

Persistence of Exponential Trichotomy for Linear Operators: A Lyapunov–Perron Approach

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Abstract In this article we revisit the perturbation of exponential trichotomy of linear difference equation in Banach space by using a Perron–Lyapunov fixed point formulation for the perturbed evolution operator. This approach allows us to directly re-construct the perturbed semiflow without using shift spectrum arguments. These arguments are presented to the case of linear autonomous discrete time dynamical system. This result is then coupled to Howland semigroup procedure to obtain the persistence of exponential trichotomy for non-autonomous difference equations.

Keywords Exponential trichotomy · Exponential dichotomy · Discrete time dynamical systems · Difference equations · Howland semigroup

1 Introduction

Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space $(X, \|\cdot\|)$. Recall that the spectral radius of A is defined by

$$r(A) := \lim_{n \rightarrow +\infty} \|A^n\|_{\mathcal{L}(X)}^{1/n}.$$

Assume that A has a **state space decomposition**, whenever A is regarded as the following discrete time dynamical system

$$\begin{cases} x_{n+1} = Ax_n, & \text{for } n \in \mathbb{N}, \\ x_0 = x \in X. \end{cases} \quad (1.1)$$

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Namely, we can find three closed subspaces X_s the **stable subspace**, X_c the **central subspace**, and X_u the **unstable subspace** such that

$$X = X_s \oplus X_c \oplus X_u \text{ and } A(X_k) \subset X_k, \quad \forall k = s, c, u.$$

Moreover if we define for each $k = s, c, u$, $A_k \in \mathcal{L}(X_k)$ the part of A in X_k (i.e. $A_k x = Ax, \quad \forall x \in X_k$). Then there exists a constant $\alpha \in (0, 1)$ such that

$$r(A_s) \leq \alpha < 1$$

the linear operator A_u on X_u is invertible and

$$r(A_u^{-1}) \leq \alpha < 1$$

and the operator A_c on X_c is invertible and

$$r(A_c) < \alpha^{-1} \text{ and } r(A_c^{-1}) < \alpha^{-1}.$$

We summarize the notion of state space decomposition into the following definition. In the context of linear dynamical system (or linear skew-product semiflow) this notion also corresponds to the notion of exponential trichotomy. The following definition corresponds to the one introduced by Hale and Lin in [13].

Definition 1.1 Let $A \in \mathcal{L}(X)$ be a bounded linear operator on a Banach space $(X, \|\cdot\|)$. We will say that A has an **exponential trichotomy** (or A is **exponentially trichotomic**) if there exist three bounded linear projectors $\Pi_s, \Pi_c, \Pi_u \in \mathcal{L}(X)$ such that

$$X = X_s \oplus X_c \oplus X_u,$$

and

$$A(X_k) \subset X_k, \quad \forall k = s, c, u,$$

where $X_k := \Pi_k(X)$, for $k = s, c, u$, and

$$X_c \oplus X_u = (I - \Pi_s)(X), X_s \oplus X_u = (I - \Pi_c)(X) \text{ and } X_s \oplus X_c = (I - \Pi_u)(X).$$

Moreover we assume that there exists a constant $\alpha \in (0, 1)$ satisfying the following properties:

- (i) Let $A_s \in \mathcal{L}(X_s)$ be the part of A in X_s (i.e. $A_s(x) = A(x), \quad \forall x \in X_s$) we assume that $r(A_s) \leq \alpha$;
- (ii) Let $A_u \in \mathcal{L}(X_u)$ be the part of A in X_u (i.e. $A_u(x) = A(x), \quad \forall x \in X_u$) we assume that A_u is invertible and $r(A_u^{-1}) \leq \alpha$;
- (iii) Let $A_c \in \mathcal{L}(X_c)$ be the part of A in X_c (i.e. $A_c(x) = A(x), \quad \forall x \in X_c$) we assume that A_c is invertible and $r(A_c) < \alpha^{-1}$ and $r(A_c^{-1}) < \alpha^{-1}$.

Let $A : D(A) \subset X \rightarrow X$ is a linear operator on a Banach space X . Let $Y \subset X$ is a subspace of X . Recall that $A_Y : D(A_Y) \subset Y \rightarrow Y$ the part of A in Y is defined by

$$D(A_Y) := \{x \in D(A) \cap Y : Ax \in Y\} \text{ and } A_Y x = Ax, \quad \forall x \in D(A_Y).$$

Note that in Definition 1.1 only the forward information are used on the stable part X_s , forward and backward for the central part X_c while only backward information are necessary on the unstable part X_u . This remark motivates the following definition of exponential trichotomy for unbounded linear operator operator that will be used throughout this work.

Definition 1.2 Let $A : D(A) \subset X \rightarrow X$ be a closed linear operator on a Banach space $(X, \|\cdot\|)$. We will say that A has an **exponential trichotomy** (or A is **exponentially tri-
chotomic**) if there exist three bounded linear projectors $\Pi_s, \Pi_c, \Pi_u \in \mathcal{L}(X)$ such that

$$X = X_s \oplus X_c \oplus X_u, \text{ where } X_k := \Pi_k(X), \quad \forall k = s, c, u, \tag{1.2}$$

and

$$X_c \oplus X_u = (I - \Pi_s)(X), X_s \oplus X_u = (I - \Pi_c)(X) \text{ and } X_s \oplus X_c = (I - \Pi_u)(X).$$

Moreover we assume that

$$D(A) = X_s \oplus X_c \oplus (D(A) \cap X_u).$$

and

$$A(D(A) \cap X_k) \subset X_k, \quad \forall k = s, c, u. \tag{1.3}$$

Furthermore we assume that there exists a constant $\alpha \in (0, 1)$ satisfying the following properties:

(i) Let $A_s \in \mathcal{L}(X_s)$ be the part of A in X_s , we assume that

$$r(A_s) \leq \alpha; \tag{1.4}$$

(ii) Let $A_u : D(A_u) \subset X_u \rightarrow X_u$ be the part of A in X_u , we assume that A_u is invertible and

$$r(A_u^{-1}) \leq \alpha; \tag{1.5}$$

(iii) Let $A_c \in \mathcal{L}(X_c)$ be the part of A in X_c , we assume that A_c is invertible and

$$r(A_c) < \alpha^{-1} \text{ and } r(A_c^{-1}) < \alpha^{-1}. \tag{1.6}$$

Remark 1.3 The above properties (1.5)–(1.7) are also equivalent to say that there exist three constants $\kappa \geq 1$ and $0 < \rho_0 < \rho$ such that

$$\|A_c^n\|_{\mathcal{L}(X_c)} \leq \kappa e^{\rho_0|n|}, \quad \forall n \in \mathbb{Z}, \tag{1.7}$$

$$\|A_s^n\|_{\mathcal{L}(X_s)} \leq \kappa e^{-\rho n}, \quad \forall n \in \mathbb{N}, \tag{1.8}$$

$0 \in \rho(A_u)$ the resolvent set of A_u and

$$\|A_u^{-n}\|_{\mathcal{L}(X_u)} \leq \kappa e^{-\rho n}, \quad \forall n \in \mathbb{N}. \tag{1.9}$$

In the sequel, the above estimates will be referred as exponential trichotomy with exponents $\rho_0 < \rho$, constant κ and associated to the projectors $\{\Pi^\alpha\}_{\alpha=s,c,u}$.

Remark 1.4 Since the linear operator A is assumed to be closed, by the closed graph theorem, Definition 1.2 coincides with Definition 1.1 if and only if $D(A_u) = X_u$.

By using the definition of exponential trichotomy we may also define the notion of exponential dichotomy.

Definition 1.5 Let $A : D(A) \subset X \rightarrow X$ be a closed linear operator on a Banach space $(X, \|\cdot\|)$. We will say that A has an **exponential dichotomy** if A has an exponential trichotomy with $X_c = \{0\}$.

The aim of this paper is to study the persistence of exponential trichotomy (according to Definition 1.2) under small bounded additive perturbation. Before going to our main result and application to non-autonomous problems, let us recall that exponential dichotomy (trichotomy and more generally invariant exponential splitting) is a basic tool to study stability for non-autonomous dynamical systems (see for instance [2, 11, 20] and the references therein). It is also a powerful ingredient to construct suitable invariant manifolds for non-linear problems (see [2, 6, 10] and the references therein). In the last decades a lot of attention has been paid to and much progress has been made in understanding invariant splitting for non-autonomous linear dynamical systems (continuous time as well as linear difference equations) as well as their persistence under small perturbations. We refer for instance to [8, 9, 13, 14, 19, 21–24] and the references cited therein.

Let us also mention the notion of non-uniform dichotomy and trichotomy in which the boundedness of the projectors is relaxed allowing unbounded linear projectors (see for instance [1–4] for non-autonomous dynamical systems and [27] for random linear difference equations).

The persistence of exponential splitting under small perturbation is also an important problem with several applications in dynamical systems such as shadowing properties. We refer to Palmer [16, 17] and the references therein.

Finally we would like to compare our definition of exponential splitting with the one recently considered by Potzsche in his monograph (see Chap. 3 in [20]). In the homogeneous case, Potzsche considers exponential splitting for a pair of linear operators $(A, B) \in \mathcal{L}(X, Y)$ where X and Y denote two Banach spaces. Note that X can be different from Y so that this framework applies to closed linear operators. Let us recall that when $(A, B) \in \mathcal{L}(X, Y)$ one may consider the corresponding (Y -valued) linear difference equation on X defined as

$$Bx_{k+1} = Ax_k, \quad k \in \mathbb{Z}.$$

We now recall the definition of exponential dichotomy used by Potzsche in [20]:

Definition 1.6 An operator pair $(A, B) \in \mathcal{L}(X, Y)^2$ acting between two Banach spaces X and Y is said to have an exponential dichotomy if there exist $\kappa > 0, \rho > 0$ and two orthogonal and complementary projectors $\Pi_s, \Pi_u \in \mathcal{L}(X)$ such that, by setting $X_k = \Pi_k(X)$ for $k = s, u$

$$\begin{aligned} X &= X_s \oplus X_u \\ \ker B|_{X_s} &= \{0\}, \quad R(A\Pi_s) \subset R(B\Pi_s) =: Y_s \\ \ker A|_{X_u} &= \{0\}, \quad R(B\Pi_u) \subset R(A\Pi_u) =: Y_u \end{aligned}$$

and $\Phi_s := \left(B|_{Y_s}^{-1} A \right) |_{X_s} \in \mathcal{L}(X_s)$ and $\Phi_u := \left(A|_{Y_u}^{-1} B \right) |_{X_u} \in \mathcal{L}(X_u)$ while

$$\|\Phi_s^n\|_{\mathcal{L}(X_s)} \leq \kappa e^{-n\rho}, \quad \text{and} \quad \|\Phi_u^n\|_{\mathcal{L}(X_s)} \leq \kappa e^{-n\rho} \quad \forall n \geq 0.$$

Note that when $A : D(A) \subset X \rightarrow X$ is a closed linear operator, the application of this theory to the pair $(A, J) \in \mathcal{L}(D(A), X)$ (where $J : D(A) \rightarrow X$ denotes the canonical embedding from $D(A)$ into X i.e. $J(x) = x, \quad \forall x \in D(A)$) would lead us to a splitting of the Banach space $D(A)$ (endowed with the graph norm). Let us also notice that when $A \in \mathcal{L}(X)$ has an exponential dichotomy (according to Definition 1.1) then the pair $(A, I_X) \in \mathcal{L}(X)^2$ has an exponential dichotomy in the above sense (Definition 1.6). In the same way when $A \in \mathcal{L}(X)$ has an exponential trichotomy (according to Definition 1.1) then the pair $(A, I_X) \in \mathcal{L}(X)^2$ has 3-exponential invariant splitting in the sense of Potzsche [20, Definition 3.4.12, p. 135].

Now let $A : D(A) \subset X \rightarrow X$ be an exponential dichotomic closed linear operator with parameters $\kappa > 0$, $\rho > 0$ and $\delta > 0$ and projectors Π_k $k = s, u$ (see Definition 1.2). Consider the linear operator $\widehat{B} \in \mathcal{L}(X)$ defined by $\widehat{B} = (A_u^{-1}\Pi_u + \Pi_s) \in \mathcal{L}(X)$. Then by applying \widehat{B} on the left side of (1.1), the linear difference equation $x_{n+1} = Ax_n$ becomes

$$\widehat{B}x_{n+1} = \widehat{B}Ax_n$$

or equivalently

$$\widehat{B}x_{n+1} = \widehat{A}x_n$$

where $\widehat{A} := \Pi_u + A_s\Pi_s \in \mathcal{L}(X)$ is a bounded extension of $\widehat{B}A : D(A) \subset X \rightarrow X$ (the unique bounded extension if A is densely defined). In order to deal with the above linear difference equation, one may consider the operator pair $(\widehat{A}, \widehat{B})$. Note that $\ker \widehat{B} = \{0\}$ and $\widehat{B}^{-1} : D(\widehat{B}^{-1}) \subset X \rightarrow X$ is the closed linear operator defined by

$$D(\widehat{B}^{-1}) = R(\widehat{B}) = X_s \oplus D(A) \cap X_u = D(A) \text{ and } \widehat{B}^{-1} = A_u\Pi_u + \Pi_s.$$

Then it is easy to check that if $A : D(A) \subset X \rightarrow X$ has an exponential dichotomy (see Definition 1.2) then the pair $(\widehat{B}, \widehat{A}) \in \mathcal{L}(X)^2$ has an exponential dichotomy according to Definition 1.6.

One can therefore try to use this operator pair framework to study the persistence of exponential dichotomy provided by the extended Definition 1.2. Let recall that Potzsche derived in his monograph a general roughness result using the operator pair framework (see Theorem 3.6.5, p. 165). Let $(A, B) \in \mathcal{L}(X, Y)^2$ be an exponential dichotomy operator pair and let $(\overline{A}, \overline{B}) \in \mathcal{L}(X, Y)^2$ be a given perturbation. Then if

$$R(A) \subset R(B), R(\overline{A}) \subset R(B) \text{ and } R(\overline{B}) \subset R(B)$$

then under suitable smallness assumptions the operator pair $(A + \overline{A}, B + \overline{B})$ has an exponential dichotomy.

Consider an exponentially dichotomic closed linear operator $A : D(A) \subset X \rightarrow X$ as well as a small perturbation $C \in \mathcal{L}(X)$. Using the above transformation the linear difference equation $x_{k+1} = (A + C)x_k$ rewrites as studying the invariant splitting for the operator pair $(\widehat{B}, \widehat{A} + \widehat{B}C) \in \mathcal{L}(X)^2$. In that context, note the compatibility condition $R(\widehat{A}) \subset R(\widehat{B})$ re-writes as $X_u \oplus R(A_s) \subset (D(A) \cap X_u) \oplus X_s$ that is true if and only if $D(A) \cap X_u = X_u$, that is $D(A) = X$ and $A \in \mathcal{L}(X)$. Here since A is closed the closed graph theorem implies that A is bounded.

As a consequence, the general persistence results of Potzsche in [20] cannot directly apply to study the persistence of the splitting for the class of linear unbounded operators.

