_1}{(\delta - \omega_A)^{\gamma-\alpha}}.\end{aligned}$$

and (4.34) follows. Now, for all $\gamma > \alpha$, for all $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) > \omega_A$ and $\delta \in \mathbb{R}$ such that $\delta > \omega_A$, one has

$$B(\lambda I - A_0)^{-1} = B(\delta I - A_0)^{-\gamma} (\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1}.$$

Using (3.15) as well as the resolvent formula, we have

$$(\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1} = (\delta - \lambda)(\delta I - A_0)^{-(1-\gamma)} (\lambda I - A_0)^{-1} + (\delta I - A_0)^{-(1-\gamma)}.$$

Therefore, one obtains that

$$\left\| (\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1} \right\| \leq |\delta - \lambda| \frac{\check{M}}{(\delta - \omega_A)^{1-\gamma}} \frac{M_A}{|\lambda - \omega_A|} + \frac{\check{M}}{(\delta - \omega_A)^{1-\gamma}}.$$

By taking now $\delta = \omega_A + |\lambda - \omega_A| > \omega_A$, one has $|\lambda - \delta| \leq 2|\lambda - \omega_A|$ and thus

$$\left\| (\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1} \right\| \leq \frac{C_2}{|\lambda - \omega_A|^{1-\gamma}},$$

and (4.35) follows.

Finally, we prove (4.36). To do so let us notice that

$$B(\lambda I - A)^{-1} = B(\delta I - A_0)^{-\gamma} (\delta I - A_0)^{\gamma} (\lambda I - A)^{-1},$$

and similarly,

$$(\delta I - A_0)^{\gamma} (\lambda I - A)^{-1} = (\delta - \lambda)(\delta I - A_0)^{-(1-\gamma)} (\lambda I - A)^{-1} + (\delta I - A)^{-(1-\gamma)}. \quad (4.37)$$

Since $\alpha < 1/p^*$, we can find $\gamma > 0$ such that

$$\alpha < \gamma < 1/p^* \text{ and } 1 - \gamma > 1 - 1/p^*.$$

Then due to Lemma 3.7 $(\delta I - A)^{-(1-\gamma)}$ is well defined. Moreover, by setting $\delta = \omega_A + |\lambda - \omega_A|$ in (4.37) we obtain

$$\left\| (\delta I - A_0)^{\gamma} (\lambda I - A)^{-1} \right\| \leq |\lambda - \omega_A| . \frac{\check{M}}{|\lambda - \omega_A|^{1-\gamma}} . \frac{\tilde{M}}{|\lambda - \omega_A|^{1/p^*}} + \frac{\tilde{M}}{|\lambda - \omega_A|^{1-\gamma}}.$$

Therefore, there exist two constants $C_3, C_4 > 0$ such that

$$\left\| B(\lambda I - A)^{-1} \right\| \leq \frac{C_3}{|\lambda - \omega_A|^{1/p^* - \alpha}} + \frac{C_4}{|\lambda - \omega_A|^{1-\alpha}}.$$

Thus for $\widetilde{\omega} > 0$ large enough, we obtain for each $\lambda \in \mathbb{C}$ such that $\mathcal{R}e\,(\lambda) - \omega_A > \widetilde{\omega}$,

$$\left\| B(\lambda I - A)^{-1} \right\| \leq \frac{C_3}{\widetilde{\omega}^{1/p^* - \alpha}} + \frac{C_4}{\widetilde{\omega}^{1-\alpha}} \leq \frac{1}{2},$$

and (4.36) follows. □

Proof of Theorem 4.2. For each $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \omega_A + \widetilde{\omega}$, where $\widetilde{\omega}$ is provided by Lemma 4.5 we have

$$\begin{aligned}(\lambda I - (A+B)_0)^{-1} &= (\lambda I - A)^{-1} \left(\sum_{k=0}^{\infty} \left[B(\lambda I - A)^{-1} \right]^k \right), \\ &= (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \sum_{k=1}^{\infty} \left[B(\lambda I - A)^{-1} \right]^k,\end{aligned}$$

thus

$$(\lambda I - (A+B)_0)^{-1} = (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \sum_{k=0}^{\infty} \left[B(\lambda I - A)^{-1} \right]^k B(\lambda I - A_0)^{-1}. \tag{4.38}$$

