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Sharp discontinuous traveling waves in a hyperbolic Keller–Segel equation

Xiaoming Fu*, Quentin Griette† and Pierre Magal‡
Institut de Mathématiques de Bordeaux, Université de Bordeaux,
CNRS, IMB, UMR 5251, Talence 33400, France
**xmfu@ecust.edu.cn*
†quentin.griette@u-bordeaux.fr
‡pierre.magal@u-bordeaux.fr

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In this work, we describe a hyperbolic model with cell–cell repulsion with a dynamics in the population of cells. More precisely, we consider a population of cells producing a field (which we call "pressure") which induces a motion of the cells following the opposite of the gradient. The field indicates the local density of population and we assume that cells try to avoid crowded areas and prefer locally empty spaces which are far away from the carrying capacity. We analyze the well-posedness property of the associated Cauchy problem on the real line. We start from bounded initial conditions and we consider some invariant properties of the initial conditions such as the continuity, smoothness and monotony. We also describe in detail the behavior of the level sets near the propagating boundary of the solution and we find that an asymptotic jump is formed on the solution for a natural class of initial conditions. Finally, we prove the existence of sharp traveling waves for this model, which are particular solutions traveling at a constant speed, and argue that sharp traveling waves are necessarily discontinuous. This analysis is confirmed by numerical simulations of the PDE problem.

Keywords: Traveling wave; hyperbolic equation; discontinuous wave profile.

AMS Subject Classification 2020: 92C17, 35L60, 35D30

1. Introduction

In this paper, we are concerned with the following diffusion equation with logistic source:

$$\begin{cases}\partial_t u(t,x) - \chi\partial_x(u(t,x)\partial_x p(t,x)) = u(t,x)(1-u(t,x)), & t>0,\ x\in\mathbb{R},\\ u(t=0,x)=u_0(x), & \end{cases} \tag{1.1}$$

†Corresponding author

where $\chi > 0$ is a *sensing coefficient* and $p(t,x)$ is an external pressure. Model (1.1) describes the behavior of a population of cells $u(t,x)$ living in a one-dimensional habitat $x \in \mathbb{R}$, which undergo a logistic birth and death population dynamics, and in which individual cells follow the gradient of a field p. The constant χ characterizes the response of the cells to the effective gradient p_x. In this work, we will consider the case where p is itself determined by the state of the population $u(t,x)$ as

$$-\sigma^2 \partial_{xx} p(t,x) + p(t,x) = u(t,x), \quad t > 0, x \in \mathbb{R}. \tag{1.2}$$

Equation (1.2) corresponds to the limit of fast diffusion $\varepsilon \to 0$ of the parabolic equation (1.8). It corresponds to a scenario in which the field $p(t,x)$ is produced by the cells, diffuses to the whole space with diffusivity σ^2 (for $\sigma > 0$), and vanishes at rate one. As a result cells are pushed away from crowded area to emptier region.

A similar model has been successfully used in our recent work[23] to describe the motion of cancer cells in a Petri dish in the context of cell co-culture experiments of Pasquier *et al.*[41] Pasquier *et al.*[41] cultivated two types of breast cancer cells to study the transfer of proteins between them in a study of multi-drug resistance. It was observed that the two types of cancer cells form segregated clusters of cells of each kind after a 7-day co-culture experiment (Fig. 1(a)). In a previous paper,[23] we studied the segregation property of a model similar to (1.1)–(1.2), set in a circular domain in two spatial dimensions $x \in \mathbb{R}^2$ representing a Petri dish. Starting from islet-like initial conditions representing cell clusters, it was numerically observed that the distribution of cells converges to a segregated state in the long run.

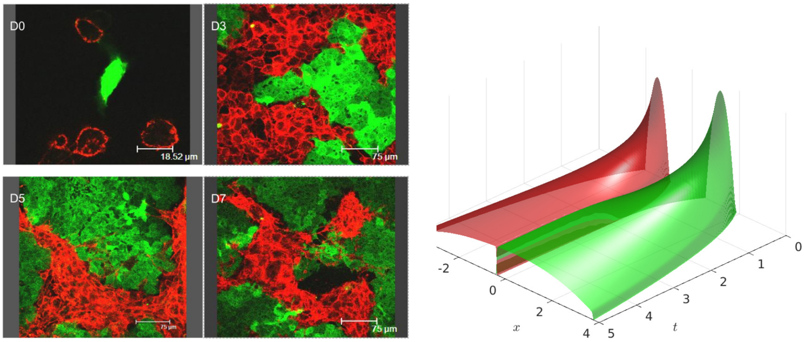
One may observe that in such an experiment the cells are well fed. So there is no limitation for food. As explained in Ref. 16, the limitations are due to space and the contact inhibition of growth is involved. Therefore, the right-hand side of (1.1), which is a logistic term (for simplicity), could possibly have the following form:

$$f(x) = \frac{\beta x}{1 + \alpha x} - \mu x,$$

where β is the division rate and μ is the mortality rate. We believe that our results hold for such a non-linearity and this is left for future work.

Strikingly, even before the two species come in contact, a sharp transition is formed between the space occupied by one species and the empty space being invaded (Fig. 1(b)) and the distribution of cells looks like a very sharp traveling front. In an attempt to better understand the spatial behavior of cell populations growing in a Petri dish, in this paper we investigate the mathematical properties of a simplified model for a single species on the real line. We are particularly interested in showing the existence of a sharp traveling front moving at a constant speed.

Our model can be included in the family of nonlocal advection models for cell–cell adhesion and repulsion. As pointed out by many biologists, cell–cell interactions do not only exist in a local scope, but a long-range interaction should be taken into account to guide the mathematical modeling. Armstrong *et al.*[1] in their early work proposed a model (APS model) in which a local diffusion is added to the nonlocal



(a) (b)

Fig. 1. (Color online) (Figs. 1 and 5(b) in Ref. 23) (a) Direct immunodetection of P-gp transfers in co-cultures of sensitive (MCF-7) and resistant (MCF-7/Doxo) variants of the human breast cancer cell line. (b) The temporal-spatial evolution of the two species in the 1D model. One can check that a discontinuity is forming near the front face of the green surface.

attraction driven by the adhesion forces to describe the phenomenon of cell mixing, full/partial engulfment and complete sorting in the cell sorting problem. Based on the APS model, Murakawa and Togashi[39] thought that the population pressure should come from the cell volume size instead of the linear diffusion. Therefore, the linear diffusion was changed into a nonlinear diffusion in order to capture the sharp fronts and the segregation in cell co-culture. Carrillo *et al.*[10] recently proposed a new assumption on the adhesion velocity field and their model showed a good agreement in the experiments in the work of Katsunuma *et al.*[30] The idea of the long-range attraction and short-range repulsion can also be seen in the work of Leverentz *et al.*[34] They considered a nonlocal advection model to study the asymptotic behavior of the solution. By choosing a Morse-type kernel which follows the attractive-repulsive interactions, they found that the solution can asymptotically spread, contract (blow-up), or reach a steady-state. Burger *et al.*[6] considered a similar nonlocal adhesion model with nonlinear diffusion, for which they investigated the well-posedness and proved the existence of a compactly supported, non-constant steady state. Dyson *et al.*[20] established the local existence of a classical solution for a nonlocal cell–cell adhesion model in spaces of uniformly continuous functions. For Turing and Turing-Hopf bifurcation due to the nonlocal effect, we refer to Refs. 15 and 48. We also refer to Refs. 37, 21, 17 and 18 for more topics on nonlocal advection equations. For the derivation of such models, we refer to the work of Bellomo *et al.*[5] and Morale *et al.*[38]

Since the pressure $p(t,x)$ is a nonlocal function of the density $u(t,x)$ in (1.2), the spatial derivative appears as a *nonlocal advection* term in (1.1). In fact, our problem (1.1)–(1.2) can be rewritten as a transport equation in which the speed of particles is nonlocal in the density,

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x (u(t,x) \partial_x (\rho \star u)(t,x)) = u(t,x)(1-u(t,x)), \\ u(t=0,x) = u_0(x), \end{cases} \tag{1.3}$$

where

$$(\rho \star u)(x) = \int_{\mathbb{R}} \rho(x-y)u(t,y)dy, \quad \rho(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}. \tag{1.4}$$

Traveling waves for a similar diffusive equation with logistic reaction have been investigated for quite general nonlocal kernels by Hamel and Henderson,[26] who considered the model

$$u_t + (u\,(K \star u))_x = u_{xx} + u(1-u), \tag{1.5}$$

where $K \in L^p(\mathbb{R})$ is odd and $p \in [1,\infty]$. Notice that the attractive parabolic–elliptic Keller–Segel model (1.9) is included in this framework by the particular choice

$$K(x) = -\chi\,\mathrm{sign}(x)e^{-|x|/\sqrt{d}}/\big(2\sqrt{d}\big).$$

They proved a spreading result for this equation (initially compactly supported solutions to the Cauchy problem propagate to the whole space with constant speed) and explicit bounds on the speed of propagation. Diffusive nonlocal advection also appears in the context of swarm formation.[36] Pattern formation for a model similar to (1.5) by Ducrot *et al.*[15] Let us mention that the inviscid equation (1.3) has been studied in a periodic cell by Ducrot and Magal.[17] Other methods have been established for conservative systems of interacting particles and their kinetic limit[4,9] based on gradient flows set on measure spaces; those are difficult to adapt here because of the logistic term. There is also related literature regarding traveling waves in nonlocal reaction–diffusion equations.[12,19,22,25,50]

Recall that a traveling wave is a special solution having the specific form:

$$u(t,x) = U(x-ct), \quad \text{for a.e. } (t,x) \in \mathbb{R}^2,$$

where the profile U has the following behavior at $\pm\infty$:

$$\lim_{z\to-\infty} U(z) = 1, \quad \lim_{z\to\infty} U(z) = 0.$$

The goal of this paper is to investigate sharp traveling waves namely,

$$U(x) = 0, \quad \text{for all } x > 0.$$

Moreover as it is represented in Fig. 2(a) we will obtain the existence of such a wave with a discontinuity at $x = 0$ for the profile U. Discontinuous traveling waves in hyperbolic partial differential equations have appeared in the literature of the recent few years. Traveling wave solutions with a shock or jump discontinuity have been found e.g. in models of malignant tumor cells (where the existence of discontinuous waves is proved by means of geometric singular perturbation theory for ODEs[27,35]) or chemotaxis (where both smooth and discontinuous traveling waves are found using phase plane analysis[33]).