In this work we propose to revisit the problem of persistence of exponential trichotomy for the class of operators described in Definition 1.2 by dealing with a direct proof based on Perron–Lyapunov fixed point argument for the perturbed semiflows and projectors. More specifically if $B \in \mathcal{L}(X)$ (with $\|B\|_{\mathcal{L}(X)}$ small enough) we aim at investigating the persistence of such the state space decomposition for a small bounded linear perturbation of an exponentially trichotomic closed linear operator $A : D(A) \subset X \rightarrow X$.

The main result of the manuscript is the following theorem.

Theorem 1.7 (Perturbation) *Let $A : D(A) \subset X \rightarrow X$ be a closed linear operator on a Banach space X , and assume that A has exponential trichotomy with exponents $\rho_0 < \rho$, constant κ and associated to the projectors $\{\Pi^\alpha\}_{\alpha=s,c,u}$ (see Remark 1.3). Then for each $B \in \mathcal{L}(X)$ with $\|B\|_{\mathcal{L}(X)}$ small enough the closed linear operator $(A + B) : D(A) \subset X \rightarrow X$*

has an exponential trichotomy, which corresponds to the following state space decomposition

$$X = \widehat{X}_s \oplus \widehat{X}_c \oplus \widehat{X}_u,$$

and which corresponds to the bounded linear projectors $\widehat{\Pi}_s, \widehat{\Pi}_c, \widehat{\Pi}_u \in \mathcal{L}(X)$ satisfying

$$\widehat{X}_k := \widehat{\Pi}_k(X), \quad \forall k = s, c, u,$$

and

$$\widehat{X}_c \oplus \widehat{X}_u = (I - \widehat{\Pi}_s)(X), \widehat{X}_s \oplus \widehat{X}_u = (I - \widehat{\Pi}_c)(X) \text{ and } \widehat{X}_s \oplus \widehat{X}_c = (I - \widehat{\Pi}_u)(X).$$

Moreover precisely, let three constants $\widehat{\rho}_0, \widehat{\rho} \in (0, +\infty)$ and $\widehat{\kappa}$ be given such that

$$0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho \text{ and } \widehat{\kappa} > \kappa.$$

There exists $\delta_0 = \delta_0(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa, \widehat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ if $\|B\|_{\mathcal{L}(X)} \leq \delta$, then $(A + B)$ has an exponential trichotomy with exponent $\widehat{\rho}_0$ and $\widehat{\rho}$ and with constant $\widehat{\kappa}$.

Moreover, the three associated projectors $\widehat{\Pi}_s, \widehat{\Pi}_c, \widehat{\Pi}_u \in \mathcal{L}(X)$ satisfy

$$\|\widehat{\Pi}_k - \Pi_k\|_{\mathcal{L}(X)} < \frac{\kappa\delta}{\delta_0 - \delta} \leq \delta_0 < \sqrt{2} - 1, \quad \forall k = s, c, u,$$

and as a consequence the subspace $\widehat{X}_k := \widehat{\Pi}_k(X)$ is isomorphic to the subspace $X_k = \Pi_k(X)$.

Furthermore the following estimates hold true for each $n \in \mathbb{N}$,

$$\begin{aligned} \|(A + B)_s^n \widehat{\Pi}_s - A_s^n \Pi_s\|_{\mathcal{L}(X)} &\leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \\ \|(A + B)_u^{-n} \widehat{\Pi}_u - A_u^{-n} \Pi_u\|_{\mathcal{L}(X)} &\leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \end{aligned}$$

and for each $n \in \mathbb{Z}$

$$\|(A + B)_c^n \widehat{\Pi}_c - A_c^n \Pi_c\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\widehat{\rho}_0|n|}.$$

In case of bounded linear operator the above result is a particular case of the result proved by Potzsche in [20] and by Pliss and Sell [18] using perturbation of exponential dichotomy for linear skew product semiflow and shifted operators. For the class of unbounded linear operator we consider in this work this result is new.

In addition, the above result has some consequence for non-autonomous discrete time linear equations by using Howland semigroup procedure to re-formulate such problems as autonomous systems.

In the next subsection we will state some consequences of Theorem 1.7. Section 2 is devoted to the proof of Theorem 1.7. Section 3 is concerned with the application of Theorem 1.7 for non-autonomous dynamical system (see Theorem 2.2). We also refer to Seydi [25] for further application in the context random dynamical systems and shadowing of normally hyperbolic dynamics.

2 Consequences of Theorem 1.7 for Discrete Time Non-autonomous Dynamical System

As mentioned above, exponential trichotomy or dichotomy play an important role in the study of the asymptotic behaviour of non-autonomous dynamical systems. Roughly speaking exponential trichotomy generalizes the usual spectral theory of linear semigroups to linear evolution operators. It ensures an invariant state space decomposition at each time into three sub-spaces: a stable, an unstable and central space in which the evolution operator has different behaviours. Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be a given sequence of bounded linear operators on the Banach space $(Y, \| \cdot \|)$. Consider the linear non-autonomous difference equation

$$x(n + 1) = A_n x(n), \text{ for } n \geq m, \quad x(m) = x_m \in Y. \tag{2.1}$$

Let us introduce the discrete evolution semigroup associated to \mathbf{A} defined as the 2-parameters linear operator on $\Delta_+ := \{(n, m) \in \mathbb{Z}^2 : n \geq m\}$ by

$$U_{\mathbf{A}}(n, m) := A_{n-1} \dots A_m, \quad \text{if } n > m, \text{ and } I_Y, \quad \text{if } n = m,$$

wherein I_Y denotes the identity operator in Y . In the following we will always use the notation $n \geq m$ as well as $U_{\mathbf{A}}(n, m)$ for the evolution semigroup. Whenever $U_{\mathbf{A}}(m, n)$ is considered this will mean that $U_{\mathbf{A}}(n, m)$ is invertible and

$$U_{\mathbf{A}}(m, n) = U_{\mathbf{A}}(n, m)^{-1}.$$

Let us observe that $U_{\mathbf{A}}$ satisfies

$$U_{\mathbf{A}}(n, k) U_{\mathbf{A}}(k, m) = U_{\mathbf{A}}(n, m) \quad \text{for each } n \geq k \geq m.$$

Then let us recall the following definition taken from Hale and Lin [13].

Definition 2.1 (*Exponential trichotomy*) Let $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be given. Then $U_{\mathbf{A}}$ has an **exponential trichotomy** (or \mathbf{A} is **exponentially trichotomic**) on \mathbb{Z} with constant κ , exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$, with $\alpha = u, s, c$ satisfying the following properties:

- (i) For all $n \in \mathbb{Z}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta, \text{ and } \Pi_n^s + \Pi_n^u + \Pi_n^c = I_Y.$$

- (ii) For all $n, m \in \mathbb{Z}$ with $n \geq m$ we have

$$U_{\mathbf{A}}^\alpha(n, m) := \Pi_n^\alpha U_{\mathbf{A}}(n, m) = U_{\mathbf{A}}(n, m) \Pi_m^\alpha, \text{ for } \alpha = u, s, c.$$

- (iii) $U_{\mathbf{A}}^\alpha(n, m)$ is invertible from $\Pi_m^\alpha(Y)$ into $\Pi_n^\alpha(Y)$ for all $n \geq m$ in \mathbb{Z} , $\alpha = u, c$ and its inverse is denoted by $U_{\mathbf{A}}^\alpha(m, n) : \Pi_n^\alpha(Y) \rightarrow \Pi_m^\alpha(Y)$.
- (iv) For each $y \in Y$ we have for all $n, m \in \mathbb{Z}$

$$\|U_{\mathbf{A}}^c(n, m) \Pi_m^c y\| \leq \kappa e^{\rho_0|n-m|} \|y\|, \tag{2.2}$$

and if $n \geq m$

$$\|U_{\mathbf{A}}^s(n, m) \Pi_m^s y\| \leq \kappa e^{-\rho(n-m)} \|y\|, \tag{2.3}$$

$$\|U_{\mathbf{A}}^u(m, n) \Pi_n^u y\| \leq \kappa e^{-\rho(n-m)} \|y\|. \tag{2.4}$$

Let us observe that the operators $U_{\mathbf{A}}^\alpha(n, p) \in \mathcal{L}(Y)$, for $n \geq p$ in \mathbb{Z} and $\alpha = u, s, c$ (resp. $U_{\mathbf{A}}^\alpha(p, n)$, for $n \geq p$ in \mathbb{Z} and $\alpha = u, c$) inherit the evolution property of $U_{\mathbf{A}}$ that reads as

$$U_{\mathbf{A}}^\alpha(n, p) U_{\mathbf{A}}^\alpha(p, m) = U_{\mathbf{A}}^\alpha(n, m), \quad \forall n \geq p \geq m \text{ in } \mathbb{Z} \text{ and } \alpha = u, s, c,$$

respectively

$$U_{\mathbf{A}}^\alpha(m, p) U_{\mathbf{A}}^\alpha(p, n) = U_{\mathbf{A}}^\alpha(m, n), \quad \forall n \geq p \geq m \text{ in } \mathbb{Z} \text{ and } \alpha = u, c.$$

Before stating our result, let us notice that since $\Pi_n^\alpha = U_{\mathbf{A}}^\alpha(n, n)$, for $\alpha = u, s, c$ and $n \in \mathbb{Z}$, property (iv) in Definition 2.1 implies that the projectors are uniformly bounded by the constant κ .

Using Howland’s semigroups like procedure (see Chicone and Latushkin [7]), as a consequence of Theorem 1.7, we obtain the following version for non-autonomous dynamical systems.

Theorem 2.2 (Perturbation) *Let $\mathbf{A} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ be given such that $U_{\mathbf{A}}$ has an exponential trichotomy on \mathbb{Z} with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three families of projectors $\{\Pi^\alpha : \mathbb{Z} \rightarrow \mathcal{L}(Y)\}_{\alpha=s,c,u}$. Let $\rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ and $\widehat{\kappa} > \kappa$ be given. Then there exists $\delta_0 := \delta_0(\rho_0, \widehat{\rho}_0, \widehat{\rho}, \rho, \kappa, \widehat{\kappa}) \in (0, \sqrt{2} - 1)$ such that for each $\delta \in (0, \frac{\delta_0^2}{\kappa + \delta_0})$ and each $\mathbf{B} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$ with*

$$\sup_{n \in \mathbb{Z}} \|B_n\| \leq \delta,$$

the evolution semigroup $U_{\mathbf{A}+\mathbf{B}}$ has an exponential trichotomy on \mathbb{Z} with constant $\widehat{\kappa}$, exponents $\widehat{\rho}, \widehat{\rho}_0$ and projectors $\{\widehat{\Pi}^\alpha\}_{\alpha=s,c,u}$. For each $n \in \mathbb{Z}$ and $\alpha = u, s, c$, the spaces $\mathcal{R}(\Pi_n^\alpha)$ and $\mathcal{R}(\widehat{\Pi}_n^\alpha)$ are isomorphic. Moreover the following perturbation estimates hold true: we have for all $n \geq p$,

$$\|U_{\mathbf{A}+\mathbf{B}}^s(n, p) - U_{\mathbf{A}}^s(n, p)\| \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho}(n-p)}, \tag{2.5}$$

$$\|U_{\mathbf{A}+\mathbf{B}}^u(p, n) - U_{\mathbf{A}}^u(p, n)\| \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho}(n-p)}, \tag{2.6}$$

and for all $(n, p) \in \mathbb{Z}$

$$\|U_{\mathbf{A}+\mathbf{B}}^c(n, p) - U_{\mathbf{A}}^c(n, p)\| \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{\widehat{\rho}_0|n-p|}. \tag{2.7}$$

The proof of the above result will be obtained as a consequence of Theorem 1.7 by using Howland procedure (we refer to the monograph of Chicone and Latushkin [7] and the references therein). To be more precise, let $q \in [1, \infty]$ be given and let us introduce the Banach space $X = l^q(\mathbb{Z}, Y)$. Let us consider the closed linear operator $\mathcal{A} : D(\mathcal{A}) \subset l^q(\mathbb{Z}; Y) \rightarrow l^q(\mathbb{Z}; Y)$ defined by

$$\begin{aligned} D(\mathcal{A}) &= \{u \in X : \{A_k u\}_{k \in \mathbb{Z}} \in X\}, \\ (\mathcal{A}u)_k &= A_{k-1} u_{k-1}, \quad \forall k \in \mathbb{Z}, \forall u \in l^q(\mathbb{Z}; Y). \end{aligned} \tag{2.8}$$

Then we will show if \mathbf{A} has an exponential trichotomy (according to Definition 2.1) then the linear operator \mathcal{A} also has an exponential trichotomy (see Definition 1.2). Note that Theorem 2.2 is not a new result (see Pötzsche [20] and the reference therein). However since we do not assume that the sequence \mathbf{A} is uniformly bounded, the Howland evolution operator is not bounded and therefore our proof of Theorem 2.2 is new.

Note that since operator $(\mathcal{A}, D(\mathcal{A}))$, one can apply Theorem 1.7 with general small perturbation $\mathcal{B} = (B_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \mathcal{L}(X)$ to obtain the persistence of exponential dichotomy or trichotomy for some advanced and retarded perturbation of (2.1) of the form:

$$x_{n+1} = A_n x_n + \sum_{j \in \mathbb{Z}} B_{n,j} x_j.$$

Of course, in such a setting the perturbed projectors $\widehat{\Pi}^\alpha$ does not a simple form but are “full” matrix operators.

To conclude this section, in view of Definition 1.2, we will introduce without any proof the new class of sequence of closed linear operators $\mathbf{A} = \{A_n\}_{n \in \mathbb{Z}}$ such that the corresponding Howland evolution operator has an exponential trichotomy and for which perturbation result (see Theorem 2.2) holds true. To do so let us consider for each $n \in \mathbb{Z}$ a closed linear operator $A_n : D(A_n) \subset Y \rightarrow Y$. We introduce the following definition:

Definition 2.3 We say that the sequence $\mathbf{A} = \{A_n : D(A_n) \subset Y \rightarrow Y\}_{n \in \mathbb{Z}}$. \mathbf{A} is **exponentially trichotomic** on \mathbb{Z} with constant κ , exponents $0 < \rho_0 < \rho$ if there exist three families of projectors $\Pi^\alpha = \{\Pi_n^\alpha\}_{n \in \mathbb{Z}} : \mathbb{Z} \rightarrow \mathcal{L}(Y)$, with $\alpha = u, s, c$ satisfying the following properties:

(i) For all $n \in \mathbb{Z}$ and $\alpha, \beta \in \{u, s, c\}$ we have

$$\Pi_n^\alpha \Pi_n^\beta = 0, \text{ if } \alpha \neq \beta, \text{ and } \Pi_n^s + \Pi_n^u + \Pi_n^c = I_Y.$$

- (ii) For all $n \in \mathbb{Z}$ we have $D(A_n) = \Pi_n^s(Y) \oplus \Pi_n^c(Y) \oplus (D(A_n) \cap \Pi_n^u(Y))$ and for each $\alpha = s, c, u$ $A_n(D(A_n) \cap \Pi_n^\alpha(Y)) \subset \Pi_{n+1}^\alpha(Y)$.
- (iii) For each $n \in \mathbb{Z}$ the operator $A_n^c := (A_n)|_{\Pi_n^c(Y)}$ is invertible from $\Pi_n^c(Y)$ onto $\Pi_{n+1}^c(Y)$ and the operator $A_n^u : D(A_n^u) \subset \Pi_n^u(Y) \rightarrow \Pi_{n+1}^u(Y)$ is invertible with $(A_n^u)^{-1} \in \mathcal{L}(\Pi_{n+1}^u(Y), \Pi_n^u(Y))$.