We infer by combining Assumption 3.4-(b), (4.35) and (4.36) that

$$\begin{aligned}\left\| (\lambda I - (A+B)_0)^{-1} \right\| &\leq \frac{\hat{M}}{|\lambda - \omega_A|} + \frac{M}{|\lambda - \omega_A|^{1/p^*}} \frac{1}{1 - 1/2} \frac{C_2}{|\lambda - \omega_A|^{1-\alpha}}, \\ &\leq \frac{1}{|\lambda - \omega_A|} \left(\hat{M} + \frac{C_2 M}{|\lambda - \omega_A|^{1/p^* - \alpha}} \right), \\ &\leq \frac{1}{|\lambda - \omega_A|} \left(\hat{M} + \frac{C_2 M}{\widetilde{\omega}^{1/p^* - \alpha}} \right).\end{aligned}$$

Thus, there exists some constant $C > 0$ such that for each $\lambda \in \mathbb{C}$ with $\mathcal{R}e\,(\lambda) > \omega_A + \widetilde{\omega}$,

$$\left\| (\lambda I - (A+B)_0)^{-1} \right\| \leq \frac{C}{|\lambda - \omega_A|},$$

which implies that $(A+B)_0$ is a sectorial operator. Next for $\lambda > \omega_A + \widetilde{\omega}$ we have

$$(\lambda I - (A+B))^{-1} = (\lambda I - A)^{-1} + (\lambda I - A)^{-1} \sum_{k=0}^{\infty} \left[B(\lambda I - A)^{-1} \right]^k.$$

Combining now Assumption 3.4-(b), (4.36) together with (4.38) lead us to

$$\left\|(\lambda I-(A+B))^{-1}\right\| \leq \frac{\hat{M}}{|\lambda-\omega_A|^{1/p^*}}+\frac{M}{|\lambda-\omega_A|^{1/p^*}}\left(\frac{1}{1-1/2}\times\frac{1}{2}\right).$$

Hence, there exist two constants $\widehat{\omega} > \omega_A + \widetilde{\omega}$ and $\widetilde{M} > 0$ such that $(\widehat{\omega}, +\infty) \subset \rho\,(A+B)$ and

$$\left\|(\lambda I-(A+B))^{-1}\right\| \leq \frac{\widetilde{M}}{(\lambda-\widehat{\omega})^{1/p^*}},\ \forall\lambda > \widehat{\omega}.$$

The result is proved. □

THEOREM 4.6. *Let Assumptions 3.4 and 4.1 be satisfied and assume that* $Y = X_0$. *Then* $A + B : D(A) \cap D(B) \subset X \to X$ *satisfies Assumption 3.4.*

Proof. For each $\lambda \in \mathbb{C}$ with $\mathcal{R}e\ (\lambda) > \omega_A + \widetilde{\omega}$, we have

$$\begin{aligned}(\lambda I-(A+B)_0)^{-1} &= (\lambda I-A)^{-1}\left(\sum_{k=0}^{\infty}\left[B\,(\lambda I-A)^{-1}\right]^k\right),\\ &= (\lambda I-A_0)^{-1}+(\lambda I-A)^{-1}\sum_{k=1}^{\infty}\left[B\,(\lambda I-A)^{-1}\right]^k,\\ &= (\lambda I-A_0)^{-1}+(\lambda I-A_0)^{-1}\sum_{k=1}^{\infty}\left[B\,(\lambda I-A)^{-1}\right]^k\\ &\quad\times B\,(\lambda I-A_0)^{-1}.\end{aligned}$$

and we obtain that for any $\lambda \in \mathbb{C}$ such that $\mathcal{R}e\ (\lambda) > \widetilde{\omega} + \omega_A$,

$$\left\|(\lambda I-(A+B)_0)^{-1}\right\| \leq \frac{C}{|\lambda-\omega_A|},$$

where $C > 0$ is some constant. Similarly, we also deduce that there exists $\widehat{\omega} > \omega_A$ such that

$$\left\|(\lambda I-(A+B))^{-1}\right\| \leq \frac{\widetilde{M}}{(\lambda-\widehat{\omega})^{1/p^*}},\quad \forall\lambda > \widehat{\omega},$$

and the result follows. □

In order to extend the linear theory to the semi-linear ones, it will be useful to find some invariant $L^{\hat{p}}$ space. We address this question in the next Proposition.