It can be noticed that, in the limit of slow diffusivity $\sigma \to 0$ (and under the simplifying assumption that $\chi = 1$), we get $u(t,x) \equiv p(t,x)$ and (1.1) is equivalent to an equation with *porous medium-type diffusion* and logistic reaction

$$u_t - \frac{1}{2}(u^2)_{xx} = u(1-u). \tag{1.6}$$

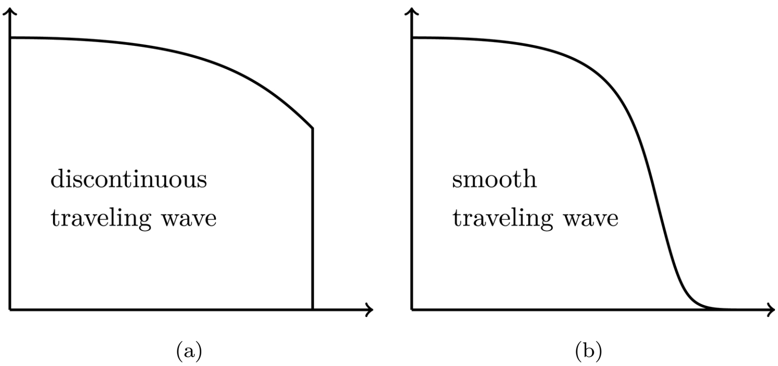


Fig. 2. An illustration of two types of traveling wave solutions.

We refer to Ref. 49 for more result about porous medium equation. The propagation dynamics for this kind of equation was first studied, to the extent of our knowledge, by Aronson,[2] Atkinson *et al.*,[3] and later by de Pablo and Vázquez,[14] in the more general context of nonlinear diffusion

$$u_t = (u^m)_{xx} + u(1-u), \quad \text{with } m > 1. \tag{1.7}$$

In Sec. 3.3, we observe that the discontinuous sharp traveling obtained in this paper converge (numerically) to the continuous profile described by Pablo and Vázquez.[14]

The particular relation between the pressure $p(t,x)$ and the density $u(t,x)$ in (1.2) strongly reminds the celebrated model of chemotaxis studied by Patlak and Keller and Segel[31,32,42] (parabolic–parabolic Keller–Segel model) and, more specifically, the parabolic–elliptic Keller–Segel model which is derived from the former by a quasi-stationary assumption on the diffusion of the chemical.[29] Indeed Eq. (1.2) can be formally obtained as the quasistatic approximation of the following parabolic equation:

$$\varepsilon \partial_t p(t,x) = \chi p_{xx}(t,x) + u(t,x) - p(t,x), \tag{1.8}$$

when $\varepsilon \to 0$. A rigorous derivation of the limit has been achieved in the case of the Keller–Segel model by Carrapatoso and Mischler.[8] We also refer to Ref. 13 for such a result in the context of linear parabolic equations. We refer to Refs. 7, 28 and 43 and the references therein for a mathematical introduction and biological applications. In these models, the field $p(t,x)$ is interpreted as the concentration of a chemical produced by the cells rather than a physical pressure. One of the difficulties in attractive chemotaxis models is that two opposite forces compete to drive the behavior of the equations: the *diffusion* due to the random motion of cells, on the one hand, and on the other hand the *nonlocal advection* due to the attractive chemotaxis; the former tends to regularize and homogenize the solution, while the latter promotes cell aggregation and may lead to the blow-up of the solution in finite time.[11,29] At this point let us mention that our study concerns *repulsive* cell–cell interaction with no diffusion, therefore no such blow-up phenomenon is expected in our study; however the absence of diffusion adds to the mathematical complexity of the study, because standard methods of reaction–diffusion equations cannot be employed here. Traveling waves for the (attractive) parabolic–elliptic Keller–Segel model were studied by Nadin *et al.*[40] who constructed these traveling wave by a bounded interval approximation of the 1D system

$$\begin{cases} u_t + \chi\,(u p_x)_x = u_{xx} + u(1-u), \\ -d\,p_{xx} + p = u, \end{cases} \tag{1.9}$$

set on the real line $x \in \mathbb{R}$, when the strength of the advection is not too strong $0 < \chi < \min(1, d)$, and gave estimates on the speed of such a traveling wave: $2 \leq c_* \leq 2 + \chi\sqrt{d}/(d-\chi)$. More recently, Salako and Shen[44–46] and Salako *et al.*[47] published a series of papers concerning the asymptotic properties and spatial dynamics of chemotaxis models.

In this paper, we focus on the particular case of (1.1)–(1.2) with $\sigma > 0$ and $\chi > 0$. The paper is organized as follows. In Sec. 2, we present our main results. In Sec. 3 we present numerical simulations to illustrate our theoretical results. In Sec. 4, we prove the propagation properties of the solution and describe the local behavior near the propagating boundary (see Proposition 2.2 for definition), including the formation of a discontinuity for time-dependent solutions. In Sec. 5 we prove the existence of sharp traveling waves. We also prove that smooth traveling waves are necessarily positive, which shows that sharp traveling waves are necessarily singular (in this case, discontinuous). In particular, a solution starting from a compactly supported initial condition with polynomial behavior at the boundary can never catch such a smooth traveling wave.

2. Main Results and Comments

We begin by defining our notion of solution to Eq. (1.1).

Definition 2.1. (Integrated solutions) Let $u_0 \in L^\infty(\mathbb{R})$. A measurable function $u(t,x) \in L^\infty([0,T] \times \mathbb{R})$ is an *integrated solution* to (1.1) if the characteristic equation

$$\begin{cases} \dfrac{d}{dt} h(t,x) = -\chi(\rho_x \star u)(t, h(t,x)), \\ h(t=0,x) = x \end{cases} \tag{2.1}$$

has a classical solution $h(t,x)$ (i.e. for each $x \in \mathbb{R}$ fixed, the function $t \mapsto h(t,x)$ is in $C^1([0,T], \mathbb{R})$ and satisfies (2.1)), and for a.e. $x \in \mathbb{R}$, the function $t \mapsto u(t, h(t,x))$ is in $C^1([0,T], \mathbb{R})$ and satisfies

$$\begin{cases} \dfrac{d}{dt} u(t,h(t,x)) = u(t,h(t,x))\big(1 + \hat{\chi}(\rho \star u)(t,h(t,x)) - (1+\hat{\chi})u(t,h(t,x))\big), \\ u(t=0,x) = u_0(x), \end{cases} \tag{2.2}$$

where $\hat{\chi} := \frac{\chi}{\sigma^2}$.

We define weighted space $L^1_\eta(\mathbb{R})$ as follows:

$$L^1_\eta(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} \,\middle|\, \int_{\mathbb{R}} |f(x)| e^{-\eta|x|} dx < \infty \right\}.$$

$L^1_\eta(\mathbb{R})$ is a Banach space endowed with the norm

$$\|f\|_{L^1_\eta} := \frac{\eta}{2} \int_{\mathbb{R}} |f(y)| e^{-\eta|y|} dy.$$

We first recall some results concerning the existence of integrated solutions for Eq. (1.1) in Theorem 2.1, Proposition 2.1 and Theorem 2.2. We prove those results in the companion paper.[24]

Theorem 2.1. (Well-posedness) *Let $u_0 \in L^\infty_+(\mathbb{R})$ and fix $\eta > 0$. There exists $\tau^*(u_0) \in (0, +\infty]$ such that for all $\tau \in (0, \tau^*(u_0))$, there exists a unique integrated solution $u \in C^0([0,\tau], L^1_\eta(\mathbb{R}))$ to* (1.1) *which satisfies $u(t=0,x) = u_0(x)$. Moreover $u(t,\cdot) \in L^\infty(\mathbb{R})$ for each $t \in [0, \tau^*(u_0))$ and the map $t \in [0, \tau^*(u_0)) \mapsto T_t u_0 := u(t,\cdot)$ is a semigroup which is continuous for the $L^1_\eta(\mathbb{R})$-topology. The map $u_0 \in L^\infty(\mathbb{R}) \mapsto T_t u_0 \in L^1_\eta(\mathbb{R})$ is continuous.*

Finally, if $0 \le u_0(x) \le 1$, then $\tau^*(u_0) = +\infty$ and $0 \le u(t,\cdot) \le 1$ for all $t > 0$.

The next result concerns the preservation properties satisfied by the solutions of (1.1) (see Proposition 2.2 in Ref. 24).

Proposition 2.1. (Regularity of solutions) *Let $u(t,x)$ be an integrated solution to* (1.1).

(1) *if $u_0(x)$ is continuous, then $u(t,x)$ is continuous for each $t > 0$.*
(2) *if $u_0(x)$ is monotone, then $u(t,x)$ has the same monotony for each $t > 0$.*
(3) *if $u_0(x) \in C^1(\mathbb{R})$, then $u \in C^1([0,T] \times \mathbb{R})$ and u is then a classical solution to* (1.1).

In this following theorem we consider the long-time behavior of some solutions to (1.1) (see Theorem 2.3 in Ref. 24).