Now for each $n \geq m$ we set for $\alpha = s, c$:

$$U_{\mathbf{A}}^\alpha(n, m) := A_{n-1}^\alpha \dots A_m^\alpha \Pi_m^\alpha, \text{ if } n > m, \text{ and } I_Y \text{ if } n = m,$$

and for each $n \geq m$ we set for $\alpha = u, c$:

$$U_{\mathbf{A}}^\alpha(m, n) := (A_m^\alpha)^{-1} \dots (A_{n-1}^\alpha)^{-1} \Pi_n^\alpha, \text{ if } n > m, \text{ and } I_Y \text{ if } n = m.$$

(iv) For each $y \in Y$ we have for all $n, m \in \mathbb{Z}$

$$\|U_{\mathbf{A}}^c(n, m) \Pi_m^c y\| \leq \kappa e^{\rho_0|n-m|} \|y\|,$$

and if $n \geq m$

$$\begin{aligned} \|U_{\mathbf{A}}^s(n, m) \Pi_m^s y\| &\leq \kappa e^{-\rho(n-m)} \|y\|, \\ \|U_{\mathbf{A}}^u(m, n) \Pi_n^u y\| &\leq \kappa e^{-\rho(n-m)} \|y\|. \end{aligned}$$

Using the above definition, one can check that if $\mathbf{A} = \{A_n : D(A_n) \subset Y \rightarrow Y\}_{n \in \mathbb{Z}}$ is an exponentially trichotomic sequence of closed linear operators then the linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \rightarrow X$ defined by

$$\begin{cases} D(\mathcal{A}) = \{u \in X : u_k \in D(A_k), \forall k \in \mathbb{Z} \text{ and } (A_n u_n)_{n \in \mathbb{Z}} \in X\}, \\ (\mathcal{A}u)_k = A_{k-1} u_{k-1}, \quad k \in \mathbb{Z}, \quad u \in D(\mathcal{A}), \end{cases}$$

has an exponential trichotomy according to Definition 1.2.

3 Proof of Theorem 1.7

3.1 A Continuity Projector Lemma

The following lemma is inspired from [5, Lemma 4.1].

Lemma 3.1 *Let $\Pi : X \rightarrow X$ and $\widehat{\Pi} : X \rightarrow X$ be two bounded linear projectors on a Banach space X . Assume that*

$$\|\Pi - \widehat{\Pi}\|_{\mathcal{L}(X)} < \delta \text{ with } 0 < \delta < \sqrt{2} - 1. \tag{3.1}$$

Then Π is invertible from $\widehat{\Pi}(X)$ into $\Pi(X)$ and

$$\left\| \left(\Pi|_{\widehat{\Pi}(X)} \right)^{-1} x \right\| \leq \frac{1}{1 - \delta} \|x\|, \quad \forall x \in \Pi(X). \tag{3.2}$$

Remark 3.2 *By symmetry, the bounded linear projector $\widehat{\Pi}$ is also invertible from $\Pi(X)$ into $\widehat{\Pi}(X)$ and*

$$\left\| \left(\widehat{\Pi}|_{\Pi(X)} \right)^{-1} x \right\| \leq \frac{1}{1 - \delta} \|x\|, \quad \forall x \in \widehat{\Pi}(X) \tag{3.3}$$

Proof We will first prove two claims.

Claim 3.3 *If $\widehat{\Pi}\Pi$ is invertible from $\widehat{\Pi}(X)$ into $\widehat{\Pi}(X)$ then $\widehat{\Pi}$ is onto from $\Pi(X)$ into $\widehat{\Pi}(X)$.*

Proof of Claim 3.3 Let $y \in \widehat{\Pi}(X)$ be given. Since the map $\widehat{\Pi}\Pi$ is invertible on $\widehat{\Pi}(X)$, there exists a unique $x \in \widehat{\Pi}(X)$ such that $\widehat{\Pi}\Pi x = y$. Therefore by setting $\bar{x} = \Pi x \in \Pi(X)$ we have $\widehat{\Pi}\bar{x} = y$, which implies the surjectivity of $\widehat{\Pi}$ from $\Pi(X)$ onto $\widehat{\Pi}(X)$. \square

Claim 3.4 *If $\Pi\widehat{\Pi}$ is invertible from $\Pi(X)$ into $\Pi(X)$ then $\widehat{\Pi}$ is one to one from $\Pi(X)$ into $\widehat{\Pi}(X)$.*

Proof of Claim 3.4 Let $x \in \Pi(X)$ be given such that $\widehat{\Pi}x = 0$. Then we have $\Pi\widehat{\Pi}x = 0$. Since $\Pi\widehat{\Pi}$ is invertible from $\Pi(X)$ into $\Pi(X)$ we deduce that $x = 0$ and the claim follows. \square

Let us now prove that $\widehat{\Pi}\Pi$ is invertible from $\widehat{\Pi}(X)$ into $\widehat{\Pi}(X)$. First note that one has

$$\widehat{\Pi}\Pi = I - (I - \widehat{\Pi}\Pi).$$

Hence it is sufficient to prove that

$$\|I - \widehat{\Pi}\Pi\|_{\mathcal{L}(\widehat{\Pi}(X))} < 1. \tag{3.4}$$

Let $x \in \widehat{\Pi}(X)$ be given. Then we have

$$x - \widehat{\Pi}\Pi x = \widehat{\Pi}x - \widehat{\Pi}\Pi x = [\widehat{\Pi} - \Pi]x + [\widehat{\Pi} - \Pi]\Pi x.$$

Thus

$$\|x - \widehat{\Pi}\Pi x\| \leq \|\widehat{\Pi} - \Pi\|_{\mathcal{L}(X)} \|x\| + \|\widehat{\Pi} - \Pi\|_{\mathcal{L}(X)} \|\Pi x\|,$$

and

$$\|x - \widehat{\Pi}\Pi x\| \leq \delta \|x\| + \delta \|\Pi x\|.$$

Since $x \in \widehat{\Pi}(X)$ we have

$$\Pi x = \Pi x - x + x = \Pi x - \widehat{\Pi}x + x, \tag{3.5}$$

hence

$$\|\Pi x\| \leq \left[\|\Pi - \widehat{\Pi}\|_{\mathcal{L}(X)} + 1 \right] \|x\| \leq (1 + \delta) \|x\|.$$

Then we obtain

$$\|x - \widehat{\Pi}\Pi x\| \leq \delta (2 + \delta) \|x\|, \quad \forall x \in \widehat{\Pi}(X).$$

Recalling that $\delta \in (0, \sqrt{2} - 1)$, that reads $\delta (2 + \delta) < 1$, and we deduce that $\widehat{\Pi}\Pi$ is invertible from $\widehat{\Pi}(X)$ into $\widehat{\Pi}(X)$. By symmetry it follows that $\Pi\widehat{\Pi}$ is also invertible from $\Pi(X)$ into $\Pi(X)$.

To conclude the proof let us estimate the norm of the inverse of $\Pi|_{\widehat{\Pi}(X)}$. Let $x \in \widehat{\Pi}(X)$ be given. From (3.5) one has

$$\|\Pi x\| \geq \|x\| - \|\widehat{\Pi}x - \Pi x\| \geq \|x\| - \|\widehat{\Pi} - \Pi\|_{\mathcal{L}(X)} \|x\| \geq (1 - \delta) \|x\|,$$

and the result follows. □

3.2 Derivation of the Fixed Point Problem

In this section we shall derive a fixed point formulation for perturbed trichotomy. All the computations we will done for bounded linear operator. However one could remark that the formulations summarized in the lemma below makes sense for unbounded exponentially trichotomic linear operator as defined in Definition 1.2. This will be used to prove Theorem 1.7 for bounded perturbation of unbounded exponentially trichotomic linear operator.

Recall the discrete time variation of constant formula for bounded linear operators $A, B \in \mathcal{L}(X)$. We have

$$\begin{aligned} (A + B)^n &= A(A + B)^{n-1} + B(A + B)^{n-1} \\ &= A^2(A + B)^{n-2} + AB(A + B)^{n-2} + B(A + B)^{n-1} \end{aligned}$$

thus by induction

$$(A + B)^n = A^n + A^{n-1}B + \dots + AB(A + B)^{n-2} + B(A + B)^{n-1}, \tag{3.6}$$

so that for each $n \geq p$ we obtain

$$(A + B)^{n-p} = A^{n-p} + \sum_{m=p}^{n-1} A^{n-m-1}B(A + B)^{m-p}. \tag{3.7}$$

In the sequel and throughout this work we shall use the following summation convention:

$$\sum_n^m = 0 \quad \text{if } m < n.$$

This notational convention is similar to the one used by Vanderbauwhede in [26] who specified this using the symbol $\sum^{(+)}$.

Then using the above constant variation formula, one obtains the following fixed point formulation for a perturbed trichotomic semiflow in the bounded case:

Lemma 3.5 *Let $A \in \mathcal{L}(X)$ be given such that it has an exponential trichotomy with constant κ , exponents $0 < \rho_0 < \rho$ and associated to the three projectors $\Pi_k, k = s, c, u$. Let $B \in \mathcal{L}(X)$ be given such that $A+B$ has an exponential trichotomy with constant $\widehat{\kappa}$, exponents*

$0 < \widehat{\rho}_0 < \widehat{\rho}$ such that $\rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ and associated to the three projectors $\widehat{\Pi}_k, k = s, c, u$. Then one has for each $n \in \mathbb{N}$,

$$(A + B)_s^n \widehat{\Pi}_s = A_s^n \Pi_s \widehat{\Pi}_s + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \widehat{\Pi}_s - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B (A + B)_s^{n+m} \widehat{\Pi}_s, \tag{3.8}$$

$$(A + B)_u^{-n} \widehat{\Pi}_u = A_u^{-n} \Pi_u \widehat{\Pi}_u - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B (A + B)_u^{m-n} \widehat{\Pi}_u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B (A + B)_u^{-m-1-n} \widehat{\Pi}_u, \tag{3.9}$$

$$(A + B)_c^n \widehat{\Pi}_c = A_c^n \Pi_c \widehat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B (A + B)_c^m \widehat{\Pi}_c - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m+n} \widehat{\Pi}_c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1+n} \widehat{\Pi}_c. \tag{3.10}$$

$$(A + B)_c^{-n} \widehat{\Pi}_c = A_c^{-n} \Pi_c \widehat{\Pi}_c - \sum_{m=0}^{n-1} A_c^{-m-1} \Pi_c B (A + B)_c^{m-n} \widehat{\Pi}_c - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m-n} \widehat{\Pi}_c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1-n} \widehat{\Pi}_c. \tag{3.11}$$

$$\widehat{\Pi}_s = \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B [(A + B)_u^{-m-1} \widehat{\Pi}_u + (A + B)_c^{-m-1} \widehat{\Pi}_c] - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B (A + B)_s^m \widehat{\Pi}_s, \tag{3.12}$$

$$\widehat{\Pi}_u = \Pi_u + \sum_{m=0}^{+\infty} [A_c^m \Pi_c + A_s^m \Pi_s] B (A + B)_u^{-m-1} \widehat{\Pi}_u + \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B [(A + B)_s^m \widehat{\Pi}_s + (A + B)_c^m \widehat{\Pi}_c], \tag{3.13}$$

and

$$\widehat{\Pi}_c = I - \widehat{\Pi}_s - \widehat{\Pi}_u \tag{3.14}$$

$$\begin{aligned}
 &= \Pi_c - \sum_{m=0}^{+\infty} [A_c^m \Pi_c + A_s^m \Pi_s] B (A + B)_u^{-m-1} \widehat{\Pi}_s \\
 &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B [(A + B)_s^m \widehat{\Pi}_s + (A + B)_c^m \widehat{\Pi}_c] \\
 &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B [(A + B)_u^{-m-1} \widehat{\Pi}_u + (A + B)_c^{-m-1} \widehat{\Pi}_c] \\
 &\quad + \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B (A + B)_s^m \widehat{\Pi}_s
 \end{aligned}$$

Proof Derivation of formula (3.12) for $\widehat{\Pi}_s$: By applying $\widehat{\Pi}_s$ on the right side of (3.7) we obtain

$$(A + B)_s^{n-p} \widehat{\Pi}_s = A^{n-p} \widehat{\Pi}_s + \sum_{m=p}^{n-1} A^{n-m-1} B (A + B)_s^{m-p} \widehat{\Pi}_s, \quad \forall n \geq p. \tag{3.15}$$

By fixing $p = 0$ and by applying $A_u^{-n} \Pi_u$ on the left side of the above formula we obtain

$$A_u^{-n} \Pi_u (A + B)_s^n \widehat{\Pi}_s = \Pi_u \widehat{\Pi}_s + \sum_{m=0}^{n-1} A_u^{-m-1} \Pi_u B (A + B)_s^m \widehat{\Pi}_s. \tag{3.16}$$

Since for each $n \geq 0$ one has

$$\|A_u^{-n} \Pi_u\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho n} \|\Pi_u\|_{\mathcal{L}(X)} \quad \text{and} \quad \|(A + B)_s^n \widehat{\Pi}_s\|_{\mathcal{L}(X)} \leq \kappa e^{-\widehat{\rho} n} \|\widehat{\Pi}_s\|_{\mathcal{L}(X)},$$

by letting n goes to $+\infty$ in (3.16) it follows that

$$\Pi_u \widehat{\Pi}_s = - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_s^m \widehat{\Pi}_s. \tag{3.17}$$

By fixing $p = 0$ and by applying $A_c^{-n} \Pi_c$ (instead of $A_u^{-n} \Pi_u$) on the left side of (3.15) we obtain

$$\Pi_c \widehat{\Pi}_s = - \sum_{m=0}^{+\infty} A_c^{-m-1} \Pi_c B (A + B)_s^m \widehat{\Pi}_s. \tag{3.18}$$

Then combining (3.17) and (3.18) leads us to

$$\begin{aligned}
 \widehat{\Pi}_s &= \Pi_s \widehat{\Pi}_s + \Pi_u \widehat{\Pi}_s + \Pi_c \widehat{\Pi}_s \\
 &= \Pi_s \widehat{\Pi}_s - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B (A + B)_s^m \widehat{\Pi}_s.
 \end{aligned} \tag{3.19}$$

It thus remains to reformulate $\Pi_s \widehat{\Pi}_s$ by using

$$\Pi_s \widehat{\Pi}_s = \Pi_s [I - \widehat{\Pi}_u - \widehat{\Pi}_c].$$

Therefore we will compute $\Pi_s \widehat{\Pi}_u$ and $\Pi_s \widehat{\Pi}_c$.