PROPOSITION 4.7. *Let Assumptions 3.4 and 4.1 be satisfied. Assume in addition that* $\omega_A < 0$. *If there exists* $\hat{p} \in [1, +\infty)$ *such that*

$$p^* < \hat{p} < \frac{1}{\alpha}. \tag{4.39}$$

Then,

(i) *The map* $x \to BT_{A_0}(.)x$ *defines a bounded linear operator from* X_0 *into* $L^{\hat{p}}\big((0, \tau), X_0\big)$.

(ii) *For each* $f \in L^{\hat{p}}((0,\tau), X)$,

$$B(S_A \diamond f)(.) \in L^{\hat{p}}((0,\tau), Y).$$

Moreover for each $\beta \in (1 - 1/p^*, 1-\alpha)$, *the following estimate holds:*

$$\|B(S_A \diamond f)(.)\|_{L^{\hat{p}}((0,\tau),Y)} \leq C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} \mathrm{d}t \, \|f(.)\|_{L^{\hat{p}}((0,\tau),X)} \tag{4.40}$$

wherein

$$C_{\alpha,\beta} =: \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathcal{L}(X_0,Y)} M_{\beta+\alpha} \left\| (\lambda I - A)^{-\beta} \right\|_{\mathcal{L}(X)}.$$

Proof. First note that since

$$\hat{p} \in \big(p^*, +\infty\big), \tag{4.41}$$

the assumptions of Theorem 3.11 are satisfied. Now let $\lambda > \omega_A$ be given. Then for any $t > 0$, we have for each $\gamma > \alpha$,

$$\begin{aligned} BT_{A_0}(t) &= B(\lambda I - A_0)^{-\gamma} (\lambda I - A_0)^{\gamma} T_{A_0}(t) \\ &= B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t). \end{aligned}$$

Therefore, for any $x \in X_0$, we have

$$\begin{aligned} \left\| BT_{A_0}(.)x \right\|_{L^{\hat{p}}((0,\tau),Y)} &= \left\| B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(.)x \right\|_{L^{\hat{p}}((0,\tau),Y)} \\ &= \left(\int_0^\tau \left\| B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t)x \right\|_Y^{\hat{p}} \mathrm{d}t \right)^{1/\hat{p}}, \\ &\leq \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathcal{L}(X_0,Y)} \\ &\quad \times \left(\int_0^\tau \left\| (\lambda I - A_0)^{\alpha} T_{A_0}(t)x \right\|_{X_0}^{\hat{p}} \mathrm{d}t \right)^{1/\hat{p}}, \\ &\leq M_\alpha \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathcal{L}(X_0,Y)} \left(\int_0^\tau t^{-\hat{p}\alpha} \mathrm{d}t \right)^{1/\hat{p}} \|x\|_{X_0}. \end{aligned}$$

Let us now fix $\hat{p}$ such that

$$\hat{p} < \frac{1}{\alpha}. \tag{4.42}$$

Then we obtain that the map $t \to t^{-\hat{p}\alpha}$ is integrable over $(0, \tau)$ and the map $x \to BT_{A_0}(.)x$ is bounded by X_0 into $L^{\hat{p}}((0,\tau), Y)$. Moreover, the following estimate holds:

$$\left\| BT_{A_0}(.)x \right\|_{L^{\hat{p}}((0,\tau),Y)} \leq C \|x\|_{X_0}$$

wherein C is defined by

$$C := M_\alpha \left\| B (\lambda I - A_0)^{-\alpha} \right\|_{\mathcal{L}(X_0,Y)} \left(\int_0^\tau t^{-\hat{p}\alpha} \right)^{1/\hat{p}}.$$