Theorem 2.2. (Long-time behavior) *Let $0 \le u_0(x) \le 1$ be a nontrivial non-negative initial condition and $u(t,x)$ be the corresponding integrated solution. Then $0 \le u(t,x) \le 1$ for all $t > 0$ and $x \in \mathbb{R}$. If moreover there exists $\delta > 0$ such that $\delta \le u_0(x) \le 1$ then*

$$u(t,x) \to 1, \quad \text{as } t \to \infty$$

and the convergence holds uniformly in $x \in \mathbb{R}$.

We now arrive at the main interest of the paper, which is to describe the spatial dynamics of solutions to (1.1)–(1.2). To get insight about the asymptotic propagation properties of the solutions, we focus on initial conditions whose support is bounded towards $+\infty$. If the behavior of the initial condition in a neighborhood of the boundary of the support is polynomial, we can establish a precise estimate of the location of the level sets relative to the position of the rightmost positive point. Our first assumption requires that the initial condition is supported in $(-\infty, 0]$.

Assumption 2.1. (Initial condition) We assume that $u_0(x)$ is a continuous function satisfying

$$\begin{aligned} 0 \le u_0(x) \le 1, &\quad \text{for all } x \in \mathbb{R}, \\ u_0(x) = 0, &\quad \text{for all } x \ge 0, \\ u_0(x) > 0, &\quad \text{for all } x \in (-\delta_0, 0), \end{aligned}$$

for some $\delta_0 > 0$.

Under this assumption we show that u is propagating to the right.

Proposition 2.2. (The separatrix) *Let $u_0(x)$ satisfy Assumption 2.1 and $h^*(t) := h(t, 0)$ be the separatrix. Then $h^*(t)$ stays at the rightmost boundary of the support of $u(t, \cdot)$, i.e.*

(i) *we have*

$$u(t, x) = 0 \quad \text{for all } x \ge h^*(t), \tag{2.3}$$

(ii) *for each $t > 0$ there exists $\delta > 0$ such that*

$$u(t, x) > 0 \quad \text{for all } x \in (h^*(t) - \delta, h^*(t)). \tag{2.4}$$

Moreover, u is propagating to the right i.e.

$$\frac{d}{dt} h^*(t) > 0 \quad \text{for all } t > 0.$$

We precise the behavior of the initial condition in a neighborhood of 0 and estimate the steepness of u in positive time.

Assumption 2.2. (Polynomial behavior near 0) In addition to Assumption 2.1, we require that there exists $\alpha \ge 1$ and $\gamma > 0$ such that

$$u_0(x) \ge \gamma |x|^\alpha, \quad \text{for all } x \in (-\delta, 0).$$

Theorem 2.3. (Formation of a discontinuity) *Let $u_0(x)$ satisfy Assumptions 2.1 and 2.2 and $u(t, x)$ solve (1.1) with $u(t = 0, x) = u_0(x)$. For all $\delta > 0$ we have*

$$\limsup_{t \to +\infty} \sup_{x \in (h^*(t) - \delta, h^*(t))} u(t, x) \ge \frac{1}{1 + \hat{\chi} + \alpha\chi} > 0. \tag{2.5}$$

More precisely, define the level set

$$\xi(t, \beta) := \sup\{x \in \mathbb{R} \mid u(t, x) = \beta\},$$

for all $t > 0$ and $0 < \beta < \frac{1}{1+\hat{\chi}+\alpha\chi}$. Then, for each $0 < \beta < \frac{1}{1+\hat{\chi}+\alpha\chi}$, the distance between $\xi(t, \beta)$ and the separatrix is decaying exponentially fast:

$$h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha} t} \le \xi(t, \beta) \le h^*(t), \tag{2.6}$$

where $\eta \in (0, 1)$ is given in Proposition 4.4 and $\hat{\chi} = \frac{\chi}{\sigma^2}$.

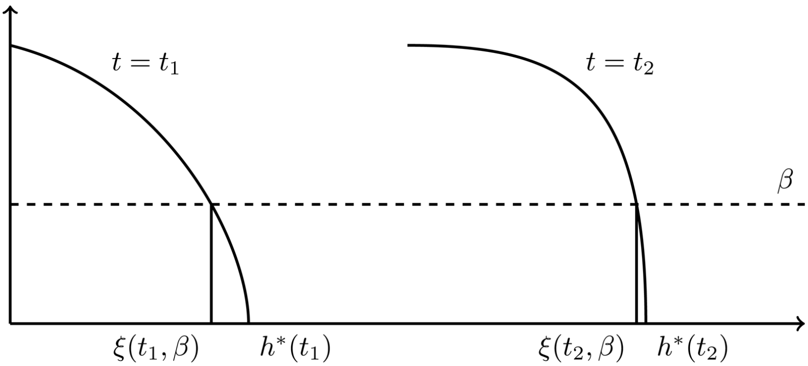


Fig. 3. A cartoon for the formation of the discontinuity. Here we choose $t_1 < t_2$ and $\xi(t, \beta), t = t_1, t_2$ are the level sets. Theorem 2.3 proves that when Assumptions 2.1 and 2.2 are satisfied, then the distance $|\xi(t, \beta) - h^*(t)|$ converges to 0 exponentially fast.

In particular, the profile $u(t, x)$ forms a discontinuity near the boundary point $h^*(t)$ as $t \to +\infty$, see Fig. 3. By considering discontinuous integrated solutions, we are able to estimate the size of the jump for nonincreasing profiles, which leads to an estimate of the asymptotic speed.

Proposition 2.3. (Asymptotic jump near the separatrix) *Let* u_0 *be a nonincreasing function satisfying* $u_0(-\infty) \leq 1$, $u_0(0) > 0$ *and* $u_0(x) = 0$ *for* $x > 0$. *Then*

$$\liminf_{t \to +\infty} u(t, h^*(t)) \geq \frac{2}{2 + \hat{\chi}}, \tag{2.7}$$

$$\liminf_{t \to +\infty} \frac{d}{dt} h^*(t) \geq \frac{\sigma \hat{\chi}}{2 + \hat{\chi}}, \tag{2.8}$$

where $\hat{\chi} = \frac{\chi}{\sigma^2}$.

We finally turn to traveling wave solutions $u(t, x) = U(x - ct)$, which are self-similar profiles traveling at a constant speed.

Definition 2.2. (Traveling wave solution) A *traveling wave* is a non-negative solution $u(t, x)$ to (1.1) such that there exists a function $U \in L^\infty(\mathbb{R})$ and a speed $c \in \mathbb{R}$ such that $u(t, x) = U(x - ct)$ for a.e. $(t, x) \in \mathbb{R}^2$. By convention, we also require that U has the following behavior at $\pm\infty$:

$$\lim_{z \to -\infty} U(z) = 1, \quad \lim_{z \to \infty} U(z) = 0.$$

The function U is the *profile* of the traveling wave.

Under a technical assumption on $\hat{\chi} = \frac{\chi}{\sigma^2}$, we can prove the existence of sharp traveling waves which present a jump at the vanishing point.

Assumption 2.3. (Bounds on $\hat{\chi}$) Let $\chi > 0$ and $\sigma > 0$ be given and define $\hat{\chi} := \frac{\chi}{\sigma^2}$. We assume that $0 < \hat{\chi} < \bar{\chi}$, where $\bar{\chi}$ is the unique root of the function

$$\hat{\chi} \mapsto \ln\left(\frac{2-\hat{\chi}}{\hat{\chi}}\right) + \frac{2}{2+\hat{\chi}}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right)$$

given in Lemma A.1.

Remark 2.1. It follows from Lemma A.1 that $\hat{\chi} = 1$ satisfies Assumption 3. Actually, numerical evidence suggest that $\bar{\chi} \approx 1.045$.

Theorem 2.4. (Existence of a sharp discontinuous traveling wave) *Let Assumption* 2.3 *be satisfied. There exists a traveling wave* $u(t,x) = U(x-ct)$ *traveling at speed*

$$c \in \left(\frac{\sigma\hat{\chi}}{2+\hat{\chi}}, \frac{\sigma\hat{\chi}}{2}\right),$$

where $\hat{\chi} = \frac{\chi}{\sigma^2}$

Moreover, the profile U *satisfies the following properties* (*up to a shift in space*):

(i) U *is* sharp *in the sense that* $U(x) = 0$ *for all* $x \geq 0$; *moreover,* U *has a discontinuity at* $x = 0$ *with* $U(0^-) \geq \frac{2}{2+\hat{\chi}}$.

(ii) U *is continuously differentiable and strictly decreasing on* $(-\infty, 0]$, *and satisfies*

$$-cU' - \chi(UP')' = U(1-U)$$

pointwise on $(-\infty, 0)$, *where* $P(z) := (\rho \star U)(z)$.

Our proof is based on a fixed-point argument. Other methods could have been imagined, like a vanishing viscosity argument. This method consists in adding a small elliptic regularization $\varepsilon\partial_{xx}u$ in the right-hand side of Eq. (1.1), prove the existence of a traveling wave for the regularized problem (similar to (1.5)), then let the regularization vanish $\varepsilon \to 0$. With the appropriate estimates, it may then be possible to prove the existence of a traveling front for the original equation. However, the implementation of this method is not without difficulties. Firstly, the vanishing viscosity process $\varepsilon \to 0$ requires a kind of compactness, which cannot be provided by the Arzelà–Ascoli here because the limiting object is discontinuous. Secondly, the traveling wave problem (1.5) is itself nontrivial. The existing constructions[40,44] are only valid for a parameter range which prevents the vanishing of the elliptic parameter. Overall the vanishing viscosity method may be as hard to implement as the present argument. Connecting the solutions to (1.5) to the ones of (1.1)–(1.2) is still an interesting problem and we plan to investigate it in a future work.

Finally, we show that continuous traveling waves cannot be sharp, i.e. are necessarily positive on $\mathbb{R}$.