Computation of $\Pi_s \widehat{\Pi}_u$: By applying $\widehat{\Pi}_u$ on the right side of (3.7) we have

$$(A + B)_u^{n-p} \widehat{\Pi}_u = A^{n-p} \widehat{\Pi}_u + \sum_{m=p}^{n-1} A^{n-m-1} B (A + B)_u^{m-p} \widehat{\Pi}_u, \quad \forall n \geq p. \tag{3.20}$$

By applying Π_s on the left side of the above formula we obtain

$$\Pi_s (A + B)_u^{n-p} \widehat{\Pi}_u = A_s^{n-p} \Pi_s \widehat{\Pi}_u + \sum_{m=p}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_u^{m-p} \widehat{\Pi}_u, \quad \forall n \geq p, \tag{3.21}$$

and by applying $(A + B)_u^{p-n} \widehat{\Pi}_u$ on the right side of (3.21) we have

$$\Pi_s \widehat{\Pi}_u = A_s^{n-p} \Pi_s (A + B)_u^{p-n} \widehat{\Pi}_u + \sum_{m=p}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_u^{m-n} \widehat{\Pi}_u, \quad \forall n \geq p, \tag{3.22}$$

and since

$$\left\| A_s^{n-p} \Pi_s \right\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)} \|\Pi_c\|_{\mathcal{L}(X)}$$

and

$$\left\| (A + B)_u^{p-n} \widehat{\Pi}_u \right\|_{\mathcal{L}(X)} \leq \kappa e^{-\widehat{\rho}(n-p)} \|\widehat{\Pi}_u\|_{\mathcal{L}(X)},$$

by taking the limit when p goes to $-\infty$ in (3.22) yields

$$\Pi_s \widehat{\Pi}_u = \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_u^{-m-1} \widehat{\Pi}_u. \tag{3.23}$$

Computation of $\Pi_s \widehat{\Pi}_c$: Starting from the equality

$$(A + B)_c^{n-p} \widehat{\Pi}_c = A^{n-p} \widehat{\Pi}_c + \sum_{m=p}^{n-1} A^{n-m-1} B (A + B)_c^{m-p} \widehat{\Pi}_c, \quad \forall n \geq p.$$

and applying $(A + B)_c^{p-n} \widehat{\Pi}_u$ on the right side of this formula we obtain for each $n \geq p$

$$\Pi_s \widehat{\Pi}_c = A_s^{n-p} \Pi_s (A + B)_c^{p-n} \widehat{\Pi}_u + \sum_{k=p}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_c^{m-n} \widehat{\Pi}_c, \quad \forall n \geq p. \tag{3.24}$$

and since

$$\left\| A_s^{n-p} \Pi_s \right\|_{\mathcal{L}(X)} \leq \kappa e^{-\rho(n-p)} \|\Pi_c\|_{\mathcal{L}(X)},$$

and

$$\left\| (A + B)_u^{p-n} \widehat{\Pi}_u \right\|_{\mathcal{L}(X)} \leq \kappa e^{\widehat{\rho}_0(n-p)} \|\widehat{\Pi}_u\|_{\mathcal{L}(X)},$$

with $\widehat{\rho}_0 < \rho$, by letting p goes to $-\infty$ into (3.24) we derive

$$\Pi_s \widehat{\Pi}_c = \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1} \widehat{\Pi}_c. \tag{3.25}$$

Computation of $\Pi_s \widehat{\Pi}_s$: By summing (3.23) and (3.25) it follows that

$$\Pi_s [\widehat{\Pi}_c + \widehat{\Pi}_u] = \sum_{m=0}^{+\infty} A_s^m \Pi_s B [(A + B)_c^{-m-1} \widehat{\Pi}_c + (A + B)_u^{-m-1} \widehat{\Pi}_u], \tag{3.26}$$

and since $\widehat{\Pi}_c + \widehat{\Pi}_u = I - \widehat{\Pi}_s$ it follows that

$$\Pi_s \widehat{\Pi}_s = \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B [(A + B)_c^{-m-1} \widehat{\Pi}_c + (A + B)_u^{-m-1} \widehat{\Pi}_u]. \tag{3.27}$$

Finally the expression of $\widehat{\Pi}_s$ in (3.12) follows by combining (3.19) and (3.27).

Computation of $\widehat{\Pi}_u$ and $\widehat{\Pi}_c$: The derivation of the formula (3.13) for $\widehat{\Pi}_u$ uses the same arguments as for $\widehat{\Pi}_s$. The formula (3.14) for $\widehat{\Pi}_c$ is obtained by using $\widehat{\Pi}_c = I - \widehat{\Pi}_s - \widehat{\Pi}_u$.

Computation of $(A + B)_s^n \widehat{\Pi}_s$: Next we derive (3.8). By applying $(A + B)_s^n \widehat{\Pi}_s$ on the right side of (3.12) we obtain

$$(A + B)_s^n \widehat{\Pi}_s = \Pi_s (A + B)_s^n \widehat{\Pi}_s - \sum_{m=0}^{+\infty} [A_u^{-m-1} + A_c^{-m-1}] B (A + B)_s^{m+n} \widehat{\Pi}_s. \tag{3.28}$$

In order to determine $\Pi_s (A + B)_s^n \widehat{\Pi}_s$, we apply Π_s on the left side of (3.15) and we obtain

$$\Pi_s (A + B)_s^n \widehat{\Pi}_s = A_s^{n-p} \Pi_s \widehat{\Pi}_s + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \widehat{\Pi}_s, \quad \forall n \in \mathbb{N}, \tag{3.29}$$

and (3.8) follows.

Computation of $(A + B)_c^n \widehat{\Pi}_c$ for $n \geq 0$: By applying $(A + B)_c^n \widehat{\Pi}_c$ on the right side of (3.14) we obtain for each $n \in \mathbb{N}$

$$\begin{aligned} (A + B)_c^n \widehat{\Pi}_c &= \Pi_c (A + B)_c^n \widehat{\Pi}_c - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m+n} \widehat{\Pi}_c \\ &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1+n} \widehat{\Pi}_c. \end{aligned} \tag{3.30}$$

Next we compute $\Pi_c (A + B)_c^n \widehat{\Pi}_c$. By using the variation of constant formula (3.7) with $p = 0$, and applying $\widehat{\Pi}_c$ on the right side and Π_c on the left side we obtain

$$\Pi_c (A + B)_c^n \widehat{\Pi}_c = A_c^n \Pi_c \widehat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B (A + B)_c^m \widehat{\Pi}_c, \tag{3.31}$$

and (3.10) follows.

Computation of $(A + B)_c^n \widehat{\Pi}_c$ for $n \leq 0$: By applying $(A + B)_c^{-n} \widehat{\Pi}_c$ on the right side of (3.14) we obtain

$$\begin{aligned} (A + B)_c^{-n} \widehat{\Pi}_c &= \Pi_c (A + B)_c^{-n} \widehat{\Pi}_c \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B (A + B)_c^{m-n} \widehat{\Pi}_c \\ &\quad + \sum_{m=0}^{+\infty} A_s^m \Pi_s B (A + B)_c^{-m-1-n} \widehat{\Pi}_c. \end{aligned} \tag{3.32}$$

Next we compute $\Pi_c (A + B)_c^{-n} \widehat{\Pi}_c$. By applying $(A + B)_c^{-n} \widehat{\Pi}_c$ on the right side of the variation of constant formula (3.7) (with $p = 0$) we obtain

$$\widehat{\Pi}_c = A^n (A + B)_c^{-n} \widehat{\Pi}_c + \sum_{m=0}^{n-1} A^{n-m-1} B (A + B)_c^{m-n} \widehat{\Pi}_c. \tag{3.33}$$

By applying $A_c^{-n} \Pi_c$ on the left side of the above formula we get

$$A_c^{-n} \Pi_c \widehat{\Pi}_c = \Pi_c (A + B)_c^{-n} \widehat{\Pi}_c + \sum_{m=0}^{n-1} A_c^{-m-1} \Pi_c B (A + B)_c^{m-n} \widehat{\Pi}_c, \tag{3.34}$$

and the result follows. □

3.3 Abstract Reformulation of the Fixed Point Problem

In this section we reformulate the fixed point problem (3.8)–(3.13) by using an abstract fixed point formulation.

Let $\eta > 0$ be given. Define

$$\mathbb{L}_{-\eta}(\mathbb{N}, \mathcal{L}(X)) := \left\{ \mathbf{u} : \mathbb{N} \rightarrow \mathcal{L}(X) : \sup_{n \in \mathbb{N}} e^{\eta n} \|u_n\| < +\infty \right\},$$

which is a Banach space endowed with the norm

$$\|\mathbf{u}\|_{\mathbb{L}_{-\eta}} := \sup_{n \in \mathbb{N}} e^{\eta n} \|u_n\|.$$

Define

$$\mathbb{L}_{\eta}(\mathbb{Z}, \mathcal{L}(X)) := \left\{ \mathbf{v} : \mathbb{Z} \rightarrow \mathcal{L}(X) : \sup_{n \in \mathbb{Z}} e^{-\eta|n|} \|v_n\|_{\mathcal{L}(X)} < +\infty \right\}$$

which is a Banach space endowed with the norm

$$\|\mathbf{v}\|_{\mathbb{L}_{\eta}} := \sup_{n \in \mathbb{Z}} e^{-\eta|n|} \|v_n\|_{\mathcal{L}(X)}.$$

Consider S_{\pm} the shift operators on $\mathbb{L}_{\pm\eta}(\mathbb{N}, \mathcal{L}(X))$

$$S_{-}(\mathbf{u})_n = u_{n+1} \text{ whenever } n \in \mathbb{Z} \text{ or } n \in \mathbb{N}.$$

Let $C \in \mathcal{L}(X)$. In the following we will use the linear operators

$$\Phi_C(\mathbf{u})_n = \sum_{m=0}^{n-1} C^m u_{n-1-m}, \text{ and } \Theta_C(\mathbf{u})_n = \sum_{m=0}^{+\infty} C^m u_{n+m}.$$

Reformulation of Eq. (3.8) on \widehat{X}_s : Set for each $n \in \mathbb{N}$: $E_n^s := (A + B)_s^n \widehat{\Pi}_s$. We require $E^s \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, where $\widehat{\rho}$ is the constant introduced in Theorem 1.7. Consider the linear operators $\Phi_s, \Theta_{cu} : \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ defined by

$$\Phi_s = \Phi_{A_s \Pi_s} \text{ and } \Theta_{cu} = \Theta_{(A_c^{-1} \Pi_c + A_u^{-1} \Pi_u)}.$$

We observe that

$$\Phi_s \circ B \circ (E^s)_n = \sum_{l=0}^{n-1} A_s^l \Pi_s B E_n^s (A + B)_s^{n-1-l} \widehat{\Pi}_s = \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B (A + B)_s^m \widehat{\Pi}_s$$

therefore the Eq. (3.8) can be rewritten for $n \in \mathbb{N}$ as

$$E_n^s = A_s^n \Pi_s \widehat{\Pi}_s + \Phi_s \circ B \circ (E^s)_n - \Theta_{cu}((A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) B E^s)_n. \tag{3.35}$$

In order to solve the fixed point problem we will use the following lemma.

Lemma 3.6 *The operators Φ_s and Θ_{cu} map $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_s(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \frac{\kappa e^{\widehat{\rho}}}{1 - e^{\widehat{\rho}-\rho}} \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_{cu}(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \left[\frac{\kappa}{1 - e^{\rho_0 - \widehat{\rho}}} + \frac{\kappa}{1 - e^{-(\rho + \widehat{\rho})}} \right] \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

Reformulation of Eq. (3.9) on \widehat{X}_u : Set for each $n \in \mathbb{N}$: $E_n^u := (A + B)_u^{-n} \widehat{\Pi}_u$. We require $E^u \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$, where $\widehat{\rho}$ is the constant introduced in Theorem 1.7. Consider the linear operators $\Phi_u, \Theta_{sc} : \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$

$$\Phi_u := \Phi_{A_u^{-1} \Pi_u} \text{ and } \Theta_{sc} := \Theta_{(A_s \Pi_s + A_c \Pi_c)}.$$

We observe that

$$\begin{aligned} \Phi_u \circ (A_u^{-1} \Pi_u B) \circ S_-(E^u)_n &= \sum_{m=0}^{n-1} A_u^{-m} \Pi_u (A_u^{-1} \Pi_u B) E_{n-m}^u \\ &= \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B (A + B)_u^{m-n} \widehat{\Pi}_u \end{aligned}$$

therefore Eq. (3.9) can be rewritten for each $n \in \mathbb{N}$ as

$$E_n^u = A_u^{-n} \Pi_u \widehat{\Pi}_u - \Phi_u \circ (A_u^{-1} \Pi_u B) \circ S_-(E^u)_n + \Theta_{sc} \circ B \circ S_-(E^u)_n. \tag{3.36}$$

Lemma 3.7 *The operators Φ_u and Θ_{sc} map $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_u(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \frac{\kappa e^{\widehat{\rho}}}{1 - e^{\widehat{\rho} - \rho}} \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_{sc}(\mathbf{u})\|_{\mathbb{L}_{-\widehat{\rho}}} \leq \left[\frac{\kappa}{1 - e^{-(\rho + \widehat{\rho})}} + \frac{\kappa}{1 - e^{\rho_0 - \widehat{\rho}}} \right] \|\mathbf{u}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{u} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

Reformulation of Eq. (3.10)–(3.11) on \widehat{X}_c : Set for each $n \in \mathbb{Z}$: $E_n^c := (A + B)_c^n \widehat{\Pi}_c$. We require $E^c \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. Define the linear operators

$$\Phi_c(\mathbf{u})_n := \begin{cases} \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c u_m, & \text{if } n \geq 0 \\ -\sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c u_{m+n}, & \text{if } n \leq 0 \end{cases}$$

and

$$\Theta_{su}(\mathbf{u})_n := -\sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u u_{m+n} + \sum_{m=0}^{+\infty} A_s^m \Pi_s u_{n-1-m}, \quad \text{for } n \in \mathbb{Z},$$

therefore Eqs. (3.10)–(3.11) can be rewritten for each $n \in \mathbb{Z}$ as

$$E_n^c := A_c^n \Pi_c [I - (\widehat{\Pi}_s + \widehat{\Pi}_u)] + \Phi_c(BE^c)_n + \Theta_{su}(BE^c)_n. \tag{3.37}$$

Lemma 3.8 *The operators Φ_c and Θ_{su} map $\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ into itself and are bounded linear operators on $\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$. More precisely we have*

$$\|\Phi_c(\mathbf{v})\|_{\mathbb{L}_{\widehat{\rho}_0}} \leq \frac{\kappa}{1 - e^{\rho_0 - \widehat{\rho}_0}} \|\mathbf{v}\|_{\mathbb{L}_{\widehat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)),$$

and

$$\|\Theta_{su}(\mathbf{v})\|_{\mathbb{L}_{\widehat{\rho}_0}} \leq \left[\frac{\kappa e^{\widehat{\rho}}}{1 - e^{\widehat{\rho}_0 - \widehat{\rho}}} + \frac{\kappa e^{-\widehat{\rho}}}{1 - e^{\widehat{\rho}_0 - \widehat{\rho}}} \right] \|\mathbf{v}\|_{\mathbb{L}_{\widehat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)).$$

