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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A HYPERBOLIC KELLER–SEGEL EQUATION

Xiaoming Fu, Quentin Griette and Pierre Magal*

Université de Bordeaux – Institut de Mathématiques de Bordeaux
351 cours de la Libération
33400 Talence, France

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Abstract. In this work we describe a hyperbolic model with cell-cell repulsion with a dynamics in the population of cells. More precisely, we consider a population of cells producing a field (which we call "pressure") which induces a motion of the cells following the opposite of the gradient. The field indicates the local density of population and we assume that cells try to avoid crowded areas and prefer locally empty spaces which are far away from the carrying capacity. We analyze the well-posed property of the associated Cauchy problem on the real line. Moreover we obtain a convergence result for bounded initial distributions which are positive and stay away from zero uniformly on the real line.

1. Introduction.

In this article we are concerned with the following diffusion equation with logistic source:

$$\begin{cases}\partial_t u(t,x) - \chi\partial_x\big(u(t,x)\partial_x p(t,x)\big) = u(t,x)(1-u(t,x)), \quad t>0,\ x\in\mathbb{R},\\ u(t=0,x) = u_0(x),\end{cases} \tag{1}$$

where $\chi > 0$ is a *sensing coefficient* and $p(t,x)$ is an external pressure. Model (1) describes the behavior of a population of cells $u(t,x)$ living in a one-dimensional habitat $x\in\mathbb{R}$, which undergo a logistic birth and death population dynamics, and in which individual cells follow the gradient of a field p. The constant χ characterizes the response of the cells to the effective gradient p_x. In this work we will consider the case where p is itself determined by the state of the population $u(t,x)$ as

$$-\sigma^2\partial_{xx}p(t,x) + p(t,x) = u(t,x), \quad t>0, x\in\mathbb{R}. \tag{2}$$

This corresponds to a scenario in which the field $p(t,x)$ is produced by the cells, diffuses to the whole space with diffusivity σ^2 (for $\sigma>0$), and vanishes at rate one. As a result cells are pushed away from crowded area to emptier region.

A similar model has been successfully used in our recent work [21] to describe the motion of cancer cells in a Petri dish in the context of cell co-culture experiments of Pasquier et al. [33]. Pasquier et al. [33] cultivated two types of breast cancer cells to study the transfer of proteins between them in a study of multi-drug resistance.

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* Corresponding author: Pierre Magal pierre.magal@u-bordeaux.fr.

It was observed that the two types of cancer cells form segregated clusters of cells of each kind after a 7-day co-culture experiment. In [21], the authors studied the segregation property of a model similar to (1)–(2), set in a circular domain in two spatial dimensions $x \in \mathbb{R}^2$ representing a Petri dish. The study aims at describing the cancer cells motion in a Petri dish [21, 33] in the context of a batch culture. The cell population should be regarded as a mono-layer attached to the bottom of the Petri dish covered a large quantity of nutritional liquid (used in the cell culture), which is constantly renewed.

Our model can be included in the family of non-local advection models for cell-cell adhesion and repulsion. As pointed out by many biologists, cell-cell interactions do not only exist in a local scope, but a long-range interaction should be taken into account to guide the mathematical modeling. Armstrong, Painter and Sherratt [1] in their early work proposed a model (APS model) in which a local diffusion is added to the non-local attraction driven by the adhesion forces to describe the phenomenon of cell mixing, full/partial engulfment and complete sorting in the cell sorting problem. Based on the APS model, Murakawa and Togashi [32] thought that the population pressure should come from the cell volume size instead of the linear diffusion. Therefore, the linear diffusion was changed into a nonlinear diffusion in order to capture the sharp fronts and the segregation in cell co-culture. Carrillo et al. [11] recently proposed a new assumption on the adhesion velocity field and their model showed a good agreement in the experiments in the work of Katsunuma et al. [25]. The idea of the long-range attraction and short-range repulsion can also be seen in the work of Leverentz, Topaz and Bernoff [28]. They considered a non-local advection model to study the asymptotic behavior of the solution. By choosing a Morse-type kernel which follows the attractive-repulsive interactions, they found that the solution can asymptotically spread, contract (blow-up), or reach a steady-state. Burger, Fetecau and Huang [8] considered a similar non-local adhesion model with nonlinear diffusion, for which they investigated the well-posedness and proved the existence of a compactly supported, non-constant steady state. Dyson et al. [19] established the local existence of a classical solution for a non-local cell-cell adhesion model in spaces of uniformly continuous functions. For Turing and Turing-Hopf bifurcation due to the non-local effect, we refer to Ducrot et al. [16] and Song et al. [37]. We also refer to Mogliner et al. [30], Eftimie et al. [20], Ducrot and Magal [17], Ducrot and Manceau [18] for more topics on non-local advection equations. For the derivation of such models, we refer to the work of Bellomo et al. [5] and Morale, Capasso and Oelschläger [31].

It can be noticed that, in the limit of slow diffusivity $\sigma \to 0$ (and under the simplifying assumption that $\chi = 1$), we get $u(t,x) \equiv p(t,x)$ and (1) is equivalent to an equation with *porous medium-type diffusion* and logistic reaction

$$u_t - \frac{1}{2}(u^2)_{xx} = u(1-u). \tag{3}$$

The propagation dynamics for this kind of equation was first studied, to the extent of our knowledge, by Aronson [2], Atkinson, Reuter and Ridler-Rowe [3], and later by de Pablo and Vázquez [15], in the more general context of nonlinear diffusion

$$u_t = (u^m)_{xx} + u(1-u), \text{ with } m > 1. \tag{4}$$

We refer to the monograph of Vázquez [38] for a detailed study of solutions to porous medium equations.

The particular relation between the pressure $p(t,x)$ and the density $u(t,x)$ in (2) strongly reminds the celebrated model of chemotaxis studied by Patlak (1953) and Keller and Segel (1970) [34, 26, 27] (parabolic-parabolic Keller-Segel model) and, more specifically, the parabolic-elliptic Keller-Segel model which is derived from the former by a quasi-stationary assumption on the diffusion of the chemical [24]. Indeed Equation (2) can be formally obtained as the quasistatic approximation of the following parabolic equation

$$\varepsilon \partial_t p(t,x) = \chi p_{xx}(t,x) + u(t,x) - p(t,x),$$

when $\varepsilon \to 0$.

A rigorous derivation of the limit has been achieved in the case of the Keller-Segel model by Carrapatoso and Mischler [10]. We refer to [9, 23, 35] and the references therein for a mathematical introduction and biological applications. In these models, the field $p(t,x)$ is interpreted as the concentration of a chemical produced by the cells rather than a physical pressure. One of the difficulties in attractive chemotaxis models is that two opposite forces compete to drive the behavior of the equations: the *diffusion* due to the random motion of cells, on the one hand, and on the other hand the *non-local advection* due to the attractive chemotaxis; the former tends to regularize and homogenize the solution, while the latter promotes cell aggregation and may lead to the blow-up of the solution in finite time [14, 24].

Since the pressure $p(t,x)$ is a non-local function of the density $u(t,x)$ in (2), the spatial derivative appears as a *non-local advection* term in (1). In fact, our problem (1)–(2) can be rewritten as a transport equation in which the speed of particles is non-local in the density,

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x (u(t,x) \partial_x (\rho \star u)(t,x)) = u(t,x)(1-u(t,x)) \\ u(t=0,x) = u_0(x), \end{cases} \tag{5}$$

where

$$(\rho \star u)(x) = \int_{\mathbb{R}} \rho(x-y) u(t,y) dy, \quad \rho(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}. \tag{6}$$

Traveling waves for a similar diffusive equation with logistic reaction have been investigated for quite general non-local kernels by Hamel and Henderson [22], who considered the model

$$u_t + (u\,(K \star u))_x = u_{xx} + u(1-u), \tag{7}$$

where $K \in L^p(\mathbb{R})$ is odd and $p \in [1,\infty]$. Notice that the attractive parabolic-elliptic Keller-Segel model is included in this framework by the particular choice

$$K(x) = -\chi \operatorname{sign}(x) e^{-|x|/\sqrt{d}} / \big(2\sqrt{d}\big).$$

They proved a spreading result for this equation (initially compactly supported solutions to the Cauchy problem propagate to the whole space with constant speed) and explicit bounds on the speed of propagation. Diffusive non-local advection also appears in the context of swarm formation [29]. Pattern formation for a model similar to (7) by Ducrot, Fu and Magal [16]. Let us mention that the inviscid equation (5) has been studied in a periodic cell by Ducrot and Magal [17]. A substantial literature has been produced for conservative systems of interacting particles and their kinetic limit (Balagué et al. [4], Carrillo et al. [12], Bernoff and Topaz [6], Bertozzi, Laurent and Rosado [7], among others).

This paper is a part of a set of two papers. Here we study the well-posed character of the Cauchy problem (1)–(2). In a forthcoming paper, we will build on these results to study the propagation dynamics of compactly supported initial conditions and the existence of sharp discontinuous traveling waves for the model (1)–(2).

In this paper we focus on the particular case of (1)–(2) with $\sigma > 0$ and $\chi > 0$. The paper is organized as follows. In Section 2, we present our main results. Section 3 is devoted to the well-posedness of the Cauchy problem for system (1)–(2).

2. Main results.

We begin by defining our notion of solution to equation (1).

Definition 2.1 (Integrated solutions)**.** Let $u_0 \in L^\infty(\mathbb{R})$. A measurable function $u(t,x) \in L^\infty([0,T]\times\mathbb{R})$ is an *integrated solution* to (1) if the characteristic equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} h(t,x) = -\chi(\rho_x \star u)(t, h(t,x)) \\ h(t=0,x) = x. \end{cases} \tag{8}$$

has a classical solution $h(t,x)$ (*i.e.* for each $x \in \mathbb{R}$ fixed, the function $t \mapsto h(t,x)$ is in $C^1([0,T],\mathbb{R})$ and satisfies (8)), and for a.e. $x \in \mathbb{R}$, the function $t \mapsto u(t,h(t,x))$ is in $C^1([0,T],\mathbb{R})$ and satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u(t,h(t,x)) = u(t,h(t,x))\big(1+\hat{\chi}(\rho\star u)(t,h(t,x)) - (1+\hat{\chi})u(t,h(t,x))\big), \\ u(t=0,x) = u_0(x), \end{cases} \tag{9}$$

where $\hat{\chi} := \frac{\chi}{\sigma^2}$.

We define weighted space $L^1_\eta(\mathbb{R})$ as follows

$$L^1_\eta(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} \,\middle|\, \int_{\mathbb{R}} |f(x)| e^{-\eta|x|} \mathrm{d}x < \infty \right\}.$$

$L^1_\eta(\mathbb{R})$ is a Banach space endowed with the norm

$$\|f\|_{L^1_\eta} := \frac{\eta}{2} \int_{\mathbb{R}} |f(y)| e^{-\eta|y|} \mathrm{d}y.$$

Our first result concerns the existence of integrated solutions to (1).

Theorem 2.2 (Well-posedness)**.** *Let $u_0 \in L^\infty_+(\mathbb{R})$ and fix $\eta > 0$. There exists $\tau^*(u_0) \in (0,+\infty]$ such that for all $\tau \in (0,\tau^*(u_0))$, there exists a unique integrated solution $u \in C^0([0,\tau], L^1_\eta(\mathbb{R}))$ to (1) which satisfies $u(t=0,x) = u_0(x)$. Moreover $u(t,\cdot) \in L^\infty(\mathbb{R})$ for each $t \in [0,\tau^*(u_0))$ and the map $t \in [0,\tau^*(u_0)) \mapsto T_t u_0 := u(t,\cdot)$ is a semigroup which is continuous for the $L^1_\eta(\mathbb{R})$-topology. The map $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R})$ is continuous.*

Finally, if $0 \le u_0(x) \le 1$, then $\tau^*(u_0) = +\infty$ and $0 \le u(t,\cdot) \le 1$ for all $t > 0$.

Next we show that the semiflow preserves some properties satisfied by the initial condition, namely the monotony, continuity and continuous differentiability. In the case of a $C^1(\mathbb{R})$ initial condition, we show that the solution integrated along the characteristics is actually a classical pointwise solution to the original problem (1)–(2).

Proposition 1 (Regularity of solutions)**.** *Let $u(t,x)$ be an integrated solution to (1).*

1. *if $u_0(x)$ is continuous, then $u(t,x)$ is continuous for each $t > 0$.*
2. *if $u_0(x)$ is monotone, then $u(t,x)$ has the same monotony for each $t > 0$.*
3. *if $u_0(x) \in C^1(\mathbb{R})$, then $u \in C^1([0,T]\times\mathbb{R})$ and u is then a classical solution to (1)–(2).*

Next we show the long-time behavior of the solutions to (1).

Theorem 2.3 (Long-time behavior). *Let $0 \leq u_0(x) \leq 1$ be a nontrivial non-negative initial condition and $u(t,x)$ be the corresponding integrated solution. Then $0 \leq u(t,x) \leq 1$ for all $t > 0$ and $x \in \mathbb{R}$. If moreover there exists $\delta > 0$ such that $\delta \leq u_0(x) \leq 1$ then*

$$u(t,x) \to 1, \text{ as } t \to \infty$$

and the convergence holds uniformly in $x \in \mathbb{R}$.

The case of bounded initial conditions which are not positively bounded from below is more complex. In the case of initial conditions which are compactly supported, we expect that the support will expand to the whole space with constant speed and that the profile of the solution reaches an asymptotic shape (traveling wave). This situation will be investigated in a forthcoming paper.

3. **Well-posedness of the Cauchy problem.** In this section we investigate the existence and uniqueness of solutions for the system (8)-(9). The idea to construct a fixed point problem is to consider the two variables

$$w(t,x) = u(t,h(t,x)) \text{ and } p(t,x) = (\rho \star u)(t,x).$$

Before we state the theorem, let us introduce some functional spaces and definitions. We introduce the following weighted L^1 space for any $\eta > 0$, as

$$L^1_\eta(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} \,\middle|\, \int_{\mathbb{R}} |f(x)| e^{-\eta|x|} \mathrm{d}x < \infty \right\},$$

endowed with the norm $\|f\|_{L^1_\eta} := \frac{\eta}{2} \int_{\mathbb{R}} |f(y)| e^{-\eta|y|} \mathrm{d}y$. Then for any $\eta > 0$ the space $L^1_\eta(\mathbb{R})$ is a Banach space and for any $0 < \eta < \eta' < +\infty$ we have

$$L^\infty(\mathbb{R}) \subset L^1_\eta(\mathbb{R}) \subset L^1_{\eta'}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R}).$$

We will say that a measurable set $\mathcal{U} \subset \mathbb{R}$ is *conull* if $|\mathbb{R} \backslash \mathcal{U}| = 0$, where $|A|$ is the Lebesgue measure of the set A. In what follows we need to work in the space of regular bounded functions on a measurable set $\mathcal{U} \subset \mathbb{R}$. Let us recall that the space

$$\mathcal{L}^\infty(\mathcal{U}) := \left\{ f : \mathcal{U} \to \mathbb{R} \,\middle|\, \sup_{x \in \mathcal{U}} |f(x)| < +\infty \right\},$$

endowed with the norm $\|f\|_{\mathcal{L}^\infty(\mathcal{U})} := \sup_{x \in \mathcal{U}} |f(x)|$, is a Banach space. If $\mathcal{U}$ is conull then $\mathcal{L}^\infty(\mathcal{U})$ is continuously embedded in $L^\infty(\mathbb{R})$ since

$$\|f\|_{L^\infty(\mathbb{R})} \leq \|f\|_{\mathcal{L}^\infty(\mathcal{U})}.$$

Finally we introduce the fixed point problem which is the key element of our proof of Theorem 2.2. Let $\tau > 0$ and $\mathcal{U} \subset \mathbb{R}$ be a conull set, we introduce the function spaces:

$$\begin{aligned} X^\tau_{\mathcal{U}} &:= C^0\big([0,\tau], \mathcal{L}^\infty(\mathcal{U})\big), \quad \tilde{X}^\tau_{\mathcal{U}} := C^0\big([0,\tau], \mathcal{L}^\infty_+(\mathcal{U})\big), \\ Y^\tau &:= C^0\big([0,\tau], W^{1,\infty}(\mathbb{R})\big), \\ \tilde{Y}^\tau &:= \{p \in Y^\tau \,|\, p(t,\cdot) \in W^{2,\infty}(\mathbb{R}) \text{ for all } t \in [0,\tau] \\ &\quad \text{and } \sup_{t \in [0,\tau]} \|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty\}, \\ Z^\tau_{\mathcal{U}} &:= X^\tau_{\mathcal{U}} \times Y^\tau, \quad \tilde{Z}^\tau_{\mathcal{U}} := \tilde{X}^\tau_{\mathcal{U}} \times \tilde{Y}^\tau. \end{aligned} \tag{10}$$

Clearly, $\tilde{X}^\tau_{\mathcal{U}}$ is closed in the Banach space $C^0\big([0,\tau],\mathcal{L}^\infty(\mathcal{U})\big)$. $\tilde{Y}^\tau$ is not closed in $C^0\big([0,\tau],W^{1,\infty}(\mathbb{R})\big)$, however for each $K>0$, the set

$$\tilde{Y}^\tau_K := \{p\in\tilde{Y}^\tau \mid \sup_{t\in[0,\tau]}\|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})}\leq K\} \tag{11}$$

is closed in Y^τ. Indeed, let $p^n(t,x)\to p(t,x)$ be a converging sequence in Y^τ. Since $C^0\big([0,\tau],W^{1,\infty}(\mathbb{R})\big)$ is a Banach space we have $p\in C^0\big([0,\tau],W^{1,\infty}(\mathbb{R})\big)$. Moreover for each $t\in[0,\tau]$ there exists a measurable set $E_t\subset\mathbb{R}$ such that $\int_{\mathbb{R}\setminus E_t}1\mathrm{d}x=0$, $p^n_x(t,x)$ and $p_x(t,x)$ are well-defined for any $x\in E_t$ and $\lim_{n\to+\infty}p^n_x(t,x)=p_x(t,x)$ for each $x\in E$. Let $x,y\in E_t$, we have:

$$\begin{aligned}|p_x(t,x)-p_x(t,y)| &\leq |p_x(t,x)-p^n_x(t,x)|+|p^n_x(t,x)-p^n_x(t,y)|+|p^n_x(t,y)-p^n_x(t,y)|\\ &\leq |p_x(t,x)-p^n_x(t,x)|+K|x-y|+|p^n_x(t,y)-p^n_x(t,y)|.\end{aligned}$$

Taking the limit $n\to\infty$, we obtain

$$|p_x(t,x)-p_x(t,y)|\leq K|x-y|$$

hence $\|p_{xx}\|_{L^\infty}\leq K$ and $p\in\tilde{Y}^\tau_K$.