Proposition 2.4. (Smooth traveling waves) *Let* $U(x)$ *be the profile of a traveling wave solution to* (1.1) *and assume that* U *is continuous. Then* $U \in C^1(\mathbb{R})$, U *is strictly positive and we have the estimate*:

$$-\chi(\rho_x \star U)(x) < c, \quad \text{for all } x \in \mathbb{R}. \tag{2.9}$$

In particular, by Theorem 2.3, any solutions starting from an initial condition satisfying Assumption 2.2 may never catch such a traveling wave.

3. Numerical Simulations

We first describe the numerical framework of this study.

- The parameters σ and χ are fixed as $\sigma = 1$ and $\chi = 1$.
- We choose a bounded interval $[-L, L]$ and an initial distribution of $\phi \in C([-L, L])$;
- We solve numerically the following PDE using the upwind scheme (p being given):

$$\begin{cases} \partial_t u(t,x) - \partial_x\big(u(t,x)\partial_x p(t,x)\big) = u(t,x)(1-u(t,x)), \\ \nabla p(t,x)\cdot\nu = 0, \\ u(0,x) = \phi(x), \end{cases} \qquad t>0,\ x\in[-L,L]. \tag{3.1}$$

- The pressure p is defined as

$$p(t,x) = (I-\Delta)_{\mathcal{N}}^{-1} u(t,x), \quad t>0, x\in[-L,L], \tag{3.2}$$

  where $(I-\Delta)_{\mathcal{N}}^{-1}$ is the Laplacian operator with Neumann boundary condition. Due to the Neumann boundary condition of the pressure p, we do not need boundary condition on u (see our related paper[23]).

Our numerical scheme is detailed in Appendix B.

3.1. *Formation of a discontinuity*

In this part, we use numerical simulations to verify the theoretical predictions in the previous sections. Firstly, we choose the initial value $\phi \in C^1([-L, L])$ as follows:

$$\phi(x) = \frac{(x-x_0)^2}{(L+x_0)^2}\mathbb{1}_{[-L,x_0]}(x), \quad L=20,\ x_0=-15. \tag{3.3}$$

Notice that this initial condition satisfies Assumptions 2.1 and 2.2. Due to Theorem 2.3, we should observe the formation of a discontinuity in space for large time.

We plot the evolution of the solution $u(t, x)$ starting from $u(0, x) = \phi(x)$ in Fig. 4. We observe that the jump is formed for large time and the height of the jump is greater than 2/3 which is in accordance with Theorem 2.4.

Next, we study the propagation speed of different level sets, namely,

$$t \longmapsto \xi(t,\beta) + L,$$

where $\xi(t, \beta) := \sup\{x \in \mathbb{R} \mid u(t, x) = \beta\}$ and $\beta = 0, 0.2, 2/3, 0.8$. Note that the case $\beta = 0$ corresponds to the rightmost characteristic. We plot the position of some level sets as a function of time in Fig. 5.

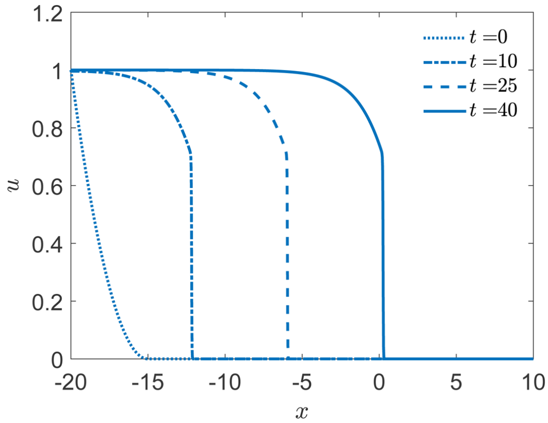


Fig. 4. We plot the propagation of the traveling waves under system (3.1) with the initial value (3.3). We plot the propagation profile at $t = 0, 10, 25, 40$ (respectively, dashed lines, dotted-dashed lines, dotted lines and solid lines).

We compute the propagation speed in the following way: for different $\beta \in [0, 1]$, we choose $t_1 = 15$ and $t_2 = 40$ where the propagation speed is almost stable after $t = t_1$. Thus, we can compute the mean propagation speed as follows:

$$\text{Propagation speed at level } \beta = \frac{\xi(t_2, \beta) - \xi(t_1, \beta)}{t_2 - t_1}. \tag{3.4}$$

Next, we want to check whether the solutions of system (3.2) starting from two different initial values converge to the same discontinuous traveling wave solution.

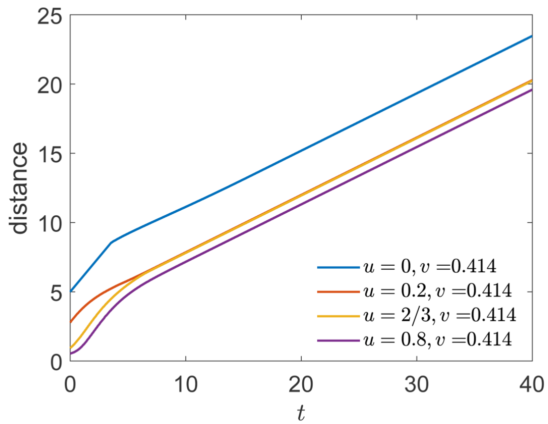


Fig. 5. We plot the evolution of different level sets $t \longmapsto \xi(t, \beta) + L$ under system (3.1). Our initial distribution is taken as (3.3). We plot the propagating speeds of the profile at $\beta = 0, 0.2, 2/3, 0.8$. The x-axis represents the time and the y-axis is the relative distance $\xi(t, \beta) + L$. The velocity is calculated by (3.4) for $t_1 = 15$ and $t_2 = 40$.

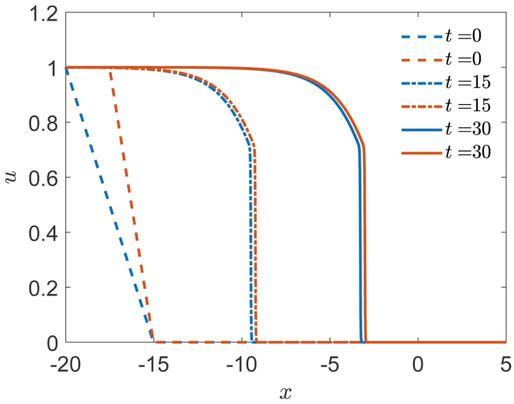


Fig. 6. (Color online) We plot the propagation of two profiles under system (3.1) with initial distributions are taken as (3.5). The blue curves represent the profile with initial distribution ϕ_1 while the red curves represent the profile with initial distribution ϕ_2. We plot the propagation profiles at $t = 0, 15$ and 30 (respectively, dashed lines, dotted-dashed lines and solid lines). The simulation shows that the two profiles converge to the same discontinuous traveling wave solution.

To that aim, given two different initial profiles ϕ_1 and ϕ_2 with $\phi_1 \leq \phi_2$ on $[-L, L]$,

$$\begin{aligned} \phi_1(x) &= -\frac{x+15}{5}\mathbb{1}_{[-20,-15]}(x), \\ \phi_2(x) &= \mathbb{1}_{[-20,-17.5]}(x) - \frac{x+15}{10}\mathbb{1}_{[-17.5,-15]}(x). \end{aligned} \tag{3.5}$$

We simulate the propagation of these two profiles in Fig. 6.

3.2. *Large speed traveling waves*

As we know for porous medium equation, the existence of large speed $c > c_*$ traveling wave solutions is known[14] and it can be observed numerically by taking the exponentially decreasing function as initial value. In this part, instead of taking a compactly supported initial value, we set the initial value

$$\phi_\alpha(x) = \frac{1}{1+e^{\alpha(x-x_0)}}, \quad x_0 = -15, \tag{3.6}$$

where $\alpha \geq 1$ is a parameter introduced to describe the decaying rate of the initial value.

We compare the following three different scenarios with different parameters $\alpha = 1, 2, 5$ in the initial value (3.6). We observe the large speed traveling waves in Fig. 7 when $\alpha = 1, 2$. We note that as the parameter α in (3.6) is increasing, the propagation speed is decreasing and $c \approx 1/\alpha$. When $\alpha = 5$, the propagation of the traveling waves is similar to the case in Fig. 4 in which we started from the compactly supported initial value. In other word, we can observe the formation of discontinuity and the critical speed $c_* \approx 0.414$ is reached.

