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A Finite-time Condition for Exponential Trichotomy in Infinite Dynamical Systems

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Abstract. In this article we study exponential trichotomy for infinite dimensional discrete time dynamical systems. The goal of this article is to prove that finite time exponential trichotomy conditions allow us to derive exponential trichotomy for arbitrary times. We present an application to the case of pseudo orbits in some neighborhood of a normally hyperbolic set.

1 Introduction

Consider the linear non-autonomous discrete time dynamical system

$$(1.1)\qquad \begin{aligned} x_{n+1} &= A_n x_n, \quad \text{for each } n \geq m,\\ x_m &= x \in X, \end{aligned}$$

where $\mathbf{x} = \{x_n\}_{n\geq m}$ is a sequence in a Banach space $(X, \|\cdot\|)$ and $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{L}(X)$, the space of bounded linear operators on X. The discrete time evolution semigroup associated with the system (1.1), or equivalently, associated to the sequence of bounded linear operator $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$, is defined as $\{U_{\mathbf{A}}(n, p)\}_{n\geq p} \subset \mathcal{L}(X)$, which is a parametrized family of bounded linear operators on X defined on

$$\Delta_+ := \{(n, p) \in \mathbb{Z}^2 : n \geq p\}$$

by

$$U_{\mathbf{A}}(n, p) := \begin{cases} A_{n-1}\cdots A_p & \text{if } n > p,\\ I, & \text{if } n = p. \end{cases}$$

In this work we will use the following notion of exponential dichotomy taken from Hale and Lin [11].

Definition 1.1 Let $\mathbb{I}$ be an interval in $\mathbb{Z}$ and let $\mathbf{A} = \{A_n\}_{n\in\mathbb{I}} \colon \mathbb{I} \to \mathcal{L}(X)$ be a map. Then $U_{\mathbf{A}}$ has an *exponential dichotomy* (or $\mathbf{A}$ is *exponentially dichotomic*) on $\mathbb{I}$ with constant κ and exponents $\rho > 0$ if there exist two families of projectors $\mathbf{\Pi}^{\alpha} = \{\Pi_n^{\alpha}\}_{n\in\mathbb{I}} \colon \mathbb{I} \to \mathcal{L}(X)$ with $\alpha = u, s$ satisfying the following properties.

(i) For $n \in \mathbb{I}$ and $\alpha, \beta \in \{u, s\}$, we have $\Pi_n^s + \Pi_n^u = I$ and if $\alpha \neq \beta$, $\Pi_n^{\alpha}\Pi_n^{\beta} = 0$.

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(ii) For all $n, m \in \mathbb{I}$ with $n \geq m$ we have

$$U_{\mathbf{A}}^{\alpha}(n,m) := \Pi_n^{\alpha} U_{\mathbf{A}}(n,m) = U_{\mathbf{A}}(n,m)\Pi_m^{\alpha} \text{ for } \alpha = u, s.$$

(iii) $U_{\mathbf{A}}^{u}(n,m)$ is invertible from $\Pi_m^u(X)$ into $\Pi_n^u(X)$ for all $n \geq m$ in $\mathbb{I}$, and its inverse is denoted by $U_{\mathbf{A}}^{\alpha}(m,n)\colon \Pi_n^{\alpha}(X) \to \Pi_m^{\alpha}(X)$.

(iv) For each $x \in X$ we have, for all $n \geq m$ in $\mathbb{I}$,

$$\|U_{\mathbf{A}}^{s}(n,m)\Pi_m^s x\| \leq \kappa e^{-\rho(n-m)}\|x\|,$$

$$\|U_{\mathbf{A}}^{u}(m,n)\Pi_n^u x\| \leq \kappa e^{-\rho(n-m)}\|x\|.$$

We also introduce the following notion of a relatively dense subset of integers taken from Palmer [15].

Definition 1.2 Let $\mathbb{D} = \{\theta_i\}_{i\in\mathbb{Z}} \subset \mathbb{Z}$ be a non-decreasing sequence of integers and let $T_0 \in \mathbb{N}\diagdown\{0\}$.

(i) We will say that $\mathbb{D}$ is a *relatively dense subset of integers for* T_0 if every interval (in $\mathbb{Z}$) of length T_0 contains at least one point of $\mathbb{D}$.

(ii) We will say that $\mathbb{D}$ is a T_0-*covering* of $\mathbb{Z}$ if

$$\mathbb{Z} = \bigcup_{i\in\mathbb{Z}} [\theta_i, \theta_i + T_0],$$

where $[\theta_i, \theta_i + T_0]$ is understood as an interval in $\mathbb{Z}$.

Lemma 1.3 *Properties* (i) *and* (ii) *in Definition* 1.2 *are equivalent.*

Proof (i) $\Rightarrow$ (ii) Let $n \in \mathbb{Z}$ be given. Then $[n - T_0, n]$ is an interval of length T_0 and there exists $\theta_i \in [n - T_0, n]$ such that $n \in [\theta_i, \theta_i + T_0]$.

(ii) $\Rightarrow$ (i) Let $[n, p]$ be an interval of $\mathbb{Z}$ of length T_0. Then one has $[n, p] = [n, n + T_0]$. By using (ii) and since $p = n + T_0 \in \mathbb{Z}$, we can find $\theta_i \in \mathbb{D}$ such that $n + T_0 \in [\theta_i, \theta_i + T_0]$. Therefore, $\theta_i \in [n, n + T_0]$. The proof is completed. ∎

Continuing a recent work by Palmer [15] on the finite time condition for exponential dichotomy, we will prove the following theorem.

Theorem 1.4 *Let $\rho > \widehat{\rho} > 0$ and $\kappa \geq 1$. Let $\mathbb{D} = \{\theta_i\}_{i\in\mathbb{Z}}$ be a T_0-covering of $\mathbb{Z}$. Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be a given sequence of bounded linear operators on a Banach space X. Assume that $\|A_n\|_{\mathcal{L}(X)} \leq K$, for some positive real constant $K > 0$.*

Then there exist two constants $\widehat{T} := \widehat{T}(T_0, K, \kappa, \rho, \widehat{\rho}) > 0$ and $\widehat{\kappa} := \widehat{\kappa}(K, \kappa, \rho) \geq \kappa$ such that for each $T \geq \widehat{T}$, if $\mathbf{A}$ is exponentially dichotomic on each interval $[\theta_i, \theta_i + T]$ of $\mathbb{Z}$ (with constant $\kappa \geq 1$ and exponent $\rho > 0$), then $\mathbf{A}$ is exponentially dichotomic on $\mathbb{Z}$ (with constant $\widehat{\kappa}$ and exponent $\widehat{\rho}$).

Compared to [15, Theorem 2.1], one may observe that in Theorem 1.4 the invertibility of the bounded linear operators A_n is not required. We also refer to Palmer [18, Lemma 2.17] for early results on this topic. In [18, Lemma 2.17], Palmer considers intervals of the form $[(i-1)m, im]$ (where m is a fixed positive integer) and assumes that the projectors

$$\left\{ \Pi_n^{i\alpha}, n \in [(i-1)m, im] \right\}$$

for $\alpha = u, s$ are close at the endpoints, *i.e.*, that the norm of $\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}$ is small enough. In Palmer [15] this condition is no longer required. Actually the closeness of the projectors follows from the exponential dichotomy property on intervals large enough (see Lemma 4.9). Nevertheless, the proof of Lemma 4.9 will be different from Palmer [15] due to the non-invertibility of the linear operators.

We should also mention that there are various perturbation results using finite time conditions. We refer to Henry [12, Theorem 7.6.8 p. 234], Sakamoto [25, Theorem 4], Palmer [15–18], and Pötzsche [20] for more results on this subject.

Theorem 1.4 has many consequences, namely shadowing in dynamical systems, robustness of hyperbolic sets (see also Sacker and Sell [23, Theorem 6] for spectral theory approach), hyperbolicity along pseudo orbits, slowly varying systems, and almost periodic systems (see for instance [15, Section 3] and [6]). This concept is also the main tools in the theory of invariant hyperbolic sets and invariant normally hyperbolic sets using Lyapunov-Perron approach. We refer the reader to Sakamoto [24, 25] and Henry [12] for more results on this topic.

The main goal of this article is to extend Palmer's [15] results from the exponential dichotomy to the exponential trichotomy (see Section 2) for infinite dimensional dynamical systems.

The plan of the article follows. In Section 2 we will present Theorem 2.3, which is the main result. In section 3 we present an application of this theorem to study the persistence of exponential trichotomy along pseudo orbits. Sections 4 and 5 are devoted to the proofs of these results. In the spirit of Palmer's work, in Section 6 we apply Theorem 2.3 in the context of slowly varying systems, almost periodic systems, and to a perturbation problem.

2 Main Results

Before presenting the main results of this article, we first need to define the notion of exponential trichotomy. The notion used here is taken from Hale and Lin [11].

Definition 2.1 Let $\mathbb{I}$ be an interval of $\mathbb{Z}$ and let $\mathbf{A} = \{A_n\}_{n\in\mathbb{I}} \colon \mathbb{I} \to \mathcal{L}(X)$ be a map. Then $U_{\mathbf{A}}$ has an *exponential trichotomy* (or $\mathbf{A}$ is *exponentially trichotomic*) on $\mathbb{I}$ with constant κ and exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\mathbf{\Pi}^{\alpha} = \{\Pi_n^\alpha\}_{n\in\mathbb{I}} \colon \mathbb{I} \to \mathcal{L}(X)$ with $\alpha = u, s, c$ satisfying the following properties.

(i) For all $n \in \mathbb{I}$ and $\alpha, \beta \in \{u, s, c\}$, we have

$$\Pi_n^\alpha \Pi_n^\beta = 0 \text{ if } \alpha \neq \beta, \quad \text{and} \quad \Pi_n^s + \Pi_n^u + \Pi_n^c = I.$$

(ii) For all $n \geq m$ in $\mathbb{I}$ we have

$$U_{\mathbf{A}}^\alpha(n,m) := \Pi_n^\alpha U_{\mathbf{A}}(n,m) = U_{\mathbf{A}}(n,m)\Pi_m^\alpha \quad \text{for } \alpha = u, s, c.$$

(iii) $U_{\mathbf{A}}^\alpha(n,m)$ is invertible from $\Pi_m^\alpha(X)$ into $\Pi_n^\alpha(X)$ for all $n \geq m$ in $\mathbb{I}$, $\alpha = u, c$ and its inverse is denoted by $U_{\mathbf{A}}^\alpha(m,n) \colon \Pi_n^\alpha(X) \to \Pi_m^\alpha(X)$.

(iv) For each $x \in X$ we have

$$\|U^c_{\mathbf{A}}(n,m)\Pi^c_m x\| \leq \kappa e^{\rho_0|n-m|}\|x\|$$

for all $n, m \in \mathbb{I}$, and

$$\|U^s_{\mathbf{A}}(n,m)\Pi^s_m x\| \leq \kappa e^{-\rho(n-m)}\|x\|,$$

$$\|U^u_{\mathbf{A}}(m,n)\Pi^u_n x\| \leq \kappa e^{-\rho(n-m)}\|x\|$$

for $n > m$.

Remark 2.2 The above definition coincides with the Definition 1.1 whenever $\Pi^c_n = 0_{\mathcal{L}(X)}$, for all $n \in \mathbb{I}$.

The main result of this article is the following theorem.

Theorem 2.3 *Let $\rho > \widehat{\rho} > \widehat{\rho}_0 > \rho_0 > 0$ and $\kappa \geq 1$. Let $\mathbb{D} = \{\theta_i\}_{i\in\mathbb{Z}}$ be a T_0-covering of $\mathbb{Z}$. Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be a given sequence of bounded linear operators on a Banach space X. Assume that*

$$\|A_n\|_{\mathcal{L}(X)} \leq K,\ n \in \mathbb{Z},$$

for some positive real constant $K > 0$.

Then there exists $\widehat{T} := \widehat{T}(T_0, K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) > 0$ and $\widehat{\kappa} := \widehat{\kappa}(K, \kappa, \rho) \geq \kappa$ such that for each $T \geq \widehat{T}$, if $\mathbf{A}$ is exponentially trichotomic on each intervals $[\theta_i, \theta_i + T]$ of $\mathbb{Z}$ (with constant κ exponents ρ and ρ_0), then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ (with constant $\widehat{\kappa}$ exponents $\widehat{\rho}$ and $\widehat{\rho}_0$).