Reformulation of Eqs. (3.12)–(3.13) for the projectors on \widehat{X}_s and \widehat{X}_u : Define the linear operator

$$\Theta_s(u) := \Theta_{A_s \Pi_s}(u)_0 = \sum_{m=0}^{+\infty} A_s^m \Pi_s u_m$$

then Eq. (3.12) becomes

$$\widehat{\Pi}_s = \Pi_s - \Theta_s \circ B \circ S_-(E^u + \chi_-(E^c)) + \Theta_{cu}((A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) B E^s)_0 \quad (3.38)$$

where $\chi_- : \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ is defined by

$$\chi_-(E^c)_n = E^c_{-n} \text{ for } n \geq 0.$$

Define

$$\Theta_u(u) := \Theta_{A_u^{-1} \Pi_u}(u)_0 = \sum_{m=0}^{+\infty} A_u^{-m} \Pi_u u_m,$$

and (3.13) re-writes as:

$$\widehat{\Pi}_u = \Pi_u + \Theta_{sc} \circ B \circ S_-(E^u)_0 + \Theta_u(A_u^{-1} \Pi_u B(E^s + \chi_+(E^c))), \quad (3.39)$$

where $\chi_+ : \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)) \rightarrow \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ is defined by

$$\chi_+(E^c)_n := E^c_n \text{ for } n \geq 0.$$

Lemma 3.9 *The operators Θ_s and Θ_u have the following properties:*

(i) Θ_s and Θ_u map $\mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ into $\mathcal{L}(X)$ with

$$\|\Theta_s(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{\widehat{\rho}_0 - \widehat{\rho}}} \|\mathbf{v}\|_{\mathbb{L}_{\widehat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X)),$$

and

$$\|\Theta_u(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{\widehat{\rho}_0 - \widehat{\rho}}} \|\mathbf{v}\|_{\mathbb{L}_{\widehat{\rho}_0}}, \quad \forall \mathbf{v} \in \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X));$$

(ii) Θ_s and Θ_u map $\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \subset \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$ into $\mathcal{L}(X)$ with

$$\|\Theta_s(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{-\widehat{\rho} - \widehat{\rho}}} \|\mathbf{v}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{v} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)),$$

$$\|\Theta_u(\mathbf{v})\|_{\mathcal{L}(X)} \leq \frac{\kappa}{1 - e^{-\widehat{\rho} - \widehat{\rho}}} \|\mathbf{v}\|_{\mathbb{L}_{-\widehat{\rho}}}, \quad \forall \mathbf{v} \in \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)).$$

By using the expressions of $\widehat{\Pi}_s$ and $\widehat{\Pi}_u$ obtained in (3.38) and (3.39), and by replacing those expressions into $A_s^n \Pi_s \widehat{\Pi}_s$ (respectively into $A_u^{-n} \Pi_u \widehat{\Pi}_u$ and $A_c^n \Pi_c [I - (\widehat{\Pi}_s + \widehat{\Pi}_u)]$) in Eq. (3.35) [(respectively in (3.36) and (3.37)] we will derive a new fixed point problem only for E^s , E^u and E^c given explicitly as follow for each $n \in \mathbb{N}$:

$$\begin{aligned} E_n^s &= A_s^n \Pi_s - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [E_{m+1}^u + E_{-m-1}^c] \\ &+ \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s, \end{aligned} \quad (3.40)$$

$$\begin{aligned}
 E_n^u &= A_u^{-n} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s + E_m^c] \\
 &\quad - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+m+1}^u, \tag{3.41}
 \end{aligned}$$

and for each $n \in \mathbb{Z}$:

$$\begin{aligned}
 E_n^c &= A_c^n \Pi_c - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B E_{m+1}^u + \sum_{m=0}^{+\infty} A_c^{-m-1} \Pi_c B E_m^s \\
 &\quad + \sum_{m=0}^{n-1} A_c^{-m-1} \Pi_c B E_m^c - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c \\
 &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^c. \tag{3.42}
 \end{aligned}$$

Moreover with this notation the explicit formulas for $\widehat{\Pi}_k$, $k = s, c, u$ reads as

$$\begin{aligned}
 \widehat{\Pi}_s &= \Pi_s - \sum_{m=0}^{+\infty} A_s^m \Pi_s B [E_{m+1}^u + E_{-m-1}^c] - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_m^s, \\
 \widehat{\Pi}_u &= \Pi_u + \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B [E_m^s + E_m^c] + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] E_{m+1}^u, \\
 \widehat{\Pi}_c &= \Pi_c - \sum_{m=0}^{+\infty} A_c^m \Pi_c B E_{m+1}^u + \sum_{m=0}^{+\infty} A_c^{-m-1} \Pi_c B E_m^s \\
 &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_m^c + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1}^c.
 \end{aligned}$$

Observe that we have the following relation $E_0^k = \widehat{\Pi}_k$ for each $k = s, c, u$, as well as the following identity

$$E_0^s + E_0^c + E_0^u = \widehat{\Pi}_s + \widehat{\Pi}_c + \widehat{\Pi}_u = I.$$

Furthermore the system (3.40)–(3.42) also re-writes as the following compact form

$$\begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} = \begin{pmatrix} A_s^+ \Pi_s \\ A_u^- \Pi_u \\ A_c^- \Pi_c \end{pmatrix} + \mathcal{J} \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix}, \text{ and } \mathcal{J} := (\mathcal{J}_{ij})_{1 \leq i, j \leq 3} \tag{3.43}$$

wherein the linear operators $\{\mathcal{J}_{ij}\}_{1 \leq i, j \leq 3}$ are given by

$$\begin{aligned}
 \mathcal{J}_{11} &:= A_s^+ \Pi_s \circ \Theta_{cu} (\cdot)_0 \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B + \Phi_s \circ B \\
 &\quad - \Theta_{cu} \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B \\
 \mathcal{J}_{12} &:= -A_s^+ \Pi_s \circ \Theta_s \circ B \circ S_- \\
 \mathcal{J}_{13} &:= -A_s^+ \Pi_s \circ \Theta_s \circ B \circ S_- \circ \chi_- \\
 \mathcal{J}_{21} &:= A_u^- \Pi_u \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \\
 \mathcal{J}_{22} &:= A_u^- \Pi_u \circ \Theta_{sc} (\cdot)_0 \circ B \circ S_- - \Phi_u \circ A_u^{-1} \Pi_u \circ B \circ S_-
 \end{aligned}$$

$$\begin{aligned}
 &+ \Theta_{sc} \circ B \circ S_- \\
 \mathcal{J}_{23} &:= A_u^- \Pi_u \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \circ \chi_+ \\
 \mathcal{J}_{31} &:= -A_c \Pi_c \circ \Theta_{cu}(\cdot)_0 \circ (A_u^{-1} \Pi_u + A_c^{-1} \Pi_c) \circ B \\
 &\quad - A_c \Pi_c \circ \Theta_u(\cdot) \circ A_u^{-1} \Pi_u \circ B \\
 \mathcal{J}_{32} &:= A_c \Pi_c \circ \Theta_s \circ B \circ S_- - A_c \Pi_c \circ \Theta_{sc}(\cdot)_0 \circ B \circ S_- \\
 \mathcal{J}_{33} &:= A_c \Pi_c \circ \Theta_s \circ B \circ S_- \circ \chi_- \\
 &\quad A_c \Pi_c \circ \Theta_u \circ A_u^{-1} \Pi_u \circ B \circ \chi_+ + \Phi_c \circ B + \Theta_{su} \circ B.
 \end{aligned}$$

In the sequel we define the Banach space

$$\mathcal{X} = \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)),$$

endowed with the usual product norm:

$$\|\mathbf{Z}\|_{\mathcal{X}} = \max \left\{ \|E^s\|_{\mathbb{L}_{-\widehat{\rho}}}, \|E^u\|_{\mathbb{L}_{-\widehat{\rho}}}, \|E^c\|_{\mathbb{L}_{\widehat{\rho}_0}} \right\}, \quad \forall \mathbf{Z} = (E^s, E^u, E^c)^T \in \mathcal{X}.$$

The following lemma holds true:

Lemma 3.10 *Let $A : D(A) \subset X \rightarrow X$ be a closed linear operator and let us assume that the conditions of Theorem 1.7 are satisfied. Then the linear operator \mathcal{J} defined in (3.43) satisfies $\mathcal{J} \in \mathcal{L}(\mathcal{X})$. More precisely there exists some constant $C := C(\kappa, \rho, \rho_0, \widehat{\rho}, \widehat{\rho}_0)$ such that*

$$\|\mathcal{J}(\mathbf{Z})\|_{\mathcal{X}} \leq C \|B\|_{\mathcal{L}(X)} \|\mathbf{Z}\|_{\mathcal{X}}, \quad \forall \mathbf{Z} \in \mathcal{X}.$$

Proof Let us notice that χ_+ and χ_- are bounded linear operator defined from $\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ into $\mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))$. Furthermore we have

$$\|\chi_+\|_{\mathcal{L}(\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)), \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))} \leq 1, \tag{3.44}$$

and

$$\|\chi_-\|_{\mathcal{L}(\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)), \mathbb{L}_{\widehat{\rho}_0}(\mathbb{N}, \mathcal{L}(X))} \leq 1. \tag{3.45}$$

We also note that $S_- \in \mathcal{L}(\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)))$ and $S_- \in \mathcal{L}(\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)))$ with

$$\|S_-\|_{\mathcal{L}(\mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X)))} \leq e^{\widehat{\rho}_0} \text{ and } \|S_-\|_{\mathcal{L}(\mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)))} \leq 1. \tag{3.46}$$

Therefore by combining (3.44)–(3.46) together with lemmas (3.6)–(3.9) the result follows easily by simple computations. \square

As a consequence of the above lemma we obtain the following result:

Proposition 3.11 *Let $A : D(A) \subset X \rightarrow X$ be given such that the conditions of Theorem 1.7 are satisfied. Then there exists $\delta_0 := \delta_0(\kappa, \rho, \rho_0, \widehat{\rho}, \widehat{\rho}_0) \in (0, C^{-1})$ such that for each $\delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$ and each $B \in \mathcal{L}(X)$ with $\|B\|_{\mathcal{L}(X)} \leq \delta$, there exists a unique $\mathbf{Z} = (E^s, E^u, E^c)^T \in \mathcal{X}$ such that (3.43) [or equivalently (3.40)–(3.42)] holds true. Moreover we have the following properties*

(i) For each $n \in \mathbb{N}$

$$\|E_n^s\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta_0}{\delta_0 - \delta} e^{-\widehat{\rho}n} \text{ and } \|E_n^u\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta_0}{\delta_0 - \delta} e^{-\widehat{\rho}n},$$

and for each $n \in \mathbb{Z}$

$$\|E_n^c\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta_0}{\delta_0 - \delta} e^{\widehat{\rho}_0 |n|}.$$

(ii) The following estimates hold:

$$\|E_n^s - A_s^n \Pi_s\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \quad n \in \mathbb{N},$$

$$\|E_n^u - A_u^{-n} \Pi_u\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{-\widehat{\rho} n}, \quad n \in \mathbb{N},$$

and

$$\|E_n^c - A_c^n \Pi_c\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} e^{\widehat{\rho}_0 |n|}, \quad n \in \mathbb{Z}.$$

(iii) One has

$$E_n^s(X) \subset D(A), \quad \forall n \in \mathbb{N},$$

$$E_n^c(X) \subset D(A), \quad \forall n \in \mathbb{Z},$$

$$E_n^u(X) \subset D(A), \quad \forall n \geq 1 \text{ and } \widehat{\Pi}_u(D(A)) \subset D(A).$$

In particular one has $\widehat{\Pi}_k(X) \subset D(A)$ for $k = s, c$ and for each $n \geq 0$ $(A + B) \circ E_n^s \in \mathcal{L}(X)$ while for each $n \in \mathbb{Z}$, $(A + B) \circ E_n^c \in \mathcal{L}(X)$.

Proof Let $\delta_0 \in (0, C^{-1})$ be given. Assume that

$$\|B\|_{\mathcal{L}(X)} \leq \delta \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right). \tag{3.47}$$

Then since $\frac{\delta_0^2}{\kappa + \delta_0} \leq \delta_0$, the existence and the uniqueness of a fixed point of (3.43) [or equivalently (3.40)–(3.42)] follows from Lemma 3.10.

In the sequel of this proof we denote by $\mathbf{Z}_0 = (A_s \Pi_s, A_u^{-1} \Pi_u, A_c \Pi_c)^T \in \mathcal{X}$ the fixed point of \mathcal{J} with $B = 0$. In order to obtain the properties (i) and (ii) we will make use of (3.43). First of all since $\mathbf{Z}_0 \in \mathcal{X}$, let us observe that using (1.7)–(1.9) we obtain

$$\|\mathbf{Z}_0\|_{\mathcal{X}} \leq \kappa. \tag{3.48}$$

Proof of (i): By using the fixed point problem (3.43) combined together with Lemma 3.10 and (3.47) we obtain (recalling the notation $\mathbf{Z} = (E^s, E^u, E^c)^T$) that

$$\|\mathbf{Z}\|_{\mathcal{X}} \leq \kappa + C\delta \|\mathbf{Z}\|_{\mathcal{X}},$$

so that (i) follows from the estimate:

$$\|\mathbf{Z}\|_{\mathcal{X}} \leq \frac{\kappa}{1 - C\delta} \leq \frac{\kappa \delta_0}{\delta_0 - \delta}. \tag{3.49}$$

Proof of (ii): By using the fixed point problem (3.43) combined together with Lemma 3.10 and (3.47) we obtain that

$$\|\mathbf{Z} - \mathbf{Z}_0\|_{\mathcal{X}} \leq C\delta \|\mathbf{Z}\|_{\mathcal{X}}, \tag{3.50}$$

so that plugging (3.49) into (3.50) yields

$$\|\mathbf{Z} - \mathbf{Z}_0\| \leq \frac{\kappa C\delta}{1 - C\delta} \leq \frac{\kappa \delta}{\delta_0 - \delta}.$$

This prove (ii).

Proof of (iii): Let us recall that since $(A, D(A))$ is exponentially trichotomic (see Definition 1.2) then one has $D(A) = X_s \oplus X_c \oplus (D(A) \cap X_u)$. Hence the result directly follows from right-hand side of (3.40)–(3.42). The boundedness of $(A + B) \circ E_n^s$ and $(A + B) \circ E_n^c$ follows from the closed graph theorem since A is closed and B bounded. \square

3.4 Regularized Semigroup Property and Orthogonality Property

Definition 3.12 A family of bounded linear operators $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is a discrete time **regularized semigroup** if

$$W_n W_p = W_{n+p}, \quad \forall n, p \in \mathbb{N}. \tag{3.51}$$

Remark 3.13 If $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is a regularized semigroup then W_0 is a bounded linear projector on X .

Remark 3.14 Observe that if $\{W_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X)$ is discrete time regularized semigroup then by setting $C := W_0$, and using (3.51) we obtain $W_n W_p = C W_{n+p}$ for all $n, p \geq 0$. These properties correspond to the notion of **C-regularized semigroup** given in [12, Definition 3.1, p. 13] for discrete time.