Given $p\in\tilde{Y}^\tau$, let h be the solution of the following equation

$$\begin{cases}\frac{\partial}{\partial t}h(t,s;x)=-\chi p_x(t,h(t,s;x)),\\ h(s,s;x)=x.\end{cases} \tag{12}$$

The existence of the solution h is ensured by $p\in\tilde{Y}^\tau$. Moreover,

(i) for any x, the mapping $t\mapsto p_x(t,x)$ is continuous;
(ii) the vector field $p_x(t,x)$ is Lipschitz continuous with respect to x and the Lipschitz coefficient is uniform with respect to t on $[0,\tau]$. *In particular the image of $\mathcal{U}$ by $h(t,s;\cdot)$ is still conull for any $t,s\in[0,\tau]$.*

We are now in the position to define the mapping $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ to which we aim at applying a fixed-point theorem:

$$\mathcal{T}^\tau_{\mathcal{U}}[u_0](w,p)(t,x)=\begin{pmatrix}u_0(x)\exp\Big(\int_0^t 1+\hat{\chi}p(l,h(l,0;x))-(1+\hat{\chi})w(l,x)\mathrm{d}l\Big)\\ \int_{\mathbb{R}}\rho(x-h(t,0;z))u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\end{pmatrix}^T, \tag{13}$$

where

$$(w,p)\in Z^\tau_{\mathcal{U}}:=X^\tau_{\mathcal{U}}\times Y^\tau.$$

Remark 1. In formula (13), the function h must be understood as the solution of (12) where p the argument of the function $\mathcal{T}^\tau_{\mathcal{U}}[u_0](w,p)$.

Remark 2. Since we only impose u_0 to be in L^∞ the time of local existence will depend on each value $u_0(x)$. That is why we are not considering the class of functions L^∞ for $w(t,\cdot)$. Instead we work in the space $\mathcal{L}^\infty(\mathcal{U})$ for $w(t,\cdot)$.

Our first result is the well-definition of $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$. We start with a series technical Lemma.

Lemma 3.1 (Lipschitz continuity of the characteristic flow)**.** *Let $\tau>0$, $K>0$ and $p\in\tilde{Y}^\tau_K$ be given (recall that by definition of $\tilde{Y}^\tau_K$, p_{xx} is uniformly bounded: $\sup_{t\in[0,\tau]}\|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})}\leq K<+\infty$). Then, the solution $h(t,s;x)$ to (12) satisfies*

$$|h(t,s;x)-h(t,s;y)|\leq e^{K\chi|t-s|}|x-y|. \tag{14}$$

Proof. The integrated form of (12) is

$$h(t,s;x) = x + \int_s^t -\chi p_x(l, h(l,x;x))\mathrm{d}l,$$

therefore

$$\begin{aligned}|h(t,s;x)-h(t,s;y)| &\le |x-y| + \chi\int_s^t |p_x(t,h(t,s;x)) - p_x(t,h(t,s;y))|\mathrm{d}y\\ &\le |x-y| + \chi \sup_{t\in[0,\tau]} \|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})}\int_s^t |h(l,s;x)-h(l,s;y)|\mathrm{d}y\\ &\le |x-y| + K\chi\int_s^t |h(l,s;x)-h(l,s;y)|\mathrm{d}y,\end{aligned}$$

since $p\in\tilde{Y}_K^\tau$. Grönwall's inequality [13, Lemma 4.2.1] implies:

$$|h(t,s;x)-h(t,s;y)| \le e^{K\chi|t-s|}|x-y|.$$

Lemma 3.1 is proved. ☐

Lemma 3.2. *Let $\tilde{p}, p \in \tilde{Y}_K^\tau$ (where $\tilde{Y}_K^\tau$ is defined as in (11)) and $\tilde{h}, h$ be the corresponding characteristic flows defined in (12) with p and $\tilde{p}$ respectively. Then for any $\tau > 0$ and $t, s \in [0,\tau]$ we have*

$$\|\tilde{h}(t,s;\cdot)-h(t,s;\cdot)\|_{L^\infty(\mathbb{R})} \le |t-s|\chi \sup_{l\in[0,\tau]}\|\tilde{p}_x(l,\cdot)-p_x(l,\cdot)\|_{L^\infty(\mathbb{R})}e^{K\chi|t-s|}$$

Proof. Without loss of generality we suppose $t\ge s$, then

$$\begin{aligned}\partial_t\big(\tilde{h}(t,s;x)-h(t,s;x)\big) &= -\chi\tilde{p}_x(t,\tilde{h}(t,s;x)) + \chi p_x(t,h(t,s;x))\\ &= -\chi\tilde{p}_x(t,\tilde{h}(t,s;x)) + \chi p_x(t,\tilde{h}(t,s;x)) - \chi p_x(t,\tilde{h}(t,s;x))\\ &\quad + \chi p_x(t,h(t,s;x)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|\tilde{h}(t,s;\cdot)-h(t,s;\cdot)\|_{L^\infty(\mathbb{R})}\\ &\qquad \le |t-s|\chi\sup_{l\in[s,t]}\|p_x(l,\cdot)-\tilde{p}_x(l,\cdot)\|_{L^\infty(\mathbb{R})}\\ &\qquad\quad + \chi\sup_{l\in[0,\tau]}\|p_{xx}(l,\cdot)\|_{L^\infty(\mathbb{R})}\int_s^t\|\tilde{h}(l,s;\cdot)-h(l,s;\cdot)\|_{L^\infty(\mathbb{R})}\mathrm{d}l.\end{aligned}$$

The result follows from Grönwall's inequality and the definition of $\tilde{Y}_K^\tau$. ☐

Lemma 3.3 (Continuity properties)**.** *Let $(w,p)\in\tilde{Z}_{\mathcal{U}}^\tau$ be given. Then, the function $u(t,x) := w(t,h(0,t;x))$, defined for each $t\in[0,\tau]$ and a.e. $x\in\mathbb{R}$, is a continuous function of time for the $L^1_\eta(\mathbb{R})$ topology (i.e., the map $t\mapsto u(t,\cdot)$ is continuous in $L^1_\eta(\mathbb{R})$). The maps $t\mapsto(\rho\star u)(t,\cdot)$ and $t\mapsto(\rho_x\star u)(t,\cdot)$ are continuous for the $C^0_b(\mathbb{R})$ topology and moreover $(\rho\star u)(t,\cdot)\in W^{2,\infty}(\mathbb{R})$ for all $t\in[0,\tau]$.*

Proof. Let $(w,p)\in\tilde{Z}_{\mathcal{U}}^\tau$ be given. We first remark that, since p_x is Lipschitz continuous, the function $h(t,s;\cdot)$ is locally Lipschitz continuous for all $t,s\in[0,\tau]$ and therefore $h(t,0;\mathcal{U})$ is conull. In particular, $u(t,x)$ is well-defined for every $x\in h(t,0;\mathcal{U})$, therefore almost everywhere, for each $t\in[0,\tau]$.

We divide the rest of the proof in two steps.

Step 1. We show the continuity of $t\mapsto u(t,\cdot)$.

Let $t \in [0,\tau]$ and $\varepsilon > 0$ be given. For $s \in [0,\tau]$, we have:

$$\begin{aligned}\|u(t,\cdot)-u(s,\cdot)\|_{L^1_\eta} &= \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - w(s,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\le \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\quad + \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,s;x)) - w(s,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x.\end{aligned}$$

By the continuity of $t \mapsto w(t,\cdot)$ in $\mathcal{L}^\infty(\mathcal{U})$, there is $\delta_0 > 0$ such that if $|t-s| \le \delta_0$, then $\|w(t,\cdot)-w(s,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \le \frac{\varepsilon}{2}$. Therefore if $|t-s| \le \delta_0$,

$$\begin{aligned}&\|u(t,\cdot)-u(s,\cdot)\|_{L^1_\eta} \\ &\quad \le \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x + \|w(t,\cdot)-w(s,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \\ &\quad \le \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x + \frac{\varepsilon}{2}.\end{aligned}$$

Next we select $R > 0$ sufficiently large, so that

$$\begin{aligned}&\min(h(s,0;R), -h(s,0;-R)) \\ &\qquad \ge \frac{-1}{\eta}\ln\left(\frac{\varepsilon}{18\sup_{t\in[0,\tau]}\|w\|_{\mathcal{L}^\infty(\mathcal{U})}}\right) \text{ for all } s \in [t-\delta_0, t+\delta_0].\end{aligned}$$

By the density of compactly supported smooth function in $L^1(-R,R)$, there is $\varphi \in C^1_c([-R,R])$ such that

$$\|w-\varphi\|_{L^1(-R,R)} \le \frac{\varepsilon}{18\eta} e^{-K\chi(t+\delta_0)}.$$

Then, we have:

$$\begin{aligned}\|u(t,\cdot)-u(s,\cdot)\|_{L^1_\eta} &\le \frac{\varepsilon}{2} + \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\le \frac{\varepsilon}{2} + \frac{\eta}{2}\int_{\mathbb{R}} |w(t,h(0,t;x)) - \varphi(h(0,t;x))| e^{-\eta|x|}\mathrm{d}x \qquad (15) \\ &\quad + \frac{\eta}{2}\int_{\mathbb{R}} |\varphi(h(0,t;x)) - \varphi(h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \qquad (16) \\ &\quad + \frac{\eta}{2}\int_{\mathbb{R}} |\varphi(h(0,s;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x. \qquad (17)\end{aligned}$$

Next we estimate (16) and (17) (remark that (15) is a particular case of (17), for $s = t$), starting with (17). We have

$$\begin{aligned}&\frac{\eta}{2}\int_{\mathbb{R}} |\varphi(h(0,s;x)) - w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\qquad = \frac{\eta}{2}\int_{-\infty}^{h(s,0;-R)} |w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\qquad\quad + \frac{\eta}{2}\int_{h(s,0;-R)}^{h(s,0;R)} |w(t,h(0,s;x)) - \varphi(h(0,s;x))| e^{-\eta|x|}\mathrm{d}x \\ &\qquad\quad + \frac{\eta}{2}\int_{h(s,0;R)}^{+\infty} |w(t,h(0,s;x))| e^{-\eta|x|}\mathrm{d}x,\end{aligned}$$

then:

$$\frac{\eta}{2}\int_{h(s,0;R)}^{+\infty} |w(t,h(0,s;x))|e^{-\eta|x|}\mathrm{d}x \leq \sup_{t\in[0,\tau]} \|w\|_{\mathcal{L}^\infty} \frac{\eta}{2}\left[\frac{e^{-\eta x}}{-\eta}\right]_{h(s,0;R)}^{+\infty}$$

$$= \sup_{t\in[0,\tau]} \|w\|_{\mathcal{L}^\infty} \frac{e^{-\eta h(s,0;R)}}{2} \leq \frac{\varepsilon}{36}.$$

Similarly, we have

$$\frac{\eta}{2}\int_{-\infty}^{h(s,0;-R)} |w(t,h(0,s;x))|e^{-\eta|x|}\mathrm{d}x \leq \frac{\varepsilon}{36}.$$

Moreover, changing the variable in the integral, we have

$$\frac{\eta}{2}\int_{h(s,0;-R)}^{h(s,0;R)} |w(t,h(0,s;x)) - \varphi(h(0,s;x))|e^{-\eta|x|}\mathrm{d}x$$

$$= \frac{\eta}{2}\int_{-R}^{R} |w(t,y)-\varphi(y)|e^{-\eta|h(s,0;y)|}|h_x(s,0;y)|\mathrm{d}y$$

$$\leq \frac{\eta}{2}e^{K\chi s}\|w-\varphi\|_{L^1(-R,R)} \leq \frac{\eta}{2}e^{K\chi(s-t-\delta_0)}\frac{\varepsilon}{18\eta} \leq \frac{\varepsilon}{36},$$

where we recall that $|h_x| \leq e^{K\chi|t-s|}$ by (14) and $s \leq t+\delta_0$. We have shown that

$$\frac{\eta}{2}\int_{\mathbb{R}} |\varphi(h(0,s;x)) - w(t,h(0,s;x))|e^{-\eta|x|}\mathrm{d}x \leq \frac{\varepsilon}{12},$$

for each $s \in (t-\delta_0, t+\delta_0)$, which is our desired estimate for (17) (and therefore for (15)).

Next we estimate (16). Let

$$R' := \sup_{s\in(t-\delta_0,t+\delta_0)} \max\big(h(s,0;R), -h(s,0;-R)\big),$$

which is well-defined by the continuity of $s \mapsto h(s,0;\pm R)$ on $[t-\delta_0, t+\delta_0]$. Then the functions $x \mapsto \varphi(h(0,s;x))$ have their support in $(-R', R')$ for any $s \in (t-\delta_0, t+\delta_0)$. In particular,

$$\frac{\eta}{2}\int_{\mathbb{R}} |\varphi(h(0,t;x)) - \varphi(h(0,s;x))|e^{-\eta|x|}\mathrm{d}x$$

$$\leq \frac{\eta}{2}\|\varphi'\|_{C^0(-R',R')}\int_{-R'}^{R'} |h(0,t;x)-h(0,s;x)|e^{-\eta|x|}\mathrm{d}x$$

$$\leq \|\varphi'\|_{C^0(-R',R')} \sup_{x\in[-R,R]} |h(t,0;x)-h(s,0;x)|.$$

Since $(s,x) \mapsto h(s,0;x)$ is continuous on the compact set $[t-\delta_0, t+\delta_0]\times[-R', R']$, it is uniformly continuous on this set and there exists $\delta_1 > 0$ such that

$$\sup_{x\in[-R,R]} |h(t,0;x)-h(s,0;x)| \leq \frac{\varepsilon}{6\|\varphi'\|_{C^0(-R',R')}}$$

whenever $|t-s| \leq \delta_1$. This finishes our estimate of (16).

Summarizing, we have found $\delta_1 > 0$ such that for all $s \in [t-\delta_1, t+\delta_1]$, the inequality

$$\|u(t,\cdot) - u(s,\cdot)\|_{L^1_\eta(\mathbb{R})} \leq \varepsilon$$

holds. This finishes the proof of the continuity of $u(t,\cdot)$ in $L^1_\eta(\mathbb{R})$.

Step 2. Define $p(t,x) := (\rho \star u)(t,x) = \int_{\mathbb{R}} \rho(x-y)u(t,y)\mathrm{d}y$ in the scope of this Step. We first show that for any $t \in [0,T]$ we have $p(t,\cdot) \in W^{2,\infty}(\mathbb{R})$. Indeed, since $\rho \in W^{1,\infty}(\mathbb{R})$ it is classical that $p_x(t,x)$ exists for each $t \in [0,T]$ and $x \in \mathbb{R}$ and

$$p_x(t,x) = \int_{\mathbb{R}} \rho_x(x-y)u(t,y)\mathrm{d}y.$$

Next we remark that for $x \leq y$ we have

$$\begin{aligned}
|p_x(t,x) - p_x(t,y)| &= \left| \int_{\mathbb{R}} \big(\rho_x(x-z) - \rho_x(y-z)\big) u(t,z)\mathrm{d}z \right| \\
&\leq \int_{\mathbb{R}} |\rho_x(x-z) - \rho_x(y-z)|\mathrm{d}z \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
&\leq \int_{\mathbb{R}} |\rho_x(z) - \rho_x(y-x+z)|\mathrm{d}z \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \\
&= \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \times \frac{1}{2\sigma^2} \left[\int_{-\infty}^{x-y} -e^{z/\sigma} + e^{(y-x+z)/\sigma}\mathrm{d}z \right. \\
&\quad + \int_{x-y}^{0} e^{z/\sigma} + e^{(x-y-z)/\sigma}\mathrm{d}z \\
&\quad \left. + \int_{0}^{+\infty} e^{-z/\sigma} - e^{(x-y-z)/\sigma}\mathrm{d}z \right] \\
&= \frac{\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}}{2} \times 4 \left(1 - e^{-\frac{|x-y|}{\sigma}}\right) \leq \frac{2}{\sigma} \|u(t,\cdot)\|_{L^\infty(\mathbb{R})} |x-y|.
\end{aligned} \tag{18}$$

We deduce that

$$|p_x(t,x) - p_x(t,y)| \leq \frac{2}{\sigma}\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}|x-y|, \text{ for all } t \in [0,T].$$

In particular $p_x(t,\cdot)$ is globally Lipschitz continuous and thus $p(t,\cdot) \in W^{2,\infty}(\mathbb{R})$.