3 Application

In this section, we present an application of Theorem 2.3 to study the persistence of the normal hyperbolicity along pseudo orbits for the discrete time dynamical system

$$x_{n+1} = F(x_n), \quad \forall n \geq 0, \quad x_0 = x \in X,$$

whenever $F\colon X \to X$ is a continuously differentiable map on the Banach space X.

Recall that a sequence $x = \{x_n\}_{n\in\mathbb{Z}} \subset X$ is called a δ*-pseudo-orbit* for F if

$$\|x_{n+1} - F(x_n)\| \leq \delta, \quad \forall n \in \mathbb{Z}, \tag{3.1}$$

while we will say that $x = \{x_n\}_{n\in\mathbb{Z}}$ is a *complete orbit* for F if

$$x_{n+1} = F(x_n),\ \forall n \in \mathbb{Z}.$$

So a complete orbit is nothing but a δ-pseudo-orbit with $\delta = 0$ in (3.1).

Next we recall the definition of normally hyperbolic invariant sets inspired by Bates, Lu, and Zeng [1]. This notion plays a crucial role in the context of the theory of geometric singular perturbation, and we refer the reader to [1,9,13] (and reference therein) for more results.

Definition 3.1 (Normally hyperbolic set) Let $F\colon X \to X$ be a continuously differentiable map on the Banach space X. Let $M \subset X$ be an invariant subset for F, that is, $F(M) = M$. Then we will say that $M \subset X$ is *normally hyperbolic* for F if the following properties are satisfied.

(i) For each $x \in M$, there exist three closed subspaces X_x^α with $\alpha = s, c, u$, such that

$$X = X_x^u \oplus X_x^s \oplus X_x^c \tag{3.2}$$

for each $\alpha = s, c, u$, $DF(x)(X_x^\alpha) \subset X_{F(x)}^\alpha$, and for $\alpha = u, c$, the map $DF(x)|_{X_x^\alpha}$ is invertible from X_x^α into $X_{F(x)}^\alpha$.

(ii) There exist a constant $\kappa \geq 1$ and rates $0 < \rho_0 < \rho$ such that for each $n \geq 0$ and each $x \in M$,

$$\begin{cases} \|DF^n(x)|_{X_x^s}\| \leq \kappa e^{-\rho n}, \\ \dfrac{1}{\kappa} e^{\rho n} \leq \inf\{\|DF^n(x)x^u\| : x^u \in X_x^u \text{ and } \|x^u\| = 1\}, \end{cases}$$

and

$$\begin{aligned} \frac{1}{\kappa} e^{-\rho_0 n} &\leq \inf\{\|DF^n(x)x^c\| : x^c \in X_x^c \text{ and } \|x^c\| = 1\} \\ &\leq \|DF^n(x)|_{X_x^c}\| \leq \kappa e^{\rho_0 n}, \end{aligned}$$

where

$$\|DF^n(x)|_{X_x^\alpha}\| = \sup\{\|DF^n(x)x\| : x \in X_x^\alpha \text{ and } \|x\| = 1\} \text{ for } \alpha = s, c.$$

By using property (i) for each $x \in M$, we can define three projectors Π_x^α for $\alpha = u, s, c$ associated with the state decomposition (3.2). For each $x \in M$ and each $\alpha \in \{s, c, u\}$, these projectors are uniquely determined by

$$\mathcal{R}(\Pi_x^\alpha) = X_x^\alpha \text{ and } \mathcal{N}(\Pi_x^\alpha) = \bigoplus_{\alpha' \in \{s,c,u\} \setminus \{\alpha\}} X_x^{\alpha'}.$$

Define an (open) ε-neighborhood of a subset $M \subset X$,

$$\mathcal{V}(M, \varepsilon) := \{x \in X : d(x, M) < \varepsilon\},$$

where

$$d(x, M) := \inf_{y \in M} \|x - y\|.$$

One may equivalently define $\mathcal{V}(M, \varepsilon)$ as

$$\mathcal{V}(M, \varepsilon) := \bigcup_{x \in M} B(x, \varepsilon),$$

where

$$B(x, \varepsilon) := \{y \in X : \|x - y\| < \varepsilon\}.$$

Now we make the following assumption.

Assumption 3.2 *Let $F\colon X \to X$ be a continuously differentiable map on the Banach space X. Let $M \subset X$ be an invariant subset for F. We assume the following:*

(i) *M is normally hyperbolic for F with the constant $\kappa \geq 1$ and the rates $0 < \rho_0 < \rho$.*

(ii) *There exists* $\kappa_0 \geq 1$ *such that*

$$\sup_{x \in M} \|\Pi_x^\alpha\|_{\mathcal{L}(X)} \leq \kappa_0, \ \alpha = s, c, u. \tag{3.3}$$

(iii) *There exist* $K > 0$ *and* $\varepsilon_0 > 0$ *such that the map* $x \to DF(x)$ *is uniformly continuous from* $\mathcal{V}(M, \varepsilon_0)$ *into* $\mathcal{L}(X)$ *and*

$$\|F(x) - F(y)\| \leq K\|x - y\| \quad \forall x, y \in \mathcal{V}(M, \varepsilon_0),$$
$$\sup_{x \in M} \|DF(x)\|_{\mathcal{L}(X)} \leq K.$$

Remark 3.3 Whenever $\mathcal{V}(M, \varepsilon_0)$ is convex, assertion (iii) can be replaced by

$$\sup_{x \in \mathcal{V}(M, \varepsilon_0)} \|DF(x)\|_{\mathcal{L}(X)} \leq K.$$

The main result of this section is the following proposition.

Proposition 3.4 *Let Assumption 3.2 be satisfied. Let* $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$. *There exist two constants*

$$\delta_0 := \delta_0(K, \kappa, \kappa_0, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}) > 0 \text{ and } \varepsilon_0 := \varepsilon_0(K, \kappa, \kappa_0, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}) > 0,$$

such that for each $\delta \in [0, \delta_0)$, $\varepsilon \in [0, \varepsilon_0)$ *if* $x = \{x_n\}_{n \in \mathbb{Z}} \subset \mathcal{V}(M, \varepsilon)$ *is a* δ*-pseudo-orbit of* F *in the neighborhood* $\mathcal{V}(M, \varepsilon)$ *of* M *then* $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ *is exponentially trichotomic on* $\mathbb{Z}$ *with a constant* $\widehat{\kappa} := \widehat{\kappa}(K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0)$ *and exponents* $\widehat{\rho}_0, \widehat{\rho}$.

Remark 3.5 Under Assumption 3.2, for each complete orbit $\mathbf{x} = \{x_n\}_{n \in \mathbb{Z}}$ of F in M, the family of bounded linear operators $DF(\mathbf{x}) = \{DF(x_n)\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widetilde{\kappa} := \kappa\kappa_0$ and exponents ρ_0, ρ.

4 Proof of Theorem 2.3

We will start this section by considering the case where $\mathbf{A}\colon \mathbb{Z} \to \mathcal{L}(X)$ is exponentially trichotomic on intervals of the form

$$[(i-1)m, im], \quad i \in \mathbb{Z},$$

where $m > 0$ is some integer. We observe that $\mathbb{Z}$ is a disjoint union of the intervals $\{[(i-1)m, im[\}_{i \in \mathbb{Z}}$. Therefore,

$$\mathbb{Z} = \bigcup_{i \in \mathbb{Z}} \big[(i-1)m, im\big[\;= \bigcup_{i \in \mathbb{Z}} \big[(i-1)m, im-1\big].$$

As we will see at the end of this section, this is not a real restriction for the proof of Theorem 2.3. In order to clarify the notion of exponential trichotomy on a family of intervals, we introduce the following definition.

Definition 4.1 Let $m > 0$ be an integer. We will say that a sequence of bounded linear operators $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on the family of intervals $\{[(i-1)m, im] : i \in \mathbb{Z}\}$ with uniform constant κ and uniform exponents

$\rho > 0$ and $\rho_0 \in (0, \rho)$ if on each interval $[(i-1)m, im]$ the following properties are satisfied:

(i) There exist three families of projectors $\{\Pi_n^{i\alpha}, n \in [(i-1)m, im]\} \subset \mathcal{L}(X)$, with $\alpha = u, s, c$, such that for each $n \in [(i-1)m, im]$

$$\Pi_n^{i\alpha}\Pi_n^{i\beta} = 0 \text{ if } \alpha \neq \beta, \quad \text{and} \quad \Pi_n^{is} + \Pi_n^{ic} + \Pi_n^{iu} = I.$$

(ii) For all $n, p \in [(i-1)m, im]$ with $n \geq p$, we have

$$U_{\mathbf{A}}^{i\alpha}(n, p) := \Pi_n^{i\alpha} U_{\mathbf{A}}(n, p) = U_{\mathbf{A}}(n, p)\Pi_p^{i\alpha} \text{ for } \alpha = u, s, c.$$

(iii) The map $U_{\mathbf{A}}^{i\alpha}(n, p)$ is invertible from $\Pi_p^{i\alpha}(X)$ into $\Pi_n^{i\alpha}(X)$ for all $n \geq p$ in $[(i-1)m, im]$ and $\alpha = u, c$, and its inverse is denoted by

$$U_{\mathbf{A}}^{i\alpha}(p, n) \colon \Pi_n^{\alpha}(X) \to \Pi_p^{\alpha}(X).$$

(iv) For each $x \in X$, we have for all $n, p \in [(i-1)m, im]$

$$\|U_{\mathbf{A}}^{ic}(n, p)\Pi_p^{ic}x\| \leq \kappa e^{\rho_0|n-p|}\|x\|, \tag{4.1}$$

and if $n \geq p$, then

$$\|U_{\mathbf{A}}^{is}(n, p)\Pi_p^{is}x\| \leq \kappa e^{-\rho(n-p)}\|x\|, \tag{4.2}$$

$$\|U_{\mathbf{A}}^{iu}(p, n)\Pi_n^{iu}x\| \leq \kappa e^{-\rho(n-p)}\|x\|. \tag{4.3}$$

Remark 4.2 Let us note that due to condition (iv), we have for each $i \in \mathbb{Z}$ and each $n \in [(i-1)m, im]$,

$$\|\Pi_n^{i\alpha}\|_{\mathcal{L}(X)} \leq \kappa, \ \alpha = u, s, c. \tag{4.4}$$

The proof of the following theorem is given in Appendix A. An extended version has been proved recently in Ducrot, Magal, and Seydi [8, Theorem 1.8].

Theorem 4.3 (Perturbation) *Let $\mathbf{A} \colon \mathbb{Z} \to \mathcal{L}(X)$ be given. Assume that $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ, exponents $0 < \rho_0 < \rho$, and associated with the three families of projectors $\{\mathbf{\Pi}^{\alpha} \colon \mathbb{Z} \to \mathcal{L}(X)\}_{\alpha=s,c,u}$. Let $\rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ and $\widehat{\kappa} > 2\kappa$ be given. Then there exists $\delta_0 := \delta_0(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa, \widehat{\kappa}) \in (0, 1)$ such that for each each $\mathbf{B} \colon \mathbb{Z} \to \mathcal{L}(X)$ with $\sup_{n \in \mathbb{Z}} \|B_n\|_{\mathcal{L}(X)} \leq \delta$, the sequence $\mathbf{A}+\mathbf{B}$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}^2$ and exponents $\widehat{\rho}$, $\widehat{\rho}_0$.*

An easy consequence of the above theorem is the following corollary.

Corollary 4.4 *Let $\mathbb{I}$ be an interval (finite or infinite) in $\mathbb{Z}$ and let $\mathbf{A} = \{A_n\}_{n \in \mathbb{I}} \colon \mathbb{I} \to \mathcal{L}(X)$ be a map. Assume that $\mathbf{A}$ is exponentially trichotomic on $\mathbb{I}$ with constant κ, exponents $0 < \rho_0 < \rho$, and associated with the three families of projectors $\{\mathbf{\Pi}^{\alpha} \colon \mathbb{Z} \to \mathcal{L}(X)\}_{\alpha=s,c,u}$. Let $\rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ and $\widehat{\kappa} > 2\kappa$ be given. Then there exists $\delta > 0$ such that for each $\mathbf{B} \colon \mathbb{Z} \to \mathcal{L}(X)$ with*

$$\sup_{n \in \mathbb{I}} \|B_n\|_{\mathcal{L}(X)} \leq \delta,$$

the sequence $\mathbf{A}+\mathbf{B}=\{A_n+B_n\}_{n\in\mathbb{I}}$ *is exponentially trichotomic on* $\mathbb{I}$ *with constant* $\widehat{\kappa}^2$ *and exponents* $\widehat{\rho}$, $\widehat{\rho}_0$.