In the next lemmas we will show that $\{E_n^k\}_{n \in \mathbb{N}}$, $k = s, c, u$, are regularized semigroup and that we have the orthogonality property namely for each $n \in \mathbb{N}$

$$E_n^k E_n^l = 0_{\mathcal{L}(X)} \text{ if } k, l = s, c, u \text{ with } k \neq l.$$

The latter equality will allows us to obtain that the bounded linear projectors $\widehat{\Pi}_k = E_0^k$, $k = s, c, u$ satisfy the orthogonality property

$$\widehat{\Pi}_k \widehat{\Pi}_l = 0_{\mathcal{L}(X)} \text{ if } k, l = s, c, u \text{ with } k \neq l.$$

Lemma 3.15 *Let the conditions of Theorem 1.7 be satisfied. If*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \quad \text{with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right) \tag{3.52}$$

where δ_0 is given in Proposition 3.11 then the following properties hold:

- (i) for each $n, p \in \mathbb{N}$ we have $E_n^u E_p^u = E_{n+p}^u$ and $E_n^s E_p^u = 0_{\mathcal{L}(X)}$, while for each $n \in \mathbb{Z}$, $p \in \mathbb{N}$ we have $E_n^c E_p^u = 0_{\mathcal{L}(X)}$.
- (ii) $\widehat{\Pi}_u \in \mathcal{L}(X)$ is a projector on X and for each $n \geq 0$ one has $E_n^u(X) \subset \widehat{\Pi}_u(X)$.

Proof First of all let us notice that since we have $\widehat{\Pi}_u = E_0^u$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (ii).

The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed)

$$\left\{ E_n^u E_p^u - E_{n+p}^u \right\}_{n \in \mathbb{N}}, \left\{ E_n^s E_p^u \right\}_{n \in \mathbb{N}} \text{ and } \left\{ E_n^c E_p^u \right\}_{n \in \mathbb{Z}}.$$

Let $p \in \mathbb{N}$ be given and fixed. Then observe that

$$\mathbf{W} := \left(E_n^s E_p^u, E_n^u E_p^u - E_{n+p}^u, E_n^c E_p^u \right)^T \in \mathcal{X}.$$

Equation for $\{E_n^u E_p^u - E_{n+p}^u\}_{n \in \mathbb{N}}$: Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$ be given. Multiplying the right side of E_n^u given in (3.41) by E_p^u leads us to

$$E_n^u E_p^u = A_u^{-n} \Pi_u E_p^u + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^u + E_m^c E_p^u] - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+m+1}^u E_p^u. \tag{3.53}$$

Next by using also (3.41) and replacing n with $n + p$ we obtain

$$E_{n+p}^u = A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] - \sum_{m=0}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B E_{n+p+m+1}^u. \tag{3.54}$$

Therefore by subtracting (3.54) from (3.53) we get

$$E_n^u E_p^u - E_{n+p}^u = A_u^{-n} \Pi_u E_p^u - A_u^{-n-p} \Pi_u - \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] + \sum_{m=0}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u - \sum_{m=0}^{n-1} [A_u^{-m-1} \Pi_u] B E_{n-m}^u E_p^u + \sum_{m=0}^{+\infty} A_u^{-n-m-1} \Pi_u B [E_m^s E_p^u + E_m^c E_p^u] + \sum_{m=0}^{+\infty} [A_s^m \Pi_s + A_c^m \Pi_c] B [E_{n+m+1}^u E_p^u - E_{n+p-m}^u]. \tag{3.55}$$

Now note that by using (3.41), replacing n with p in order to obtain E_p^u and multiply its left side by $A_u^{-n} \Pi_u$ we obtain

$$A_u^{-n} \Pi_u E_p^u = A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] - \sum_{m=0}^{p-1} [A_u^{-n-m-1} \Pi_u] B E_{p-m}^u,$$

and since we have

$$- \sum_{m=0}^{p-1} [A_u^{-n-m-1} \Pi_u] B E_{p-m}^u = - \sum_{m=n}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u,$$

it follows that

$$A_u^{-n} \Pi_u E_p^u = A_u^{-n-p} \Pi_u + \sum_{m=0}^{+\infty} A_u^{-n-p-m-1} \Pi_u B [E_m^s + E_m^c] \tag{3.56}$$

$$- \sum_{m=n}^{n+p-1} [A_u^{-m-1} \Pi_u] B E_{n+p-m}^u.$$

Therefore by plugging the expression of $A_u^{-n} \Pi_u E_p^u$ given by (3.56) into (3.55) and recalling (3.43) we obtain that

$$E_{\cdot}^u E_p^u - E_{\cdot+p}^u = (\mathcal{J}_{21}, \mathcal{J}_{22}, \mathcal{J}_{23}) \mathbf{W}. \tag{3.57}$$

Equation for $\{E_n^s E_p^u\}_{n \in \mathbb{N}}$: Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$ be given. Then by using (3.40) and multiply the right side of E_n^s by E_p^u we obtain

$$\begin{aligned} E_n^s E_p^u &= A_s^n \Pi_s E_p^u - \sum_{m=0}^{+\infty} A_s^{m+n} \Pi_s B [E_{m+1}^u E_p^u + E_{-m-1}^c E_p^u] \\ &\quad + \sum_{m=0}^{n-1} A_s^{n-m-1} \Pi_s B E_m^s E_p^u - \sum_{m=0}^{+\infty} [A_u^{-m-1} \Pi_u + A_c^{-m-1} \Pi_c] B E_{n+m}^s E_p^u. \end{aligned} \tag{3.58}$$

Next by replacing n by p in (3.41) we obtain E_p^u and by multiplying its left side by $A_s^n \Pi_s$ we get

$$A_s^n \Pi_s E_p^u = \sum_{m=0}^{+\infty} A_s^{n+m} \Pi_s E_{p+m+1}^u. \tag{3.59}$$

Then plugging (3.59) into (3.58) yields

$$E_{\cdot}^s E_p^u = (\mathcal{J}_{11}, \mathcal{J}_{12}, \mathcal{J}_{13}) \mathbf{W}. \tag{3.60}$$

Equation for $\{E_n^c E_p^u\}_{n \in \mathbb{Z}}$: Let $n \in \mathbb{Z}$ and $p \in \mathbb{N}$ be given. By multiplying the right side of (3.42) by E_p^u we get

$$\begin{aligned} E_n^c E_p^u &= A_c^n \Pi_c E_p^u - \sum_{m=0}^{+\infty} A_c^{m+n} \Pi_c B E_{m+1}^u E_p^u + \sum_{m=0}^{+\infty} A_c^{n-m-1} \Pi_c B E_m^s E_p^u \\ &\quad + \sum_{m=0}^{n-1} A_c^{n-m-1} \Pi_c B E_m^c E_p^u - \sum_{m=0}^{-n-1} A_c^{-m-1} \Pi_c B E_{n+m}^c E_p^u \\ &\quad - \sum_{m=0}^{+\infty} A_u^{-m-1} \Pi_u B E_{n+m}^c E_p^u + \sum_{m=0}^{+\infty} A_s^m \Pi_s B E_{-m-1+n}^u E_p^u. \end{aligned} \tag{3.61}$$

Next by replacing n by p in (3.41) we obtain E_p^u and by multiplying its left side by $A_c^n \Pi_c$ we get

$$A_c^n \Pi_c E_p^u = \sum_{m=0}^{+\infty} A_c^{n+m} \Pi_c E_{p+m+1}^u, \tag{3.62}$$

Therefore by plugging (3.62) into (3.61) we get

$$E_{\cdot}^c E_p^u = (\mathcal{J}_{31}, \mathcal{J}_{32}, \mathcal{J}_{33}) \mathbf{W}. \tag{3.63}$$

Recalling (3.43), it follows that \mathbf{W} satisfies $\mathbf{W} = \mathcal{J}(\mathbf{W})$. Hence we infer from Lemma 3.10 that since $C \|B\|_{\mathcal{L}(X)} \leq C\delta < 1$, one has $\mathbf{W} = 0_X$ and this completes the proof of the lemma. □

Remark 3.16 The arguments for the proof of the next two lemmas are similar to the arguments used for the proof of Lemma 3.15.

Lemma 3.17 *Let the conditions of Theorem 1.7 and (3.52) be satisfied. Then the following properties hold true:*

- (i) *for each $n, p \in \mathbb{N}$ we have $E_n^s E_p^s = E_{n+p}^s$ and $E_n^u E_p^s = 0_{\mathcal{L}(X)}$ while for each $n \in \mathbb{Z}, p \in \mathbb{N}$ we have $E_n^c E_p^s = 0_{\mathcal{L}(X)}$.*
- (ii) *$\widehat{\Pi}_s$ is a bounded linear projector on X and for each $n \geq 0$ one has $E_n^s(X) \subset \widehat{\Pi}_s(X)$.*

Proof First of all let us notice that since we have $\widehat{\Pi}_s = E_0^s$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (ii).

The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed):

$$\left\{ E_n^s E_p^s - E_{n+p}^s \right\}_{n \in \mathbb{N}}, \left\{ E_n^u E_p^s \right\}_{n \in \mathbb{N}} \text{ and } \left\{ E_n^c E_p^s \right\}_{n \in \mathbb{Z}}.$$

Let $p \in \mathbb{N}$ be given and fixed and let us observe that

$$\mathbf{W} := \left(E^s E_p^s - E_{\cdot+p}^s, E^u E_p^s, E^c E_p^s \right)^T \in \mathcal{X}.$$

By proceeding as in the proof of Lemma 3.15 we obtain the following closed system of equations $\mathbf{W} = \mathcal{J}(\mathbf{W})$, that ensures that $\mathbf{W} = 0_{\mathcal{X}}$. This ends the proof of this lemma. \square

Lemma 3.18 *Let the conditions of Theorem 1.7 and (3.52) be satisfied. Then the following properties hold true:*

- (i) *for each $n, p \in \mathbb{Z}$ we have $E_n^c E_p^c = E_{n+p}^c$ and for each $n \in \mathbb{N}, p \in \mathbb{Z}$ we have $E_n^s E_p^c = E_n^u E_p^c = 0_{\mathcal{L}(X)}$.*
- (ii) *$\widehat{\Pi}_c$ is a bounded linear projector on X and for each $n \in \mathbb{Z}$ one has $E_n^c(X) \subset \widehat{\Pi}_c(X)$.*

Proof First of all let us notice that since we have $\widehat{\Pi}_c = E_0^c$ the property (ii) is a direct consequence of the property (i). Therefore we will focus on the property (ii).

The idea of this proof is to derive a suitable closed system of equations for the following three quantities (wherein $p \in \mathbb{N}$ is fixed):

$$\left\{ E_n^c E_p^c - E_{n+p}^c \right\}_{n \in \mathbb{Z}}, \left\{ E_n^u E_p^c \right\}_{n \in \mathbb{N}} \text{ and } \left\{ E_n^s E_p^c \right\}_{n \in \mathbb{N}}.$$

Let $p \in \mathbb{Z}$ be given and fixed and observe that:

$$\mathbf{W} := \left(E^s E_p^c, E^u E_p^c, E^c E_p^c - E_{\cdot+p}^c \right)^T \in \mathcal{X}.$$

By proceeding as in the proof of Lemma 3.15 we obtain the following closed system of equations $\mathbf{W} = \mathcal{J}(\mathbf{W})$. This completes the proof of this lemma. \square

3.5 Proof of Theorem 1.7

In this section we complete the proof of Theorem 1.7. The main points are summarized in the following lemma. Note that the proof of Theorem 1.7 becomes a direct consequence of Proposition 3.11 and Lemma 3.19 below.

Lemma 3.19 *Let us assume that the conditions of Theorem 1.7 are satisfied. Up to reduce the value of δ_0 provided by Proposition 3.11 so that $\delta_0 < \min\left(C^{-1}, \frac{1-C^{-1}}{6\kappa^3+1-C^{-1}}, \sqrt{2}-1\right)$, if $B \in \mathcal{L}(X)$ satisfies*

$$\|B\|_{\mathcal{L}(X)} \leq \delta, \text{ with } \delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$$

then the following properties hold:

- (i) *The three bounded linear projectors $\widehat{\Pi}_s, \widehat{\Pi}_u$ and $\widehat{\Pi}_c$ provided by Lemmas 3.15, 3.17 and 3.18 satisfy*

$$\widehat{\Pi}_k \widehat{\Pi}_l = 0_{\mathcal{L}(X)} \text{ if } k \neq l, \text{ with } k, l = s, u, c, \tag{3.64}$$

and

$$\|\widehat{\Pi}_k - \Pi_k\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} \leq \delta_0. \tag{3.65}$$

- (ii) *For each $n \in \mathbb{N}$ and $k = s, c$ we have $E_n^k = (A + B)^n \widehat{\Pi}_k \in \mathcal{L}(X, \widehat{\Pi}_k(X))$.*
- (iii) *For each $n \in \mathbb{N}$ $(A + B)^n \widehat{\Pi}_c$ is invertible from $\widehat{\Pi}_c(X)$ into $\widehat{\Pi}_c(X)$ with*

$$E_{-n}^c (A + B)^n \widehat{\Pi}_c = (A + B)^n \widehat{\Pi}_c E_{-n}^c = \widehat{\Pi}_c. \tag{3.66}$$

- (iv) *One has $(A + B)(D(A) \cap \widehat{\Pi}_u(X)) \subset \widehat{\Pi}_u(X)$. Consider $(A + B)_u : D(A) \cap \widehat{\Pi}_u(X) \subset \widehat{\Pi}_u(X) \rightarrow \widehat{\Pi}_u(X)$ the part of $(A + B)$ in $\widehat{\Pi}_u(X)$. Then one has $0 \in \rho((A + B)_u)$ and for each $n \geq 0$:*

$$E_n^u = ((A + B)_u)^{-n} \widehat{\Pi}_u. \tag{3.67}$$

- (v) *For $k = s, u, c$, the projector $\widehat{\Pi}_k$ satisfies $\widehat{\Pi}_k(D(A)) \subset D(A)$ and*

$$(A + B)\widehat{\Pi}_k x = \widehat{\Pi}_k(A + B)x, \quad \forall x \in D(A). \tag{3.68}$$

Proof Proof of (i): By recalling that $E_0^k = \widehat{\Pi}_k$ it follows from Lemmas 3.15, 3.17 and 3.18 that (3.64) holds true. Moreover the condition (ii) of Proposition 3.11 together with $\delta \in \left(0, \frac{\delta_0^2}{\kappa + \delta_0}\right)$ provide that

$$\|\widehat{\Pi}_k - \Pi_k\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta}{\delta_0 - \delta} \leq \delta_0 \in \left(0, \sqrt{2}-1\right). \tag{3.69}$$

This completes the proof of (i).

Proof of (ii): Let $n \in \mathbb{N} \setminus \{0\}$ be given. We will first prove that $E_n^s = (A + B)^n \widehat{\Pi}_s$. By replacing n by $n - 1$ in (3.40), recalling that $E_{n-1}^s(X) \subset \widehat{\Pi}_s(X) \cap D(A)$ [see Proposition 3.11 (iii)] and multiplying the left side of E_{n-1}^s by A it follows that

$$AE_{n-1}^s = E_n^s - BE_{n-1}^s \iff E_n^s = (A + B)E_{n-1}^s.$$

Hence by induction [see Proposition 3.11 (iii)] one obtains that for each $n \geq 0$: $(A + B)^n \widehat{\Pi}_s(X) \subset D(A)$, $(A + B)^n \widehat{\Pi}_s \in \mathcal{L}(X, \widehat{\Pi}_s(X))$ and

$$E_n^s = (A + B)^n E_0^s = (A + B)^n \widehat{\Pi}_s.$$

Next we prove that $E_n^c = (A + B)^n \widehat{\Pi}_c$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N} \setminus \{0\}$ be given. By replacing n by $n - 1$ in (3.42), recalling that $E_{n-1}^c(X) \subset D(A)$ and multiplying the left side of E_{n-1}^c by A we obtain

$$AE_{n-1}^c = E_n^c - BE_{n-1}^c \iff E_n^c = (A + B)E_{n-1}^c,$$

providing that

$$E_n^c = (A + B)^n \widehat{\Pi}_c \in \mathcal{L}(X, \widehat{\Pi}_c(X)). \tag{3.70}$$

This completes the proof of (ii).