Next we prove that $p_x(t,x) = (\rho_x \star u)(t,x) \in C^0([0,T] \times \mathbb{R})$. Let $\varepsilon > 0$ and $R := \ln\left(\frac{6\|u\|_{L^\infty([0,T]\times\mathbb{R})}}{\varepsilon}\right)$, then we have $\|\rho_x\|_{L^1(\mathbb{R}\setminus(-R,R))} = \varepsilon/\big(6\|u\|_{L^\infty([0,T]\times\mathbb{R})}\big)$. Let $0 < s < t$, we have

$$\begin{aligned}
|p_x(t,x) - p_x(s,y)| &\leq |p_x(t,x) - p_x(t,y)| + |p_x(t,y) - p_x(s,y)| \\
&\leq \frac{2}{\sigma}\|u\|_{L^\infty([0,T]\times\mathbb{R})}|x-y| + \int_{(-R,R)} |\rho_x(y-z)|\big|u(t,z) - u(s,z)\big|\mathrm{d}z \\
&\quad + \int_{\mathbb{R}\setminus(-R,R)} \big|\rho_x(y-z)u(t,z) - \rho_x(y-z)u(s,z)\big|\mathrm{d}z \\
&\leq \frac{2}{\sigma}\|u\|_{L^\infty([0,T]\times\mathbb{R})}|x-y| + \|\rho\|_{L^\infty}\|u(t,\cdot) - u(s,\cdot)\|_{L^1\left((-R,R)\right)} \\
&\quad + \|\rho_x\|_{L^1(\mathbb{R}\setminus(-R,R))} \times 2\|u\|_{L^\infty([0,T]\times\mathbb{R})} \\
&\leq \frac{2}{\sigma}\|u\|_{L^\infty([0,T]\times\mathbb{R})}|x-y| + \|\rho\|_{L^\infty}\|u(t,\cdot) - u(s,\cdot)\|_{L^1\left((-R,R)\right)} + \frac{\varepsilon}{3}.
\end{aligned}$$

Hence, choosing $|x-y| \leq \frac{\sigma\varepsilon}{6\|u\|_{L^\infty([0,T]\times\mathbb{R})}}$ and $|t-s|$ sufficiently small so that the norm $\|u(t,.) - u(s,.)\|_{L^1\left((-R,R)\right)}$ is controlled by $\frac{\varepsilon}{3\|\rho\|_{L^\infty}}$ we have

$$|p_x(t,x) - p_x(s,y)| \leq \varepsilon.$$

Hence p_x is continuous. The continuity of $t \mapsto p(t,\cdot)$ in $L^\infty(\mathbb{R})$ can be shown similarly. □

Theorem 3.4 (Local existence and uniqueness of solutions)**.** *Let $\mathcal{U}$ be conull and $u_0 \in \mathcal{L}^\infty(\mathcal{U})$ be given. There exists $\tau > 0$ such that $\mathcal{T}_{\mathcal{U}}^\tau[u_0]$ has a unique fixed point in $\tilde{Z}^\tau$. Moreover τ can be chosen as a continuous function $\tau\left(\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}\right)$ of $\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}$ and the mapping $u_0 \in \mathcal{L}^\infty(\mathcal{U}) \mapsto (w(t,x), p(t,x)) \in \tilde{Z}^\tau$ is continuous in a neighborhood of u_0.*

Proof. We divide the proof in three steps.

Step 1. Stability of $\tilde{Z}_{\mathcal{U}}^\tau$ by $T_{\mathcal{U}}^\tau[u_0]$. We show that $\mathcal{T}_{\mathcal{U}}^\tau[u_0](\tilde{Z}_{\mathcal{U}}^\tau) \subset \tilde{Z}_{\mathcal{U}}^\tau$. Define $(w^1, p^1) := \mathcal{T}_{\mathcal{U}}^\tau[u_0](w,p)$. We first prove $w^1 \in X^\tau = C([0,\tau], \mathcal{L}^\infty(\mathcal{U}))$. By definition we have

$$
\begin{aligned}
w^1(t,\cdot) - w^1(s,\cdot) = u_0(\cdot) \exp\left(\int_0^t 1 + \hat{\chi} p(l, h(l,0;\cdot)) - (1+\hat{\chi}) w(l,\cdot) \mathrm{d}l\right) \\
- u_0(\cdot) \exp\left(\int_0^s 1 + \hat{\chi} p(l, h(l,0;\cdot)) - (1+\hat{\chi}) w(l,\cdot) \mathrm{d}l\right).
\end{aligned}
$$

Let us denote $\Theta[u] := |u| e^{|u|},\ u \in \mathbb{R}$ and recall the inequality $e^u - 1 \le |u| e^{|u|} = \Theta[u]$ for all $u \in \mathbb{R}$. We have

$$
\begin{aligned}
&\Bigg\| u_0(\cdot) \exp\left(\int_0^t 1 + \hat{\chi} p(l, h(l,0;\cdot)) - (1+\hat{\chi}) w(l,\cdot) \mathrm{d}l\right) \\
&\quad - u_0(\cdot) \exp\left(\int_0^s 1 + \hat{\chi} p(l, h(l,0;\cdot)) - (1+\hat{\chi}) w(l,\cdot) \mathrm{d}l\right) \Bigg\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\qquad = \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} e^{s\left(1+\hat{\chi}\|p\|_{L^\infty((0,\tau)\times\mathbb{R})}\right)} \\
&\quad \times \left\| \exp\left(\int_s^t 1 + \hat{\chi} p(l, h(l,0;\cdot)) - (1+\hat{\chi}) w(l,\cdot) \mathrm{d}l\right) - 1 \right\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\qquad \le \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} e^{s\left(1+\hat{\chi}\|p\|_{L^\infty((0,\tau)\times\mathbb{R})}\right)} \\
&\quad \times \Theta\Big[(t-s)\big(1 + \hat{\chi}\|p\|_{L^\infty((0,\tau)\times\mathbb{R})} + (1+\hat{\chi}) \sup_{l\in[0,\tau]} \|w(l,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}\big)\Big].
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|w^1(t,\cdot) - w^1(s,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\quad \le \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} e^{s\left(1+\hat{\chi}\|p\|_{Y^\tau}\right)} \Theta\Big[(t-s)\big(1 + \hat{\chi}\|p\|_{Y^\tau} + (1+\hat{\chi})\|w\|_{X^\tau}\big)\Big]. \quad (19)
\end{aligned}
$$

Since $\chi[u] \to 0$ as $u \to 0$, the continuity of w^1 is proved.

Next we prove $p^1 \in \tilde{Y}^\tau$. Recall that, by definition of $\tilde{Y}^\tau$ (see (10)), the second derivative of p is uniformly bounded: $\sup_{t\in[0,\tau]} \|p_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} =: K < +\infty$. For any $t, s \in [0,\tau]$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|p^1(t,x) - p^1(s,x)\right| \\
&= \left| \int_{\mathbb{R}} \left(\rho(x - h(t,0;z)) e^{\int_0^t 1 - w(l,z)\mathrm{d}l} - \rho(x - h(s,0;z)) e^{\int_0^s 1 - w(l,z)\mathrm{d}l} \right) u_0(z) \right| \\
&\leq \|u_0\|_{L^\infty(\mathbb{R})} \left(\left\| e^{\int_0^t 1 - w(l,\cdot)\mathrm{d}l} - e^{\int_0^s 1 - w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho(x - h(t,0;z))| \mathrm{d}z \right. \\
&\quad \left. + \left\| e^{\int_0^s 1 - w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho(x - h(t,0;z)) - \rho(x - h(s,0;z))| \mathrm{d}z \right).
\end{aligned} \tag{20}$$

Since $p \in \tilde{Y}^\tau$ we have $\|p_{xx}\|_{L^\infty((0,\tau)\times\mathbb{R})} \leq K$ and thus, recalling the Lipschitz property of h (14),

$$\begin{aligned}
\left\| e^{\int_0^t 1 - w(l,\cdot)\mathrm{d}l} - e^{\int_0^s 1 - w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty} &\leq |t - s|(e^t + e^s) \left(1 + \sup_{t \in [0,\tau]} \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \right) \\
&\leq |t - s| 2e^\tau \left(1 + \|w\|_{X^\tau_{\mathcal{U}}} \right), \\
\int_{\mathbb{R}} |\rho(x - h(t,0;z))| \mathrm{d}z = \int_{\mathbb{R}} \rho(x - y) \partial_x h(0,t;y) \mathrm{d}y &\leq e^{K\chi t}.
\end{aligned} \tag{21}$$

where we have used the classical inequality

$$|e^x - e^y| \leq (e^x + e^y)|x - y| \text{ for all } x, y \in \mathbb{R}. \tag{22}$$

There remains to estimate the second term in the right-hand side of (20). Using (22) we have

$$\begin{aligned}
&\int_{\mathbb{R}} |\rho(x - h(t,0;z)) - \rho(x - h(s,0;z))| \mathrm{d}z \\
&\qquad = \frac{1}{2\sigma} \int_{\mathbb{R}} \left| e^{-\frac{|x - h(t,0;z)|}{\sigma}} - e^{-\frac{|x - h(s,0;z)|}{\sigma}} \right| \mathrm{d}z \\
&\qquad \leq \frac{1}{2\sigma} \int_{\mathbb{R}} \left(e^{-\frac{|x - h(t,0;z)|}{\sigma}} + e^{-\frac{|x - h(s,0;z)|}{\sigma}} \right) \sigma^{-1} |h(t,0;z) - h(s,0;z)| \mathrm{d}z \\
&\qquad \leq \frac{1}{2\sigma^2} \|h(t,0;\cdot) - h(s,0;\cdot)\|_{L^\infty(\mathbb{R})} \\
&\qquad\quad \times \left(\int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} h_x(0,t;y) \mathrm{d}y + \int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} h_x(0,s;y) \mathrm{d}y \right) \\
&\qquad \leq \sigma^{-1} \|h(t,0;\cdot) - h(s,0;\cdot)\|_{L^\infty(\mathbb{R})} (e^{K\chi t} + e^{K\chi s}) \\
&\qquad \leq 2\sigma^{-1} e^{K\chi\tau} \|h(t,0;\cdot) - h(s,0;\cdot)\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{23}$$

Moreover, since

$$h(t,0;x) - h(s,0;x) = -\int_s^t \chi p_x(l, h(l,0;x)) \mathrm{d}l, \tag{24}$$

we have $\|h(t,0;\cdot) - h(s,0;\cdot)\|_{L^\infty(\mathbb{R})} \leq |t - s| \chi \sup_{l \in [0,\tau]} \|p_x(t,\cdot)\|_{L^\infty(\mathbb{R})}$. Combining (20) and (23) we have

$$\left\|p^1(t,\cdot) - p^1(s,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq |t - s| \times 2e^{(K\chi+1)\tau} \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} \left(1 + \|w\|_{X^\tau_{\mathcal{U}}} + \sigma^{-1}\chi\|p\|_{Y^\tau} \right). \tag{25}$$

This proves $p^1 \in C([0,\tau], L^\infty(\mathbb{R}))$.

Similarly, we compute for any $t,s\in[0,\tau]$ and $x\in\mathbb{R}$:

$$\begin{aligned}\left|p_x^1(t,x)-p_x^1(s,x)\right| \leq |t-s|\times 2\sigma^{-1}e^{(K\chi+1)\tau}\|u\|_{L^\infty(\mathbb{R})}\left(1+\|w\|_{X_{\mathcal{U}}^\tau}\right)\\ +\|u\|_{L^\infty(\mathbb{R})}e^s\int_{\mathbb{R}}|\rho_x(x-h(t,0;z))-\rho_x(x-h(s,0;z))|\mathrm{d}z.\end{aligned} \tag{26}$$

In order to estimate the last term in (26), suppose first that $h(0,t;x)\leq h(0,s;x)$. We have

$$\begin{aligned}&\int_{\mathbb{R}}|\rho_x(x-h(t,0;z))-\rho_x(x-h(s,0;z))|\mathrm{d}z\\ &\qquad=\frac{1}{2\sigma^2}\int_{-\infty}^{h(0,t;x)}\left|-e^{-\frac{x-h(t,0;z)}{\sigma}}+e^{-\frac{x-h(s,0;z)}{\sigma}}\right|\mathrm{d}z\\ &\qquad\quad+\frac{1}{2\sigma^2}\int_{h(0,s;x)}^{\infty}\left|e^{\frac{x-h(t,0;z)}{\sigma}}-e^{\frac{x-h(s,0;z)}{\sigma}}\right|\mathrm{d}z\\ &\qquad\quad+\frac{1}{2\sigma^2}\int_{h(0,t;x)}^{h(0,s;x)}\left|e^{\frac{x-h(t,0;z)}{\sigma}}+e^{-\frac{x-h(s,0;z)}{\sigma}}\right|\mathrm{d}z.\end{aligned}$$

Using (22) and (21) we have

$$\begin{aligned}&\int_{\mathbb{R}}|\rho_x(x-h(t,0;z))-\rho_x(x-h(s,0;z))|\mathrm{d}z\\ &\leq\frac{1}{2\sigma^2}\int_{-\infty}^{h(0,t;x)}\left(e^{-\frac{|x-h(t,0;z)|}{\sigma}}+e^{-\frac{|x-h(s,0;z)|}{\sigma}}\right)|h(t,0;z)-h(s,0;z)|\mathrm{d}z\\ &\quad+\frac{1}{2\sigma^2}\int_{h(0,s;x)}^{\infty}\left(e^{-\frac{|x-h(t,0;z)|}{\sigma}}+e^{-\frac{|x-h(s,0;z)|}{\sigma}}\right)|h(t,0;z)-h(s,0;z)|\mathrm{d}z\\ &\quad+\frac{1}{2\sigma^2}\int_{h(0,t;x)}^{h(0,s;x)}\left|e^{\frac{x-h(t,0;z)}{\sigma}}+e^{-\frac{x-h(s,0;z)}{\sigma}}\right|\mathrm{d}z\\ &\leq\frac{1}{2\sigma^2}\|h(t,0;\cdot)-h(s,0;\cdot)\|_{L^\infty}\int_{\mathbb{R}}\left(e^{-\frac{|x-h(t,0;z)|}{\sigma}}+e^{-\frac{|x-h(s,0;z)|}{\sigma}}\right)\mathrm{d}z\\ &\quad+\frac{1}{2\sigma^2}\int_{h(0,t;x)}^{h(0,s;x)}2\mathrm{d}z\\ &\leq 2\sigma^{-1}e^{K\chi\tau}\|h(t,0;\cdot)-h(s,0;\cdot)\|_{L^\infty(\mathbb{R})}+\sigma^{-2}\|h(0,t;\cdot)-h(0,s;\cdot)\|_{L^\infty(\mathbb{R})}.\end{aligned} \tag{27}$$

Moreover by (24) we have $\|h(0,t;\cdot)-h(0,s;\cdot)\|_{L^\infty(\mathbb{R})}\leq|t-s|\chi\|p\|_{Y^\tau}$. Combining (26) and (27) we have

$$\begin{aligned}&\left\|p_x^1(t,\cdot)-p_x^1(s,\cdot)\right\|_{L^\infty(\mathbb{R})}\\ &\leq|t-s|\times\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}\sigma^{-1}\left(2e^{(K\chi+1)\tau}(1+\|w\|_{X^\tau})+\chi e^\tau(2e^{K\chi\tau}+\sigma^{-1})\|p\|_{Y^\tau}\right).\end{aligned} \tag{28}$$

This proves $p_x^1\in C([0,\tau],L^\infty(\mathbb{R}))$. According to (25) and (28) we have

$$\left\|p^1(t,\cdot)-p^1(s,\cdot)\right\|_{W^{1,\infty}(\mathbb{R})}\leq C|t-s|\times\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}e^{(K\chi+1)\tau}, \tag{29}$$

where C is a constant depending on σ, χ, $\|w\|_{X^\tau}$ and $\|p\|_{Y^\tau}$. Therefore $p^1\in Y^\tau$.

There remains to show that $\sup_{t\in[0,\tau]} \|p^1_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} < +\infty$. Let $t, s \in [0,\tau]$ and $x \in \mathbb{R}$. We have

$$\begin{aligned}
|p^1_x(t,x) - p^1_x(t,y)| &= \left| \int_{\mathbb{R}} \big(\rho_x(x - h(t,0;z)) - \rho_x(y - h(t,0;z))\big) u_0(z) e^{\int_0^t 1 - w(l,z)\mathrm{d}l} \mathrm{d}z \right| \\
&\leq \|u_0\|_{L^\infty(\mathbb{R})} e^t \int_{\mathbb{R}} |\rho_x(x-z) - \rho_x(y-z)| h_x(0,t;z) \mathrm{d}z \\
&\leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{L^\infty(\mathbb{R})} |x-y|.
\end{aligned}$$

Therefore

$$\sup_{t\in[0,\tau]} \|p^1_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} < +\infty. \tag{30}$$

We have shown the stability of $\tilde{Z}^\tau_{\mathcal{U}}$.

Step 2. Local stability of a vicinity. We show the stability of the set

$$\overline{B}_r := \{(w,p) \in \tilde{Z}^\tau_{\mathcal{U}} \mid \sup_{t\in[0,\tau]} \|u_0 - w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \leq r \text{ and } p \in \tilde{Y}^\tau_K \text{ and } \|p - (\rho \star u_0)\|_{Y^\tau} \leq r\}, \tag{31}$$

for any $r > 0$ and $\tau > 0$ sufficiently small, where $K := 4\sigma^{-1}\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}$. Note that $\overline{B}_r$ is closed in $Z^\tau_{\mathcal{U}}$ for any $r > 0$.