Proof Define

$$n_- := \inf\{k : k \in \mathbb{I}\} \text{ and } n_+ := \sup\{k : k \in \mathbb{I}\}.$$

We will give the arguments for the case $n_+ < +\infty$ and $n_- = -\infty$. The remaining cases hold similarly. Define the following sequences of bounded linear operators

$$\overline{A}_n := \begin{cases} A_n, & \text{if } n < n_+, \\ e^{-\rho}\Pi^s_{n_+} + e^{\rho}\Pi^u_{n_+} + \Pi^c_{n_+}, & \text{if } n \geq n_+, \end{cases}$$

and

$$\overline{B}_n := \mathbf{1}_{\mathbb{I}}(n)B_n,\ \forall n \in \mathbb{Z},$$

where $\mathbf{1}_{\mathbb{I}}(\cdot)$ is the characteristic function on $\mathbb{I}$. Then the sequences $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ trivially satisfy the conditions of Theorem 4.3. Hence the result follows by applying Theorem 4.3 with $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ and using the fact that $\overline{A}_n+\overline{B}_n = A_n+B_n$ for all $n \in (-\infty, n_+ - 1]$. ∎

In the next lemma we will show that if $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic on the family of intervals $\{[i-1m, im]\}_{i\in\mathbb{Z}}$ (m a positive integer large enough) and the norm of $\Pi^{i\alpha}_{im} - \Pi^{(i+1)\alpha}_{im}$, $i \in \mathbb{Z}$ is small, then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$. This lemma generalizes [15, Lemma 2.3].

Lemma 4.5 *Let* $\rho > \widehat{\rho} > \widehat{\rho}_0 > \rho_0 > 0$, $\kappa \geq 1$ *and* $K \geq 1$ *be fixed. Define*

$$\begin{aligned} \tilde{\kappa} &:= \max\{2\kappa^3 e^{\rho+\rho_0}, (3\kappa K + K)\kappa\} \\ m_0 &:= \max\left\{2, \frac{2}{\rho-\widehat{\rho}}\ln\tilde{\kappa}, \frac{2}{\widehat{\rho}_0-\rho_0}\ln\tilde{\kappa}\right\}. \end{aligned} \tag{4.5}$$

Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ *be a sequence of bounded linear operators on* X. *Assume that*

(i)

$$\sup_{n\in\mathbb{Z}} \|A_n\|_{\mathcal{L}(X)} \leq K. \tag{4.6}$$

(ii) *There exists an integer* $m \geq m_0$ *such that* $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ *is exponentially trichotomic on the family of intervals* $\{[(i-1)m, im] : i \in \mathbb{Z}\}$ *with uniform constant* κ *and uniform exponents* ρ *and* ρ_0.

Then there exist two constants

$$\widehat{\kappa} := \widehat{\kappa}(K,\kappa,\rho,\rho_0) > 2\tilde{\kappa}^2 \quad \textit{and} \quad \eta_0 := \eta_0(\rho_0,\widehat{\rho}_0,\widehat{\rho},\rho,\kappa,\widehat{\kappa},K) \in (0, \sqrt{2}-1)$$

such that

$$\|\Pi^{i\alpha}_{im} - \Pi^{(i+1)\alpha}_{im}\|_{\mathcal{L}(X)} \leq \eta_0, \quad \forall i \in \mathbb{Z},\ \forall \alpha = u,s,c,$$

implies that $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ *is exponentially trichotomic on* $\mathbb{Z}$ *with constant* $\widehat{\kappa}^2$ *and exponents* $\widehat{\rho}$ *and* $\widehat{\rho}_0$.

In order to prove Lemma 4.5, we will need the following auxiliary lemma (see [8, Lemma 2.1] or [1, Lemma 4.1]).

Lemma 4.6 *Let $\Pi\colon X \to X$ and $\widehat{\Pi}\colon X \to X$ be two bounded linear projectors on a Banach space X. Assume that*

$$\|\Pi - \widehat{\Pi}\|_{\mathcal{L}(X)} < \delta \text{ with } 0 < \delta < \sqrt{2} - 1.$$

Then Π is invertible from $\widehat{\Pi}(X)$ into $\Pi(X)$ and

$$\|(\Pi|_{\widehat{\Pi}(X)})^{-1}x\| \le \frac{1}{1-\delta}\|x\|, \quad \forall x \in \Pi(X).$$

Remark 4.7 By symmetry, the bounded linear projector $\widehat{\Pi}$ is also invertible from $\Pi(X)$ into $\widehat{\Pi}(X)$ and

$$\left\| (\widehat{\Pi}|_{\Pi(X)})^{-1}x \right\| \le \frac{1}{1-\delta}\|x\|, \quad \forall x \in \widehat{\Pi}(X).$$

Proof of Lemma 4.5 The principle of the proof is to construct an auxiliary sequence of bounded linear operators $\overline{\mathbf{A}}\colon \mathbb{Z} \to \mathcal{L}(X)$ and three families of projectors $\{\overline{\mathbf{\Pi}}^{\alpha}\}_{\alpha=u,s,c} \subset \mathcal{L}(X)$, such that $\overline{\mathbf{A}}$ is exponentially trichotomic on $\mathbb{Z}$, and $\overline{\mathbf{A}}$ is close to $\mathbf{A}$.

To do so let $m \ge m_0$ be the positive integer defined in the assumption of Lemma 4.5. Recall that for each $i \in \mathbb{Z}$, A is exponentially trichotomic on $[(i-1)m, im]$, with uniform constant κ and uniform exponents $\rho > 0$, $\rho_0 \in (0,\rho)$ and projectors $\{\Pi_n^{i\alpha}, n \in [(i-1)m, im]\} \subset \mathcal{L}(X)$, $\alpha = u, s, c$ satisfying properties (i)–(iv) in Definition 4.1.

We define a family of bounded linear projectors $\{\overline{\Pi}_n^{\alpha}\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$, $\alpha = u, s, c$, given on each interval $[(i-1)m, im[= [(i-1)m, im-1]$ of $\mathbb{Z}$ by

$$\overline{\Pi}_n^{\alpha} := \Pi_n^{i\alpha}, \text{ for } (i-1)m \le n \le im - 1. \tag{4.7}$$

It follows that

$$\overline{\Pi}_{im}^{\alpha} = \Pi_{im}^{(i+1)\alpha}, \quad \forall i \in \mathbb{Z}. \tag{4.8}$$

We define $\overline{\mathbf{A}}\colon \mathbb{Z} \to \mathcal{L}(X)$ on each interval $[(i-1)m, im[= [(i-1)m, im-1]$ of $\mathbb{Z}$ by

$$\overline{A}_n := \begin{cases} A_n, & \text{if } n \in [(i-1)m, im-2], \\ \sum\limits_{\alpha=u,s,c} \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} & \text{if } n = im - 1. \end{cases} \tag{4.9}$$

Next we will prove that $\overline{\mathbf{A}} = \{\overline{A}_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ is exponentially trichotomic with projectors $\{\overline{\mathbf{\Pi}}^{\alpha}\}_{\alpha=u,s,c}$ and that $\overline{\mathbf{A}}$ is close to $\mathbf{A}$.

To do so, we verify properties (i)–(iv) stated in Definition 2.1. Without loss of generality, we can assume that $\eta_0 \in (0, 1/2)$, and that

$$\|\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}\|_{\mathcal{L}(X)} \le \eta_0, \ \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c. \tag{4.10}$$

Proof of (i) From (4.7) for each $n \in \mathbb{Z}$ we have

$$\overline{\Pi}_n^u + \overline{\Pi}_n^s + \overline{\Pi}_n^c = I_{\mathcal{L}(X)} \text{ and } \overline{\Pi}_n^\beta \overline{\Pi}_n^\alpha = 0_{\mathcal{L}(X)} \text{ for } \alpha, \beta \in \{u, s, c\} \text{ with } \alpha \neq \beta,$$

and property *(i)* is satisfied.

Proof of (ii) We will prove that

$$\overline{\Pi}_{n+1}^\alpha \overline{A}_n = \overline{A}_n \overline{\Pi}_n^\alpha, \ \forall n \in \mathbb{Z} \text{ and } \alpha = u, s, c, \tag{4.11}$$

or equivalently

$$U_{\overline{\mathbf{A}}}^\alpha(n, p) := U_{\overline{\mathbf{A}}}(n, p)\overline{\Pi}_p^\alpha = \overline{\Pi}_n^\alpha U_{\overline{\mathbf{A}}}(n, p), \ \forall n \geq p \text{ and } \alpha = u, s, c. \tag{4.12}$$

Let $n \in \mathbb{Z}$ be given. Let $i \in \mathbb{Z}$ be given such that $n \in [(i-1)m, im[$. If $(i-1)m \leq n+1 < im$, that is, $n \neq im - 1$, then by using (4.7) and (4.9), property (4.11) is clearly verified.

If $n+1 = im$ (*i.e.*, $n = im - 1$), one has from (4.8)–(4.9) that for each $\alpha = u, s, c$,

$$\begin{aligned}
\overline{\Pi}_{n+1}^\alpha \overline{A}_n &= \overline{\Pi}_{im}^\alpha \overline{A}_{im-1} \\
&= \Pi_{im}^{(i+1)\alpha} \sum_{\beta=u,s,c} \Pi_{im}^{(i+1)\beta} A_{im-1} \Pi_{im-1}^{i\beta} \\
&= \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} \\
&= \left[\sum_{\beta=u,s,c} \Pi_{im}^{(i+1)\beta} A_{im-1} \Pi_{im-1}^{i\beta} \right] \Pi_{im-1}^{i\alpha} \\
&= \overline{A}_{im-1} \Pi_{im-1}^{i\alpha},
\end{aligned}$$

so we obtain

$$\overline{\Pi}_{n+1}^\alpha \overline{A}_n = \overline{A}_{im-1} \Pi_{im-1}^{i\alpha} = \overline{A}_{im-1} \overline{\Pi}_{im-1}^\alpha = \overline{A}_n \overline{\Pi}_n^\alpha, \quad \forall \alpha = u, s, c.$$

Proof of (iii) We need to prove that for each $n \geq p$ and each $\alpha = u, c$, the linear operator $U_{\overline{\mathbf{A}}}^\alpha(n, p)$ is invertible from $\overline{\Pi}_p^\alpha(X)$ into $\overline{\Pi}_n^\alpha(X)$. Due to the definition of the evolution semigroup $U_{\overline{\mathbf{A}}}^\alpha$ in (4.12) it is sufficient to prove that for each $n \in \mathbb{Z}$ and each $\alpha = u, c$, the operator $\overline{A}_n \overline{\Pi}_n^\alpha$ is invertible from $\overline{\Pi}_n^\alpha(X)$ into $\overline{\Pi}_{n+1}^\alpha(X)$. But on each interval $[(i-1)m, im[$, $i \in \mathbb{Z}$, the operator

$$\overline{A}_n \overline{\Pi}_n^\alpha = A_n \Pi_n^{i\alpha}, \text{ with } n \in [(i-1)m, im[\text{ and } n \neq im - 1,$$

is invertible from $\overline{\Pi}_n^\alpha(X) = \Pi_n^{i\alpha}(X)$ into $\overline{\Pi}_{n+1}^\alpha(X) = \Pi_{n+1}^{i\alpha}(X)$. Therefore, it is sufficient to prove that $\overline{A}_{im-1} \overline{\Pi}_{im-1}^\alpha$ is invertible from $\overline{\Pi}_{im-1}^\alpha(X) = \Pi_{im-1}^{i\alpha}(X)$ into $\overline{\Pi}_{im}^\alpha(X) = \Pi_{im}^{(i+1)\alpha}(X)$. But due to (4.9), we have

$$\overline{A}_{im-1} \overline{\Pi}_{im-1}^\alpha = \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c,$$

hence

$$\overline{A}_{im-1} \overline{\Pi}_{im-1}^\alpha = \Pi_{im}^{(i+1)\alpha} \overline{A}_{im-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, s, c.$$