Note that due to (4.41), (4.42) one can find β such that

$$\beta > 1 - 1/p^* \text{ and } \alpha + \beta < 1. \tag{4.43}$$

Let us now fix such a value β. Then since $\beta > 1 - 1/p^*$, the fractional power of $(\lambda I - A)^{-\beta}$ is well defined for λ large enough and we have:

$$\begin{aligned} &B (S_A \diamond f)(t) \\ &= B (\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} \int_0^t (\lambda I - A_0)^{\beta} T_{A_0}(t-s) (\lambda I - A)^{-\beta} f(s)\mathrm{d}s, \\ &= B (\lambda I - A_0)^{-\alpha} \int_0^t (\lambda I - A_0)^{\beta+\alpha} T_{A_0}(t-s) (\lambda I - A)^{-\beta} f(s)\mathrm{d}s. \end{aligned}$$

Thus

$$\begin{aligned} \| B (S_A \diamond f) \|_{L^{\hat{p}}((0,\tau),Y)} &\le C_{\alpha,\beta} \left(\left(\int_0^\tau \int_0^t (t-s)^{-(\beta+\alpha)} \| f(s) \| \,\mathrm{d}s \,\mathrm{d}t \right)^{\hat{p}} \right)^{1/\hat{p}} \\ &= C_{\alpha,\beta} \left\| \left((.)^{-(\beta+\alpha)} * \| f(.) \| \right) (.) \right\|_{L^{\hat{p}}((0,\tau),\mathbb{R})} \\ &\le C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} \mathrm{d}t \, \| f(.) \|_{L^{\hat{p}}((0,\tau),X)} \end{aligned}$$

where

$$C_{\alpha,\beta} =: \left\| B (\lambda I - A_0)^{-\alpha} \right\|_{\mathcal{L}(X_0,Y)} M_{\beta+\alpha} \left\| (\lambda I - A)^{-\beta} \right\|_{\mathcal{L}(X)}.$$

Due to (4.43) the map $t \to t^{-(\beta+\alpha)}$ is integrable over $(0, \tau)$ and the result follows. □

The following result is another formulation of the perturbation result.

THEOREM 4.8. *With the same assumptions as in Proposition 4.7,* $\left\{ T_{(A+B)_0}(t) \right\}_{t \ge 0}$ *the* C_0 *-semigroup generated by* $(A + B)_0$ *is the unique solution of the fixed point problem*

$$T_{(A+B)_0}(t) = T_{A_0}(t) + (S_A \diamond V)(t) \tag{4.44}$$

where $V(.)x \in L^{\widehat{p}}_{\omega^*}((0, +\infty), X)$ *(for* $\omega^* > 0$ *large enough) is the solution of*

$$V(t)x = BT_{A_0}(t)x + B (S_A \diamond V(.)x)(t), \text{ for } t > 0, \tag{4.45}$$

wherein $L^{\widehat{p}}_{\omega^*}((0, +\infty), X)$ *is the space of map* $f : (0, +\infty) \to X$ *Bochner's measurable such that*

$$\| f \|_{L^{\widehat{p}}_{\omega^*}} := \left(\int_0^{+\infty} \left\| e^{-\omega^* t} f(t) \right\|^{\widehat{p}} \mathrm{d}t \right)^{1/\widehat{p}} < +\infty.$$

Proof. Let $\lambda \in (\omega_A, +\infty)$ be fixed. Multiplying (4.45) by the map $t \to e^{-\widehat{\omega} t}$, and using the same arguments as in the proof of Proposition 4.7, we obtain

$$\begin{aligned} e^{-\widehat{\omega} t} V(t)x &= e^{-\widehat{\omega} t} B T_{A_0}(t)x + e^{-\widehat{\omega} t} B \left(S_A \diamond V(.)x\right)(t) \\ &= e^{-\widehat{\omega} t} B \left(\lambda I - A_0\right)^{-\alpha} \left(\lambda I - A_0\right)^{\alpha} T_{A_0}(t) \\ &\quad + B \left(\lambda I - A_0\right)^{-\alpha} \int_0^t e^{-\widehat{\omega}(t-s)} \left(\lambda I - A_0\right)^{\beta+\alpha} T_{A_0}(t-s) \\ &\quad \times \left(\lambda I - A\right)^{-\beta} e^{-\widehat{\omega} s} V(s)x \mathrm{d}s. \end{aligned}$$