Let $(w,p) \in \overline{B}_r$, and define $\kappa := \|(u_0, \rho \star u_0)\|_{Z^\tau} + r$. By definition, we have

$$\|(w,p)\|_{\tilde{Z}^\tau} \leq \|u_0, \rho \star u_0\|_{\tilde{Z}^\tau} + r = \kappa.$$

On the one hand by (19) (with $s = 0$) we find that

$$\begin{aligned}
\sup_{t\in[0,\tau]} \|w^1(t,\cdot) - u_0(\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} &= \sup_{t\in[0,\tau]} \left\|w^1(t,\cdot) - w^1(0,\cdot)\right\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\leq \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} \Theta\Big[\tau\big(1 + \hat{\chi}\|p\|_{Y^\tau} + (1+\hat{\chi})\|w\|_{X^\tau}\big)\Big] \\
&\leq \kappa\chi\Big[\tau(1 + (1+2\hat{\chi})\kappa)\Big] \xrightarrow[\tau\to 0]{} 0 < r,
\end{aligned}$$

where $\Theta[u] = |u|e^{|u|}$. On the other hand, by (29) (with $s = 0$), for all $t \in [0,\tau]$,

$$\begin{aligned}
\|p^1(t,x) - (\rho \star u_0)(x)\|_{Y^\tau} &= \sup_{t\in[0,\tau]} \|p^1(t,\cdot) - p^1(0,\cdot)\|_{W^{1,\infty}(\mathbb{R})} \\
&\leq C\tau \times \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} e^{(K\chi+1)\tau}. \\
&\leq C\tau\kappa e^{(K\chi+1)\tau} \xrightarrow[\tau\to 0]{} 0 < r.
\end{aligned}$$

Finally by (30),

$$\begin{aligned}
\sup_{t\in[0,\tau]} \|p^1_{xx}(t,\cdot)\|_{L^\infty(\mathbb{R})} &\leq 2\sigma^{-1} e^{(K\chi+1)\tau} \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\xrightarrow[\tau\to 0]{} 2\sigma^{-1}\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} < 4\sigma^{-1}\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})} = K.
\end{aligned}$$

We conclude that for any $r > 0$ there is $\tau > 0$ sufficiently small so that the inclusion $\mathcal{T}^\tau_{\mathcal{U}}[u_0](\overline{B}_r) \subset \overline{B}_r$ holds.

Step 3. $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ is a contraction. More precisely, we show that $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ is contracting for τ sufficiently small.

Let $r>0$ be given and $\tau>0$ be sufficiently small so that $\overline{B}_r$ is left stable by $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$, and define $\kappa := \|(u_0,\rho\star u_0)\|_{Z^\tau}+r$ as in Step 2. Let $(w,p)\in\overline{B}_r$ and $(\tilde w,\tilde p)\in\overline{B_r}$ be given, we observe that for any $t,s\in[0,\tau]$ and $x\in\mathcal{U}$,

$$\begin{aligned}
&|\tilde w^1(t,x)-w^1(t,x)|\\
&\le \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}\left|e^{\int_0^t 1+\hat\chi p(l,h(l,0;x))-(1+\hat\chi)w(l,x)\mathrm{d}l}-e^{\int_0^t 1+\tilde p(l,\tilde h(l,0;x))-(1+\hat\chi)\tilde w(l,x)\mathrm{d}l}\right|\\
&\le \|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}e^{t(1+\hat\chi\|p\|_{Y^\tau})}\left|1-e^{\int_0^t \hat\chi\tilde p(l,\tilde h(l,0;x))-\hat\chi p(l,h(l,0;x))-(1+\hat\chi)(\tilde w(l,x)-w(l,x))\mathrm{d}l}\right|\\
&\le \kappa e^{\tau(1+\kappa\chi)}\left|1-e^{\int_0^t \hat\chi\tilde p(l,\tilde h(l,0;x))-\hat\chi p(l,h(l,0;x))-(1+\hat\chi)(\tilde w(l,x)-w(l,x))\mathrm{d}l}\right|\\
&\le \kappa e^{\tau(1+\kappa\chi)}\Theta\Big[\tau\Big(\hat\chi\sup_{l\in[0,\tau]}|\tilde p(l,\tilde h(l,0;x))-p(l,h(l,0;x))|+(1+\hat\chi)\|\tilde w-w\|_{X^\tau}\Big)\Big],
\end{aligned}$$

where we have used the inequality $|e^u-1|\le |u|e^{|u|}=:\Theta[u]$, $\forall u\in\mathbb{R}$. Moreover, we have

$$\begin{aligned}
&\sup_{l\in[0,\tau]}|\tilde p(l,\tilde h(l,0;x))-p(l,h(l,0;x))|\\
&\qquad\le \sup_{l\in[0,\tau]}\|\tilde p(l,\cdot)-p(l,\cdot)\|_{L^\infty(\mathbb{R})}+\sup_{l\in[0,\tau]}|p(l,\tilde h(l,0;x))-p(l,h(l,0;x))|\\
&\qquad\le \|\tilde p-p\|_{Y^\tau}+\sup_{l\in[0,\tau]}\|p_x(l,\cdot)\|_{L^\infty(\mathbb{R})}\sup_{l\in[0,\tau]}\|\tilde h(l,0;\cdot)-h(l,0;\cdot)\|_{L^\infty(\mathbb{R})}\\
&\qquad\le \|\tilde p-p\|_{Y^\tau}+\kappa\sup_{l\in[0,\tau]}\|\tilde h(l,0;\cdot)-h(l,0;\cdot)\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

According to Lemma 3.2 we have

$$\|\tilde h(t,0;\cdot)-h(t,0;\cdot)\|_{L^\infty(\mathbb{R})}\le \tau\chi\sup_{l\in[0,\tau]}\|\tilde p_x(l,\cdot)-p_x(l,\cdot)\|_{L^\infty(\mathbb{R})}e^{K\chi\tau},$$

which yields

$$\sup_{l\in[0,\tau]}|\tilde p(l,\tilde h(l,0;x))-p(l,h(l,0;x))|\le \|\tilde p-p\|_{\tilde Y^\tau}(1+\kappa\chi\tau e^{K\chi\tau}).$$

This implies

$$\|\tilde w^1-w^1\|_{\tilde X^\tau}\le e^{\tau(1+\kappa\chi)}\Theta\Big[\tau\Big(\hat\chi\|\tilde p-p\|_{\tilde Y^\tau}(1+\kappa\chi\tau e^{K\chi\tau})+(1+\hat\chi)\|\tilde w-w\|_{\tilde X^\tau}\Big)\Big]. \tag{32}$$

On the other hand, we have

$$\begin{aligned}
&\left|\tilde p^1(t,x)-p^1(t,x)\right|\\
&=\left|\int_{\mathbb{R}}\Big(\rho(x-\tilde h(t,0;z))e^{\int_0^t 1-\tilde w(l,z)\mathrm{d}l}-\rho(x-h(t,0;z))e^{\int_0^t 1-w(l,z)\mathrm{d}l}\Big)u_0(z)\mathrm{d}z\right|\\
&=\left|\int_{\mathbb{R}}\Big(\rho(x-\tilde h(t,0;z))\Big(e^{\int_0^t 1-\tilde w(l,z)\mathrm{d}l}-e^{\int_0^t 1-w(l,z)\mathrm{d}l}\Big)\right.\\
&\qquad\left.-\big(\rho(x-\tilde h(t,0;z))-\rho(x-h(t,0;z))\big)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\Big)u_0(z)\mathrm{d}z\right|\\
&\le \|u_0\|_{L^\infty(\mathbb{R})}\Big(\Big\|e^{\int_0^t 1-\tilde w(l,\cdot)\mathrm{d}l}-e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l}\Big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\rho(x-\tilde h(t,0;z))|\mathrm{d}z\\
&\qquad+\Big\|e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l}\Big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}|\rho(x-\tilde h(t,0;z))-\rho(x-h(t,0;z))|\mathrm{d}z\Big).
\end{aligned}$$

In order to estimate the term $\left\| e^{\int_0^t 1-\tilde{w}(l,\cdot)\mathrm{d}l} - e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})}$, we write

$$\begin{aligned}\left\| e^{\int_0^t 1-\tilde{w}(l,\cdot)\mathrm{d}l} - e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})} &\leq 2e^\tau \left\| \int_0^t \tilde{w}(l,\cdot) - w(l,\cdot)\mathrm{d}l \right\|_{\mathcal{L}^\infty(\mathcal{U})} \\ &\leq 2\tau e^\tau \left\| \tilde{w} - w \right\|_{X^\tau},\end{aligned}$$

where we have used (22). Next we notice that that $\tilde{p} \in \tilde{Y}^\tau$ implies the inequality $\|\tilde{p}_{xx}\|_{L^\infty((0,\tau)\times\mathbb{R})} \leq K$, thus we obtain by a change of variable (recall the Lipschitz continuity of $\tilde{h}$ by Lemma 3.1)

$$\int_{\mathbb{R}} |\rho(x - \tilde{h}(t,0;z))|\mathrm{d}z = \int_{\mathbb{R}} \rho(x-z)\partial_x \tilde{h}(0,t;z)\mathrm{d}z \leq e^{K\chi\tau}.$$

Finally we have

$$\begin{aligned}&\int_{\mathbb{R}} |\rho(x - \tilde{h}(t,0;z)) - \rho(x - h(t,0;z))|\mathrm{d}z \\ &\qquad = \frac{1}{2\sigma} \int_{\mathbb{R}} \left| e^{-\frac{|x-\tilde{h}(t,0;z)|}{\sigma}} - e^{-\frac{|x-h(t,0;z)|}{\sigma}} \right| \mathrm{d}z \\ &\qquad \leq \frac{1}{2\sigma} \int_{\mathbb{R}} \left(e^{-\frac{|x-\tilde{h}(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}} \right) |\tilde{h}(t,0;z) - h(t,0;z)|\mathrm{d}z \\ &\qquad \leq \|\tilde{h}(t,0;\cdot) - h(t,0;\cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|x-\tilde{h}(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}} \mathrm{d}z \\ &\qquad \leq \|\tilde{h}(t,0;\cdot) - h(t,0;\cdot)\|_{L^\infty(\mathbb{R})} (e^{K\chi t} + e^{K\chi t}) \\ &\qquad \leq 2e^{K\chi\tau} \|\tilde{h}(t,0;\cdot) - h(t,0;\cdot)\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Applying Lemma 3.2 yields

$$\begin{aligned}\int_{\mathbb{R}} |\rho(x - \tilde{h}(t,0;z)) - \rho(x - h(t,0;z))|\mathrm{d}z &\leq 2e^{K\tau} \|\tilde{h}(t,0;\cdot) - h(t,0;\cdot)\|_{L^\infty(\mathbb{R})} \\ &\leq 2\chi\tau e^{2K\chi\tau} \|\tilde{p} - p\|_{\tilde{Y}^\tau}.\end{aligned}$$

We have shown the following estimate on p:

$$\sup_{t\in[0,\tau]} \left\| \tilde{p}^1(t,\cdot) - p^1(t,\cdot) \right\|_{L^\infty(\mathbb{R})} \leq 2\kappa\tau e^{(K\chi+1)\tau} \left\| \tilde{w} - w \right\|_{\tilde{X}^\tau} + 2\kappa\chi\tau e^{(2K\chi+1)\tau} \|\tilde{p} - p\|_{\tilde{Y}^\tau}. \tag{33}$$

Next we estimate the gradient of p. We have:

$$\begin{aligned}&\left|\tilde{p}_x^1(t,x) - p_x^1(t,x)\right| \\ &= \left| \int_{\mathbb{R}} \left(\rho_x(x - \tilde{h}(t,0;z)) e^{\int_0^t 1-\tilde{w}(l,z)\mathrm{d}l} - \rho_x(x - h(t,0;z)) e^{\int_0^t 1-w(l,z)\mathrm{d}l} \right) u_0(z) \right| \\ &\leq \|u_0\|_{L^\infty(\mathbb{R})} \Bigg(\left\| e^{\int_0^t 1-\tilde{w}(l,\cdot)\mathrm{d}l} - e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho_x(x - \tilde{h}(t,0;z))|\mathrm{d}z \\ &\quad + \left\| e^{\int_0^t 1-w(l,\cdot)\mathrm{d}l} \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\rho_x(x - \tilde{h}(t,0;z)) - \rho_x(x - h(t,0;z))|\mathrm{d}z \Bigg) \\ &\leq 2\sigma^{-1}\kappa\tau e^{(K\chi+1)\tau} \left\| \tilde{w} - w \right\|_{X^\tau} + \kappa e^\tau \int_{\mathbb{R}} |\rho_x(x - \tilde{h}(t,0;z)) - \rho_x(x - h(t,0;z))|\mathrm{d}z.\end{aligned}$$

For the need of this computation, let us introduce $h^- := \min\big(\tilde h(0,t;x), h(0,t;x)\big)$ and $h^+ := \max\big(\tilde h(0,t;x), h(0,t;x)\big)$. We have:

$$\begin{aligned}
&\int_{\mathbb{R}} |\rho_x(x-\tilde h(t,0;z)) - \rho_x(x-h(t,0;z))|\mathrm{d}z \\
&\quad\le \frac{1}{2\sigma^2}\int_{-\infty}^{h^-}\big(e^{-\frac{|x-\tilde h(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}}\big)|\tilde h(t,0;z)-h(t,0;z)|\mathrm{d}z \\
&\qquad + \frac{1}{2\sigma^2}\int_{h^+}^{\infty}\big(e^{-\frac{|x-\tilde h(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}}\big)|\tilde h(t,0;z)-h(t,0;z)|\mathrm{d}z \\
&\qquad + \frac{1}{2\sigma^2}\int_{h^-}^{h^+}\big|e^{-\frac{|x-\tilde h(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}}\big|\mathrm{d}z \\
&\quad\le \frac{1}{2\sigma^2}\|\tilde h(t,0;\cdot)-h(t,0;\cdot)\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\big(e^{-\frac{|x-\tilde h(t,0;z)|}{\sigma}} + e^{-\frac{|x-h(t,0;z)|}{\sigma}}\big)\mathrm{d}z \\
&\qquad + \frac{1}{2\sigma^2}\int_{h^-}^{h^+} 2\mathrm{d}z \\
&\quad\le 2\sigma^{-1}e^{K\chi\tau}\|\tilde h(t,0;\cdot)-h(t,0;\cdot)\|_{L^\infty(\mathbb{R})} + \sigma^{-2}\|\tilde h(0,t;\cdot)-h(0,t;\cdot)\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

According to Lemma 3.2 we have then

$$\begin{aligned}
&\int_{\mathbb{R}} |\rho_x(x-\tilde h(t,0;z)) - \rho_x(x-h(t,0;z))|\mathrm{d}z \\
&\quad\le 2\sigma^{-1}e^{K\chi\tau}\|\tilde h(t,0;\cdot)-h(t,0;\cdot)\|_{L^\infty(\mathbb{R})} + \sigma^{-2}\|\tilde h(0,t;\cdot)-h(0,t;\cdot)\|_{L^\infty(\mathbb{R})} \\
&\quad\le \big(2\tau\sigma^{-1}\chi e^{2K\chi\tau} + \sigma^{-2}\chi\tau e^{K\chi\tau}\big)\|\tilde p - p\|_{Y^\tau}.
\end{aligned}$$

This implies

$$\begin{aligned}
\sup_{t\in[0,\tau]}\big\|\tilde p_x^1(t,\cdot) - p_x^1(t,\cdot)\big\|_{L^\infty(\mathbb{R})} &\le 2\sigma^{-1}\kappa\tau e^{(K\chi+1)\tau}\,\|\tilde w - w\|_{X^\tau} \\
&\quad + \big(2\kappa\chi\sigma^{-1}\tau e^{(2K\chi+1)\tau} + \kappa\chi\sigma^{-2}\tau e^{(K\chi+1)\tau}\big)\|\tilde p - p\|_{Y^\tau}. \qquad (34)
\end{aligned}$$

Combining (32), (33) and (34), there exists a mapping $\tau\mapsto L(\tau)$ with $L(\tau)\to 0$ as $\tau\to 0$ such that

$$\|\mathcal{T}_{\mathcal{U}}^\tau[u_0](\tilde w,\tilde p) - \mathcal{T}_{\mathcal{U}}^\tau[u_0](w,p)\|_{Z^\tau} \le L(\tau)\,\|(\tilde w,\tilde p)-(w,p)\|_{Z^\tau}. \qquad (35)$$

Thus for $\tau > 0$ sufficiently small we have $L(\tau) < 1$ in which case $\mathcal{T}_{\mathcal{U}}^\tau[u_0]$ is a contraction on the complete metric space $\overline{B}_r$ equipped with the topology induced by Z_τ. By the Banach contraction principle, there exists then a unique fixed point to $\mathcal{T}_{\mathcal{U}}^\tau[u_0]$. Moreover τ can be chosen as a continuous function of $\|u_0\|_{\mathcal{L}^\infty(\mathcal{U})}$.

Finally, the continuous dependency of (w,p) with respect to u_0 is a direct application of the continuous dependency of the fixed point with respect to a parameter [39, Proposition 1.2]. □

In order to show the semigroup property satisfied by (w,p) and to make the link with the integrated solutions to (1), we need the following technical Lemma.

Lemma 3.5 (The derivatives of p and h)**.** *Let $\mathcal{U}\subset\mathbb{R}$ be conull and $\tau>0$ be given. Let $(w,p)\in\tilde Z_{\mathcal{U}}^\tau$ be a fixed point of $\mathcal{T}_{\mathcal{U}}^\tau[u_0]$. Then there exists a conull set $\mathcal{U}'$ such that*

(i) *for any* $t, s \in [0, \tau]$, *the solution* $h(t, s; x)$ *to* (12) *is differentiable for each* $x \in h(s, 0; \mathcal{U}')$ *(therefore for almost every* $x \in \mathbb{R}$*) and we have*

$$h_x(t, s; x) = \exp\left(\hat{\chi}\int_s^t w(l, x) - p(l, h(l, s; x))\mathrm{d}l\right) \text{ for a.e. } x \in \mathcal{U}. \tag{36}$$

(ii) *for every* $t \in [0, \tau]$ *and* $x \in \mathbb{R}$ *we have*

$$p(t, x) = \int_{\mathbb{R}} \rho(x-y)w(t, h(0, t; y))\mathrm{d}y \text{ and } p_x(t, x) = \int_{\mathbb{R}} \rho_x(x-y)w(t, h(0, t; y))\mathrm{d}y.$$

(iii) *for every* $x \in \mathcal{U}'$, *the function* $p_x(t, \cdot)$ *is differentiable at* $h(t, 0; x)$ *and we have*

$$\sigma^2 p_{xx}(t, h(t, 0; x)) = p(t, h(t, 0; x)) - w(t, x).$$

Proof. We divide the proof in three steps.