But by assumption, A_{im-1} is invertible from $\overline{\Pi}_{im-1}^\alpha(X) = \Pi_{im-1}^{i\alpha}(X)$ into $\Pi_{im}^{i\alpha}(X)$, and since

$$\|\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}\|_{\mathcal{L}(X)} < \sqrt{2} - 1,$$

we deduce from Lemma 4.6, that $\Pi_{im}^{(i+1)\alpha}$ is invertible from $\Pi_{im}^{i\alpha}(X)$ into $\Pi_{im}^{(i+1)\alpha}(X)$. Therefore, $\overline{A}_{im-1}$ is invertible from $\Pi_{im-1}^{i\alpha}(X)$ into $\Pi_{im}^{(i+1)\alpha}(X)$, and

$$(\overline{A}_{im-1}\overline{\Pi}_{im-1}^{\alpha})^{-1} = (A_{im-1}\Pi_{im-1}^{i\alpha})^{-1}[\Pi_{im}^{(i+1)\alpha}]^{-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, c;$$

$[\Pi_{im}^{(i+1)\alpha}]^{-1}\colon \Pi_{im}^{(i+1)\alpha}(X) \to \Pi_{im}^{i\alpha}(X)$ is the inverse of $\Pi_{im}^{(i+1)\alpha}\colon \Pi_{im}^{i\alpha}(X) \to \Pi_{im}^{(i+1)\alpha}(X)$. So we deduce that for each $n \geq p$ and each $\alpha = u, c$, the operator $U_{\overline{\mathbf{A}}}^{\alpha}(n, p)$ is invertible from $\mathcal{R}(\overline{\Pi}_p^{\alpha})$ into $\mathcal{R}(\overline{\Pi}_n^{\alpha})$, and its inverse is defined by

$$U_{\overline{\mathbf{A}}}^{\alpha}(p, n)\colon \mathcal{R}(\overline{\Pi}_n^{\alpha}) \to \mathcal{R}(\overline{\Pi}_p^{\alpha}).$$

More precisely, for each $i \in \mathbb{Z}$ and each $\alpha = u, c$ the inverse of $U_{\overline{\mathbf{A}}}^{\alpha}(im, im-1)$ is given by

$$U_{\overline{\mathbf{A}}}^{\alpha}(im-1, im)\colon \mathcal{R}(\overline{\Pi}_{im}^{\alpha}) \to \mathcal{R}(\overline{\Pi}_{im-1}^{\alpha}),$$

which is defined by

$$U_{\overline{\mathbf{A}}}^{\alpha}(im-1, im) = U_{\mathbf{A}}^{i\alpha}(im-1, im)[\Pi_{im}^{(i+1)\alpha}]^{-1}, \quad \forall i \in \mathbb{Z} \text{ and } \alpha = u, c,$$

and by using Lemma 4.6 again combined with (4.10), we also deduce that for each $i \in \mathbb{Z}$ and each $\alpha = u, c$

$$\|U_{\overline{\mathbf{A}}}^{\alpha}(im-1, im)\overline{\Pi}_{im}^{\alpha}\|_{\mathcal{L}(X)} \leq \frac{\kappa^2 e^{\rho_0}}{1-\eta_0} \leq 2\kappa^2 e^{\rho_0}, \quad \forall \eta_0 \in (0, 1/2). \tag{4.13}$$

Closeness of A and $\overline{\mathbf{A}}$ Let us now give some estimate of the supremum norm of $\mathbf{A} - \overline{\mathbf{A}}$. By using (4.9) we have

$$\sup_{n\in\mathbb{Z}} \|\overline{A}_n - A_n\|_{\mathcal{L}(X)} = \sup_{i\in\mathbb{Z}} \|\overline{A}_{im-1} - A_{im-1}\|_{\mathcal{L}(X)}.$$

But

$$\begin{aligned}
\|A_{im-1} - \overline{A}_{im-1}\|_{\mathcal{L}(X)} &= \Big\| \sum_{\alpha=u,s,c} \Pi_{im}^{i\alpha} A_{im-1} - \sum_{\alpha=u,s,c} \Pi_{im}^{(i+1)\alpha} \overline{A}_{im-1} \Big\|_{\mathcal{L}(X)} \\
&= \Big\| \sum_{\alpha=u,s,c} [\Pi_{im}^{i\alpha} A_{im-1} \Pi_{im-1}^{i\alpha} - \Pi_{im}^{(i+1)\alpha} A_{im-1} \Pi_{im-1}^{i\alpha}] \Big\|_{\mathcal{L}(X)} \\
&\leq \sum_{\alpha=u,s,c} \|\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}\|_{\mathcal{L}(X)} \|A_{im-1}\Pi_{im-1}^{i\alpha}\|_{\mathcal{L}(X)} \\
&\leq \sum_{\alpha=u,s,c} \|\Pi_{im}^{i\alpha} - \Pi_{im}^{(i+1)\alpha}\|_{\mathcal{L}(X)} \|A_{im-1}\|_{\mathcal{L}(X)} \|\Pi_{im-1}^{i\alpha}\|_{\mathcal{L}(X)}.
\end{aligned}$$

So by using (4.4), (4.6), and (4.10) we deduce that

$$\sup_{n\in\mathbb{Z}} \|A_n - \overline{A}_n\|_{\mathcal{L}(X)} \leq 3\kappa K \eta_0, \quad \forall \eta_0 \in (0, \sqrt{2}-1), \tag{4.14}$$

and by using (4.6) and (4.14) we obtain

$$\sup_{n\in\mathbb{Z}} \|\overline{A}_n\|_{\mathcal{L}(X)} \leq \sup_{n\in\mathbb{Z}} \|A_n - \overline{A}_n\|_{\mathcal{L}(X)} + \sup_{n\in\mathbb{Z}} \|A_n\|_{\mathcal{L}(X)} \leq 3\kappa K + K. \tag{4.15}$$

Proof of (iv) We will proceed in three steps to provide some growth rate estimates for $U_{\overline{\mathbf{A}}}$.

Step 1. Let $i \in \mathbb{Z}$ be given. Let $n, p \in [(i-1)m, im]$ be such that

$$(i-1)m \leq p \leq n < im.$$

Then we have

$$U^{\alpha}_{\overline{\mathbf{A}}}(n,p) = U_{\overline{\mathbf{A}}}(n,p)\overline{\Pi}^{\alpha}_{p} = U_{\mathbf{A}}(n,p)\Pi^{i\alpha}_{p} = U^{i\alpha}_{\mathbf{A}}(n,p), \text{ for } \alpha = u,s,c.$$

Hence we deduce from (4.1), (4.2), and (4.3) that for $(i-1)m \leq p \leq n < im$,

$$\|U^{s}_{\overline{\mathbf{A}}}(n,p)\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)}, \qquad \|U^{u}_{\overline{\mathbf{A}}}(p,n)\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)},$$
$$\|U^{c}_{\overline{\mathbf{A}}}(n,p)\|_{\mathcal{L}(X)} \leq \kappa e^{\rho_0(n-p)}, \qquad \|U^{c}_{\overline{\mathbf{A}}}(p,n)\|_{\mathcal{L}(X)} \leq \kappa e^{\rho_0(n-p)}.$$

Step 2. Let $i \in \mathbb{Z}$ be given. Let $n, p \in [(i-1)m, im]$ be such that $(i-1)m \leq p \leq n = im$. First note that we have

$$U^{\alpha}_{\overline{\mathbf{A}}}(im,p) = \overline{\Pi}^{\alpha}_{im} = \overline{\Pi}^{(i+1)\alpha}_{im} \text{ if } p = im,$$
$$U^{\alpha}_{\overline{\mathbf{A}}}(im,p) = U^{\alpha}_{\overline{\mathbf{A}}}(im, im-1)U^{i\alpha}_{\mathbf{A}}(im-1,p) = \overline{A}_{im-1}U^{i\alpha}_{\mathbf{A}}(im-1,p), \text{ if } p < im.$$

Therefore it follows from the Step 1 and (4.15) that

$$\|U^{s}_{\overline{\mathbf{A}}}(n,p)\|_{\mathcal{L}(X)} \leq (3\kappa K + K)e^{\rho}\kappa e^{-\rho(im-p)},$$
$$\|U^{c}_{\overline{\mathbf{A}}}(n,p)\| \leq (3\kappa K + K)\kappa e^{-\rho_0}e^{\rho_0(im-p)}.$$

Note that one has for $\alpha = u, c$,

$$U^{\alpha}_{\overline{\mathbf{A}}}(p,im) = \begin{cases} \overline{\Pi}^{\alpha}_{im} = \overline{\Pi}^{(i+1)\alpha}_{im} & \text{if } p = im, \\ U^{i\alpha}_{\mathbf{A}}(p, im-1)U^{\alpha}_{\overline{\mathbf{A}}}(im-1, im) & \text{if } p < im, \end{cases}$$

so that we deduce from Step 1 and (4.13) that

$$\|U^{u}_{\overline{\mathbf{A}}}(p,n)\| \leq 2\kappa^2 e^{\rho_0}\kappa e^{\rho}e^{-\rho(im-p)} \quad \text{and} \quad \|U^{u}_{\overline{\mathbf{A}}}(p,n)\| \leq 2\kappa^2 e^{\rho_0}\kappa e^{-\rho_0}e^{-\rho(im-p)}.$$

Step 3. We can summarize Steps 1 and 2 as follows for each $i \in \mathbb{Z}$ and each $n, p \in [(i-1)m, im]$ with $n \geq p$, we have

$$\|U^{s}_{\overline{\mathbf{A}}}(n,p)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{-\rho(n-p)}, \qquad \|U^{u}_{\overline{\mathbf{A}}}(p,n)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{-\rho(n-p)},$$
$$\|U^{c}_{\overline{\mathbf{A}}}(n,p)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{\rho_0(n-p)}, \qquad \|U^{c}_{\overline{\mathbf{A}}}(p,n)\|_{\mathcal{L}(X)} \leq \tilde{\kappa} e^{\rho_0(n-p)},$$

where $\tilde{\kappa}$ is defined in (4.5).
Now let $n, p \in \mathbb{Z}$ with $n > p$ be given. Then there exists $j \leq i$ such that

$$(j-1)m \leq p \leq jm \quad \text{and} \quad (i-1)m \leq n \leq im.$$

Since the case $j = i$ is already studied in Step 1 and Step 2, it is sufficient to study the case $j \leq i-1$. In fact if $j \leq i-1$, one has for $\alpha = s$

$$\begin{aligned} \|U^{s}_{\overline{\mathbf{A}}}(n,p)\| &= \|U^{s}_{\overline{\mathbf{A}}}(n,(i-1)m)U^{s}_{\overline{\mathbf{A}}}((i-1)m, jm)U^{s}_{\overline{\mathbf{A}}}(jm,p)\| \\ &\leq \|U^{s}_{\overline{\mathbf{A}}}(n,(i-1)m)\|\|U^{s}_{\overline{\mathbf{A}}}((i-1)m, jm)\|\|U^{s}_{\overline{\mathbf{A}}}(jm,p)\| \\ &\leq \tilde{\kappa} e^{-\rho(n-(i-1)m)}(\tilde{\kappa} e^{-\rho})^{(i-1)m-jm}\tilde{\kappa} e^{-\rho(jm-p)} \\ &\leq \tilde{\kappa}^2\tilde{\kappa}^{i-1-j}e^{-\rho(n-p)}, \end{aligned}$$

and since $(i-1-j)m \le n-p$ and $\tilde{\kappa} \ge 1$, one obtains

$$\|U^{s}_{\overline{\mathbf{A}}}(n,p)\| \le \tilde{\kappa}^2 (e^{-\rho}\tilde{\kappa}^{\frac{1}{m}})^{n-p} \le \tilde{\kappa}^2 e^{-(\rho-\frac{1}{m}\ln\tilde{\kappa})(n-p)}.$$

Similarly, we also obtain

$$\begin{aligned}
\|U^{c}_{\overline{\mathbf{A}}}(n,p)\| &\le \tilde{\kappa}^2 e^{(\rho_0+\frac{1}{m}\ln\tilde{\kappa})(n-p)},\\
\|U^{c}_{\overline{\mathbf{A}}}(p,n)\| &\le \tilde{\kappa}^2 e^{(\rho_0+\frac{1}{m}\ln\tilde{\kappa})(n-p)},\\
\|U^{u}_{\overline{\mathbf{A}}}(n,p)\| &\le \tilde{\kappa}^2 e^{-(\rho-\frac{1}{m}\ln\tilde{\kappa})(n-p)}.
\end{aligned}$$

Now note that since $m \ge m_0$ with m_0 defined in (4.5), one has the inequality

$$\rho_0 + \frac{1}{m}\ln\tilde{\kappa} < \frac{\widehat{\rho}_0+\rho_0}{2} < \frac{\widehat{\rho}+\rho}{2} < \rho - \frac{1}{m}\ln\tilde{\kappa}.$$

Therefore, one can claim that $\overline{\mathbf{A}}$ is exponentially trichotomic with constant $\tilde{\kappa}^2$, exponents $\frac{\widehat{\rho}_0+\rho_0}{2}$ and $\frac{\widehat{\rho}+\rho}{2}$. The proof is complete. ■

Remark 4.8 Note that in Lemma 4.5, condition (4.6) can be replaced by

$$\|A_n\Pi^{iu}_n\|_{\mathcal{L}(X)} \le K, \quad \forall i \in \mathbb{Z} \quad \text{and} \quad n \in [(i-1)m, im[. \tag{4.16}$$

In fact by using (4.1), (4.3) and (4.16), we deduce that for each $i \in \mathbb{Z}$ and $n \in [(i-1)m, im[$,

$$\begin{aligned}
\|A_n\|_{\mathcal{L}(X)} &\le \|A_n\Pi^{iu}_n\|_{\mathcal{L}(X)} + \|A_n\Pi^{is}_n\|_{\mathcal{L}(X)} + \|A_n\Pi^{ic}_n\|_{\mathcal{L}(X)}\\
&\le K + \kappa e^{-\rho} + \kappa e^{\rho_0}.
\end{aligned}$$

The next lemma will allow us to derive the closeness of the projectors. This lemma generalizes [15, Lemma 2.2], but this proof is different, since the linear operators are not invertible.