Proof of (iii): Let us prove that for each $n \in \mathbb{N}$ the bounded linear operator $(A + B)^n \widehat{\Pi}_c$ is invertible from $\widehat{\Pi}_c(X)$ into $\widehat{\Pi}_c(X)$.

In fact each $n \in \mathbb{N}$ by using Lemma 3.18 combined together with (3.70) we obtain

$$E_{-n}^c (A + B)^n \widehat{\Pi}_c = E_{-n}^c E_n^c = E_0^c = \widehat{\Pi}_c = E_n^c E_{-n}^c = (A + B)^n \widehat{\Pi}_c E_{-n}^c.$$

This prove that $(A + B)^n \widehat{\Pi}_c$ is invertible from $\widehat{\Pi}_c(X)$ into $\widehat{\Pi}_c(X)$ and (3.66) holds true.

Proof of (iv): In order to prove this point we claim that

Claim 3.20 *The following holds true:*

- (a) *Recalling that $E_1^u(X) \subset D(A)$ one has $(A + B)E_1^u = \widehat{\Pi}_u$.*
- (b) *Consider the closed linear operator $C_u : D(C_u) \subset \widehat{\Pi}_u(X) \rightarrow \widehat{\Pi}_u(X)$ defined by $D(C_u) = D(A) \cap \widehat{\Pi}_u(X)$ and $C_u = \widehat{\Pi}_u(A + B)$. Then it satisfies $0 \in \rho(C_u)$.*

Before proving this claim let us complete the proof of (3.67). To do so let us first notice that (a) and (b) implies that

$$(A + B)(D(A) \cap \widehat{\Pi}_u(X)) = (A + B)(C_u^{-1}(\widehat{\Pi}_u(X))) = \widehat{\Pi}_u(X).$$

Hence the linear operator $(C_u, D(C_u))$ coincide the part $(A + B)_u$ of $(A + B)$ in $\widehat{\Pi}_u(X)$. Therefore $0 \in \rho((A + B)_u)$ and using (a) and the orthogonality of the perturbed projectors one gets:

$$E_1^u = ((A + B)_u)^{-1} \widehat{\Pi}_u.$$

Finally due to the semiflow property for E_n^u one gets

$$E_n^u = ((A + B)_u)^{-n} \widehat{\Pi}_u, \quad n \geq 0,$$

and (3.67) follows.

It remains to prove Claim 3.20.

Proof of (a): Let us first recall that $E_1(X) \subset D(A)$ and let us multiply the left side of E_1^u given in (3.41) by A to obtain

$$AE_1^u = E_0^u - BE_1^u \iff (A + B)E_1^u = E_0^u,$$

that completes the proof of (a).

Proof of (b): Before proceeding to the proof of this statement let us notice that since we have $E_0^k = \widehat{\Pi}_k, k = s, u, c$ it follows from the condition (i) of Proposition 3.11 that

$$\|\widehat{\Pi}_k\|_{\mathcal{L}(X)} \leq \frac{\kappa \delta_0}{\delta_0 - \delta}, \quad k = s, u, c. \tag{3.71}$$

Now recalling that $D(A) = X_s \oplus X_c \oplus (X_u \cap D(A))$ one has that for each $x \in D(A) \cap \widehat{\Pi}_u(X)$:

$$x = \Pi_s x + \Pi_c x + \Pi_u x,$$

so that $\Pi_u x \in X_u \cap D(A)$. This re-writes as $\Pi_u(D(A) \cap \widehat{\Pi}_u(X)) \subset D(A) \cap \Pi_u(X)$. Hence one has

$$C_u = \widehat{\Pi}_u A [\Pi_u + \Pi_s + \Pi_c] + \widehat{\Pi}_u B.$$

This re-writes as

$$C_u = \tilde{A}_u + L_u, \tag{3.72}$$

wherein we have set $\tilde{A}_u : D(C_u) \subset \hat{\Pi}_u(X) \rightarrow \hat{\Pi}_u(X)$ defined as

$$\tilde{A}_u x = \hat{\Pi}_u A \Pi_u x, \quad \forall x \in D(A) \cap \hat{\Pi}_u(X),$$

and $L_u \in \mathcal{L}(\hat{\Pi}_u(X))$ defined by:

$$L_u := \hat{\Pi}_u A_s \Pi_s \hat{\Pi}_u + \hat{\Pi}_u A_c \Pi_c \hat{\Pi}_u + \hat{\Pi}_u B \hat{\Pi}_u.$$

Next observe that due to (3.69) Lemma 3.1 applies to Π_u and $\hat{\Pi}_u$ and provides that $\Pi_u|_{\hat{\Pi}_u(X)}$ is an isomorphism from $\hat{\Pi}_u(X)$ onto $\Pi_u(X)$ while $\hat{\Pi}_u|_{\Pi_u(X)}$ is an isomorphism from $\Pi_u(X)$ onto $\hat{\Pi}_u(X)$. One furthermore has

$$\left\| (\hat{\Pi}_u|_{\Pi_u(X)})^{-1} x \right\| \leq \frac{1}{1 - \delta_0} \|x\|, \quad \forall x \in \Pi_u(X), \tag{3.73}$$

and

$$\left\| (\Pi_u|_{\hat{\Pi}_u(X)})^{-1} x \right\| \leq \frac{1}{1 - \delta_0} \|x\|, \quad \forall x \in \hat{\Pi}_u(X). \tag{3.74}$$

Then due to the above isomorphism one has

$$\Pi_u(D(A) \cap \hat{\Pi}_u(X)) = D(A) \cap \Pi_u(X).$$

Indeed first note that inclusion \subset has already been observed. Consider $x \in D(A) \cap \Pi_u(X)$. Then there exists a unique $y \in \hat{\Pi}_u(X)$ such that $\Pi_u(y) = x$. Then we write $y = \Pi_s y + \Pi_c y + \Pi_u y$. Since $D(A) = X_s \oplus X_c \oplus (D(A) \cap X_u)$ and $\Pi_u y = x \in D(A) \cap \Pi_u(X)$ one obtains that $y \in D(A) \cap \hat{\Pi}_u(X)$ and $x \in \Pi_u(D(A) \cap \hat{\Pi}_u(X))$ and the equality follows.

As a consequence one gets:

$$D(C_u) = D(A) \cap \hat{\Pi}_u(X) = \left(\Pi_u|_{\hat{\Pi}_u(X)} \right)^{-1} (D(A) \cap \Pi_u(X)).$$

Using this relation and recalling that $0 \in \rho(A_u)$ it is easy to check that $0 \in \rho(\tilde{A}_u)$ and

$$(\tilde{A}_u)^{-1} = \left(\Pi_u|_{\hat{\Pi}_u(X)} \right)^{-1} \circ A_u^{-1} \circ (\hat{\Pi}_u|_{\Pi_u(X)})^{-1}.$$

Finally due to (3.72), in order to complete the proof of point (b) it is sufficient to check that

$$\|L_u\|_{\mathcal{L}(\hat{\Pi}_u(X))} \|\tilde{A}_u^{-1}\|_{\mathcal{L}(\hat{\Pi}_u(X))} < 1.$$

To do so let us first notice that due to (3.73)–(3.74) one has

$$\|\tilde{A}_u^{-1}\|_{\mathcal{L}(\hat{\Pi}_u(X))} \leq \frac{1}{(1 - \delta_0)^2} \|A_u^{-1}\|_{\mathcal{L}(\Pi_u(X))} \leq \frac{1}{(1 - \delta_0)^2} \kappa e^{-\rho}. \tag{3.75}$$

On the other one has

$$\begin{aligned} L_u &= \hat{\Pi}_u A \Pi_s \hat{\Pi}_u + \hat{\Pi}_u A \Pi_c \hat{\Pi}_u + \hat{\Pi}_u B \hat{\Pi}_u \\ &= \hat{\Pi}_u A_s \Pi_s [\hat{\Pi}_u - \Pi_u] + \hat{\Pi}_u A_c \Pi_c [\hat{\Pi}_u - \Pi_u] + \hat{\Pi}_u B \hat{\Pi}_u. \end{aligned}$$

Then by using (3.65) and (3.71) and recalling that $\|B\|_{\mathcal{L}(X)} \leq \delta_0$, it follows that

$$\|L_u\|_{\mathcal{L}(\hat{\Pi}_u(X))} \leq 2\kappa^2 \delta_0 + 2\kappa^2 e^{\rho_0} + 2\kappa \delta_0 \leq 6\kappa^2 e^{\rho_0} \delta_0. \tag{3.76}$$

Now combining (3.75) together with (3.76) provides that

$$\|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \|\widetilde{A}_u^{-1}\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \leq \frac{6\kappa^3}{(1 - \delta_0)^2} \delta_0.$$

Hence up to reduce δ_0 such that

$$\delta_0 < \min \left\{ C^{-1}, \frac{1 - C^{-1}}{6\kappa^3 + 1 - C^{-1}} \right\}, \tag{3.77}$$

where $C > 1$ is the constant provided by Lemma 3.10 we obtain that

$$\|L_u\|_{\mathcal{L}(\widehat{\Pi}_u(X))} \|\widetilde{A}_u^{-1}\|_{\mathcal{L}(\widehat{\Pi}_u(X))} < 1.$$

This completes the proof of Claim 3.20 (b) and also the proof of (iv).

Proof of (v): Let us first notice that the inclusions $\widehat{\Pi}_k(X) \subset D(A)$ for any $k = s, c$ and $\widehat{\Pi}_u(D(A)) \subset D(A)$ have been observed in Proposition 3.11 (iii). Next recall that by (ii) we have

$$E_1^k = (A + B) \widehat{\Pi}_k, \quad \forall k = s, c,$$

so that

$$\widehat{\Pi}_k (A + B) \widehat{\Pi}_k = \widehat{\Pi}_k E_1^k = E_0^k E_1^k = E_1^k = (A + B) \widehat{\Pi}_k,$$

that is

$$(A + B) \widehat{\Pi}_k = \widehat{\Pi}_k (A + B) \widehat{\Pi}_k, \quad k = s, c. \tag{3.78}$$

Moreover the property (iii) implies that $(A + B) \widehat{\Pi}_u$ maps $D(A)$ into $\widehat{\Pi}_u$, that is for any $x \in D(A)$:

$$(A + B) \widehat{\Pi}_u x = \widehat{\Pi}_u (A + B) \widehat{\Pi}_u x. \tag{3.79}$$

Therefore for each $k = s, c, u$ by using (3.78) and (3.79) combined together with the orthogonality property in (3.64) we obtain for each $x \in D(A)$:

$$\begin{aligned} \widehat{\Pi}_k (A + B) x &= \widehat{\Pi}_k (A + B) [\widehat{\Pi}_s + \widehat{\Pi}_u + \widehat{\Pi}_c] x \\ &= \widehat{\Pi}_k (A + B) \widehat{\Pi}_k x = (A + B) \widehat{\Pi}_k x. \end{aligned}$$

This completes the proof of this lemma. □

4 Proof of Theorem 2.2

The aim of this section is to complete the proof of Theorem 2.2.

Let $q \in [1, \infty]$ be given. Recall that we denote the Banach space $X = l^q(\mathbb{Z}; Y)$. Recall also the definition of the linear operator $(\mathcal{A}, D(\mathcal{A}))$ in (2.8). Next let us consider the three bounded linear operators $\mathcal{P}_\alpha \in \mathcal{L}(X)$ defined for $\alpha = s, c, u$ by

$$(\mathcal{P}_\alpha u)_k = \Pi_k^\alpha u_k, \quad \forall k \in \mathbb{Z}, \quad \forall u \in X.$$

Using the above notations let us notice that for each $\alpha = s, c, u$, \mathcal{P}_α is a projector on X that satisfies

- $\mathcal{P}_\alpha \mathcal{P}_\beta = 0_{\mathcal{L}(X)}$ for all $\alpha \neq \beta$.
- $\mathcal{P}_s + \mathcal{P}_c + \mathcal{P}_u = I_{\mathcal{L}(X)}$.
- for each $\alpha = s, c, u$, one has $\mathcal{A}(D(\mathcal{A}) \cap \mathcal{P}_\alpha(X)) \subset \mathcal{P}_\alpha(X)$.

Next we set $X^\alpha = \mathcal{P}_\alpha(X)$ for $\alpha = s, c, u$ and the following straightforward lemma holds true:

Lemma 4.1 *The following holds true:*

- (i) *The part \mathcal{A}_s of \mathcal{A} in X^s satisfies $D(\mathcal{A}_s) = X^s$ and $r(\mathcal{A}_s) \leq e^\rho$. We furthermore have for each $u \in X^s$ and each $(n, k) \in \mathbb{N} \times \mathbb{Z}$:*

$$(\mathcal{A}_s^n u)_k = U_{\mathbf{A}}^s(k, k - n) \Pi_{k-n}^s u_{k-n}.$$

- (ii) *The part \mathcal{A}_u of \mathcal{A} in X^u satisfies $0 \in \rho(\mathcal{A}_u)$ and satisfies $r(\mathcal{A}_u^{-1}) \leq e^{-\rho}$. We furthermore have for each $u \in X^u$ and each $(n, k) \in \mathbb{N} \times \mathbb{Z}$:*

$$(\mathcal{A}_u^{-n} u)_k = U_{\mathbf{A}}^u(k, k + n) \Pi_{k+n}^u u_{k+n}$$

- (iii) *The part \mathcal{A}_c of \mathcal{A} in X^c satisfies $D(\mathcal{A}_c) = X^c$. It is invertible on X^c and satisfies:*

$$r(\mathcal{A}_c) \leq e^{\rho_0} \text{ and } r(\mathcal{A}_c^{-1}) \leq e^{\rho_0}.$$

We furthermore have for each $u \in X^u$ and each $(n, k) \in \mathbb{Z} \times \mathbb{Z}$:

$$(\mathcal{A}_c^n u)_k = U_{\mathbf{A}}^c(k, k - n) \Pi_{k-n}^c u_{k-n}$$

Remark 4.2 The above discussion and the above lemma imply that the closed linear operator \mathcal{A} has an exponential trichotomy according to Definition 1.2.