Step 1. We prove item (i).
Let $x \le y$ and $t, s \in [0, \tau]$ be given, we first remark that

$$\begin{aligned}
&p_x(t, h(t, 0; y)) - p_x(t, h(t, 0; x))\\
&= \int_{\mathbb{R}} \big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&= \int_{-\infty}^{x} \big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&\quad + \int_{y}^{+\infty} \big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&\quad - \frac{1}{2\sigma^2}\int_x^y \big(e^{\frac{h(t,0;y)-h(t,0;z)}{\sigma}} + e^{\frac{-h(t,0;x)+h(t,0;z)}{\sigma}}\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&= \int_{-\infty}^{x} \big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&\quad + \int_{y}^{+\infty} \big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z\\
&\quad - \frac{1}{2\sigma^2}\int_x^y \big(e^{\frac{h(t,0;y)-h(t,0;z)}{\sigma}} + e^{\frac{-h(t,0;x)+h(t,0;z)}{\sigma}}\big)u_0(z)e^{\int_0^t w(l,z)\mathrm{d}l}\\
&\quad - 2u_0(x)e^{\int_0^t 1-w(l,x)\mathrm{d}l}\mathrm{d}z - \frac{(y-x)}{\sigma^2}u_0(x)e^{\int_0^t 1-w(l,x)\mathrm{d}l}\\
&=: f(t; x, y)(h(t, 0; y) - h(t, 0; x)) - g(t; x, y)
\end{aligned}$$

where

$$\begin{aligned}
f(t; x, y) := \left(\int_{-\infty}^{x} + \int_{y}^{+\infty}\right) &\frac{\big(\rho_x(h(t, 0; y) - h(t, 0; z)) - \rho_x(h(t, 0; x) - h(t, 0; z))\big)}{h(t, 0; y) - h(t, 0; x)}\\
&\times u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z
\end{aligned}$$

and

$$\begin{aligned}
g(t; x, y) := \frac{1}{2\sigma^2}\int_x^y &\big(e^{\frac{h(t,0;y)-h(t,0;z)}{\sigma}} + e^{\frac{-h(t,0;x)+h(t,0;z)}{\sigma}}\big)u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\\
&- 2u_0(x)e^{\int_0^t 1-w(l,x)\mathrm{d}l}\mathrm{d}z + \frac{(y-x)}{\sigma^2}u_0(x)e^{\int_0^t 1-w(l,x)\mathrm{d}l}.
\end{aligned}$$

Next we remark that, with those functions f and g, we have

$$\begin{aligned}
h(t,0;y)-h(t,0;x) &= y-x-\chi\int_0^t p_x(l,h(l,0;y))-p_x(l,h(l,0;x))\mathrm{d}l \\
&= y-x-\chi\int_s^t f(l;x,y)(h(l,0;y)-h(l,0;x))-g(l;x,y)\mathrm{d}l \\
&= (y-x)e^{-\chi\int_0^t f(l;x,y)\mathrm{d}l}+\chi\int_0^t g(\sigma;x,y)e^{-\chi\int_\sigma^t f(l;x,y)\mathrm{d}l}\mathrm{d}\sigma
\end{aligned}$$

For a given $x\in\mathbb{R}$, we have

$$f(t;x,y)\xrightarrow[y\to x]{}\frac{1}{\sigma^2}p(t,h(t,0;x))$$

uniformly in t, because of Lebesgue's dominated convergence theorem.

Next we remark that, given $t\in[0,\tau]$, if x is a Lebesgue point of the function $z\mapsto u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\in C^0([0,\tau],\mathcal{L}^\infty(\mathcal{U}))$, then $\frac{g(t;x,y)}{y-x}$ has a limit as $y\to x$ and

$$\lim_{y\to x}\frac{g(t;x,y)}{y-x}=\frac{1}{\sigma^2}u_0(x)e^{\int_0^t 1-w(l,x)\mathrm{d}x}.$$

Applying Lemma A.1, we conclude that there exists a conull set $\mathcal{U}'\subset\mathcal{U}$ on which $h(t,0;\cdot)$ is differentiable at every point $x\in\mathcal{U}'$ for all $t>0$ and we have

$$\begin{aligned}
h_x(t,0;x) &= e^{-\hat\chi\int_0^t p(l,h(l,0;x))\mathrm{d}l}+\chi\sigma^{-2}\int_0^t u_0(x)e^{\int_0^\sigma 1-w(l,x)\mathrm{d}l}e^{-\hat\chi\int_\sigma^t p(l,h(l,0;x))\mathrm{d}l}\mathrm{d}\sigma \\
&= e^{-\hat\chi\int_0^t p(l,h(l,0;x))\mathrm{d}l}\left(1+\hat\chi\int_0^t u_0(x)e^{\int_0^\sigma 1+\hat\chi p(l,h(l,0;x))-w(l,x)\mathrm{d}l}\mathrm{d}\sigma\right) \\
&= e^{-\hat\chi\int_0^t p(l,h(l,0;x))\mathrm{d}l}\left(1+\int_0^t \hat\chi w(\sigma,x)e^{\hat\chi\int_0^\sigma w(l,x)\mathrm{d}x}\mathrm{d}\sigma\right) \\
&= e^{-\int_0^t p(l,h(l,0;x))\mathrm{d}l}\left(1+\int_0^t\left(e^{\int_0^\sigma \hat\chi w(l,x)\mathrm{d}x}\right)'\mathrm{d}\sigma\right) \\
&= \exp\left(\hat\chi\int_0^t w(l,x)-p(l,h(l,0;x))\mathrm{d}l\right).
\end{aligned}$$

Since $h(0,t;x)=[h(t,0;\cdot)]^{-1}(x)$, the function $h(0,t;\cdot)$ is differentiable at each point $x\in h(t,0;\mathcal{U}')$ and

$$h_x(0,t;x)=\frac{1}{h_x(t,0;h(0,t;x))}=\exp\left(-\hat\chi\int_0^t w(l,h(0,t;x))-p(l,h(l,t;x))\mathrm{d}l\right).$$

The formula (36) can be deduced from the remark $h(t,s;x)=h(t,0;h(0,s;x))$, where the right-hand side is differentiable for all $x\in h(s,0;\mathcal{U}')$.

Step 2. We show item (ii).

We have, by definition,

$$p(t,x)=\int_{\mathbb{R}}\rho(x-h(t,0;z))u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z$$

and item (i) allows a change of variables which yields

$$\begin{aligned} p(t,x) &= \int_{\mathbb{R}} \rho(x-y)u_0(h(0,t;y))e^{\int_0^t 1-w(l,h(0,t;z))\mathrm{d}l} h_x(0,t;z)\mathrm{d}z \\ &= \int_{\mathbb{R}} \rho(x-y)u_0(h(0,t;y))e^{\int_0^t 1-w(l,h(0,t;z))\mathrm{d}l} e^{-\hat{\chi}\int_0^t w(l,h(0,t;y))-p(l,h(l,t;y))\mathrm{d}l}\mathrm{d}z \\ &= \int_{\mathbb{R}} \rho(x-y)u_0(h(0,t;y))e^{\int_0^t 1+\hat{\chi}p(l,h(l,t;x))-(1+\hat{\chi})w(l,h(0,t;z))\mathrm{d}l}\mathrm{d}z \\ &= \int_{\mathbb{R}} \rho(x-y)w(t,h(0,t;y))\mathrm{d}y. \end{aligned}$$

The formula for p_x is proven similarly. Item (ii) is proved.

Step 3. We show item (iii).

Using the formula for p_x established in item (ii), we have

$$\begin{aligned} p_x(t,y)-p_x(t,x) &= \int_{\mathbb{R}} (\rho_x(y-z)-\rho_x(x-z))w(t,h(0,t;z))\mathrm{d}z \\ &= \left(\int_{-\infty}^{x} + \int_{y}^{+\infty}\right)(\rho_x(y-z)-\rho_x(x-z))w(t,h(0,t;z))\mathrm{d}z \\ &\quad - \frac{1}{2\sigma^2}\int_x^y (e^{\frac{y-z}{\sigma}} + w^{\frac{-x+z}{\sigma}})w(t,h(0,t;z))\mathrm{d}z, \end{aligned}$$

therefore $p_x(t,\cdot)$ is differentiable each time x is a Lebesgue point of $z\mapsto w(t,h(0,t;z))$ and we have

$$p_{xx}(t,x) = p(t,x) - w(t,h(0,t;x)).$$

To finish our statement, we show that there exists $\mathcal{U}''\subset\mathcal{U}'$ (see the definition of $\mathcal{U}'$ given in item (i)) such that every $x=h(t,0;x_0)$ with $x_0\in\mathcal{U}''$ is a Lebesgue point of $z\mapsto w(t,h(0,t;z))$. Indeed, let $\mathcal{U}''$ be the set given by Lemma A.1 applied to the function $w\in C^0([0,\tau],\mathcal{L}^\infty(\mathcal{U}'))$. If $x=h(t,0;x_0)$ we have:

$$\begin{aligned} &\frac{1}{y-x}\int_x^y |w(t,h(0,t;z))-w(t,h(0,t;x))|\mathrm{d}z \\ &\qquad = \frac{1}{y-x}\int_{h(0,t;x)}^{h(0,t;y)} |w(t,z)-w(t,x_0)|h_x(t,0;z)\mathrm{d}z \\ &\qquad \le \frac{h(0,t;y)-h(0,t;x)}{y-x}\,\frac{1}{h(0,t;y)-h(0,t;x)} \\ &\qquad\qquad \times \int_{h(0,t;x)}^{h(0,t;y)} |w(t,z)-w(t,x_0)|\mathrm{d}z \|h_x(t,0;\cdot)\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Since $h(0,t;x)$ is differentiable for each $x\in h(t,0;\mathcal{U}')\supset h(t,0;\mathcal{U}'')$, the right-hand side converges to 0 as $y\to x$ when $x_0\in\mathcal{U}''$ is a Lebesgue point of $w(t,\cdot)$. Lemma 3.5 is proved. □

Unfortunately, the solution (w,p) constructed in Theorem 3.4 does not satisfy a semigroup property. The reason is that, for a semigroup property to hold, the property $p(t,x)=\int_{\mathbb{R}}\rho(x-y)w(t,y)\mathrm{d}y$ would have to hold so that the vector $(w(t,\cdot),p(t,\cdot))$ can be taken as an initial condition; however, this is very unlikely in view of Lemma 3.5. In order to continue our construction of the integrated solutions, we first show that the solution can be defined in L^∞ with little modification.

Given $u_0 \in L^\infty(\mathbb{R})$, we define the operator induced by the family $\mathcal{T}^\tau_{\mathcal{U}}[u_0] : \tilde{Z}^\tau \to Z^\tau$ (for $\mathcal{U} \subset \mathbb{R}$ conull) as

$$\mathcal{T}^\tau[u_0](w,p) = \mathcal{T}^\tau_{\mathbb{R}}[u_0](w,p) \tag{37}$$

where $\mathcal{T}^\tau_{\mathbb{R}}[u_0]$ is obtained by (13) with an initial condition equal to u_0 a.e. and $Z^\tau := C^0([0,\tau], L^\infty(\mathbb{R})) \times Y^\tau$, $\tilde{Z}^\tau := C^0([0,\tau], L^\infty_+(\mathbb{R})) \times \tilde{Y}^\tau$. The fact that $\mathcal{T}^\tau[u_0]$ is well-defined is shown in the following Corollary.

Corollary 1 (Well-posedness in $L^\infty(\mathbb{R})$)**.** *Let $u_0 \in L^\infty(\mathbb{R})$ be given. Let $\mathcal{U}$ and $\mathcal{U}'$ be two conull set and $u_0^{\mathcal{U}} \in \mathcal{L}^\infty(\mathcal{U})$ and $u_0^{\mathcal{U}'} \in \mathcal{L}^\infty(\mathcal{U}')$ be such that $u_0 = u_0^{\mathcal{U}} = u_0^{\mathcal{U}'}$ almost everywhere. There exists $\tau = \tau(\|u_0\|_{L^\infty(\mathbb{R})}) > 0$ and a conull set $\mathcal{U}'' \subset \mathcal{U} \cap \mathcal{U}'$ such that the solutions $w^{\mathcal{U}} \in C^0([0,\tau^{\mathcal{U}}], \mathcal{L}^\infty(\mathcal{U}))$ and $w^{\mathcal{U}'} \in C^0([0,\tau^{\mathcal{U}'}], \mathcal{L}^\infty(\mathcal{U}'))$ given by Theorem 3.4 coincide for all $t \in [0,\tau^{\mathcal{U}}] \cap [0,\tau^{\mathcal{U}'}]$ and $x \in \mathcal{U}''$. Moreover we have $\tau \geq \max(\tau^{\mathcal{U}}, \tau^{\mathcal{U}'})$.*

In particular, let $\tilde{u}_0 \in \mathcal{L}^\infty(\mathbb{R})$ be such that $u_0 = \tilde{u}_0$ almost everywhere and $\|\tilde{u}_0\|_{\mathcal{L}^\infty(\mathbb{R})} = \|u_0\|_{L^\infty(\mathbb{R})}$ and define $w(t,\cdot)$ as the L^∞ class of the solution $\tilde{w} \in C^0([0,\tau], \mathcal{L}^\infty(\mathbb{R}))$ given by Theorem 3.4. Then $w \in C^0([0,\tau], L^\infty(\mathbb{R}))$ and w is the unique fixed point on the operator $\mathcal{T}^\tau[u_0]$ induced by the operator $\mathcal{T}^\tau_{\mathbb{R}}[\tilde{u}_0]$ defined in (13).

Proof. Most of the arguments involved in the proof of Corollary 1 are very classical therefore we concentrate on the most important point which is the well-definition of w in L^∞. The set $\mathcal{U}'' \subset \mathcal{U} \cap \mathcal{U}'$ mentioned in the corollary can be defined as

$$\mathcal{U}'' = \mathcal{U} \cap \mathcal{U}' \cap \{u_0^{\mathcal{U}}(x) \leq \|u_0\|_{L^\infty}\}.$$

Since the existence time given by Theorem 3.4 depends only on $\|u_0^{\mathcal{U}}\|_{\mathcal{L}^\infty(\mathcal{U}'')}$, we have $\tau^{\mathcal{U}''} \geq \max(\tau^{\mathcal{U}}, \tau^{\mathcal{U}'})$. Moreover since $\mathcal{U}'' \subset \mathcal{U}$ it follows from the uniqueness of the fixed point of $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ that $w^{\mathcal{U}}$ and $w^{\mathcal{U}''}$ coincide on $\mathcal{U}''$, and similarly $w^{\mathcal{U}'} = w^{\mathcal{U}''}$ on $\mathcal{U}''$. The remaining statements are classical. ☐

We are now equipped with a family of operators T_t defined for $u \in L^\infty(\mathbb{R})$ and $t \in [0, \tau(\|u_0\|_{L^\infty})]$ as

$$T_t u_0(x) := w(t, h(0,t;x)) \in L^\infty(\mathbb{R}), \tag{38}$$

where w and $\tau(\|u_0\|_{L^\infty})$ are given by Corollary 1. Next we show that the family T_t satisfies a semigroup property. We deduce the existence of a maximal solution for each $u_0 \in L^\infty(\mathbb{R})$.

Theorem 3.6 (Maximal solutions)**.** *Let $u_0 \in L^\infty(\mathbb{R})$ be given. The number*

$$\tau^*(u_0) := \sup\{\tau > 0 \,|\, \mathcal{T}^\tau[u_0] \text{ has a unique fixed point}\}$$

is well-defined and belongs to $(0,+\infty]$, where $\mathcal{T}^\tau[u_0]$ is the operator defined in (37). Moreover, there exists a conull set $\mathcal{U} \subset \mathbb{R}$ and $\tilde{u}_0 \in \mathcal{L}^\infty(\mathcal{U})$ such that the operator $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ has a unique fixed point $\tilde{w} \in C^0([0,\tau], \mathcal{L}^\infty(\mathcal{U}))$ for each $\tau \in (0, \tau^*(u_0))$ and

$$\tilde{w}(t,x) = w(t,x) \text{ for a.e. } x \in \mathbb{R}.$$

The map $u_0 \in L^\infty(\mathbb{R}) \mapsto (\tilde{w}, p) \in Z^\tau_{\mathcal{U}}$ (and therefore $u_0 \in L^\infty(\mathbb{R}) \mapsto (w,p) \in Z^\tau$) is continuous for each $\tau \in (0, \tau^*(u_0))$.

Finally, the map $t \in [0, \tau^*(u_0)) \mapsto T_t u_0 \in L^\infty(\mathbb{R})$ is a semigroup which is continuous for the $L^1_\eta(\mathbb{R})$ topology for any $\eta \in (0,1)$, where T_t is defined by (38), and *if* $\tau^*(u_0) < +\infty$ *then we have*

$$\limsup_{t\to\tau^*(u_0)^-} \|T_t u_0\|_{L^\infty(\mathbb{R})} = +\infty.$$

The map $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R})$ *is continuous for each* $t \in (0, \tau^*(u_0))$.