Lemma 4.9 *Let $\kappa > 0$, $0 < \rho_0 < \rho$ be given and let $l > 0$ be an integer. Let $a, b \in \mathbb{Z}$ such that $b - a \ge 2l$. Assume that there exist two families of projectors $\{\mathbf{\Pi}^{\alpha}\}_{\alpha=u,s,c}$ and $\{\overline{\mathbf{\Pi}}^{\alpha}\}_{\alpha=u,s,c}$ such that $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ has two exponential trichotomies on $[a,b]\cap\mathbb{Z}$, with constant κ, exponents $0 < \rho_0 < \rho$, with respect to both families of projectors. Then we have*

$$\sup_{n\in[a+l,b-l]\cap\mathbb{Z}} \|\overline{\Pi}^{\alpha}_n - \Pi^{\alpha}_n\|_{\mathcal{L}(X)} \le 6\kappa^3 e^{-(\rho-\rho_0)l}. \tag{4.17}$$

Proof Let $n \in [a+l, b-l]$ be given. Denote by $U_{\mathbf{A}}$ the evolution semigroup associated to $\mathbf{A}$. For each $\alpha = u, c$ and each $n, p \in [a,b]$ with $n \ge p$, we defined $\overline{U}^{\alpha}_{\mathbf{A}}(p,n)\colon \overline{\Pi}^{\alpha}_n(X) \to \overline{\Pi}^{\alpha}_p(X)$ the inverse of the bounded linear operator

$$U_{\mathbf{A}}(n,p)\overline{\Pi}^{\alpha}_p\colon \overline{\Pi}^{\alpha}_p(X) \to \overline{\Pi}^{\alpha}_n(X).$$

We claim that

$$\|\overline{\Pi}^{s}_n x\| \le \kappa^2 e^{-(\rho-\rho_0)l}\|x\|, \quad \forall x \in \Pi^{\alpha}_n(X), \ \forall \alpha = u, c.$$

Let $x \in \Pi_n^\alpha(X)$ with $\alpha = u, c$. We have

$$x = \Pi_n^\alpha x = U_{\mathbf{A}}(n, n-l)U_{\mathbf{A}}^\alpha(n-l, n)x,$$

which implies

$$\overline{\Pi}_n^s x = U_{\mathbf{A}}(n, n-l)\overline{\Pi}_{n-l}^s U_{\mathbf{A}}^\alpha(n-l, n)x,$$

hence

$$\|\overline{\Pi}_n^s x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|.$$

Similarly, by using Π_n^s and $\overline{U}_{\mathbf{A}}^\alpha(n-l, n)$ instead of $\overline{\Pi}_n^s$ and $U_{\mathbf{A}}^\alpha(n-l, n)$, we obtain that

$$\|\Pi_n^s x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|, \quad \forall x \in \overline{\Pi}_n^\alpha(X), \ \forall \alpha = u, c. \tag{4.18}$$

Next we claim that

$$\|\Pi_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|, \quad \forall x \in \overline{\Pi}_n^\alpha(X), \ \forall \alpha = s, c. \tag{4.19}$$

Let $x \in \overline{\Pi}_n^\alpha(X)$ be given with $\alpha = s, c$. Then

$$\Pi_n^u x = U_{\mathbf{A}}^u(n, n+l)U_{\mathbf{A}}(n+l, n)x = U_{\mathbf{A}}^u(n, n+l)U_{\mathbf{A}}(n+l, n)\overline{\Pi}_n^\alpha x,$$

so that

$$\|\Pi_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|,$$

and we obtain in a similar way that

$$\|\overline{\Pi}_n^u x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|, \quad \forall x \in \Pi_n^\alpha(X), \ \forall \alpha = u, c.$$

Finally we claim that

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|, \quad \forall x \in \overline{\Pi}_n^\alpha(X), \ \forall \alpha = u, c. \tag{4.20}$$

Let $x \in \overline{\Pi}_n^u(X)$ be given. Then one has

$$x = \Pi_n^u x = U_{\mathbf{A}}(n, n-l)\overline{U}_{\mathbf{A}}^u(n-l, n)x,$$

which implies that

$$\Pi_n^c x = U_{\mathbf{A}}^c(n, n-l)\overline{U}_{\mathbf{A}}^u(n-l, n)x,$$

thus

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|.$$

Let $x \in \overline{\Pi}_n^s(X)$ be given. Then

$$\Pi_n^c x = U_{\mathbf{A}}^c(n, n+l)U_{\mathbf{A}}(n+l, n)x = U_{\mathbf{A}}^c(n, n+l)U_{\mathbf{A}}(n+l, n)\overline{\Pi}_n^s x,$$

and we obtain

$$\|\Pi_n^c x\| \leq \kappa^2 e^{-(\rho-\rho_0)l}\|x\|.$$

Now we prove inequality (4.17).

Let $x \in X$ be given. Then we have

$$\begin{aligned}
\|[\overline{\Pi}_n^s - \Pi_n^s]x\| &= \|[I - \Pi_n^s]\overline{\Pi}_n^s x - \Pi_n^s[I - \overline{\Pi}_n^s]x\| \\
&= \|[\Pi_n^c + \Pi_n^u]\overline{\Pi}_n^s x - \Pi_n^s[\overline{\Pi}_n^c + \overline{\Pi}_n^u]x\| \\
&\leq \|\Pi_n^c\overline{\Pi}_n^s x\| + \|\Pi_n^u\overline{\Pi}_n^s x\| + \|\Pi_n^s\overline{\Pi}_n^c x\| + \|\Pi_n^s\overline{\Pi}_n^u x\|.
\end{aligned}$$

Hence by (4.18), (4.19), and (4.20), we obtain

$$\begin{aligned}\|[\overline{\Pi}_n^s - \Pi_n^s]x\| &\le \kappa^2 e^{-(\rho-\rho_0)l}\big[\|\overline{\Pi}_n^s x\| + \|\overline{\Pi}_n^s x\| + \|\overline{\Pi}_n^c x\| + \|\overline{\Pi}_n^u x\|\big] \\ &\le 3\kappa^3 e^{-(\rho-\rho_0)l}\|x\|.\end{aligned} \tag{4.21}$$

Similarly we have

$$\begin{aligned}\|[\overline{\Pi}_n^u - \Pi_n^u]x\| &= \|\Pi_n^c\overline{\Pi}_n^u x + \Pi_n^s\overline{\Pi}_n^u x - \Pi_n^u\overline{\Pi}_n^c x - \Pi_n^u\overline{\Pi}_n^s x\| \\ &\le \|\Pi_n^c\overline{\Pi}_n^u x\| + \|\Pi_n^s\overline{\Pi}_n^u x\| + \|\Pi_n^u\overline{\Pi}_n^c x\| + \|\Pi_n^u\overline{\Pi}_n^s x\| \\ &\le \kappa^2 e^{-(\rho-\rho_0)l}[\|\overline{\Pi}_n^u x\| + \|\overline{\Pi}_n^u x\| + \|\overline{\Pi}_n^c x\| + \|\overline{\Pi}_n^s x\|] \\ &\le 3\kappa^3 e^{-(\rho-\rho_0)l}\|x\|.\end{aligned} \tag{4.22}$$

Since $\overline{\Pi}_n^c = I - \overline{\Pi}_n^u - \overline{\Pi}_n^s$ and $\Pi_n^c = I - \Pi_n^u - \Pi_n^s$, we obtain from (4.21) and (4.22) that

$$\|[\overline{\Pi}_n^c - \Pi_n^c]x\| \le 6\kappa^3 e^{-(\rho-\rho_0)l}\|x\|.$$

The proof is complete. ∎

Proof of Theorem 2.3 Let $\mathbb{D} = \{\theta_i\}_{i\in\mathbb{Z}}$ be a T_0-covering of $\mathbb{Z}$. Therefore from Definition 1.2 one knows that for each interval of $\mathbb{Z}$ of length T_0 there exists some θ_i in this interval. Let $T > 3T_0$ be an integer that is assumed to be divisible by 6. Assume that $\mathbf{A}$ is exponentially trichotomic on each interval $[\theta_j, \theta_j + T]$.

Let $n \in \mathbb{Z}$ be given. Since $T > 3T_0$, it follows that the interval $[n - \frac{T}{3}, n]$ contains at least one $\theta_j \in \mathbb{D}$, and

$$\left[n, n + \frac{T}{2}\right] \subset [\theta_j, \theta_j + T], \quad \left[n + \frac{T}{6}, n + \frac{2T}{6}\right] \subset [\theta_j, \theta_j + T].$$

Since n is an arbitrary integer, by choosing $n = i\frac{T}{6}$ for some integer $i \in \mathbb{Z}$, there exists an integer $j \in \mathbb{Z}$ such that

$$\left[i\frac{T}{6}, i\frac{T}{6} + \frac{T}{2}\right] = \left[i\frac{T}{6}, i\frac{T}{6} + \frac{3T}{6}\right] \subset [\theta_j, \theta_j + T].$$

Assuming that $\mathbf{A}$ is exponentially trichotomic on $[\theta_j, \theta_j + T]$, it follows that $\mathbf{A}$ is also exponentially trichotomic on

$$\left[(i+1)\frac{T}{6}, (i+2)\frac{T}{6}\right], \quad \forall i \in \mathbb{Z},$$

and the exponential trichotomy can be extended to the interval

$$\left[i\frac{T}{6}, (i+2)\frac{T}{6} + \frac{T}{6}\right].$$

Now since $T > 3T_0$ can be chosen arbitrarily large, the result follows from Lemma 4.5 and Lemma 4.9. ∎

5 Proof of Proposition 3.4

Before proving Proposition 3.4, we first prove Remark 3.5. This remark will be useful during the proof of Proposition 3.4.