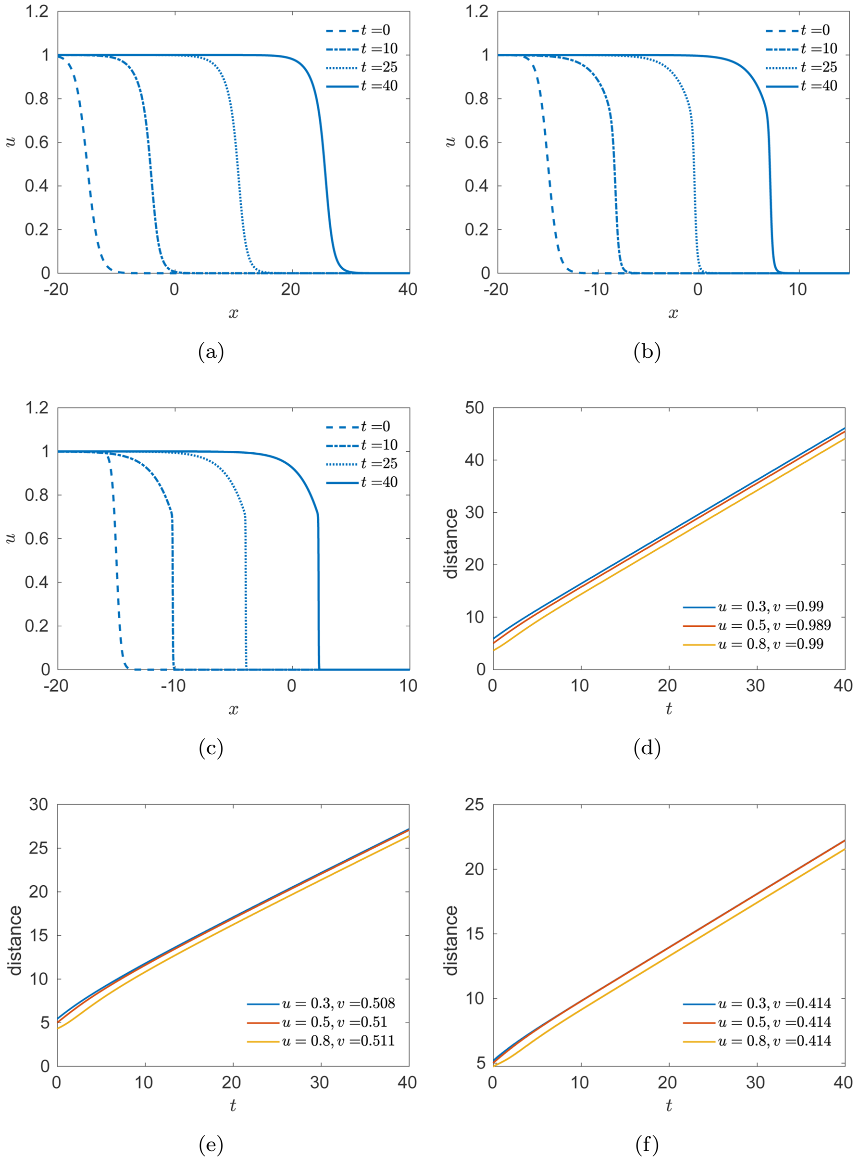


Fig. 7. We plot the propagation of the traveling waves under system (3.1) with the initial values (3.6) and the corresponding evolution of different level sets $t \longmapsto \xi(t, \beta) + L$. (a) and (d) represent the evolution of the traveling wave and its level sets when $\alpha = 1$. (b) and (e) correspond to the case when $\alpha = 2$. (c) and (f) correspond to the case when $\alpha = 5$.

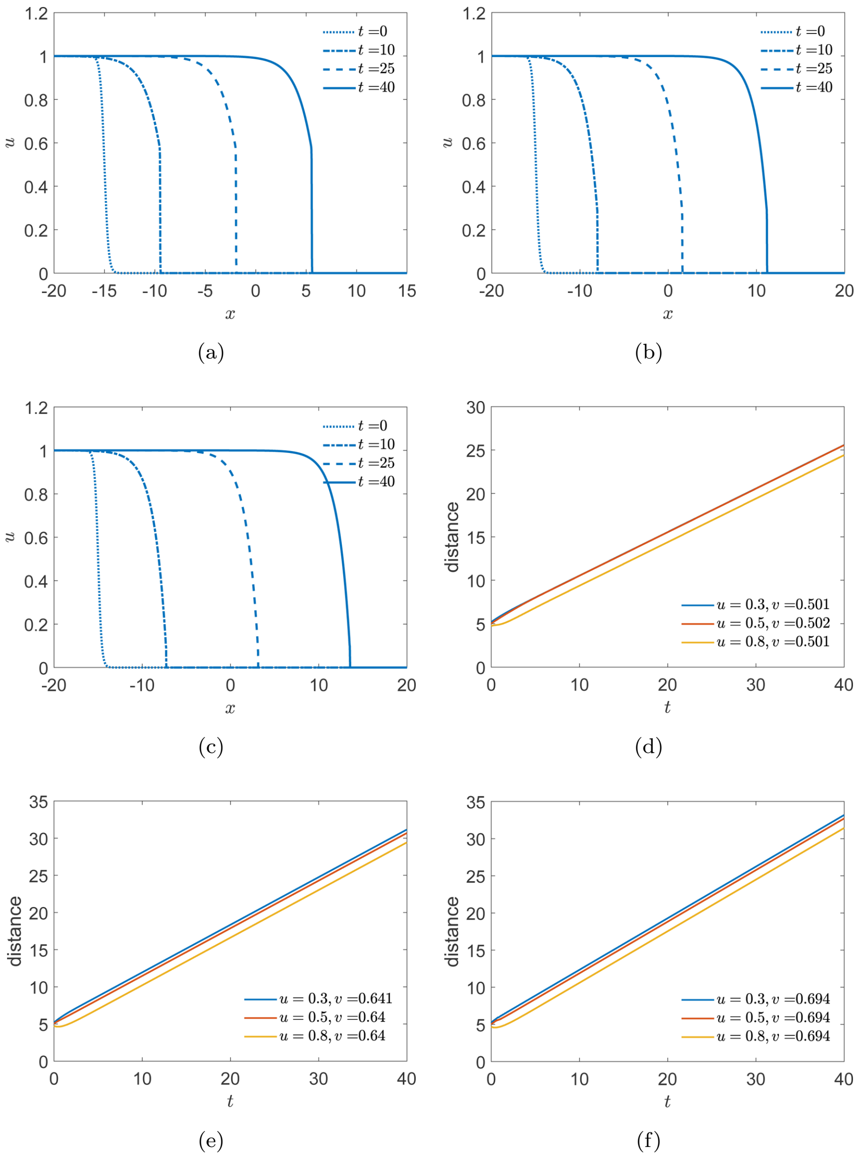


Fig. 8. We plot the propagation of the traveling waves for system (3.1) with the kernel (3.7) and the corresponding evolution of different level sets $t \longmapsto \xi(t, \beta) + L$. (a) and (d) represent the evolution of the traveling wave and its level sets when $\sigma^2 = 0.5$. (b) and (e) correspond to the case when $\sigma^2 = 0.1$. (c) and (f) correspond to the case when $\sigma^2 = 0.01$. Our initial value is taken as in (3.6) with $\alpha = 5$.

3.3. *Comparison with porous medium equations: The vanishing jump*

In this part, we compare the nonlocal advection model with the porous medium equation by varying the parameter σ

$$p(t,x) = (I - \sigma^2 \Delta)_{\mathcal{N}}^{-1} u(t,x). \tag{3.7}$$

Thus, if $\sigma \to 0$ then formally we have $p(t,x) \to u(t,x)$. Thus, the first equation of (3.1) becomes

$$u_t - \frac{1}{2}(u^2)_{xx} = u(1-u),$$

which is the classical porous medium equation. It is well-known that this equation has the explicit traveling wave solution $U(z) = (1 - e^{z/\sqrt{2}})_+$ with critical speed $c_* = 1/\sqrt{2}$.

We consider the transition from the discontinuous traveling wave solution to the continuous sharp-type traveling wave solution by letting $\sigma \to 0$. Moreover, we want to see if the critical traveling speed of the discontinuous wavefront $c(\sigma)$ converges to $c_* = 1/\sqrt{2} \approx 0.707$ as $\sigma \to 0$. Our initial value is taken as $1/(1+\exp(5*(x+15)))$, $x \in [-20, 20]$ in (3.6). We compare the following three different scenarios with different parameters $\sigma^2 = 0.5, 0.1, 0.01$ in kernel (3.7).

In Fig. 8 we can observe that as $\sigma \to 0$ in the kernel, the discontinuous jump is gradually vanishing from (a) to (c). Moreover, the critical speed $c(\sigma)$ is increasing as $\sigma \to 0$ and is approaching the critical speed $c_* = 1/\sqrt{2} \approx 0.707$ for the porous medium case.

To explore more about the relationship between parameter σ^2 and the critical speed $c(\sigma)$, we plot Fig. 9.

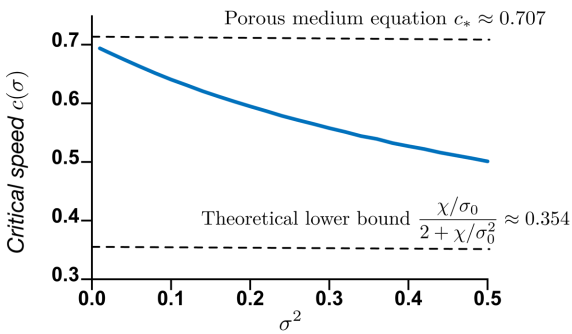


Fig. 9. The relationship between parameter σ^2 and the critical propagation speed $c(\sigma)$ by numerical simulations. Concerning the computation for the critical speed c, we take the speed on three level sets $t \to \xi(t,\beta) + L$ for $\beta = 0.3, 0.5, 0.8$ and we plot the mean value (the standard variation is negligible $\approx 10^{-3}$). As the critical speed $c(\sigma)$ is decreasing with respect to σ, the theoretical lower bound is obtained by setting $\chi = 1, \sigma_0 = 0.5$ in Theorem 2.4.

4. Properties of the Time-Dependent Solutions

4.1. *The separatrix*

In this section, we study the qualitative properties of solutions to (1.1) starting from an initial condition supported in $(-\infty, 0]$.

Proposition 4.1. (The separatrix) *Let u be a solution integrated along the characteristics to* (1.1), *starting from $u_0(x)$ satisfying Assumption* 2.1. *Let $h^*(t) := h(t, 0)$ be the separatrix* (*as in Proposition* 2.2). *Then $h^*(t)$ stays at the rightmost boundary of the support of $u(t, \cdot)$, i.e.*

(i) *we have*

$$u(t, x) = 0 \quad \textit{for all } x \geq h^*(t), \tag{4.1}$$

(ii) *for each $t > 0$ there exists $\delta > 0$ such that*

$$u(t, x) > 0 \quad \textit{for all } x \in (h^*(t) - \delta, h^*(t)). \tag{4.2}$$

Proof. By definition the characteristics are well-defined by (2.1) as the flow of an ODE. In particular, if $x \geq h^*(t) = h(t, 0)$ there exists $x_0 \geq 0$ such that $x = h(t, x_0)$. Since $u_0(x_0) = 0$ and in view of (2.2), we have indeed $u(t, x) = 0$. This proves item 4.1.