Proof. The positiveness of $\tau^*(u_0)$ is a consequence of Corollary 1. We show the existence of $\mathcal{U}$ as defined in the Theorem. Let $\mathcal{U}^0 := \mathbb{R}$ and let $\tilde{u}_0 \in \mathcal{L}^\infty(\mathbb{R})$ be a bounded measurable function on $\mathbb{R}$ such that $\|\tilde{u}_0\|_{\mathcal{L}^\infty(\mathbb{R})} = \|u_0\|_{L^\infty(\mathbb{R})}$. In the rest of the proof we identify u_0 and $\tilde{u}_0$ and consequently drop the tilde. We recursively construct a sequence of conull sets $\mathcal{U}^n$, $n \in \mathbb{R}$, such that $\mathcal{U}^{n+1} \subset \mathcal{U}^n$, and a sequence of functions $u_0^n \in L^\infty(\mathcal{U}^n)$, such that:

(i) $u_0^{n+1}(x) := w^n(\tau_n, h^n(0, \tau_n; x))$ where $\tau_n := \tau(\|u_0^n\|_{L^\infty})$, (w^n, p^n) is the unique fixed point of the operator $\mathcal{T}^{\tau_n}_{\mathcal{U}^n}$ (given by Theorem 3.4) with initial condition u_0^n and h^n is the solution of (12) corresponding to p^n.
(ii) $\mathcal{U}^{n+1} = \mathcal{U}^n \cap h^n(0, \tau_n; \mathcal{U}^n) \cap \{x \mid u_0^{n+1}(x) \leq \|u_0^{n+1}\|_{L^\infty}\}$.

We let $\mathcal{U} := \bigcap_{n\in\mathbb{N}} \mathcal{U}^n$. Remark that, since each $\mathcal{U}^n$ is conull, the set $\mathcal{U}$ is still conull. Next we show that $\mathcal{T}^\tau[u_0]$ has a unique fixed point for each $\tau \in [0, \sum_{n\in\mathbb{N}} \tau_n)$.

Let $T_0 = 0$ and $T_n := \sum_{k=0}^{n-1} \tau_{n+1}$, for all $t \in [T_n, T_n + 1)$ we define

$$\begin{aligned} w(t,x) &:= w^n(t - T_n, h_{n-1}(\tau_n, 0; x)) \text{ for all } x \in \mathcal{U}, \\ p(t,x) &:= p^n(t - T_n, h_{n-1}(\tau_n, 0; x)) \text{ for all } x \in \mathbb{R}. \end{aligned}$$

We show that (w, p) is the unique fixed point of $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ for all $\tau \in [0, T_\infty)$ by induction. Indeed, the property is a consequence of Theorem 3.4 for all $\tau \leq T_1$. Suppose that (w, p) is the unique fixed point of $\mathcal{T}^\tau_{\mathcal{U}}[u_0]$ for all $\tau \leq T_n$, $n \geq 1$. The formula

$$w(t,x) = u_0(x) \exp\left(\int_0^t 1 + \hat{\chi} p(l, h(l, 0; x)) - (1 + \hat{\chi}) w(l, x) \mathrm{d}l\right)$$

is valid for all $t \leq T_n$. For $t \in [T_n, T_{n+1}]$ we have

$$\begin{aligned} w^n(t - T_n, x) &= u_0^n(x) \exp\left(\int_0^{t-T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) \mathrm{d}l\right) \\ &= w(T_n, h(T_n, 0; x)) \exp\left(\int_0^{t-T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) \mathrm{d}l\right) \\ &= u_0(h(0, T_n; x)) \\ &\quad \times \exp\left(\int_0^{T_n} 1 + \hat{\chi} p(l, h(l, 0; h(0, T_n; x))) - (1 + \hat{\chi}) w(l, h(0, T_n; x)) \mathrm{d}l\right) \\ &\quad \times \exp\left(\int_0^{t-T_n} 1 + \hat{\chi} p^n(l, h^n(l, 0; x)) - (1 + \hat{\chi}) w^n(l, x) \mathrm{d}l\right), \end{aligned}$$

so that

$$w^n(t-T_n,h(T_n,0;x)) = u_0(x)\exp\left(\int_0^{T_n} 1+\hat{\chi}p(l,h(l,0;x))-(1+\hat{\chi})w(l,x)\mathrm{d}l\right)$$
$$\times\exp\left(\int_{T_n}^t 1+\hat{\chi}p^n(l-T_n,h^n(l-T_n,0;h(T_n,0;x)))\right.$$
$$\left.-(1+\hat{\chi})w^n(l-T_n,h(T_n,0;x))\mathrm{d}l\right). \quad (39)$$

Next we remark that, by Lemma 3.5, the formula

$$p(T_n,x) = \int_{\mathbb{R}}\rho(x-y)w(T_n,h(0,T_n;y))\mathrm{d}y = \int_{\mathbb{R}}\rho(x-y)u_0^n(y)\mathrm{d}y = p^n(0,x)$$
$$p_x(T_n,x) = \int_{\mathbb{R}}\rho_x(x-y)w(T_n,h(0,T_n;y))\mathrm{d}y = \int_{\mathbb{R}}\rho_x(x-y)u_0^n(y)\mathrm{d}y = p^n(0,x)$$

hold, therefore $p(t,x)$ can be extended to a function $p\in C^0([0,T_{n+1}],W^{1,\infty}(\mathbb{R}))$ by defining $p(t,x)=p^n(t-T_n,x)$ when $t\geq T_n$, and moreover the extended function $h(t,s;x)$ defined on $[0,T_{n+1}]\times[0,T_{n+1}]\times\mathbb{R}$ by

$$h(t,s;x) = \begin{cases} h(t,s;x) & \text{if } t,s\leq T_n \\ h^n(t-T_n,0;h(T_n,s;x)) & \text{if } s\leq T_n\leq t \\ h(t,T_n;h^n(0,s-T_n;x)) & \text{if } t\leq T_n\leq s \\ h^n(t,s;x) & \text{if } T_n\leq t,s \end{cases}$$

solves (12). Therefore (39) can be rewritten as:

$$w^n(t-T_n,h(0,T_n;x)) = u_0(x)\exp\left(\int_0^{T_n} 1+\hat{\chi}p(l,h(l,0;x))-(1+\hat{\chi})w(l,x)\mathrm{d}l\right.$$
$$\left.+\int_{T_n}^t 1+\hat{\chi}p(l,h^n(l-T_n,0;h(T_n,0;x))-(1+\hat{\chi})w^n(l-T_n,h(T_n,0;x))\mathrm{d}l\right)$$
$$= u_0(x)\exp\left(\int_0^t 1+\hat{\chi}p(l,h(l,0;x))-(1+\hat{\chi})w(l,x)\mathrm{d}l\right),$$

where we have used the function $w\in C^0([0,T_{n+1}],\mathcal{L}^\infty(\mathcal{U}))$ defined by the equality $w(t,x):=w^n(t-T_n,h(0,T_n;x))$ when $t\geq T_n$. Finally

$$p(t,x) = \int_{\mathbb{R}}\rho(x-y)w(t,h(0,t;y))\mathrm{d}y = \int_{\mathbb{R}}\rho(x-h(t,0;x)w(t,z)h_x(t,0;z)\mathrm{d}z$$
$$= \int_{\mathbb{R}}\rho(x-h(t,0;z))u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z.$$

We have shown that (w,p) is a fixed point of $\mathcal{T}_{\mathcal{U}}^t[u_0]$, for all $t\leq T_{n+1}$. Uniqueness follows from the remark: let $w,\tilde{w}$ of $\mathcal{T}_{\mathcal{U}}^{T_{n+1}}[u_0]$ be two fixed points of $\mathcal{T}_{\mathcal{U}}^{T_{n+1}}$. Then w and $\tilde{w}$ coincide in $[0,T_n]$ (by uniqueness of the fixed point) therefore $w(T_n,x)=\tilde{w}(T_n,x)$, $w(T_n,h(0,T_n;x))=\tilde{w}(T_n,h(0,T_n;x))$ and by the uniqueness of the fixed point in the interval $[T_n,T_{n+1}]$ we conclude $w(t,\cdot)=\tilde{w}(t,\cdot)$. The uniqueness is proved. We have shown by induction that $\mathcal{T}_{\mathcal{U}}^\tau[u_0]$ has a unique fixed point for all $\tau\in[0,T_\infty]$. As a by-product, this is also true for $\mathcal{T}^\tau[u_0]$ and therefore $T_\infty\leq\tau^*(u_0)$.

Next we remark that $\tau_n = \tau(\|u_0^n\|_{L^\infty})$ is a positive continuous function of $\|u_0^n\|_{L^\infty}$ and therefore $T_\infty = \sum \tau_n < +\infty$ implies $\|w(T_n,\cdot)\|_{L^\infty} = \|u_0^n\|_{L^\infty} \to +\infty$. This shows that $\tau^*(u_0) \leq T_\infty$ and therefore

$$\tau^*(u_0) = T_\infty.$$

Obviously if $T_\infty = +\infty$ then we have $\tau^*(u_0) \geq T_\infty = +\infty$. We have shown the equality between the quantities.

Finally, the continuity of $u_0 \in \mathcal{L}^\infty(\mathcal{U}) \mapsto (w,p) \in Z^\tau_{\mathcal{U}}$ is a consequence of the continuity of the continuity of the map $u_0^n \mapsto (w^n,p^n) \in Z^\tau_{\mathcal{U}}$ given by Theorem 3.4.

Next we prove the semigroup property of $t \mapsto T_t u_0$. This follows from a direct computation: let $0 \leq t \leq s < \tau^*(u_0)$, then for almost all $x \in \mathbb{R}$ we have

$$\begin{aligned}
T_{t+s}u_0(x) &= u_0(h(0,t+s;x)) \exp\bigg(\int_0^{t+s} 1 + \hat{\chi}p(l,h(l,t+s;x)) \\
&\qquad - (1+\hat{\chi})w(l,h(0,t+s;x))\mathrm{d}l\bigg) \\
&= \bigg[u_0(h(0,t;h(t,t+s))) \exp\bigg(\int_0^t 1 + \hat{\chi}p(l,h(l,t;h(t,t+s;x))) \\
&\qquad - (1+\hat{\chi})w(l,h(0,t;h(t,t+s;x))\mathrm{d}l\bigg)\bigg] \\
&\qquad \times e^{\int_t^{t+s} 1+\hat{\chi}p(l,h(l,t+s;x))-(1+\hat{\chi})w(l,h(0,t+s;x))\mathrm{d}l} \\
&= T_t u_0(h(t,t+s;x)) \exp\bigg(\int_0^s 1 + \hat{\chi}p(t+l,h(t+l,t+s;x)) \\
&\qquad - (1+\hat{\chi})w(t+l,h(0,t;h(t,t+s;x)))\mathrm{d}l\bigg).
\end{aligned}$$

Let $\tilde{p}(t,x)$, $\tilde{h}(t,s;x)$, $\tilde{w}(t,x)$ be the quantities corresponding to the initial condition $\tilde{u}_0 = T_t u_0(x)$. By Lemma 3.5 we have

$$p(t,x) = \int_{\mathbb{R}} \rho(x-y)w(t,h(0,t;y))\mathrm{d}y = \int_{\mathbb{R}} \rho(x-y)T_t(u_0)(y)\mathrm{d}y,$$

therefore by the uniqueness of the fixed point we have

$$\tilde{p}(l,y) = p(t+l,y), \quad \tilde{h}(l,\sigma;x) = h(t+l,t+\sigma;x), \quad \tilde{w}(l,x) = w(t+l,h(0,t;x)).$$

We conclude that

$$\begin{aligned}
T_{t+s}u_0(x) &= T_t u_0(\tilde{h}(0,s;x)) \exp\bigg(\int_0^s 1 + \hat{\chi}\tilde{p}(l,\tilde{h}(l,s;x)) - (1+\hat{\chi})\tilde{w}(l,\tilde{h}(0,s;x))\mathrm{d}l\bigg). \\
&= T_s T_t u_0(x).
\end{aligned}$$

The continuity of $t \mapsto T_t u_0$ in the L^1_η topology follows directly from Lemma 3.5 and 3.3.

What remains to show is the continuity of $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R})$. We use the sequential characterization of continuity. Let $u_0, u_0^n \in L^\infty(\mathbb{R})$ be such that

$$\|u_0^n - u_0\|_{L^\infty(\mathbb{R})} \xrightarrow[n\to\infty]{} 0,$$

and let $0 < t < \tau^*(u_0)$. Let us recall that the map $u_0 \in L^\infty \mapsto (w,p) \in Z^\tau$ is continuous, therefore we have $\tau^*(u_n) > t$ for n sufficiently large and

$$\|w^n(t,\cdot) - w(t,\cdot)\|_{L^\infty(\mathbb{R})} \xrightarrow[n\to\infty]{} 0,$$

where (w^n, p_n) is the fixed point of $\mathcal{T}^t[u_0^n]$. Define h^n as the solution to (12) associated with u^n, then we have

$$\begin{aligned}\|u(t,\cdot)-u^n(t,\cdot)\|_{L^1_\eta(\mathbb{R})} &= \frac{\eta}{2}\int_{\mathbb{R}} e^{-\eta|x|}|u(t,x)-u^n(t,x)|\mathrm{d}x\\ &= \frac{\eta}{2}\int_{\mathbb{R}} e^{-\eta|x|}|w(t,h(t,0;x))-w^n(t,h^n(t,0;x))|\mathrm{d}x\\ &\le \frac{\eta}{2}\int_{\mathbb{R}} e^{-\eta|x|}|w(t,h(t,0;x))-w(t,h^n(t,0;x))|\mathrm{d}x\\ &\quad+\frac{\eta}{2}\int_{\mathbb{R}} e^{-\eta|x|}|w(t,h^n(t,0;x))-w(t,h^n(t,0;x))|\mathrm{d}x\\ &\le \frac{\eta}{2}\int_{\mathbb{R}} e^{-\eta|x|}|w(t,h(t,0;x))-w(t,h^n(t,0;x))|\mathrm{d}x\\ &\quad+\|w(t,\cdot)-w^n(t,\cdot)\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

Next we remark that the function $w(t,h^n(t,0;x))$ converges to $w(t,h(t,0;x))$ for a.e. $x\in\mathbb{R}$. Indeed, let $x\in\mathbb{R}$ be a Lebesgue point of $w(t,h(t,0;\cdot))$, then we have

$$\begin{aligned}&\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;z))-w(t,h^n(t,0;z))|\mathrm{d}z\\ &\qquad\le \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;z))-w(t,h(t,0;x))|\mathrm{d}z\\ &\qquad\quad+\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;x))-w(t,h^n(t,0;z))|\mathrm{d}z\\ &\qquad= \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;z))-w(t,h(t,0;x))|\mathrm{d}z\\ &\qquad\quad+\frac{1}{2\varepsilon}\int_{h(0,t;h^n(t,0;x-\varepsilon))}^{h(0,t;h^n(t,0;x+\varepsilon))}|w(t,h(t,0;x))-w(t,h(t,0;y))|\\ &\qquad\quad\times h^n_x(t,0;h(t,0;y))h_x(t,0;y)\mathrm{d}y\\ &\qquad\le \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;z))-w(t,h(t,0;x))|\mathrm{d}z\\ &\qquad\quad+\frac{C}{2\varepsilon}\int_{h(0,t;h^n(t,0;x-\varepsilon))}^{h(0,t;h^n(t,0;x+\varepsilon))}|w(t,h(t,0;x))-w(t,h(t,0;y))|\mathrm{d}y,\end{aligned}$$

where $C:=\|h^n_x(t,0;\cdot)\|_{L^\infty}\|h_x(t,0;\cdot)\|_{L^\infty}$, so that

$$\limsup_{n\to+\infty}\int_{x-\varepsilon}^{x+\varepsilon}|w(t,h(t,0;z))-w(t,h^n(t,0;z))|\mathrm{d}z=o(\varepsilon).$$

Define

$$E_\delta:=\{x\in\mathbb{R}\,|\,\limsup_{n\to\infty}|w(t,h(t,0;x))-w(t,h^n(t,0;x))|\ge\delta\},$$

and take a compact set $\mathcal{K}\subset E_\delta$ which is contained in a open set $\mathcal{O}$ with finite Lebesgue measure. Then $\mathcal{K}$ can be covered by a finite union of the interval in the family Ω_μ of intervals $I_{x,\varepsilon,\mu}:=(x-\varepsilon,x+\varepsilon)$ such that x is a Lebesgue point of $w(t,h(t,0;\cdot))$, $I\subset\mathcal{O}$ and

$$\limsup_{n\to+\infty}\int_{I_{x,\varepsilon,\mu}}|w(t,h(t,0;z))-w(t,h^n(t,0;z))|\mathrm{d}z\le 2\mu\varepsilon.$$

Applying the Vitali covering lemma [36, Theorem 8.5 p. 154], there is a finite disjoint subcollection $I_{x_k,\varepsilon_k,\mu} = (x_k, \varepsilon_k)$ $(1 \leq k \leq n < +\infty)$ such that $|\mathcal{K} \setminus \bigcup I_{x_n,\varepsilon_n,\mu}| = 0$ and therefore

$$\begin{aligned}
\delta|\mathcal{K}| &\leq \int_{\mathcal{K}} \limsup_{n\to+\infty} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))| \mathrm{d}x \\
&\leq \sum_{k=1}^{n} \int_{I_{x_k,\varepsilon_k,\mu}} \limsup_{n\to+\infty} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))| \mathrm{d}x \\
&\leq \sum_{k=1}^{n} \limsup_{n\to+\infty} \int_{I_{x_k,\varepsilon_k,\mu}} |w(t, h(t, 0; x)) - w(t, h^n(t, 0; x))| \mathrm{d}x \\
&\leq \sum_{k=1}^{n} 2\mu\varepsilon_k = \mu \sum_{k=1}^{n} |I_{x_k,\varepsilon_k,\mu}| \leq \mu|\mathcal{O}|.
\end{aligned}$$

Since $\mathcal{O}$ is independent of μ we take the limit $\mu \to 0$ to find $|\mathcal{K}| = 0$ and therefore

$$|E_\delta| = \sup_{\mathcal{K} \text{ compact}, \ \mathcal{K}\subset E_\delta} |\mathcal{K}| = 0.$$

Since $\delta > 0$ arbitrary we have shown that the set of where $w(t, h^n(t, 0; x))$ does not converge to $w(t, h(t, 0; x))$ is included in $\bigcup_{n\geq 0} E_{1/n}$, which is still negligible for the Lebesgue measure. We have shown the convergence of $w(t, h^n(t, 0; \cdot))$ to $w(t, h(t, 0; \cdot))$ almost everywhere in $\mathbb{R}$. The convergence of $u^n(t, \cdot)$ to $u(t, \cdot)$ in $L^1_\eta(\mathbb{R})$ is then a consequence of Lebesgue's dominated convergence Theorem. □

We are now in the position to link the constructed maximal solution with the integrated solutions to (1).