Thus for each $\widehat{\omega} > 0$ large enough we obtain, for any $\hat{p} \in \left(p^*, \frac{1}{\alpha}\right)$, that

$$\begin{aligned} &\left\| e^{-\widehat{\omega}.} B \left(\lambda I - A_0\right)^{-\alpha} \left(\lambda I - A_0\right)^{\alpha} T_{A_0}(.)x \right\|_{L^{\hat{p}}} \\ &\quad \leq \left\| e^{-\widehat{\omega}.} B \left(\lambda I - A_0\right)^{-\alpha} \left(\lambda I - A_0\right)^{\alpha} T_{A_0}(.) \right\|_{L^{\hat{p}}} \\ &\quad \leq M_\alpha \left\| B \left(\lambda I - A_0\right)^{-\alpha} \right\|_{\mathcal{L}(X_0, Y)} \left(\int_0^{+\infty} t^{-\hat{p}\alpha} e^{-\hat{p}\widehat{\omega} t} \mathrm{d}t \right)^{1/\hat{p}} \|x\|_{X_0}, \end{aligned}$$

and for each $\beta \in (1 - 1/p^*, 1 - \alpha)$,

$$\begin{aligned} &\left\| e^{-\widehat{\omega}.} B \left(S_A \diamond V(.)x\right)(.) \right\|_{L^{\hat{p}}} \\ &\quad \leq C_{\alpha,\beta} \left(\left(\int_0^\tau \int_0^t e^{-\widehat{\omega}(t-s)} (t-s)^{-(\beta+\alpha)} e^{-\widehat{\omega} s} \|V(s)x\| \mathrm{d}s \mathrm{d}t \right)^{\hat{p}} \right)^{1/\hat{p}} \\ &\quad \leq C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} e^{-\widehat{\omega} t} \mathrm{d}t \left\| e^{-\widehat{\omega}.} V(.)x \right\|_{L^{\hat{p}}}. \end{aligned}$$

Thus, taking $\omega^* > 0$ large enough leads to

$$C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} e^{-\omega^* t} \mathrm{d}t < 1.$$

From some fixed point argument, we conclude that (4.45) has a unique solution. Therefore, we can define $\{L(t)\}_{t \geq 0}$ a strongly continuous family of linear operators such that

$$L(t) = T_{A_0}(t) + \left(S_A \diamond V\right)(t),$$

wherein $V(.) \in L^{\widehat{p}}_{\omega^*}\left((0, +\infty), X\right)$ satisfies (4.45). Due to Theorem 2.1 in Arendt [3], to complete the proof of the result, it is sufficient to check that the following equality holds for each $\lambda > 0$ large enough:

$$\left(\lambda I - (A + B)_0\right)^{-1} = \int_0^{+\infty} e^{-\lambda t} L(t) \mathrm{d}t.$$

On the one hand, one has

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} L(t)\mathrm{d}t &= \int_0^{+\infty} e^{-\lambda t} T_{A_0}(t)\mathrm{d}t + \int_0^{+\infty} e^{-\lambda t} \left(S_A \diamond V\right)(t)\mathrm{d}t \\ &= (\lambda I - A_0)^{-1} + \lambda \int_0^{+\infty} e^{-\lambda t} \left(S_A * V\right)(t)\mathrm{d}t \\ &= (\lambda I - A_0)^{-1} + \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t)\mathrm{d}t \int_0^{+\infty} e^{-\lambda t} V(t)\mathrm{d}t \\ &= (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \int_0^{+\infty} e^{-\lambda t} V(t)\mathrm{d}t.\end{aligned}$$

On the other hand, we infer from (4.45) that

$$\int_0^{+\infty} e^{-\lambda t} V(t)\mathrm{d}t = B \int_0^{+\infty} e^{-\lambda t} L(t)\mathrm{d}t.$$

Thus, we get

$$\int_0^{+\infty} e^{-\lambda t} L(t)\mathrm{d}t = (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} B \int_0^{+\infty} e^{-\lambda t} L(t)\mathrm{d}t,$$

and finally

$$\int_0^{+\infty} e^{-\lambda t} L(t)\mathrm{d}t = (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \sum_{k=0}^{\infty} \left[B\,(\lambda I - A)^{-1}\right] B\,(\lambda I - A_0)^{-1}.$$

The result follows from Theorem 4.2-(i). □

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