By Assumption 2.1, there exists $\delta_0 > 0$ such that $u_0(x) > 0$ for $x \in (-\delta_0, 0)$. We remark that

$$\begin{aligned}
\frac{d}{dt}u(t, h(t, x)) &= \hat{\chi}\, u(t, h(t, x))((\rho * u)(t, h(t, x)) - u(t, h(t, x))) \\
&\quad + u(t, h(t, x))(1 - u(t, h(t, x))) \\
&\geq u(t, h(t, x))(1 - (1 + \hat{\chi})u(t, h(t, x))).
\end{aligned}$$

By comparison with the solution to the ODE $v'(t) = v(t)(1 - (1 + \hat{\chi})v(t))$ starting from $v(t = 0) = u_0(x) > 0$, we deduce that $u(t, x) \geq v(t) > 0$ for each $x \in (h(t, -\delta_0), h^*(t))$. Since $h(t, -\delta_0) < h(t, 0) = h^*(t)$, this proves item 4.1. □

Next, we investigate the propagation of u.

Proposition 4.2. (u is propagating) *Let u_0 satisfy Assumption* 2.1 *and let u be the solution integrated along the characteristics to* (1.1) *starting from $u(t = 0, x) = u_0(x)$. Then u is propagating to the right, i.e.*

$$\frac{d}{dt}h^*(t) > 0. \tag{4.3}$$

Moreover, we have the estimate:

$$\frac{d}{dt}h^*(t) \leq \frac{\chi}{2\sigma}. \tag{4.4}$$

Proof. We have the following estimates:

$$\begin{aligned}\frac{d}{dt}h^*(t) &= -\chi(\rho_x * u)(t, h^*(t)) \\ &= -\chi \int_{-\infty}^{+\infty} \rho_x(y)u(t, h^*(t) - y)dy \\ &= \chi \int_{-\infty}^{+\infty} \frac{\text{sign}(y)}{2\sigma^2} e^{-\frac{|y|}{\sigma}} u(t, h^*(t) - y)dy \\ &= \frac{\chi}{\sigma} \int_0^{+\infty} \rho(y)u(t, h^*(t) - y)dy \\ &> 0,\end{aligned}$$

since $u(t, x) = 0$ for all $x > h^*(t)$. (4.3) is proved.

Then, since $0 \leq u \leq 1$, we have

$$\begin{aligned}\frac{d}{dt}h^*(t) &= \frac{\chi}{\sigma} \int_0^{+\infty} \rho(y)u(t, h^*(t) - y)dy \\ &\leq \frac{\chi}{\sigma} \int_0^{+\infty} \rho(y)dy = \frac{\chi}{2\sigma},\end{aligned}$$

which proves (4.4). □

These first two propositions together yield a proof of Proposition 2.2.

Proof of Proposition 2.2. Items 2.2 and 2.2 have been proved in Proposition 4.1, and the propagating property follows from Proposition 4.2. □

We continue with a technical lemma that will be used in the proof of Theorem 2.3.

Lemma 4.1. (Divergence speed near the separatrix) *Let* $u_0(x)$ *satisfy Assumptions* 2.1 *and* 2.2 *and* $u(t, x)$ *be the corresponding solution to* (1.1)*. Let* $h(t, x)$ *be the characteristic flow of* u *and* $h^*(t)$ *be the separatrix of* u*, as defined in Proposition* 2.2*. For all* $t \geq 0$ *and* $x < 0$ *we have*

$$\frac{d}{dt}(h^*(t) - h(t, x)) \leq \chi\,(h^*(t) - h(t, x)) \sup_{y \in (h(t,x), h^*(t))} u(t, y). \tag{4.5}$$

Proof. Recall that, by Proposition 4.1, $u(t, x) = 0$ for each $x \geq h^*(t)$. For $x < 0$, we notice that

$$\begin{aligned}\frac{d}{dt}(h^*(t) - h(t, x)) &= -\chi(\rho_x \star u)(t, h^*(t)) + \chi(\rho_x \star u)(h(t, x)) \\ &= \chi \int_{\mathbb{R}} (\rho_x(h(t, x) - y) - \rho_x(h^*(t) - y))u(t, y)dy\end{aligned}$$

$$= \chi \int_{-\infty}^{h(t,x)} (\rho_x(h(t,x)-y) - \rho_x(h^*(t)-y))u(t,y)dy$$

$$+ \chi \int_{h(t,x)}^{h^*(t)} (\rho_x(h(t,x)-y) - \rho_x(h^*(t)-y))u(t,y)dy.$$

Therefore,

$$\frac{d}{dt}(h^*(t) - h(t,x)) \leq \chi \int_{-\infty}^{h(t,x)} \big(\rho_x(h(t,x)-y) - \rho_x(h^*(t)-y)\big)u(t,y)dy$$
$$+ \chi(h^*(t) - h(t,x)) \times \sup_{y\in(h(t,x),h^*(t))} u(t,y).$$

Since $\rho_x(y) = -\frac{1}{2\sigma^2}\mathrm{sign}(y)e^{-\frac{|y|}{\sigma}}$ is increasing on $(0,+\infty)$, we have

$$\rho_x(h(t,x)-y) - \rho_x(h^*(t)-y) \leq 0$$

for each $y \leq h(t,x)$, which shows (4.5). Lemma 4.1 is proved. □

Proposition 4.3. (Formation of a discontinuity) *Let $u_0(x)$ satisfy Assumptions* 2.1 *and* 2.2 *and $u(t,x)$ be the corresponding solution to* (1.1)*. For all $\delta > 0$ we have*

$$\limsup_{t\to+\infty} \sup_{x\in(h^*(t,x)-\delta,h^*(t))} u(t,x) \geq \frac{1}{1+\hat{\chi}+\alpha\chi} > 0. \tag{4.6}$$

Proof. We divide the proof in two steps.

Step 1. We show that for all $\delta > 0$,

$$\sup_{t>0} \sup_{x\in(h^*(t)-\delta,h^*(t))} u(t,x) \geq \frac{1}{1+\hat{\chi}+\alpha\chi}. \tag{4.7}$$

Assume by contradiction that there exists $\delta > 0$ such that

$$\text{for all } t > 0, \quad \sup_{x\in(h^*(t)-\delta,h^*(t))} u(t,x) \leq \eta < \frac{1}{1+\hat{\chi}+\alpha\chi}, \tag{4.8}$$

where $\alpha \geq 1$ is the constant from Assumption 2.2.

We remark that the following inequality holds for $x \in (h^*(t)-\delta, h^*(t))$:

$$\begin{aligned}\frac{d}{dt}u(t,h(t,x)) &= \hat{\chi}\, u(t,h(t,x))(\rho\star u)(t,h(t,x)) \\ &\quad + u(t,h(t,x))\big(1-(1+\hat{\chi})u(t,h(t,x))\big) \\ &\geq u(t,h(t,x))\big(1-(1+\hat{\chi})u(t,h(t,x))\big) \\ &\geq u(t,h(t,x))\left(1-(1+\hat{\chi})\eta\right),\end{aligned} \tag{4.9}$$

therefore

$$u(t,h(t,x)) \geq u(0,x)\exp((1-(1+\hat{\chi})\eta)t),$$

provided the characteristic $h(t,x)$ does not leave the cylinder $(h^*(s)-\delta, h^*(s))$ for any $0 \le s \le t$.

Next by (4.5) and (4.8), we have

$$\frac{d}{dt}(h^*(t)-h(t,x)) \le \chi(h^*(t)-h(t,x)) \times \eta$$

for each $x \in (h^*(t)-\delta, h^*(t))$. Hence by Grönwall's lemma

$$(h^*(t)-h(t,x)) \le -xe^{\eta\chi t},$$

provided the characteristic $h(t,x)$ does not leave the cylinder $(h^*(s)-\delta, h^*(s))$ for any $0 \le s \le t$. In particular for $0 > -\frac{1}{2}\delta e^{-\eta\chi t} \ge x \ge -\delta e^{-\eta\chi t}$, we find

$$\begin{aligned} u(t,h(t,x)) &\ge u(0,x)\exp\left(\left(1-\frac{1+\hat{\chi}}{1+\hat{\chi}+\alpha\chi}\right)t\right) \\ &\ge \gamma(-x)^\alpha \exp\left(\left(1-\frac{1+\hat{\chi}}{1+\hat{\chi}+\alpha\chi}\right)t\right) \\ &\ge \frac{1}{2^\alpha}\gamma\delta^\alpha \exp((1-(1+\hat{\chi}+\alpha\chi)\eta)t) \xrightarrow[t\to+\infty]{} +\infty, \end{aligned}$$

by our assumption that $\eta < \frac{1}{1+\hat{\chi}+\alpha\chi}$. This is a contradiction.

Step 2. We show (4.6).