Proposition 2 (Integrated solutions)**.** *Let $\tau > 0$ and $u_0 \in L^\infty(\mathbb{R})$.*

- (i) *If $u \in C^0([0, \tau], L^1_{loc}(\mathbb{R}))$ is an integrated solution to (1), then $\tau^*(u_0) \geq \tau$ and $u(t, \cdot) = T_t u_0$ for all $t \in [0, \tau]$.*
- (ii) *Conversely, if $u(t, x) := T_t u_0(x)$ for all $t \in [0, \tau]$, then $u(t, x)$ is an integrated solution to (1).*

Proof. We first prove Item (i). Assume $u(t, x) \in C^0([0, \tau], L^1_{loc}(\mathbb{R})$ is an integrated solution. Define $p(t, x) := \int_{\mathbb{R}} \rho(x - y)u(t, y)\mathrm{d}y$. We first show that $p \in C^0([0, \tau], W^{1,\infty}(\mathbb{R}))$. We have:

$$\begin{aligned}
|p(t, x) - p(s, x)| &\leq \int_{\mathbb{R}} \rho(x - y)|u(t, y) - u(s, y)|\mathrm{d}y, \\
|p_x(t, x) - p_x(s, x)| &\leq \int_{\mathbb{R}} |\rho_x(x - y)||u(t, y) - u(s, y)|\mathrm{d}y,
\end{aligned}$$

and since $t \mapsto u(t, \cdot)$ is bounded in L^∞ and continuous in L^1_{loc} both right-hand sides can be made arbitrarily small (recall ρ and ρ_x are in L^1). This shows $p \in C^0([0, \tau], W^{1,\infty}(\mathbb{R}))$.

Next we show that $p(t,\cdot) \in W^{2,\infty}(\mathbb{R})$ for all $t \in [0,\tau]$ and that the inequality $\sup_{t\in[0,\tau]} \|p_{xx}(t,\cdot)\|_{L^\infty} < +\infty$ holds. Indeed, take $x \le y$, we have

$$\begin{aligned} p_x(t,x) - p_x(t,y) &= \int_{\mathbb{R}} (\rho_x(x-z) - \rho_x(y-z))\, u(t,z)\mathrm{d}z \\ &= \left(\int_{-\infty}^{x} + \int_{y}^{+\infty}\right) \big(\rho_x(x-z) - \rho_x(y-z)\big) u(t,z)\mathrm{d}z \\ &\quad - \frac{1}{2\sigma^2}\int_x^y (e^{\frac{y-z}{\sigma}} + e^{\frac{-x+z}{\sigma}})u(t,z) - 2u(t,x)\mathrm{d}z + \sigma^{-2}u(t,x), \end{aligned}$$

therefore p_x is differentiable at each Lebesgue point of u and we have

$$\sigma^2 p_{xx}(t,x) = p(t,x) - u(t,x) \text{ for a.e. } x \in \mathbb{R}.$$

Next, define the solution h to (12). According to Definition 2.1, there exists a conull set $\mathcal{U}$ on which $t \mapsto u(t,h(t,0;x))$ is a classical solution to (9). Therefore, by a direct integration, we have

$$w(t,x) = u_0(x)\exp\left(\int_0^t 1 + \hat{\chi}p(l,h(l,0;x)) - (1+\hat{\chi})w(l,x)\mathrm{d}l\right),$$

where $w(t,x) := u(t,h(t,0;x))$. In particular, $w(t,x) \in C^0([0,\tau],\mathcal{L}^\infty(\mathcal{U}))$. By Lemma A.1, there exists a subset $\mathcal{U}' \subset \mathcal{U}$ such that for each $x \in \mathcal{U}'$ and all $t \in [0,\tau]$, x is a Lebesgue point of $w(t,x)$. Since $h(t,s;\cdot)$ is Lipschitz continuous for all $t,s \in [0,\tau]$, we have

$$\begin{aligned} \int_{-1}^{1} |u(t,x+\varepsilon y) - u(t,x)|\mathrm{d}y &= \int_{h(0,t;x-\varepsilon)}^{h(0,t;x+\varepsilon)} |u(t,h(t,0;z)) - u(t,x)| h_x(t,0;z)\mathrm{d}z \\ &\le \int_{h(0,t;x-\varepsilon)}^{h(0,t;x+\varepsilon)} |w(t,z) - w(t,h(0,t;x))| h_x(t,0;z)\mathrm{d}z \\ &\le K\int_{h(0,t;x)-K\varepsilon}^{h(0,t;x)+K\varepsilon} |w(t,z) - w(t,h(0,t;x))|\mathrm{d}z, \end{aligned}$$

where K is the Lipschitz constant of $h(t,0;\cdot)$. Therefore x is a Lebesgue point of u whenever $h(0,t;x)$ is a Lebesgue point of w. In particular, for $x \in \mathcal{U}'$, $p_{xx}(t,h(t,0;x))$ is the derivative of p_x and we have

$$\sigma^2 p_{xx}(t,h(t,0;x)) = p(t,h(t,0;x)) - w(t,x).$$

In particular, writing

$$\begin{aligned} h(t,0;,x) - h(t,0;y) &= x - y - \chi\int_0^t p_x(l,h(l,x)) - p_x(l,h(l,y))\mathrm{d}l \\ &= x - y - \chi\int_0^t \frac{p_x(l,h(l,0;x)) - p_x(l,0;h(l,y))}{h(l,0;x) - h(l,0;y)}(h(l,0;x) - h(l,0;y))\mathrm{d}l \\ &= (x-y)\exp\left(-\chi\int_0^t \frac{p_x(l,h(l,0;x)) - p_x(l,0;h(l,y))}{h(l,0;x) - h(l,0;y)}\mathrm{d}l\right), \end{aligned}$$

we find that the formula

$$h_x(t,0;x) = e^{\hat{\chi}\int_0^t w(l,x) - p(l,h(l,0;x))\mathrm{d}l}$$

holds for all $x \in \mathcal{U}'$. Therefore

$$\begin{aligned} p(t,x) &= \int_{\mathbb{R}} \rho(x-y)u(t,y)\mathrm{d}y = \int_{\mathbb{R}} \rho(x-h(t,0;z))u(t,h(t,0;z))h_x(t,0;z)\mathrm{d}z \\ &= \int_{\mathbb{R}} \rho(x-h(t,0;z))w(t,z)e^{\hat{\chi}\int_0^t w(l,z)-p(l,h(l,0;z))\mathrm{d}l}\mathrm{d}z \\ &= \int_{\mathbb{R}} \rho(x-h(t,0;z))u_0(z)e^{\int_0^t 1-w(l,z)\mathrm{d}l}\mathrm{d}z. \end{aligned}$$

Therefore (w,p) is a fixed point of $\mathcal{T}_{\mathcal{U}}^{\tau}[u_0]$.

Conversely if $u(t,x) = T_t u_0(x)$ for all $t \in [0,\tau]$ then by definition u is a fixed point of $\mathcal{T}^{\tau}[u_0]$ and we have see in Theorem 3.6 that there exists $\mathcal{U} \subset \mathbb{R}$ conull such that $\mathcal{T}_{\mathcal{U}}^{\tau}[u_0](w,p) = (w,p)$ for a $p \in \tilde{Y}^{\tau}$, with $w(t,x) = u(t,h(t,0;x))$. It then follows from Lemma 3.5 that $p = \rho \star u$ and elementary computation then show that u is indeed a classical solution to (9) for all $x \in \mathcal{U}$. This proves Item (ii).

This finishes the proof of Proposition 2. □

Now we prove Lemma 3.2 which is used in the proof of Lemma 3.4. Next we prove that the solutions remain bounded by 0 and 1.

Lemma 3.7 (Boundedness of the solutions)**.** *Let $\tau > 0$ be given and let $u_0 \in L^\infty(\mathbb{R})$ satisfy $0 \le u_0(x) \le 1$. Let $u(t,x)$ be the corresponding integrated solution to (1). Then*

$$0 \le u(t,x) \le 1.$$

Proof. Let $w(t,x) := u(t,h(t,0;x)) \in C^0\big([0,T];\mathcal{L}^\infty(\mathcal{U})\big)$ for some $T > 0$ and a conull set $\mathcal{U} \subset \mathbb{R}$ (the continuity of $t \mapsto w(t,\cdot)$ follows from Theorem 3.6) be such that $t \mapsto w(t,x)$ is a classical solution to (9) for each $x \in \mathcal{U}$. We prove the uniform bound:

$$\|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \le 1. \tag{40}$$

Let $\varepsilon > 0$ and assume by contradiction that there exists $t \in [0,T)$ with

$$\|w(t,.)\|_{\mathcal{L}^\infty(\mathcal{U})} > 1 + \varepsilon.$$

Define

$$t^* := \inf\big\{t > 0 \,|\, \|w(t,.)\|_{L^\infty} > 1 + \varepsilon\big\} < T.$$

Then by the continuity of $t \mapsto \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}$ we have $\|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} = 1 + \varepsilon$. In particular there exists a sequence (t_n, x_n) with $t_n > t^*$, $t_n \to t^*$ as $n \to +\infty$ and $x \in \mathcal{U}$ which satisfies

$$\begin{aligned} & w(t_n,x_n) \to \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}, && \text{as } n \to \infty, \\ & w(t_n,x_n) > 1 + \varepsilon && \forall n \in \mathbb{N}. \end{aligned} \tag{41}$$

We claim that there exists a N such that for any $n \ge N$ and $t \in [t^*, t_n]$, we have

$$w(t,x_n) > \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \frac{\varepsilon}{2(1+\hat{\chi})} \text{ and } \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \ge \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \frac{\varepsilon}{2\hat{\chi}}. \tag{42}$$

Indeed, for $t \in [t^*, t_n]$ we have

$$\begin{aligned}
&\big|w(t,x_n) - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}\big| \\
&\qquad\le |w(t,x_n) - w(t^*,x_n)| + |w(t^*,x_n) - w(t_n,x_n)| \\
&\qquad\quad + |w(t_n,x_n) - \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}| + |\|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}| \\
&\qquad\le \|w(t,\cdot) - w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} + \|w(t^*,\cdot) - w(t_n,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \\
&\qquad\quad + |w(t_n,x_n) - \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}| + |\|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}|.
\end{aligned}$$

Due to the continuity of w in $\mathcal{L}^\infty(\mathcal{U})$ there exists $\delta_0 > 0$ such that $\|w(t,\cdot) - w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \le \frac{\varepsilon}{8(1+\hat\chi)}$ if $|t-t^*| \le \delta_0$ and by the continuity of $t \mapsto \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}$ there exists $\delta_1 > 0$ such that $|\|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}| \le \frac{\varepsilon}{8(1+\hat\chi)}$ if $|t-t^*| \le \delta_1$. Since $t_n \to t^*$ as $n \to +\infty$ and $w(t_n,x_n) \to \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}$ we can choose $N > 0$ such that for all $n \ge N$, we have $|t_n - t^*| \le \min(\delta_0,\delta_1)$ and $|w(t_n,x_n) - \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}| \le \frac{\varepsilon}{8(1+\hat\chi)}$, in which case we have the inequality $\big|\|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}\big| \le \frac{\varepsilon}{8(1+\hat\chi)} \le \frac{\varepsilon}{1+\hat\chi}$ and

$$\big|w(t,x_n) - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}\big| \le \frac{\varepsilon}{2(1+\hat\chi)}, \qquad \text{for all } t \in [t^*, t_n].$$

Finally, using (42) we have for all $t \in [t^*, t_n]$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w(t,x_n) &= w(t,x_n)\big(1 + \hat\chi(\rho\star u)(t,h(t,0;x_n)) - (1+\hat\chi)w(t,x_n)\big) \\
&\le w(t,x_n)\left(1 + \hat\chi\|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} - (1+\hat\chi)\|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} + \frac{\varepsilon}{2}\right) \\
&\le w(t,x_n)\left(1 + \frac{\varepsilon}{2} - \|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})}\right) \\
&\le w(t,x_n)\left(1 + \frac{\varepsilon}{2} - \|w(t^*,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} + \frac{\varepsilon}{2}\right) \le 0.
\end{aligned}$$

This implies

$$w(t,x_n) \le w(t^*,x_n) \le 1+\varepsilon, \quad \forall t \in [t^*, t_n].$$

On the other hand, due to (41) we have

$$w(t_n,x_n) > 1+\varepsilon.$$

This is a contradiction. Thus for any $t > 0$, $\|w(t,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})} \le 1+\varepsilon$. Since ε is arbitrary, (40) holds. □

In particular, the solution constructed in Step 1 and 2 can be extended up to $T = +\infty$. We are now in the position to prove Theorem 2.2.

Proof of Theorem 2.2. Let $u_0 \in L^\infty(\mathbb{R})$.

Existence and uniqueness. The existence and uniqueness of the integrated solution follows directly from Theorem 3.6 (existence and uniqueness of a fixed-point problem) and Proposition 2 (consistency between the fixed-point problem and the integrated solutions).

Continuity. The continuity in the space $L^1_\eta(\mathbb{R})$ and the continuity of $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R})$ have been shown in Theorem 3.6.

Other properties. The semigroup property follows directly from the form of the operator has been shown in Theorem 3.6. The uniform bound when $0 \le u_0(x) \le 1$

has been shown in Lemma 3.7 and the fact that $\tau^*(u_0) = +\infty$ from the fact that the L^∞ norm of $u(t,\cdot)$ cannot blow-up in finite time.

This ends the proof of Theorem 2.2. □

Next we show that our model preserves certain properties of the initial condition.

Proposition 3 (Properties of the solutions)**.** *Let $u(t,x)$ be an integrated solution to (1) and suppose $u_0 \in L^\infty(\mathbb{R})$ with $0 \le u_0 \le 1$. Then*

(i) *if $u_0(x)$ is continuous, then $u \in C^0([0,T]\times\mathbb{R})$.*
(ii) *if $u_0(x) \in C^1(\mathbb{R})$, then $u \in C^1([0,T]\times\mathbb{R})$ and u is then a classical solution to (1).*
(iii) *if $u_0(x)$ is monotone, then $u(t,x)$ has the same monotony for each $t>0$.*

Proof. From (9) we can directly solve the solution $w(t,x) = u(t,h(t,0;x))$ as

$$w(t,x) = \frac{u_0(x)\exp\big(\int_0^t 1+\hat\chi(\rho\star u)(l,h(l,0;x))\mathrm{d}l\big)}{1+(1+\hat\chi)u_0(x)\int_0^t \exp\big(\int_0^l 1+\hat\chi(\rho\star u)(\sigma,h(\sigma,0;x))\mathrm{d}\sigma\big)\mathrm{d}l},$$

for all $t>0$ and almost all $x\in\mathbb{R}$, which is equivalent to

$$u(t,x) = \frac{u_0(h(0,t;x))\exp\big(\int_0^t 1+\hat\chi(\rho\star u)(l,h(l,t;x))\mathrm{d}l\big)}{1+(1+\hat\chi)u_0(h(0,t;x))\int_0^t \exp\big(\int_0^l 1+\hat\chi(\rho\star u)(\sigma,h(\sigma,t;x))\mathrm{d}\sigma\big)\mathrm{d}l}.$$

Since $(t,x)\to h(t,s;x)$ is continuous, the right-hand side is a continuous function. This shows *(i)*.

Let us show *(ii)*. By *(i)* we have $u\in C^0([0,T]\times\mathbb{R})$. Thus, the spatial derivative of the vector field of (8) satisfies

$$-\sigma^2(\rho_x\star u)_x(t,x) = u(t,x) - (\rho\star u)(t,x) \in C^0([0,T]\times\mathbb{R}).$$

Therefore, the characteristic flow $(t,s,x)\mapsto h(t,s;x)\in C^1([0,T]\times[0,T]\times\mathbb{R})$. If we denote

$$\phi(t,x) := e^{\int_0^t 1+\hat\chi(\rho\star u)(l,h(l,0,x))dl}, \tag{43}$$

then $(t,x)\mapsto\phi(t,x)$ is C^1, which implies $w\in C^1([0,T]\times\mathbb{R})$. Since $u(t,x) = w(t,h(0,t;x))$ we have $u\in C^1([0,T]\times\mathbb{R})$.

Finally we show *(iii)*. We will assume that $u_0(x)$ is decreasing (the increasing case can be treated with a similar argument). We let $w(t,x) := u(t,h(t,x))$ where u is the solution to (1) starting from $u(t=0,x)\equiv u_0(x)$, and $h(t,s;x)$ be the corresponding characteristic flow, i.e. the solution to (12) with $p(t,x) := \int_{\mathbb{R}}\rho(x-z)w(t,h(0,t;z))\mathrm{d}z$. Our aim is to show that w is a fixed point of the map

$$\begin{aligned}\tilde{\mathcal{T}}_\tau :\quad & C^0\big([0,\tau],L^\infty(\mathbb{R})\big) \longrightarrow C^0\big([0,\tau],L^\infty(\mathbb{R})\big)\\ & \tilde w \longmapsto \frac{u_0(x)\exp\left(\displaystyle\int_0^t 1+\hat\chi\tilde p(s,h(s,0;x))\mathrm{d}s\right)}{1+(1+\hat\chi)u_0(x)\displaystyle\int_0^t \exp\left(\int_0^l 1+\tilde p(s,h(s,0;x))\mathrm{d}s\right)\mathrm{d}l},\end{aligned}$$

where $\tilde p(t,x)$ is defined in the above formula by

$$\tilde p(t,x) := \int_{\mathbb{R}}\rho(x-z)\tilde w(t,h(0,t;z))\mathrm{d}z$$

we stress that h is the characteristic flow corresponding to the "real" solution to (1) and is independent of $\tilde w$.