Claim 5.1 *We claim that if* $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ *is a complete orbit of* F *in* M*, then the evolution semigroup* $U_{\mathbf{x}} := U_{DF(\mathbf{x})}$ *associated with* $DF(\mathbf{x}) = \{DF(x_n)\}_{n\in\mathbb{Z}}$ *has an exponential trichotomy with constant* $\kappa\kappa_0$ *and exponents* ρ_0, ρ*.*

Let $\mathbf{x}$ be a complete orbit of F in M. By (4.2) and Definition 3.1(i) we can find three families of projectors $\{\Pi^{\alpha}_{x_n}\}_{n\in\mathbb{Z}},\ \alpha = u, s, c$ satisfying

$$\Pi^{\alpha}_{x_n}\Pi^{\alpha'}_{x_n} = 0 \text{ if } \alpha' \neq \alpha \quad \text{and} \quad \Pi^{u}_{x_n} + \Pi^{s}_{x_n} + \Pi^{c}_{x_n} = I. \tag{5.1}$$

Recall that the evolution semigroup $U_{\mathbf{x}}$ associated with $DF(\mathbf{x})$ is defined by

$$U_{\mathbf{x}}(n, p) := \begin{cases} DF(x_{n-1})\cdots DF(x_p) & \text{if } n > p, \\ I & \text{if } n = p. \end{cases}$$

Observe that since $\mathbf{x}$ is a complete orbit of F, by Definition 3.1(i), one has that for each $n \in \mathbb{Z}$,

$$\Pi^{\alpha}_{F(x_n)}DF(x_n)\Pi^{\alpha}_{x_n} = DF(x_n)\Pi^{\alpha}_{x_n},$$

$$\Pi^{\alpha}_{x_{n+1}}DF(x_n)\Pi^{\alpha}_{x_n} = DF(x_n)\Pi^{\alpha}_{x_n}. \tag{5.2}$$

Hence (5.2) combined with (5.1) implies that for each $n \in \mathbb{Z}$ and each $\alpha = u, s, c$,

$$\Pi^{\alpha}_{x_{n+1}}DF(x_n) = \Pi^{\alpha}_{x_{n+1}}DF(x_n)\Pi^{\alpha}_{x_n} \tag{5.3}$$

and (5.2) combined with (5.3) yields

$$\Pi^{\alpha}_{x_{n+1}}DF(x_n) = DF(x_n)\Pi^{\alpha}_{x_n}, \quad \forall n \in \mathbb{Z}. \tag{5.4}$$

Therefore by using (5.4) one can deduce that for each $n \geq p$ and each $\alpha = u, s, c$

$$U^{\alpha}_{\mathbf{x}}(n, p) := \Pi^{\alpha}_{x_n}U_{\mathbf{x}}(n, p) = U_{\mathbf{x}}(n, p)\Pi^{\alpha}_{x_p} = DF(x_{n-1})\Pi^{\alpha}_{x_{n-1}}\cdots DF(x_p)\Pi^{\alpha}_{x_p}.$$

By using Definition 3.1(i) again we have for each $n \geq p$ and each $\alpha = u, c$ that the bounded linear operator $U^{\alpha}_{\mathbf{x}}(n, p)\colon \Pi^{\alpha}_{x_p}(X) \to \Pi^{\alpha}_{x_n}(X)$ is invertible. Define its inverse as

$$U^{\alpha}_{\mathbf{x}}(p, n)\colon \Pi^{\alpha}_{x_n}(X) \to \Pi^{\alpha}_{x_p}(X).$$

By observing that

$$U_{\mathbf{x}}(n, p) = DF^{n-p}(x_p), \quad \forall n \geq p,$$

we obtain from Definition 3.1(ii) and (3.3) that for each $n \geq p$,

$$\|U^{s}_{\mathbf{x}}(n, p)\|_{\mathcal{L}(X)} \leq \kappa\kappa_0 e^{-\rho(n-p)},$$
$$\|U^{u}_{\mathbf{x}}(p, n)\|_{\mathcal{L}(X)} \leq \kappa\kappa_0 e^{-\rho(n-p)},$$

and for each $(n, p) \in \mathbb{Z}^2$,

$$\|U^{c}_{\mathbf{x}}(n, p)\|_{\mathcal{L}(X)} \leq \kappa\kappa_0 e^{\rho_0|n-p|}.$$

This proves the exponential trichotomy for $U_{\mathbf{x}}$.

We now turn to the proof of Proposition 3.4.

Claim 5.2 *Let $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ be given. We claim that if $\mathbf{z} = \{z_n\}_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit of F in a neighborhood $\mathcal{V}(M, \varepsilon)$ of M for some ε and δ small enough, then the evolution semigroup $U_{\mathbf{z}} := U_{DF(\mathbf{z})}$ associated with $DF(\mathbf{z}) = \{DF(z_n)\}_{n\in\mathbb{Z}}$ has an exponential trichotomy with constant $\hat{\kappa}$ and exponents $\widehat{\rho}_0$ and $\widehat{\rho}$.*

Proof of Claim 5.2 First recall that, by Claim 5.1, for each complete orbit $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ of F in M, the evolution semigroup $U_{\mathbf{x}}$ associated with $DF(\mathbf{x})$ has an exponential trichotomy with constant $\kappa \geq 0$ and exponents $0 < \rho_0 < \rho$. Let $\overline{\mathbf{z}} = \{\overline{z}_n\}_{n\in\mathbb{Z}}$ be a δ-pseudo orbit of F lying in a neighborhood $\mathcal{V}(M, \varepsilon)$ of M. Then there exists a sequence $\mathbf{z} = \{z_n\}_{n\in\mathbb{Z}} \subset M$ such that

$$\|\overline{z}_n - z_n\| \leq \varepsilon, \quad \forall n \in \mathbb{Z}.$$

Let us prove that $\mathbf{z} = \{z_n\}_{n\in\mathbb{Z}} \subset M$ is a pseudo orbit of F in M. In fact by using Assumption 3.2(iii) we have

$$\begin{aligned}\|z_{n+1} - F(z_n)\| &\leq \|z_{n+1} - \overline{z}_{n+1}\| + \|\overline{z}_{n+1} - F(\overline{z}_n)\| + \|F(\overline{z}_n) - F(z_n)\| \\ &\leq \varepsilon + \delta + K\varepsilon,\end{aligned}$$

so that $\mathbf{z} = \{z_n\}_{n\in\mathbb{Z}}$ is a $\varepsilon(1 + K) + \delta$-pseudo orbit of F in M. Next let $\theta \in \mathbb{Z}$ and T be a positive integer. Consider the complete orbit $\mathbf{x}$ given by $x_n = F^{n-\theta}(z_\theta)$ for each $n \in \mathbb{Z}$ with the notation $z_\theta = F^0(z_\theta)$. Then since $\mathbf{z}$ is a $\varepsilon(1 + K) + \delta$-pseudo orbit of F in M, one has

$$\begin{aligned}\|x_{\theta+2} - z_{\theta+2}\| &= \|F^2(z_\theta) - z_{\theta+2}\| \\ &\leq \|F(F(z_\theta)) - F(z_{\theta+1})\| + \|F(z_{\theta+1}) - z_{\theta+2}\| \\ &\leq K\|F(z_\theta) - z_{\theta+1}\| + \|F(z_{\theta+1}) - z_{\theta+1}\| \\ &\leq K[\varepsilon(1 + K) + \delta] + \varepsilon(1 + K) + \delta.\end{aligned}$$

By induction one can easily derive that for each $k \in [0, T]$

$$\begin{aligned}\|x_{\theta+k} - z_{\theta+k}\| &\leq (1 + K + \cdots + K^{k-1})[\varepsilon(1 + K) + \delta] \\ &\leq (1 + K + \cdots + K^{T-1})[\varepsilon(1 + K) + \delta].\end{aligned}$$

Since by Assumption 3.2 the map $x \to DF(x)$ is uniformly continuous on $\mathcal{V}(M, \varepsilon_0)$, we can define the modulus of continuity of $x \to DF(x)$, which is a map $\omega\colon [0, \varepsilon_0] \to [0, +\infty)$ defined by

$$\omega(\varepsilon) := \sup_{\substack{x,y\in\mathcal{V}(M,\varepsilon_0) \\ \|x-y\|\leq\varepsilon}} \|DF(x) - DF(y)\|.$$

Thus one gets for each $k \in [0, T]$,

$$\|DF(x_{\theta+k}) - DF(z_{\theta+k})\| \leq \omega((1 + K + \cdots + K^{T-1})[\varepsilon(1 + K) + \delta]).$$

Therefore we obtain that

$$\begin{aligned}&\|DF(x_{\theta+k}) - DF(\overline{z}_{\theta+k})\| \\ &\quad\leq \|DF(x_{\theta+k}) - DF(z_{\theta+k})\| + \|DF(z_{\theta+k}) - DF(\overline{z}_{\theta+k})\| \\ &\quad\leq \omega(\varepsilon) + \omega((1 + K + \cdots + K^{T-1})[\varepsilon(1 + K) + \delta]).\end{aligned} \tag{5.5}$$

Next observe that

$$DF(\overline{\mathbf{z}}) = DF(\mathbf{x}) + [DF(\overline{\mathbf{z}}) - DF(\mathbf{x})].$$

Recalling that the evolution semigroup $U_{\mathbf{x}}$ has an exponential trichotomy on $\mathbb{Z}$ and writing (5.5) as

$$\sup_{k\in[\theta,\theta+T]} \|DF(x_k) - DF(z_k)\| \leq \omega(\varepsilon) + \omega((1 + K + \cdots + K^{T-1})[\varepsilon(1 + K) + \delta]),$$

Corollary 4.4 applies and ensures, for δ and ε sufficiently small, depending only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$, and T, that $U_{\mathbf{z}}$ has an exponential trichotomy on $[\theta, \theta + T]$ with constant $\overline{\kappa} = (2\kappa + 1)^2$, exponents $\frac{\rho+\widehat{\rho}}{2}$, $\frac{\rho_0+\widehat{\rho}_0}{2} \in (0, \frac{\rho+\widehat{\rho}}{2})$ and projectors $\{\overline{\Pi}_n^{\alpha}\}_{n\in\mathbb{Z}}$ with $\alpha = u, s, c$.

We now complete the proof by applying Theorem 2.3. To do so, first note that since $U_{\mathbf{z}}$ has exponential trichotomy on each $[\theta, \theta + T]$ with $\theta \in \mathbb{Z}$ whenever δ and ε sufficiently small, one can use $\mathbb{Z}$ as a relative dense subset of integers or equivalently a 1-covering. Furthermore, we also note that

$$\frac{\rho_0 + \widehat{\rho}_0}{2} < \widehat{\rho}_0 < \widehat{\rho} < \frac{\rho + \widehat{\rho}}{2},$$

and since the choice of δ and ε depend only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$, and T, one can choose T large enough (depending only on $K, \kappa, \rho_0, \rho, \widehat{\rho}_0, \widehat{\rho}$) in the previous lines such that Theorem 2.3 holds for T with the constant of trichotomy $(2\kappa + 1)^2$ and exponents $\frac{\rho_0+\widehat{\rho}_0}{2}$ and $\frac{\rho+\widehat{\rho}}{2}$. The proof is complete. ■

6 Further Consequences

In this section we present more consequences of Theorem 2.3. We use some examples presented by Palmer [15], but for finite time exponential trichotomy instead of exponential dichotomy.

6.1 Slowly Varying Systems

Proposition 6.1 *Let $\rho > \widehat{\rho} > \widehat{\rho}_0 > \rho_0 > 0$ and $\kappa \geq 1$. Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be a given uniformly bounded sequence with*

$$\|A_n\|_{\mathcal{L}(X)} \leq K,\ n \in \mathbb{Z},$$

for some positive real constant $K > 0$. There exists $\delta := \delta(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa) > 0$ and $\widehat{\kappa}(K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) \geq \kappa$ such that if the following two properties hold, then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}$, exponents $\widehat{\rho}$ and $\widehat{\rho}_0$.

(i) *For each $k \in \mathbb{Z}$, the constant sequence $B_n^k := A_k,\ n \in \mathbb{Z}$, is exponentially trichotomic on $\mathbb{Z}$ with constant κ, exponents ρ and ρ_0.*

(ii) *The sequence $\{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfies*

$$\|A_{n+1} - A_n\|_{\mathcal{L}(X)} \leq \delta, \quad \forall n \in \mathbb{Z}.$$

Proof Let us first observe that if

$$\|A_{n+1} - A_n\|_{\mathcal{L}(X)} \leq \delta, \quad \forall n \in \mathbb{Z},$$

then for any given and fixed $T \in \mathbb{N}$ one has for each $n \in [\theta, \theta + T]$

$$\|A_n - B_n^\theta\|_{\mathcal{L}(X)} = \|A_n - A_\theta\|_{\mathcal{L}(X)} \le \sum_{k=\theta}^{\theta+T-1} \|A_{k+1} - A_k\|_{\mathcal{L}(X)} \le T\delta.$$

Thus due to Corollary 4.4, there exists $\delta > 0$ depending only on $T, \rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho$, and κ such that if conditions (i) and (ii) hold, then $\mathbf{A}$ is exponentially trichotomic on each interval $[\theta, \theta + T]$, $\theta \in \mathbb{Z}$ (T an arbitrary fixed integer) with constant $(2\kappa + 1)^2$, exponents $\frac{\rho+\widehat{\rho}}{2}$ and $\frac{\rho_0+\widehat{\rho}_0}{2}$. Next observe that $\frac{\rho_0+\widehat{\rho}_0}{2} < \widehat{\rho}_0 < \widehat{\rho} < \frac{\rho+\widehat{\rho}}{2}$ so that the result follows by applying Theorem 2.3 by first taking T large enough depending only $K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0$ and secondly choosing δ. ■

6.2 Almost Periodic Systems

Consider $l^\infty(\mathbb{Z}, \mathcal{L}(X))$ the space of bounded sequences of bounded linear operators endowed with the usual supremum norm $\|\mathbf{A}\|_\infty := \sup_{n\in Z} \|A_n\|$. Define the shift operator $S\colon l^\infty(\mathbb{Z}, \mathcal{L}(X)) \to l^\infty(\mathbb{Z}, \mathcal{L}(X))$, $S(\mathbf{A})_n = A_{n+1}, \forall n \in \mathbb{Z}$. In the following definition, property (i) corresponds to the notion of almost periodic map in the sense of Bohr [4], and (ii) corresponds to the notion of almost periodic function introduced by Bochner [3] for continuous time maps. We also refer the reader to Corduneanu [7, p. 93] (see also [14]) for more results about the discrete time case.