Assume by contradiction that there exists $T > 0$ and $\delta > 0$ such that

$$\sup_{t\ge T}\sup_{x\in[h^*(t)-\delta,h^*(t)]} u(t,x) < \frac{1}{1+\hat{\chi}+\alpha\chi}.$$

Since the function $u(t,x+h^*(t))$ is continuous on the compact set $[0,T]\times[-\delta,0]$, it is uniformly continuous on this set and hence (recall that $u(t,h^*(t))=0$) there exists $0<\delta_0\le\delta$ such that

$$\begin{aligned} \sup_{t\in[0,T],x\in[-\delta_0,0]} u(t,x+h^*(t)) &= \sup_{t\in[0,T],x\in[-\delta_0,0]} \big(u(t,x+h^*(t))-u(t,h^*(t))\big) \\ &\le \frac{1}{1+\hat{\chi}+\alpha\chi}. \end{aligned}$$

Hence we conclude

$$\sup_{t>0,x\in[-\delta_0,0]} u(t,x-h^*(t)) \le \frac{1}{1+\hat{\chi}+\alpha\chi}.$$

This is in contradiction with Step 1. Proposition 4.3 is proved. ◻

Proposition 4.4. (Refined estimate on the level sets) *Let $u_0(x)$ satisfy Assumption* 2.1 *and* 2.2. *Define*

$$\xi(t,\beta) := \sup\{x\in\mathbb{R} \mid u(t,x)=\beta\}$$

for any $0 < \beta < \frac{1}{1+\hat{\chi}+\alpha\chi}$. *Then, the level set function* $\xi(t,\beta)$ *converges exponentially fast to* $h^*(t)$

$$h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha}t} \leq \xi(t,\beta) \leq h^*(t) \tag{4.10}$$

for each $0 < \beta < \frac{1}{1+\hat{\chi}+\alpha\chi}$, *where* η *is given by*

$$\eta := 1 - \frac{1+\hat{\chi}+\alpha\chi}{\beta} \in (0,1).$$

Proof. Let $\eta \in (0,1)$ be given and set $\beta^* := \frac{1-\eta}{1+\hat{\chi}+\alpha\chi}$. Let us first remark that for any $\beta \in (0,\beta^*)$, $\xi(t,\beta)$ is well-defined by the continuity of $x \mapsto u(t,x)$ and Assumption 2.2, that $u(t,\xi(t,\beta)) = \beta$ and that $\sup_{x\in(\xi(t,\beta),h^*(t))} u(t,x) \leq \beta$. Moreover $\xi(0,\beta) < 0$ and $u_0(\xi(0,\beta)) = \beta \geq \gamma|\xi(0,\beta)|^\alpha$, therefore

$$\xi(0,\beta) \geq -\left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} \tag{4.11}$$

for each $0 < \beta \leq \beta^* = \frac{1-\eta}{1+\hat{\chi}+\alpha\chi}$.

Step 1. We show that if u_0 satisfies Assumption 2.1 and (4.11), then

$$\xi(t,\beta) \geq h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{\frac{\eta}{2\alpha}t}, \tag{4.12}$$

for all $0 \leq t \leq t^* := \frac{1}{1+\hat{\chi}} \ln(1 + \frac{\eta}{2(1-\eta)})$.

Let $0 < \beta \leq \beta^*$. We remark that, by Assumption 2.1, we have $0 \leq u(t,x) \leq 1$ hence $0 \leq (\rho \star u)(t,x) \leq 1$. It follows that, for all $t \geq 0$:

$$\frac{d}{dt}u(t,h(t,x)) = u(t,h(t,x))\big(1 + \hat{\chi}\rho\star u - (1+\hat{\chi})u(t,h(t,x))\big) \leq (1+\hat{\chi})u(t,h(t,x)).$$

In the remaining part of Step 1 we consider $t \in [0,t^*]$. Using (4.5) from Lemma 4.1 we establish the following estimates on u and h for $0 \leq t \leq t^*$ and $\xi(0,\beta^*) \leq x \leq 0$:

- Since $\frac{d}{dt}u(t,h(t,x)) \leq (1+\hat{\chi})u(t,h(t,x))$ we have $u(t,h(t,x)) \leq u_0(x)e^{(1+\hat{\chi})t}$ for all $t \leq t^*$ and hence if $x \geq \xi(0,\beta^*)$,

$$\begin{aligned} u(t,h(t,x)) \leq \beta^* e^{\ln(1+\frac{\eta}{2(1-\eta)})} &= \frac{1-\eta}{1+\hat{\chi}+\alpha\chi}\left(1 + \frac{\eta}{2(1-\eta)}\right) \\ &= \frac{1-\frac{\eta}{2}}{1+\hat{\chi}+\alpha\chi}. \end{aligned} \tag{4.13}$$

- Using (4.13) in the equation along the characteristic (2.2):

$$\begin{aligned} \frac{d}{dt}u(t,h(t,x)) &= u(t,h(t,x))\big(1 + \hat{\chi}(\rho\star u)(t,h(t,x)) - (1+\hat{\chi})u(t,h(t,x))\big) \\ &\geq \left(1 - \frac{(1+\hat{\chi})(1-\frac{\eta}{2})}{1+\hat{\chi}+\alpha\chi}\right) u(t,h(t,x)), \end{aligned}$$

we get

$$u(t,h(t,x)) \geq u_0(x)\exp\left[\left(1-\frac{(1+\hat{\chi})\left(1-\frac{\eta}{2}\right)}{1+\hat{\chi}+\alpha\chi}\right)t\right] \tag{4.14}$$

- For all $x \in (\xi(0,\beta^*),0)$, since

$$\sup_{y\in(h(t,x),h^*(t))} u(t,y) \leq \sup_{y\in(h(t,\xi(0,\beta^*)),h^*(t)} u(t,y) \leq \frac{1-\frac{\eta}{2}}{1+\hat{\chi}+\alpha\chi},$$

we have by (4.5):

$$h^*(t)-h(t,x) \leq \exp\left(\frac{(1-\frac{\eta}{2})\chi}{1+\hat{\chi}+\alpha\chi}t\right)(h^*(0)-h(0,x)),$$

hence

$$h(t,x) \geq h^*(t) + x\exp\left(\frac{(1-\frac{\eta}{2})\chi}{1+\hat{\chi}+\alpha\chi}t\right). \tag{4.15}$$

Since $\beta \leq \beta^*$, we have $\xi(0,\beta) \geq \xi(0,\beta^*)$. Using (4.14) with $x=\xi(0,\beta)$ we find that

$$u(t,h(t,\xi(0,\beta))) \geq \beta\exp\left[\left(1-\frac{(1+\hat{\chi})(1-\frac{\eta}{2})}{1+\hat{\chi}+\alpha\chi}\right)t\right],$$

which implies

$$\xi\left(t,\beta\exp\left[\left(1-\frac{(1+\hat{\chi})(1-\frac{\eta}{2})}{1+\hat{\chi}+\alpha\chi}\right)t\right]\right) \geq h(t,\xi(0,\beta)).$$

Now by using $x=\xi(0,\beta)$ in (4.15), we obtain

$$h(t,\xi(0,\beta)) \geq h^*(t) + \xi(0,\beta)\exp\left(\frac{(1-\frac{\eta}{2})\chi}{1+\hat{\chi}+\alpha\chi}t\right).$$

Using (4.11) we find that

$$\begin{aligned}\xi&\left(0,\beta\exp\left[-\left(1-\frac{(1+\hat{\chi})\left(1-\frac{\eta}{2}\right)}{1+\hat{\chi}+\alpha\chi}\right)t\right]\right)\\ &\geq -\left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}}\exp\left[-\frac{1}{\alpha}\left(1-\frac{(1+\hat{\chi})(1-\frac{\eta}{2})}{1+\hat{\chi}+\alpha\chi}\right)t\right],\end{aligned}$$

which leads to

$$\begin{aligned}\xi(t,\beta) &\geq h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}}\exp\left[-\frac{1}{\alpha}\left(1-\frac{(1+\hat{\chi})(1-\frac{\eta}{2})}{1+\hat{\chi}+\alpha\chi}\right)t + \frac{(1-\frac{\eta}{2})\chi}{1+\hat{\chi}+\alpha\chi}t\right]\\ &= h^*(t) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}}\exp\left[-\frac{\eta}{2\alpha}t\right]\end{aligned}$$

and this estimate holds for each $0 \leq t \leq t^*$ and $0 < \beta \leq \beta^*$.

Step 2. We show that the estimate (4.12) can be extended by induction.

Define $\bar{u}_0(x) := u(t^*, x + h(t^*))$ and $\bar{\xi}(t,\beta) = \xi(t + t^*, \beta) - h^*(t^*)$. We have for each $0 < \beta \le \beta^*$

$$\bar{\xi}(0,\beta) \ge -\left(\frac{\beta}{\bar{\gamma}}\right)^{\frac{1}{\alpha}},$$

where $\bar{\gamma} = \gamma e^{\frac{\eta}{2}t^*}$. In particular the inequality (4.11) is satisfied by $\bar{u}_0(x)$, as well as Assumption 2.1. We can apply Step 1 and (4.12) gives

$$\begin{aligned}\bar{\xi}(t,\beta) &\ge \bar{h}^*(t) - \left(\frac{\beta}{\bar{\gamma}}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha}t} = h(t, h^*(t)) - h^*(t^*) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha}(t+t^*)}\\ &= h^*(t+t^*) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha}(t+t^*)},\end{aligned}$$

which yields

$$\xi(t+t^*,\beta) \ge h^*(t+t^*) - \left(\frac{\beta}{\gamma}\right)^{\frac{1}{\alpha}} e^{-\frac{\eta}{2\alpha}(t+t^*)}.$$

The proof is completed. □

We are now in the position to prove Theorem 2.3

Proof of Theorem 2.3. The first part, Eq. (2.5), has been shown in Proposition 4.3, while the second part (Eq. (2.6)) has been shown in Proposition 4.4. □

We conclude this section by the proof of Proposition 2.3.