As the proof is more involved, we subdivide it in four steps.

Step one: Let $r > 0$, we show that there exists τ_0 such that the ball

$$B_r := \left\{ w \in C^0\big([0,\tau], L^\infty(\mathbb{R})\big) \,|\, \|w(t,x) - u_0(x)\|_{C^0\left([0,\tau],L^\infty(\mathbb{R})\right)} \leq r \right\}$$

is left stable by $\tilde{\mathcal{T}}_\tau$ for $0 < \tau \leq \tau_0$.

Let $w_0 \in B_r$. We compute:

$$\begin{aligned}
|\tilde{\mathcal{T}}_\tau(\tilde{w}) - u_0(x)| &= \left| \frac{u_0(x) e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}}{1 + (1+\hat{\chi})u_0(x)\int_0^t e^{\int_0^l 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}\mathrm{d}l} - u_0(x) \right| \\
&\leq |u_0(x)| \left| \frac{e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s} - 1 - (1+\hat{\chi})u_0(x)\int_0^t e^{\int_0^l 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}\mathrm{d}l}{1 + (1+\hat{\chi})u_0(x)\int_0^t e^{\int_0^l 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}\mathrm{d}l} \right| \\
&\leq \|u_0\|_{L^\infty(\mathbb{R})} \Bigg(e^{1+\hat{\chi}\|u_0\|_{L^\infty(\mathbb{R})}+\hat{\chi}r} \left| \int_0^t 1 + \hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s \right| \\
&\quad + (1+\hat{\chi})\|u_0\|_{L^\infty(\mathbb{R})} t e^{t(1+\hat{\chi}\|u_0\|_{L^\infty(\mathbb{R})}+\hat{\chi}r)} \Bigg) \\
&\leq C\tau,
\end{aligned}$$

where C depends on $\|u_0\|_{L^\infty(\mathbb{R})}$, r, and $\hat{\chi}$. The existence of τ_0 is proved.

Step two: Let $r > 0$, we show that there exists $\tau_1 > 0$ such that $\tilde{\mathcal{T}}_\tau$ is contracting on B_r for $0 < \tau < \tau_1$.

Let $\tilde{w}_1, \tilde{w}_2 \in B_r$, and let $\kappa := 1 + r$ so that $\|w_1\|_{L^\infty(\mathbb{R})} \leq \kappa$ and $\|w_2\|_{L^\infty(\mathbb{R})} \leq \kappa$. For notational compactness we define in advance

$$\begin{aligned}
\tilde{p}_i(t,x) &:= \int_{\mathbb{R}} \rho(x-z)\tilde{w}_i(t,h(0,t;z))\mathrm{d}z, & i \in \{1,2\}, \\
D_i(t,x) &:= 1 + (1+\hat{\chi})u_0(x)\int_0^t \exp\left(\int_0^l 1 + \tilde{p}_i(s,h(s,0;x))\mathrm{d}s \right)\mathrm{d}l, & i \in \{1,2\}.
\end{aligned}$$

We compute:

$$\begin{aligned}
&\left| \tilde{\mathcal{T}}_\tau(w_1)(t,x) - \tilde{\mathcal{T}}_\tau(w_2)(t,x) \right| \\
&\quad = \left| \frac{u_0(x)e^{\int_0^t 1+\hat{\chi}\tilde{p}_1(s,h(s,0;x))\mathrm{d}s} D_2(t,x) - u_0(x)e^{\int_0^t 1+\hat{\chi}\tilde{p}_2(s,h(s,0;x))\mathrm{d}s} D_1(t,x)}{D_1(t,x)D_2(t,x)} \right| \\
&\quad \leq u_0(x) \left| e^{\int_0^t 1+\hat{\chi}\tilde{p}_1(s,h(s,0;x))\mathrm{d}s} - e^{\int_0^t 1+\hat{\chi}\tilde{p}_2(s,h(s,0;x))\mathrm{d}s} \right| \\
&\qquad + (1+\hat{\chi})u_0(x)e^{(\kappa\chi+1)t} \int_0^t \Big| e^{\int_0^l 1+\hat{\chi}\tilde{p}_1(s,h(s,0;x))\mathrm{d}s + \int_0^t 1+\hat{\chi}\tilde{p}_2(s,h(s,0;x))\mathrm{d}s} \\
&\qquad - e^{\int_0^l 1+\hat{\chi}\tilde{p}_2(s,h(s,0;x))\mathrm{d}s + \int_0^t 1+\hat{\chi}\tilde{p}_1(s,h(s,0;x))\mathrm{d}s} \Big| \,\mathrm{d}l \\
&\quad \leq \left(te^{(1+\kappa\chi)} + 2(1+\hat{\chi})t^2 e^{(1+\kappa\chi)(t+1)} \right) \|\chi\tilde{p}_1 - \chi\tilde{p}_2\|_{C^0([0,\tau],L^\infty(\mathbb{R}))} \\
&\quad \leq \chi\tau \left(e^{1+\kappa\chi} + 2(1+\hat{\chi})\tau e^{(1+\kappa\chi)(\tau+1)} \right) \|\tilde{w}_1 - \tilde{w}_2\|_{C^0([0,\tau],L^\infty(\mathbb{R}))},
\end{aligned}$$

where we have used the inequalities $\|u_0\|_{L^\infty(\mathbb{R})} \leq 1$ and $\|\tilde{p}_1 - \tilde{p}_2\|_{C^0([0,\tau],L^\infty(\mathbb{R}))} \leq \|\tilde{w}_1 - \tilde{w}_2\|_{C^0([0,\tau],L^\infty(\mathbb{R}))}$.

The existence of τ_1 is proved.

Step three: We show that the map $\tilde{\mathcal{T}}_\tau$ preserves the monotony of u_0, i.e. the set

$$\mathcal{D} := \{w \in C^0\big([0,\tau], L^\infty(\mathbb{R})\big) \,|\, w(t,\cdot) \text{ is nonincreasing}\}$$

is left stable by $\tilde{\mathcal{T}}_\tau$.

Indeed, let $\tilde{w}$ be nonincreasing with respect to x. Let $\tilde{w}^1(t,x) := \tilde{\mathcal{T}}_\tau(w)(t,x)$. We first show that $\tilde{P}$ is nonincreasing:

$$\tilde{p}(t,x) - \tilde{p}(t,y) = \int_{\mathbb{R}} \rho(z)\big(\tilde{w}(t,h(0,t;x-z)) - \tilde{w}(t,h(0,t;y-z))\big)\mathrm{d}z \le 0,$$

since the characteristic flow $h(t,s;\cdot)$ is increasing. Next we let

$$D(t,x) := 1 + (1+\hat{\chi}u_0(x)\int_0^t \exp\left(\int_0^l 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s\right)\mathrm{d}l.$$

We compute:

$$\begin{aligned}
&\tilde{w}^1(t,x) - \tilde{w}^1(t,y)\\
&\quad= \frac{u_0(x)e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}D(t,y) - u_0(y)e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;y))\mathrm{d}s}D(t,x)}{D(t,x)D(t,y)}\\
&\quad= \frac{u_0(x)e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s} - u_0(y)e^{\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;y))\mathrm{d}s}}{D(t,x)D(t,y)}\\
&\qquad+ \frac{u_0(x)u_0(y)}{D(t,x)D(t,y)}\int_0^t e^{\int_0^l 1+\hat{\chi}\tilde{p}(t,h(s,0;y))\mathrm{d}y + \int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;x))\mathrm{d}s}\\
&\qquad- e^{\int_0^l 1+\hat{\chi}\tilde{p}(t,h(s,0;x))\mathrm{d}y+\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;y))\mathrm{d}s}\mathrm{d}l\\
&\quad\le \frac{u_0(x)u_0(y)}{D(t,x)D(t,y)}\int_0^t e^{\int_0^l 1+\hat{\chi}\tilde{p}(t,h(s,0;x))\mathrm{d}y+\int_0^t 1+\hat{\chi}\tilde{p}(s,h(s,0;y))\mathrm{d}s}\\
&\qquad\times\left(e^{\hat{\chi}\int_l^t \tilde{p}(s,h(s,0;x))-\tilde{p}(s,h(s,0;y))\mathrm{d}s} - 1\right)\mathrm{d}l \le 0,
\end{aligned}$$

since $\tilde{P}$ is nonincreasing. This shows the stability of $\mathcal{D}$.

Step four: We conclude.

Let $\tau := \min(\tau_0,\tau_1)$ where τ_0, τ_1 are as in Step 1 and 2. By a direct application of the Banach contraction principle, $\tilde{\mathcal{T}}_\tau$ has a unique fixed point in B_r, which is w (since w happens to be a fixed point). Moreover w can be obtained as the limit of the iteration scheme:

$$w^0(t,x) := u_0(x), \qquad\qquad w^{n+1}(t,x) := \tilde{\mathcal{T}}_\tau(w^n)(t,x).$$

Since u_0 is nonincreasing and $\tilde{\mathcal{T}}_\tau$ preserves the monotony, it follows that w is nonincreasing ($\mathcal{D}$ is closed for the considered topology).

Since τ does not depend on u_0, the monotony of $u(t,\cdot)$ for all $t>0$ follows from an induction argument. □

Theorem 3.8 (Long-time behavior)**.** *Let $\delta \in (0,1)$ and $u_0(x)$ be such that $\delta \le u_0(x) \le 1$. Let $u(t,x)$ be the corresponding integrated solution to (1). Then*

$$\lim_{t\to\infty} \|1-u(t,\cdot)\|_{L^\infty(\mathbb{R})} = 0.$$

Proof. Let θ be defined as

$$\theta := \liminf_{t\to+\infty}\inf_{x\in\mathbb{R}} u(t,x),$$

and assume by contradiction that $\theta < 1$. We first remark that for any $x \in \mathbb{R}$ we have

$$\begin{cases} \partial_t w(t,x) = w(t,x)\left(1+\hat{\chi}(\rho\star u)(t,h(t,0;x)) - (1+\hat{\chi})w(t,x)\right) \\ \qquad\qquad \geq w(t,x)\big(1-(1+\hat{\chi})w(t,x)\big) \\ w(0,x) \quad \geq \delta. \end{cases} \quad t>0,$$

Thus, for each $x \in \mathbb{R}$,

$$w(t,x) \geq \delta, \quad x\in\mathbb{R}, t>0.$$

In particular $(\rho\star u)(t,h(t,0;x)) = \int_{\mathbb{R}} \rho(h(t,0;x)-y)u(t,y)\mathrm{d}y \geq \delta\int_{\mathbb{R}}\rho(h(t,0;x)-y)\mathrm{d}y = \delta$. We deduce that

$$\begin{cases} \partial_t w(t,x) = w(t,x)\left(1+\hat{\chi}(\rho\star u)(t,h(t,0;x)) - (1+\hat{\chi})w(t,x)\right) \\ \qquad\qquad \geq w(t,x)\big(1+\hat{\chi}\delta-(1+\hat{\chi})w(t,x)\big) \\ w(0,x) \quad \geq \delta. \end{cases} \quad t>0,$$

This implies for any $t>0, x\in\mathbb{R}$

$$w(t,x) \geq \frac{\delta e^{t(1+\hat{\chi}\delta)}}{1+\frac{(1+\hat{\chi})\delta}{1+\hat{\chi}\delta}\left(e^{t(1+\hat{\chi}\delta)}-1\right)} \xrightarrow{t\to\infty} \frac{1+\hat{\chi}\delta}{1+\hat{\chi}}.$$

In particular

$$\theta \geq \frac{1+\hat{\chi}\delta}{1+\delta} > \frac{1}{1+\hat{\chi}}. \tag{44}$$

It is not difficult to see that for each $\alpha\in(0,1)$ there exists T_α such that, for all $t\geq T_\alpha$, we have

$$\inf_{x\in\mathbb{R}} w(t,x) \geq \alpha\theta.$$

Therefore for all $t \geq T_\alpha$,

$$(\rho\star u)(t,h(t,0;x)) \geq \alpha\theta\int_{\mathbb{R}}\rho(h(t,0;x)-y)\mathrm{d}y = \alpha\theta,$$

which yields

$$\begin{cases} \partial_t w(t,x) = w(t,x)\left(1+(\rho\star u)(t,h(t,0;x)) - (1+\hat{\chi})w(t,x)\right) \\ \qquad\qquad \geq w(t,x)\left(1+\alpha\theta-(1+\hat{\chi})w(t,x)\right) \\ w(T_1,x) \ \geq \frac{1+\hat{\chi}\frac{\delta}{2}}{1+\hat{\chi}} \end{cases} \quad t>T_1, x\in\mathbb{R}$$

and finally

$$\theta = \liminf_{t\to+\infty}\inf_{x\in\mathbb{R}} w(t,x) \geq \frac{1+\hat{\chi}\alpha\theta}{1+\hat{\chi}}.$$

This is a contradiction if α is chosen as

$$\alpha = 1-\frac{1}{\hat{\chi}}\left(\frac{1}{\theta}-1\right),$$

and this choice is admissible because

$$\frac{1}{\hat{\chi}}\left(\frac{1}{\theta}-1\right) < \frac{1}{\hat{\chi}}(1+\hat{\chi}-1) = 1$$

by (44). This concludes the proof of Theorem 3.8. □

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- Key Laboratory of Advanced Control and Optimization for Chemical Processes, Ministry of Education, East China University of Science and Technology, Shanghai 200237, China.
- School of Biological Sciences, the University of Edinburgh, Max Born Crescent, Edinburgh, EH9 3JH, Scotland, United Kingdom.

Appendix

Appendix A. **Lebesgue points along continuous trajectories.** Here we show that the space $\mathcal{L}^\infty(\mathcal{U})$ is well-behaved with respect to Lebesgue points when $\mathcal{U}$ is a subset of $\mathbb{R}$.

Lemma A.1 (Lebesgue points along continuous trajectories). *Let $\mathcal{U} \subset \mathbb{R}$ be conull. Let $w \in \mathcal{C}^0([0,\tau], \mathcal{L}^\infty(\mathcal{U}))$ be given, then there exists a conull set $\mathcal{U}' \subset \mathcal{U}$ such that each $x \in \mathcal{U}'$ is a Lebesgue points of $w(t,\cdot)$ for all $t \in [0,\tau]$.*

Proof. Recall that a Lebesgue point of a measurable function $f : \mathcal{U} \to \mathbb{R}$ is characterized by the property

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |f(z) - f(x)| \mathrm{d}z = 0$$

or, equivalently,

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_{-1}^{1} |f(x+\varepsilon y) - f(x)| \mathrm{d}z = 0.$$

Let $w \in C^0([0,\tau], \mathcal{L}^\infty(\mathcal{U}))$ be given. Given $q \in \mathbb{Q} \cap [0,\tau]$ we define the failure set

$$\mathcal{F}_q := \{x \in \mathcal{U} \,|\, x \text{ is a not a Lebesgue point of } w(q,\cdot)\}.$$

It is classical that for each q the set $\mathcal{F}_q$ is negligible for the Lebesgue measure λ, *i.e.* $\lambda(\mathcal{F}_q) = 0$. Since the family $(\mathcal{F}_q)_{q \in \mathbb{Q} \cap [0,\tau]}$ is countable, we have

$$\lambda\left(\bigcup_{q \in \mathbb{Q} \cap [0,\tau]} \mathcal{F}_q\right) = 0$$

therefore the set $\mathcal{U}' := \mathcal{U} \setminus \bigcup_{q \in \mathbb{Q} \cap [0,\tau]} \mathcal{F}_q$ is conull.

Let us show that $\mathcal{U}'$ is composed of Lebesgue points of $w(t,\cdot)$. Let $x \in \mathcal{U}'$ and $t \in [0,\tau]$, then there exists a sequence of rational numbers $t_n \in \mathbb{Q}$ such that $t_n \to t$. By definition of $\mathcal{U}'$, x is not in any $\mathcal{F}_{t_n}$ and therefore x is a Lebesgue point of the functions $w(t_n,\cdot)$ for all $n \in \mathbb{N}$. We have:

$$\begin{aligned}
&\int_{-1}^{1} |w(t, x+\varepsilon y) - w(t,x)| \mathrm{d}y \\
&\quad \le \int_{-1}^{1} |w(t, x+\varepsilon y) - w(t_n, x+\varepsilon y)| \mathrm{d}y + \int_{-1}^{1} |w(t_n, x+\varepsilon y) - w(t_n, x)| \mathrm{d}y \\
&\qquad + \int_{-1}^{1} |w(t_n, x) - w(t,x)| \mathrm{d}y \\
&\quad \le \int_{-1}^{1} |w(t_n, x+\varepsilon y) - w(t_n, x)| \mathrm{d}y + 2\|w(t,\cdot) - w(t_n,\cdot)\|_{\mathcal{L}^\infty(\mathcal{U})},
\end{aligned}$$

therefore the right-hand side is arbitrarily small when $\varepsilon \to 0$. We conclude that x is a Lebesgue point of $w(t, \cdot)$. Lemma A.1 is proved. □

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Received xxxx 20xx; revised xxxx 20xx.

E-mail address: xmfu@ecust.edu.cn
E-mail address: quentin.griette@u-bordeaux.fr
E-mail address: pierre.magal@u-bordeaux.fr