Definition 6.2 A sequence of bounded linear operators $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ is *almost periodic* if one of the two following (equivalent) properties is satisfied:

(i) For each $\delta > 0$ there exists $\mathbb{D}_\delta = \{\theta_i\}_{i\in\mathbb{Z}}$ a $T_\delta-$covering of $\mathbb{Z}$ such that

$$\|A_{n+\theta_i} - A_n\|_{\mathcal{L}(X)} \le \delta, \quad \forall n \in \mathbb{Z}, \ \ \forall i \in \mathbb{Z}.$$

(ii) The sequence $\{S^n(\mathbf{A})\}_{n\in\mathbb{Z}}$ is relatively compact in the Banach space $l^\infty(\mathbb{Z}, \mathcal{L}(X))$.

Let us recall that if $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}}$ is almost periodic, then $\mathbf{A} \in l^\infty(\mathbb{Z}, \mathcal{L}(X))$ (see for example [4]). Thus our proposition reads as

Proposition 6.3 *Let $\rho > \widehat{\rho} > \widehat{\rho}_0 > \rho_0 > 0$ and $\kappa \ge 1$. Let $p \in \mathbb{Z}$ be given. Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be given. Assume that $\mathbf{A}$ is almost periodic.*

There exists $\widehat{T} := \widehat{T}(T_\delta, \|\mathbf{A}\|_\infty, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) > 0$ (where T_δ is given by Definition 6.2) and $\widehat{\kappa} := \widehat{\kappa}(\|\mathbf{A}\|_\infty, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) \ge \kappa$ such that for each $T \ge \widehat{T}$ if $\mathbf{A}$ is exponentially trichotomic on $[p, p + T]$ with constant κ, exponents ρ and ρ_0, then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}$, exponents $\widehat{\rho}$ and $\widehat{\rho}_0$.

Proof In order to prove this proposition, we will apply Theorem 2.3. Hence we look for a $\overline{\mathbb{D}} = \{\overline{\theta}_i\}_{i\in\mathbb{Z}}$ a $\overline{T}$-covering of $\mathbb{Z}$ such that $\mathbf{A}$ is exponentially trichotomic on each interval of the form $[\overline{\theta}_i, \overline{\theta}_i + T]$ for some $T \in \mathbb{N}$ large enough.

Let $\delta \in (0, 1)$ be given. Then since $\mathbf{A}$ is almost periodic, there exists $\mathbb{D}_\delta = \{\theta_i\}_{i\in\mathbb{Z}}$ a T_δ-covering of $\mathbb{Z}$ such that

$$\|A_{n+\theta_i} - A_n\|_{\mathcal{L}(X)} \le \delta, \ \forall n \in \mathbb{Z}, i \in \mathbb{Z}. \tag{6.1}$$

Let $T \in \mathbb{N}$ such that $\mathbf{A}$ is exponentially trichotomic on $[p, p+T]$ with constant κ exponents ρ and ρ_0. Next note that due to (6.1), for each $n \in [p, p+T]$, the operator

$$A_{n+\theta_i} = A_n + A_{n+\theta_i} - A_n, \text{ for each } n \in [p, p+T],$$

is a small perturbation of A_n of order δ. Therefore by Corollary 4.4 for δ small enough depending only on κ, ρ, ρ_0, $\widehat{\rho}$ and $\widehat{\rho}_0$, the sequence $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}}$ is exponentially trichotomic on each $[p+\theta_i, p+\theta_i+T]$ with constant $(2\kappa+1)^2$, exponents $\frac{\rho+\widehat{\rho}}{2}$ and $\frac{\rho_0+\widehat{\rho}_0}{2}$.

It is follows that $\mathbf{A}$ is exponentially trichotomic on each interval of the form $[\overline{\theta}_i, \overline{\theta}_i + T]$, with $\overline{\theta}_i = p + \theta_i$. Clearly $\overline{\mathbb{D}}_\delta = \{p+\theta_i\}_{i\in\mathbb{Z}}$ is a T_δ-covering of $\mathbb{Z}$.

To complete the proof of the proposition it remains to apply Theorem 2.3. Since we chose

$$\frac{\rho_0 + \widehat{\rho}_0}{2} < \widehat{\rho}_0 < \widehat{\rho} < \frac{\rho + \widehat{\rho}}{2},$$

the result follows from Theorem 2.3 by taking T large enough depending only on T_δ, $\|\mathbf{A}\|_\infty$, κ, ρ, $\widehat{\rho}$, ρ_0 and $\widehat{\rho}_0$. ∎

6.3 A Perturbation Theorem

Proposition 6.4 *Let $\rho > \widehat{\rho} > \widehat{\rho}_0 > \rho_0 > 0$ and $\kappa \geq 1$. Let I be an index set and let $\mathbf{A}^i = \{A_n^i\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$, $i \in I$ be given. Let $\mathbf{A} = \{A_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be a given uniformly bounded sequence with*

$$\|A_n\|_{\mathcal{L}(X)} \leq K, \ n \in \mathbb{Z},$$

for some positive real constant $K > 0$. There exists $\delta := \delta(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa) > 0$, $\widehat{T} := \widehat{T}(K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) > 0$ and $\widehat{\kappa} := \widehat{\kappa}(K, \kappa, \rho, \widehat{\rho}, \rho_0, \widehat{\rho}_0) \geq \kappa$ such that if the following two properties hold, then $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}$, exponents $\widehat{\rho}$ and $\widehat{\rho}_0$.

(i) *For each $i \in I$, $\mathbf{A}^i$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ, exponents ρ and ρ_0;*

(ii) *For each $\theta \in \mathbb{Z}$ and some fixed $T \geq \widehat{T}$, there exists $i \in I$ such that*

$$\|A_n - A_n^i\|_{\mathcal{L}(X)} \leq \delta, \quad \forall n \in [\theta, \theta + T].$$

Proof Due to Corollary 4.4, there exists δ depending only on $\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho$, and κ such that if conditions (i) and (ii) hold, then $\mathbf{A}$ is exponentially trichotomic on each interval $[\theta, \theta+T]$, $\theta \in \mathbb{Z}$ (T an arbitrary fixed integer) with constant $(2\kappa+1)^2$, exponents $\frac{\rho+\widehat{\rho}}{2}$ and $\frac{\rho_0+\widehat{\rho}_0}{2}$. Next observe that $\frac{\rho_0+\widehat{\rho}_0}{2} < \widehat{\rho}_0 < \widehat{\rho} < \frac{\rho+\widehat{\rho}}{2}$ so that the result follows by applying Theorem 2.3 up to T large enough depending only $K, \kappa, \rho, \widehat{\rho}, \rho_0$, and $\widehat{\rho}_0$. ∎

Appendix A Persistence of Exponential Trichotomy

This section is devoted to a short proof of Theorem 4.3. It will be derived from the usual results for perturbation of exponential dichotomy coupled with spectral shift arguments. Theorem 4.3 could be considered as a classical result, but we did not find an appropriate reference for this statement. The arguments based on spectral shift in the proof below have been mentioned in Pliss and Sell [19]. Such ideas have also been used in Hale and Lin [11] with additional finite dimensional assumption for center and unstable spaces. We also refer to Barreira and Valls [2], where an additional invertibility assumption has been crucially used.

Define for each $\lambda \in \mathbb{R}$ and $\mathbf{L} = \{L_n\}_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ the operator $\mathbf{L}_\lambda := e^\lambda \mathbf{L}$.The associated evolution semigroup reads as

$$U_{\mathbf{L}_\lambda}(n,p) = e^{\lambda(n-p)} U_{\mathbf{L}}(n,p), \quad \forall n \geq p.$$

Recalling Definition 1.1 of exponential dichotomy, the following lemma holds true.

Lemma A.1 *Let $\mathbf{A}\colon \mathbb{Z}\to\mathcal{L}(X)$ be given. Assume that $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ, exponents $0 < \rho_0 < \rho$ and associated with the projectors $\{\mathbf{\Pi}^\alpha\colon \mathbb{Z}\to\mathcal{L}(X)\}_{\alpha=s,c,u}$. If we set $\lambda = \frac{\rho+\rho_0}{2}$, then the following properties hold true.*

(i) *$\mathbf{A}_\lambda$ is exponentially dichotomic on $\mathbb{Z}$ with constant 2κ, exponent $\frac{\rho-\rho_0}{2} > 0$ and associated with the projectors $\mathbf{\Pi}^s$ and $\mathbf{\Pi}^{cu} := \mathbf{\Pi}^c + \mathbf{\Pi}^u$.*
(ii) *$\mathbf{A}_{-\lambda}$ is exponentially dichotomic on $\mathbb{Z}$ with constant 2κ, exponent $\frac{\rho-\rho_0}{2} > 0$ and associated with the projectors $\mathbf{\Pi}^u$ and $\mathbf{\Pi}^{cs} := \mathbf{\Pi}^c + \mathbf{\Pi}^s$.*

By using the persistence result for exponential dichotomy in Henry [12, p. 232, Theorem 7.6.7] or in Zhou, Lu and Zhang [26, p. 4027, Theorem 1] (see also Pötzsche [21] for further results) combined with Lemma A.1, one obtains the following result.

Lemma A.2 *Let $\mathbf{A}\colon \mathbb{Z}\to\mathcal{L}(X)$ be given. Assume that $\mathbf{A}$ is exponentially trichotomic on $\mathbb{Z}$ with constant κ, exponents $0 < \rho_0 < \rho$ and associated with the projectors $\{\mathbf{\Pi}^\alpha\colon \mathbb{Z}\to\mathcal{L}(X)\}_{\alpha=s,c,u}$. Let $\eta \in (0, \frac{\rho-\rho_0}{2})$ and $\widehat{\kappa} > 2\kappa$ be given. Then, setting $\lambda = \frac{\rho+\rho_0}{2}$, there exists $\delta := \delta(\rho_0,\rho,\kappa,\eta,\widehat{\kappa}) \in (0,1)$ such that if the sequence $\mathbf{B}\colon \mathbb{Z}\to\mathcal{L}(X)$ satisfies*

$$\sup_{n\in\mathbb{Z}} \|B_n\|_{\mathcal{L}(X)} \leq \delta,$$

then the following properties hold:

(i) *The sequence of operators $\mathbf{A}_\lambda + \mathbf{B}_\lambda$ is exponentially dichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}$, exponent $\eta > 0$ and associated with the projectors $\widehat{\mathbf{\Pi}}^s\colon \mathbb{Z}\to\mathcal{L}(X)$ and $\widehat{\mathbf{\Pi}}^{cu}\colon \mathbb{Z}\to\mathcal{L}(X)$.*
(ii) *The sequence of operators $\mathbf{A}_{-\lambda} + \mathbf{B}_{-\lambda}$ is exponentially dichotomic on $\mathbb{Z}$ with constant $\widehat{\kappa}$, exponent $\eta > 0$ and associated with the projectors $\widehat{\mathbf{\Pi}}^u\colon \mathbb{Z}\to\mathcal{L}(X)$ and $\widehat{\mathbf{\Pi}}^{cs}\colon \mathbb{Z}\to\mathcal{L}(X)$.*

Remark A.3 Under the assumptions of the above lemma, one also has a characterization for the range of the projectors $\widehat{\mathbf{\Pi}}^s$ and $\widehat{\mathbf{\Pi}}^{cu}$ defined in (i), respectively $\widehat{\mathbf{\Pi}}^u$ and $\widehat{\mathbf{\Pi}}^{cs}$ defined in (ii). Following Pötzsche [22], the characterization reads as follows:

for all $n \in \mathbb{Z}$, one has

$$\widehat{\Pi}_n^s(X) := \{x \in X : \sup_{k \geq n} \|e^{\lambda(k-n)} U_{\mathbf{A+B}}(k,n)x\| < +\infty\},$$

$$\widehat{\Pi}_n^{cu}(X) := \left\{ \begin{array}{c} x \in X : \exists \{x_k\}_{k \leq n} \text{ with } x_k = (A_{k-1} + B_{k-1})x_{k-1},\ x_n = x \\ \text{and } \sup_{k \leq n} e^{-\lambda(n-k)} \|x_k\| < +\infty \end{array} \right\},$$

and, for all $n \in \mathbb{Z}$ one has

$$\widehat{\Pi}_n^{cs}(X) := \{x \in X : \sup_{k \geq n} \|e^{-\lambda(k-n)} U_{\mathbf{A+B}}(k,n)x\| < +\infty\},$$

$$\widehat{\Pi}_n^{u}(X) := \left\{ \begin{array}{c} x \in X : \exists \{x_k\}_{k \leq n} \text{ with } x_k = (A_{k-1} + B_{k-1})x_{k-1},\ x_n = x \\ \text{and } \sup_{k \leq n} e^{\lambda(n-k)} \|x_k\| < +\infty \end{array} \right\}.$$

Using the characterization of Remark A.3, we are able to derive some basic properties of the perturbed projectors provided by the above lemma. The following lemma will be used to complete the proof of Theorem 4.3.