Let $\mathbf{B} = \{B_n\}_{n \in \mathbb{Z}}$ be a bounded sequence in $\mathcal{L}(Y)$. Then let us consider the bounded linear operator $\mathcal{B} \in \mathcal{L}(X)$ defined by

$$(\mathcal{B}u)_k = B_{k-1}u_{k-1}, \quad \forall k \in \mathbb{Z}, \quad \forall u \in X.$$

Then note that one has:

$$\|\mathcal{B}\|_{\mathcal{L}(X)} \leq \sup_{k \in \mathbb{Z}} \|B_k\|_{\mathcal{L}(Y)}. \tag{4.1}$$

We are now interesting in the spectral properties of $\mathcal{A} + \mathcal{B}$ by applying Theorem 1.7. We fix $0 < \rho_0 < \widehat{\rho}_0 < \widehat{\rho} < \rho$ and $\widehat{\kappa} > \kappa$. Using the constant $\delta_0 > 0$ provided by Theorem 1.7, we fix a bounded sequence $\mathbf{B} = \{B_n\}_{n \in \mathbb{Z}}$ in $\mathcal{L}(Y)$ such that

$$\sup_{n \in \mathbb{Z}} \|B_n\|_{\mathcal{L}(Y)} \leq \frac{\delta_0^2}{\kappa + \delta_0}.$$

In view of (4.1), Theorem 1.7 applies to the perturbation problem $\mathcal{A} + \mathcal{B}$ and operator $(\mathcal{A} + \mathcal{B})$ has an exponential trichotomy with exponent $\widehat{\rho}_0$ and $\widehat{\rho}$ and with constant $\widehat{\kappa}$. If we denote the three corresponding projectors by $\widehat{\mathcal{P}}_s, \widehat{\mathcal{P}}_c, \widehat{\mathcal{P}}_u \in \mathcal{L}(X)$ and $\widehat{X}^\alpha = \widehat{\mathcal{P}}_s(X)$ we have:

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B})_s; \widehat{\mathcal{P}}_s \\ (\mathcal{A} + \mathcal{B})_u; \widehat{\mathcal{P}}_u \\ (\mathcal{A} + \mathcal{B})_c; \widehat{\mathcal{P}}_c \end{pmatrix} = (I - \mathcal{J})^{-1} \begin{pmatrix} \mathcal{A}_s; \mathcal{P}_s \\ \mathcal{A}_u; \mathcal{P}_u \\ \mathcal{A}_c; \mathcal{P}_c \end{pmatrix}, \tag{4.2}$$

wherein the bounded linear operator \mathcal{J} acting on the Banach space $\mathcal{X} := \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{-\widehat{\rho}}(\mathbb{N}, \mathcal{L}(X)) \times \mathbb{L}_{\widehat{\rho}_0}(\mathbb{Z}, \mathcal{L}(X))$ is defined in (3.43). One furthermore has the following estimates

$$\|(\mathcal{A} + \mathcal{B})_c^n \widehat{\mathcal{P}}_c\|_{\mathcal{L}(X)} \leq \widehat{\kappa} e^{\widehat{\rho}_0 |n|}, \quad \forall n \in \mathbb{Z}, \tag{4.3}$$

$$\|(\mathcal{A} + \mathcal{B})_s^n \widehat{\mathcal{P}}_s\|_{\mathcal{L}(X)} \leq \widehat{\kappa} e^{-\widehat{\rho} n}, \quad \forall n \in \mathbb{N}, \tag{4.4}$$

and

$$\|(\mathcal{A} + \mathcal{B})_u^{-n} \widehat{\mathcal{P}}_u\|_{\mathcal{L}(X)} \leq \widehat{\kappa} e^{-\widehat{\rho}n}, \quad \forall n \in \mathbb{N}, \tag{4.5}$$

as well as the following estimates for each $n \in \mathbb{N}$,

$$\|(\mathcal{A} + \mathcal{B})_s^n \widehat{\mathcal{P}}_s - \mathcal{A}_s^n \mathcal{P}_s\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \tag{4.6}$$

$$\|(\mathcal{A} + \mathcal{B})_u^{-n} \widehat{\mathcal{P}}_u - \mathcal{A}_u^{-n} \mathcal{P}_u\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{-\widehat{\rho}n}, \tag{4.7}$$

and for each $n \in \mathbb{Z}$

$$\|(\mathcal{A} + \mathcal{B})_c^n \widehat{\mathcal{P}}_c - \mathcal{A}_c^n \mathcal{P}_c\|_{\mathcal{L}(X)} \leq \frac{\kappa\delta}{\delta_0 - \delta} e^{\widehat{\rho}_0|n|}. \tag{4.8}$$

In order to prove our perturbation result, namely Theorem 2.2, we will show that the perturbed projectors exhibit a suitable structure inherited from the one of the shift operators \mathcal{A} and \mathcal{B} that reads as

$$(\widehat{\mathcal{P}}_\alpha u)_k = \widehat{\Pi}_k^\alpha u_k, \quad \forall u \in X, \quad \alpha = s, c, u,$$

and wherein for each $k \in \mathbb{Z}$ and $\alpha = s, c, u$, $\widehat{\Pi}_k^\alpha$ denotes a projector of Y . To do so, let us introduce for each $p \in \mathbb{Z}$ the linear bounded operator $\mathcal{D}_p \in \mathcal{L}(X)$ defined for each $u \in X$ and $k \in \mathbb{Z}$ by

$$(\mathcal{D}_p u)_k = \begin{cases} u_p & \text{if } k = p \\ 0 & \text{if } k \neq p \end{cases}$$

Together with this notation, let us notice that for each $p \in \mathbb{Z}$ the following commutativity properties hold true:

$$\begin{aligned} \mathcal{D}_p \mathcal{A}_s^n \mathcal{P}_s &= \mathcal{A}_s^n \mathcal{P}_s \mathcal{D}_{p-n}, \quad \forall n \geq 0, \quad \forall p \in \mathbb{Z}, \\ \mathcal{D}_p \mathcal{A}_u^{-n} \mathcal{P}_u &= \mathcal{A}_u^{-n} \mathcal{P}_u \mathcal{D}_{p+n}, \quad \forall n \geq 0, \quad \forall p \in \mathbb{Z}, \\ \mathcal{D}_p \mathcal{A}_c^n \mathcal{P}_c &= \mathcal{A}_c^n \mathcal{P}_c \mathcal{D}_{p-n}, \quad \forall n \in \mathbb{Z}, \quad \forall p \in \mathbb{Z}. \end{aligned} \tag{4.9}$$

One may also notice that \mathcal{B} satisfies:

$$\mathcal{D}_p \mathcal{B} = \mathcal{B} \mathcal{D}_{p-1}, \quad \forall p \in \mathbb{Z}. \tag{4.10}$$

If one considers the closed subspace $\mathcal{Z} \subset \mathcal{X}$ defined by

$$\mathcal{Z} = \left\{ \begin{pmatrix} E^s \\ E^u \\ E^c \end{pmatrix} \in \mathcal{X} : \begin{pmatrix} \mathcal{D}_p E^s - E^s \mathcal{D}_{p-} \\ \mathcal{D}_p E^u - E^u \mathcal{D}_{p+} \\ \mathcal{D}_p E^c - E^c \mathcal{D}_{p-} \end{pmatrix} = 0_{\mathcal{X}}, \quad \forall p \in \mathbb{Z} \right\},$$

then we claim that

Claim 4.3 *The linear bounded operator $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ satisfies $\mathcal{J}\mathcal{Z} \subset \mathcal{Z}$.*

We postpone the proof of this claim and complete the proof of Theorem 2.2.

Using the above claim and recalling that \mathcal{Z} is a closed subspace of \mathcal{X} lead us to

$$(I - \mathcal{J})^{-1} \mathcal{Z} \subset \mathcal{Z}.$$

Indeed since $\|\mathcal{J}\|_{\mathcal{L}(X)} < 1$ then $(I - \mathcal{J})^{-1} = \sum_{k=0}^\infty \mathcal{J}^k$. Hence due to (4.9) one obtains that

$$(\mathcal{A}_s \mathcal{P}_s, \mathcal{A}_u^{-1} \mathcal{P}_u, \mathcal{A}_c \mathcal{P}_c)^T \in \mathcal{Z},$$

while (4.2) ensures that

$$\begin{pmatrix} (\mathcal{A} + \mathcal{B})_s^+ \widehat{\mathcal{P}}_s \\ (\mathcal{A} + \mathcal{B})_u^- \widehat{\mathcal{P}}_u \\ (\mathcal{A} + \mathcal{B})_c^+ \widehat{\mathcal{P}}_c \end{pmatrix} \in \mathcal{Z}.$$

The above statement completes the proof of Theorem 2.2. Indeed let us first notice that the above result implies that for each $p \in \mathbb{Z}$ and $\alpha = s, c, u$,

$$\mathcal{D}_p \widehat{\mathcal{P}}_\alpha = \widehat{\mathcal{P}}_\alpha \mathcal{D}_p.$$

This means that for each $p \in \mathbb{Z}$ there exists three projectors $\widehat{\Pi}_p^\alpha \in \mathcal{L}(Y)$ for $\alpha = s, c, u$ such that:

$$(\widehat{\mathcal{P}}_\alpha u)_p = \widehat{\Pi}_p^\alpha u_p \quad \text{for all } p \in \mathbb{Z}, u \in X \text{ and } \alpha = s, c, u.$$

Note that the properties $\widehat{\mathcal{P}}_\alpha \widehat{\mathcal{P}}_\beta = 0$ for $\alpha \neq \beta$ and $\widehat{\mathcal{P}}_s + \widehat{\mathcal{P}}_c + \widehat{\mathcal{P}}_u = I_X$ directly re-write as for each $p \in \mathbb{Z}$:

$$\widehat{\Pi}_p^\alpha \widehat{\Pi}_p^\beta = 0 \text{ for } \alpha \neq \beta \text{ and } \widehat{\Pi}_p^s + \widehat{\Pi}_p^c + \widehat{\Pi}_p^u = I_Y.$$

It remains to check that $\mathbf{A} + \mathbf{B}$ has an exponential trichotomy with constant $\widehat{\kappa}$, exponents $\widehat{\rho}_0 < \widehat{\rho}$ and associated to the projectors $\{\widehat{\Pi}_k^\alpha\}_{k \in \mathbb{Z}}$ with $\alpha = s, c, u$.

Property (ii) of Definition 2.1: For each $k \in \mathbb{Z}$ and $\alpha = s, c, u$ one has for each $u \in D(\mathcal{A})$:

$$[\widehat{\mathcal{P}}_\alpha (\mathcal{A} + \mathcal{B}) u]_k = \widehat{\Pi}_k^\alpha [(\mathcal{A} + \mathcal{B}) u]_k = \widehat{\Pi}_k^\alpha U_{\mathbf{A}+\mathbf{B}}(k, k - 1) u_{k-1}.$$

Since for each $u \in D(\mathcal{A})$ one has $\widehat{\mathcal{P}}_\alpha (\mathcal{A} + \mathcal{B}) u = (\mathcal{A} + \mathcal{B}) \widehat{\mathcal{P}}_\alpha u$ and for each $k \in \mathbb{Z}$ and each $u \in Y$ the sequences $\mathbf{u}^k = \{u_p\}_{p \in \mathbb{Z}}$ defined by $u_p = 0$ for $p \neq k - 1$ and $u_{k-1} = u$ belongs to $D(\mathcal{A})$, one obtains:

$$\widehat{\Pi}_k^\alpha U_{\mathbf{A}+\mathbf{B}}(k, k - 1) u = U_{\mathbf{A}+\mathbf{B}}(k, k - 1) [\widehat{\mathcal{P}}_\alpha \mathbf{u}^k]_{k-1} = U_{\mathbf{A}+\mathbf{B}}(k, k - 1) \widehat{\Pi}_{k-1}^\alpha u.$$

As a consequence one gets that for each $k \in \mathbb{Z}$ and $\alpha = s, c, u$:

$$\widehat{\Pi}_k^\alpha U_{\mathbf{A}+\mathbf{B}}(k, k - 1) = U_{\mathbf{A}+\mathbf{B}}(k, k - 1) \widehat{\Pi}_{k-1}^\alpha.$$

This proves statement (ii). □

Proof of (iii) in Definition 2.1: Let us set for each $k \in \mathbb{Z}$ the subspaces $\widehat{Y}_k^\alpha = \widehat{\Pi}_k^\alpha(Y)$. Recall that $0 \in \rho((\mathcal{A} + \mathcal{B})_u)$. Hence for each $v \in \widehat{X}_u$ there exists a unique $u \in D(\mathcal{A}) \cap \widehat{X}_u$ such that $(\mathcal{A} + \mathcal{B}) u = v$. This re-writes as for each $k \in \mathbb{Z}$:

$$U_{\mathbf{A}+\mathbf{B}}(k, k - 1) u_{k-1} = v_k.$$

This proves that for each $k \in \mathbb{Z}$ the linear operator $U_{\mathbf{A}+\mathbf{B}}(k, k - 1)$ is invertible from \widehat{Y}_{k-1}^u onto \widehat{Y}_k^u . Due to composition argument for each $k \in \mathbb{Z}$ and $n \geq 1$, the linear operator $U_{\mathbf{A}+\mathbf{B}}(k, k - n)$ is invertible from \widehat{Y}_{k-n}^u onto \widehat{Y}_k^u . Furthermore one has for each $n \geq 0$ and $p \in \mathbb{Z}$:

$$[(\mathcal{A} + \mathcal{B})^{-n} \widehat{\mathcal{P}}_u v]_p = U_{\mathbf{A}+\mathbf{B}}(p, p + n) \widehat{\Pi}_{p+n}^u v_{p+n}.$$

The same arguments hold true for the central part and one obtains that for each $k \in \mathbb{Z}$ and $n \geq 0$ the linear operator $U_{\mathbf{A}+\mathbf{B}}(k, k - n)$ is invertible from \widehat{Y}_{k-n}^c onto \widehat{Y}_k^c . Furthermore one has for each $n \in \mathbb{Z}$ and $p \in \mathbb{Z}$:

$$[(\mathcal{A} + \mathcal{B})^n \widehat{\mathcal{P}}_c v]_p = U_{\mathbf{A}+\mathbf{B}}(p, p - n) \widehat{\Pi}_{p-n}^c v_{p-n}.$$

This proves that (iii) is true. □

Proof of (iv) in Definition 2.1: The proof of the growth estimates directly follow from the trichotomy estimates for $(A + B)_\alpha$ recalled in (4.3)–(4.5).

Finally the perturbed estimates for projected evolution semiflow stated in Theorem 2.2 directly follows from (4.6) to (4.8). This completes the proof of the result.

To complete the proof of Theorem 2.2 it remains to prove Claim 4.3. □

Proof of Claim 4.3 Let $(E^s, E^u, E^c)^T \in \mathcal{X}$ be given. Let us set $(F^s, F^u, F^c)^T = \mathcal{J}(E^s, E^u, E^c)^T$. Then according to the definition of \mathcal{J} [see (3.43) and (3.40)–(3.42)] for each $n \geq 0$ one has

$$F_n^s = \mathcal{A}_s^n \mathcal{P}_s - \sum_{m=0}^{+\infty} \mathcal{A}_s^{m+n} \mathcal{P}_s \mathcal{B} [E_{m+1}^u + E_{-m-1}^c] \\ + \sum_{m=0}^{n-1} \mathcal{A}_s^{n-m-1} \mathcal{P}_s \mathcal{B} E_m^s - \sum_{m=0}^{+\infty} [\mathcal{A}_u^{-m-1} \mathcal{P}_u + \mathcal{A}_c^{-m-1} \mathcal{P}_c] \mathcal{B} E_{n+m}^s.$$

Recalling (4.9) and (4.10) one directly checks that for each $n \geq 0$ and $p \in \mathbb{Z}$: $\mathcal{D}_p F_n^s = F_n^s \mathcal{D}_{p-n}$. Using the formula described in (3.41) and (3.42) one may directly check the claim. □

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