Lemma A.4 *Under the assumptions of Lemma A.2, the perturbed projectors satisfy the following properties:*

(i) *For all $n \in \mathbb{Z}$,*

$$\widehat{\Pi}_n^s(X) \subset \widehat{\Pi}_n^{cs}(X) \text{ and } \widehat{\Pi}_n^u(X) \subset \widehat{\Pi}_n^{cu}(X). \tag{A.1}$$

(ii) *For all $n \in \mathbb{Z}$,*

$$(\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^{cu}(X)) \oplus \widehat{\Pi}_n^s(X) \oplus \widehat{\Pi}_n^u(X) = X, \tag{A.2}$$

and

$$\widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu}x = \widehat{\Pi}_n^{cu}\widehat{\Pi}_n^{cs}x,\ \forall x \in X.$$

Proof Property (i) is a direct consequence of Remark A.3. Let us now prove (ii). In the remaining part of this proof, $n \in \mathbb{Z}$ denotes a given and fixed integer. Recall that due to Lemma A.2 one has

$$\widehat{\Pi}_n^u(X) \oplus \widehat{\Pi}_n^{cs}(X) = X \quad \text{and} \quad \widehat{\Pi}_n^s(X) \oplus \widehat{\Pi}_n^{cu}(X) = X. \tag{A.3}$$

Hence on one hand we have from (A.3)

$$(\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^{cu}(X)) \cap \widehat{\Pi}_n^u(X) = \{0\},$$

$$(\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^{cu}(X)) \cap \widehat{\Pi}_n^s(X) = \{0\}.$$

On the other hand by using (A.1) combined with (A.3), one gets

$$(\widehat{\Pi}_n^s(X) \cap \widehat{\Pi}_n^u(X)) \subset (\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^u(X)) = \{0\}.$$

To achieve the proof, it remains to show that

$$(\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^{cu}(X)) + \widehat{\Pi}_n^s(X) + \widehat{\Pi}_n^u(X) = X.$$

Let $x \in X$ be given. By using (A.3), we have

$$x = \widehat{\Pi}_n^{cu}x + \widehat{\Pi}_n^s x, \tag{A.4}$$

and

$$x = \widehat{\Pi}_n^u x + \widehat{\Pi}_n^{cs} x. \tag{A.5}$$

Hence by using (A.1) and by applying $\widehat{\Pi}_n^{cu}$ (respectively $\widehat{\Pi}_n^{cs}$) on the right-hand side of (A.5) (respectively of (A.4)) yields

$$\widehat{\Pi}_n^{cu} x = \widehat{\Pi}_n^u x + \widehat{\Pi}_n^{cu}\widehat{\Pi}_n^{cs} x, \tag{A.6}$$

respectively

$$\widehat{\Pi}_n^{cs} x = \widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu} x + \widehat{\Pi}_n^s x. \tag{A.7}$$

Next, plugging the right side of (A.6) (resp. (A.7)) into (A.4)) (resp. (A.5)) provides that

$$x = \widehat{\Pi}_n^u x + \widehat{\Pi}_n^s x + \widehat{\Pi}_n^{cu}\widehat{\Pi}_n^{cs} x = \widehat{\Pi}_n^u x + \widehat{\Pi}_n^s x + \widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu} x.$$

This implies that

$$\widehat{\Pi}_n^{cu}\widehat{\Pi}_n^{cs} x = \widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu} x \in (\widehat{\Pi}_n^{cs}(X) \cap \widehat{\Pi}_n^{cu}(X)),$$

and the result follows. ■

We are now able to complete the proof of Theorem 4.3.

Proof of Theorem 4.3 Set $\lambda = \frac{\rho+\rho_0}{2}$ and fix

$$\eta = \max\{\widehat{\rho} - \lambda; \lambda - \widehat{\rho}_0\} \in \left(0, \frac{\rho - \rho_0}{2}\right) \quad \text{and} \quad \widehat{\kappa} > 2\kappa. \tag{A.8}$$

Let $\delta := \delta(\rho, \rho_0, \widehat{\rho}, \widehat{\rho}_0, \widehat{\kappa})$ be the value provided by Lemma A.2. Let us fix $\mathbf{B} = \{B_n\}_{n\in\mathbb{Z}}$ such that

$$\sup_{n\in\mathbb{Z}} \|B_n\|_{\mathcal{L}(X)} \le \delta.$$

In the sequel we will denote by $\widehat{\mathbf{\Pi}}^s$ and $\widehat{\mathbf{\Pi}}^{cu}$ the associated projectors for $\mathbf{A}_\lambda + \mathbf{B}_\lambda$ and by $\widehat{\mathbf{\Pi}}^u$ and $\widehat{\mathbf{\Pi}}^{cs}$ the projectors associated with $\mathbf{A}_{-\lambda} + \mathbf{B}_{-\lambda}$.

Next using Lemma A.4(ii) let us define

$$\widehat{\Pi}_n^c = \widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu} = \widehat{\Pi}_n^{cu}\widehat{\Pi}_n^{cs}, \quad \forall n \in \mathbb{Z}, \tag{A.9}$$

and observe that for all $n \in \mathbb{Z}$, $\widehat{\Pi}_n^c$, is a projector on X. Next observe that due to Lemma A.2 combined with (A.1)) and (A.9) one has for all $n \in \mathbb{Z}$

$$\widehat{\Pi}_n^\alpha \widehat{\Pi}_n^\beta = 0_{\mathcal{L}(X)}, \ \forall \alpha, \beta \in \{u, s, c\} \text{ and } \alpha \ne \beta.$$

Furthermore note that by using (A.2) combined with (A.9), one also has for all $n \in \mathbb{Z}$

$$\widehat{\Pi}_n^c = I - \widehat{\Pi}_n^s - \widehat{\Pi}_n^u.$$

Finally let us notice that Lemma A.2 ensures that

$$\sup_{n\in\mathbb{Z}} \|\widehat{\Pi}_n^\alpha\|_{\mathcal{L}(X)} \le \widehat{\kappa}, \ \alpha = s, u, cu, cs. \tag{A.10}$$

To complete the proof of Theorem 4.3, we will show that $\mathbf{A} + \mathbf{B}$ is exponentially trichotomic with constant $\widehat{\kappa}^2$, exponents $0 < \widehat{\rho}_0 < \widehat{\rho}$ and associated to the family of projectors $\{\widehat{\mathbf{\Pi}}^\alpha : \mathbb{Z} \to \mathcal{L}(X)\}_{\alpha=s,c,u}$. We will split the argument into three parts to investigate the behaviour of the perturbed evolution semigroup respectively on the stable, unstable and center spaces.

Estimate along the stable space Let $x \in X$ be given. Then by using Lemma A.2 one obtains for all $n \geq p$

$$\begin{aligned}\|U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^s x\| &= \|e^{-\lambda(n-p)}U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^s x\| \\ &\leq \widehat{\kappa}e^{-(\lambda+\eta)(n-p)}\|x\| \\ &\leq \widehat{\kappa}e^{-\widehat{\rho}(n-p)}\|x\|.\end{aligned} \tag{A.11}$$

Estimate along the unstable space By using Lemma A.2 it follows that for all $n \geq p$ the operator $U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^u$ is invertible from $\widehat{\Pi}_p^u(X)$ into $\widehat{\Pi}_n^u(X)$. The inverse is denoted by $U^u_{\mathbf{A}+\mathbf{B}}(p,n)$. Moreover for all $x \in X$ we have for all $n \geq p$

$$\begin{aligned}\|U^u_{\mathbf{A}+\mathbf{B}}(p,n)\widehat{\Pi}_n^u x\| &= \|e^{-\lambda(n-p)}U^u_{\mathbf{A}_{-\lambda}+\mathbf{B}_{-\lambda}}(p,n)\widehat{\Pi}_n^u x\| \\ &\leq \widehat{\kappa}e^{-(\lambda+\eta)(n-p)}\|x\| \\ &\leq \widehat{\kappa}e^{-\widehat{\rho}(n-p)}\|x\|.\end{aligned} \tag{A.12}$$

Estimate along the center space Before proceeding to the estimates, we will first prove that for $n \geq p$, the operator $U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^c$ is invertible from $\widehat{\Pi}_p^c(X)$ into $\widehat{\Pi}_n^c(X)$.

Let $n, p \in \mathbb{Z}$ be given such that $n \geq p$. Let $x \in X$ be given. Then recall that due to Lemma A.2 the operator $U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^{cu}|_{\widehat{\Pi}_p^{cu}(X)} : \widehat{\Pi}_p^{cu}(X) \to \widehat{\Pi}_n^{cu}(X)$ is invertible. We will denote its inverse by

$$U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n) : \widehat{\Pi}_n^{cu}(X) \to \widehat{\Pi}_p^{cu}(X).$$

Then on the one hand one has

$$U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^c x = U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^{cu}\widehat{\Pi}_p^{cs} x.$$

Multiplying the left hand side of the above equality by $U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)$ implies that

$$U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^c x = \widehat{\Pi}_p^{cu}\widehat{\Pi}_p^{cs} x = \widehat{\Pi}_p^c x. \tag{A.13}$$

On the other hand, we have

$$\begin{aligned}U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^c U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)x &= U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^{cs}\widehat{\Pi}_p^{cu}U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)x \\ &= \widehat{\Pi}_n^{cs}U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^{cu}U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)x \\ &= \widehat{\Pi}_n^{cs}\widehat{\Pi}_n^{cu}x = \widehat{\Pi}_n^c x.\end{aligned}$$

Hence (A.13) and the above equality ensures that the operator $U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^c = e^{-\lambda(n-p)}U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^c$ is invertible from $\widehat{\Pi}_p^c(X)$ into $\widehat{\Pi}_n^c(X)$ for all $n \geq p$ with inverse $U^c_{\mathbf{A}+\mathbf{B}}(p,n) := e^{\lambda(n-p)}U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(p,n)$.

Let us now derive the estimates along the center space. Let $n, p \in \mathbb{Z}$ be given such that $n \geq p$ and let $x \in X$ be given. Then we have

$$\|U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^c x\| = \|e^{\lambda(n-p)}U_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}_p^{cs}\widehat{\Pi}_p^{cu}x\| \leq \widehat{\kappa}e^{(\lambda-\eta)(n-p)}\|\widehat{\Pi}_p^{cu}x\|,$$

and by using (A.8)–(A.10) it follows that

$$\|U_{\mathbf{A}+\mathbf{B}}(n,p)\widehat{\Pi}_p^c x\| \leq \widehat{\kappa}^2 e^{\widehat{\rho}_0(n-p)}\|x\|. \tag{A.14}$$

Proceeding similarly one obtains

$$\|U^{c}_{\mathbf{A}+\mathbf{B}}(p,n)\widehat{\Pi}^{c}_{p}x\| = \|e^{\lambda(n-p)}U^{cu}_{\mathbf{A}_\lambda+\mathbf{B}_\lambda}(n,p)\widehat{\Pi}^{cu}_{p}\widehat{\Pi}^{cs}_{p}x\| \leq \widehat{\kappa}^{2}e^{\widehat{\rho}_0(n-p)}\|x\|. \tag{A.15}$$

The proof is completed by combining (A.11), (A.12), (A.14), and (A.15) together with Lemmas A.2 and A.4. ∎

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