Proof of Proposition 2.3. Since $x \mapsto u(t,x)$ is nonincreasing, we have $u(t,x) \ge u(t,h^*(t))$ for each $x \le h^*(t)$. Hence $(\rho \star u)(t,h^*(t)) \ge \frac{1}{2}u(t,h^*(t))$ and

$$\begin{aligned}\frac{d}{dt}u(t,h^*(t)) &= u(t,h^*(t))(1 + \hat{\chi}\,\rho \star u - (1+\hat{\chi})u(t,h^*(t)))\\ &\ge u(t,h^*(t))\left(1 - \left(1 + \frac{\hat{\chi}}{2}\right)u(t,h^*(t))\right).\end{aligned}$$

This yields

$$u(t,h^*(t)) \ge \frac{u_0(0)}{\left(1+\frac{\hat{\chi}}{2}\right)u_0(0) + e^{-t}\left(1 - \left(1+\frac{\hat{\chi}}{2}\right)u_0(0)\right)} \underset{t\to+\infty}{\longrightarrow} \frac{1}{1+\frac{\hat{\chi}}{2}} = \frac{2}{2+\hat{\chi}}.$$

Equation (2.7) is shown. Next, we have $\frac{d}{dt}h^*(t) = -(\rho_x \star u)(t,h^*(t))$ which gives

$$\frac{d}{dt}h^*(t) = \frac{\chi}{\sigma}\int_0^\infty \rho(y)u(t,h^*(t)-y)dy \ge u(t,h^*(t)) \times \frac{\chi}{2\sigma} \underset{t\to+\infty}{\longrightarrow} \frac{\sigma\hat{\chi}}{2+\hat{\chi}}.$$

This proves (2.8) and finishes the proof of Proposition 2.3. □

5. Traveling Wave Solutions

In this section, we investigate the existence of particular solutions which consist in a fixed profile traveling at a constant speed c (traveling waves). We are particularly interested in profiles which connect the stationary state 1 near $-\infty$ to the stationary solution 0 at a finite point of space, say, for any $x \geq 0$.

5.1. *Existence of sharp traveling waves*

We study the traveling wave solutions of Eq. (1.1):

$$\begin{cases} \partial_t u(t,x) - \chi \partial_x \big(u(t,x)\partial_x p(t,x)\big) = u(t,x)(1-u(t,x)), \\ -\sigma^2 \partial_x^2 p(t,x) + p(t,x) = u(t,x), \end{cases} \qquad t>0,\ x\in\mathbb{R}.$$

Let us formally derive an equation for the traveling wave solutions to (1.1). We consider the traveling wave solution $U(x-ct) = u(t,x)$. By using the resolvent formula of the second equation of (1.1) formula we deduce that

$$p(t,x) = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} u(t,y)dy = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|x-ct-l|}{\sigma}} U(l)dl = P(x-ct)$$

and the first equation in (1.1) becomes

$$\begin{aligned} &-cU'(x-ct) - \chi\,\partial_x(U(x-ct)\,\partial_x P(x-ct)) \\ &\quad = U(x-ct)(1-U(x-ct)), \quad t>0,\ x\in\mathbb{R}. \end{aligned} \tag{5.1}$$

By developing the derivative in (5.1) we obtain

$$(-c-\chi P'(x-ct))U'(x-ct) = U(x-ct)(1+\hat{\chi}P(x-ct) - (1+\hat{\chi})U(x-ct)),$$

for all $t>0$ and $x\in\mathbb{R}$, where $\hat{\chi} = \frac{\chi}{\sigma^2}$. Therefore, by letting $z = x-ct$, the traveling wave solutions of system (1.1) satisfy the following equation:

$$\begin{cases} (-c-\chi P'(z))U'(z) = U(z)\big(1+\hat{\chi}P(z) - (1+\hat{\chi})U(z)\big), \\ -\sigma^2 P''(z) + P(z) = U(z). \end{cases} \tag{5.2}$$

Let us finally remark that

$$P(z) = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U(z-y)dy = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|z-y|}{\sigma}} U(y)dy. \tag{5.3}$$

In particular if U is non-constant and nonincreasing, then $z \mapsto P(z)$ is strictly decreasing.

The goal of this section is to show that Eq. (5.2) can solved on the half-line $(-\infty, 0)$ which, as we will see later, will give a proof of Theorem 2.4. We begin by defining a set of admissible profiles, which is the set of function on which an appropriate fixed-point theorem will be used. The properties we impose are those who we suspect will be satisfied by the real profile of the traveling wave.

Definition 5.1. We say that the profile $U : \mathbb{R} \to [0,1]$ is *admissible* if

(i) $U \in C((-\infty,0),\mathbb{R})$ and $\lim_{z\to 0^-} U(z)$ exists and belongs to $[\frac{2}{2+\hat{\chi}}, 1]$;
(ii) $0 \le U(z) \le 1$ for any $z \in \mathbb{R}$;
(iii) the map $z \mapsto U(z)$ is nonincreasing on $\mathbb{R}$;
(iv) $U(z) \equiv 0$ for any $z \ge 0$.

We denote $\mathcal{A}$ the set of all admissible functions.

Lemma 5.1. *Let Assumption* 2.3 *hold and suppose that* U *is admissible* (*as in Definition* 5.1). *Then the function* P *defined by* $P = (\rho \star U)$ *satisfies*

$$P'(0) < P'(z) \le 0, \quad \text{for all } z \in \mathbb{R}\backslash\{0\}.$$

Moreover, this estimate is locally uniform in U *on* $(-\infty,0)$ *in the sense that for each* $L > 1$ *there is* $\epsilon > 0$ independent of $U \in \mathcal{A}$ *such that*

$$P'(z) - P'(0) \ge \epsilon > 0, \quad \text{for all } z \in \left[-L, -\frac{1}{L}\right].$$

Proof. We divide the proof in five steps.

Step 1. We prove $P'(0) < P'(z)$ for any $z > 0$. Notice that, for $z > 0$, we have

$$P(z) = \frac{1}{2\sigma}\int_{-\infty}^{z} e^{-\frac{z-y}{\sigma}}U(y)dy + \frac{1}{2\sigma}\int_{z}^{\infty} e^{\frac{z-y}{\sigma}}U(y)dy = \frac{1}{2\sigma}e^{-\frac{z}{\sigma}}\int_{-\infty}^{0} e^{\frac{y}{\sigma}}U(y)dy.$$

Thus, taking derivative gives

$$P'(z) = -\frac{1}{\sigma}e^{-\frac{z}{\sigma}}\frac{1}{2\sigma}\int_{-\infty}^{0} e^{y}U(y)dy = e^{-\frac{z}{\sigma}}P'(0)$$

and since U is strictly positive for negative values of z, we deduce that $P'(0) < P'(z)$ for any $z > 0$.

Step 2. We prove that $P'(0) < P'(z)$ for any $-\sigma\ln(\frac{\hat{\chi}}{2}) < z < 0$. In fact, we prove the stronger result

$$P''(z) < 0 \quad \text{if } \sigma \ln\left(\frac{\hat{\chi}}{2}\right) < z < 0.$$

For any $z < 0$, we have

$$\begin{aligned} P''(z) &= \frac{1}{2\sigma^3}\int_{-\infty}^{z} e^{-\frac{z-y}{\sigma}}U(y)dy + \frac{1}{2\sigma^3}\int_{z}^{\infty} e^{\frac{z-y}{\sigma}}U(y)dy - \frac{1}{\sigma^2}U(z) \\ &= \frac{1}{2\sigma^3}\int_{-\infty}^{z} e^{-\frac{z-y}{\sigma}}U(y)dy + \frac{1}{2\sigma^3}\int_{z}^{0} e^{\frac{z-y}{\sigma}}U(y)dy - \frac{1}{\sigma^2}U(z). \end{aligned}$$

Due to the assumption $U \le 1$ and the fact that U is decreasing we have

$$\sigma^2 P''(z) \le \frac{1}{2\sigma}\int_{-\infty}^{z} e^{-\frac{z-y}{\sigma}}dy + \frac{1}{2\sigma}\int_{z}^{0} e^{\frac{z-y}{\sigma}}U(y)dy - U(z)$$

$$= \frac{1}{2} + \frac{1}{2\sigma}\int_z^0 e^{\frac{z-y}{\sigma}} U(y)dy - U(z) \le \frac{1}{2} + \frac{1}{2\sigma}\int_z^0 e^{\frac{z-y}{\sigma}} dy U(z) - U(z)$$

$$= \frac{1}{2} - \frac{1}{2}\left(1 + e^{\frac{z}{\sigma}}\right) U(z) \le \frac{1}{2}\frac{2+\hat{\chi} - 2(1+e^{\frac{z}{\sigma}})}{2+\hat{\chi}} = \frac{\hat{\chi} - 2e^{\frac{z}{\sigma}}}{2(2+\hat{\chi})} < 0,$$

provided $z \in (\sigma \ln(\hat{\chi}/2), 0)$. In particular

$$P'(z) - P'(0) = -\int_z^0 P''(y)dy \ge \frac{1}{\sigma(2+\hat{\chi})}\left(\frac{\hat{\chi}}{2\sigma} z + 1 - e^{\frac{z}{\sigma}}\right) > 0. \qquad (5.4)$$

Step 3. We prove that $P'(0) < P'(z)$ for any $z < \sigma \ln(1 - \frac{\hat{\chi}}{2})$. For any $z < 0$, we have

$$\sigma P'(z) = -\frac{1}{2\sigma}\int_{-\infty}^z e^{-\frac{z-y}{\sigma}} U(y)dy + \frac{1}{2\sigma}\int_z^0 e^{\frac{z-y}{\sigma}} U(y)dy,$$

$$\sigma P'(0) = -\frac{1}{2\sigma}\int_{-\infty}^0 e^{\frac{y}{\sigma}} U(y)dy$$

and

$$\sigma(P'(z) - P'(0)) = \frac{1}{2\sigma}\int_{-\infty}^0 e^{\frac{y}{\sigma}} U(y)dy - \frac{1}{2\sigma}\int_{-\infty}^z e^{-\frac{z-y}{\sigma}} U(y)dy + \frac{1}{2\sigma}\int_z^0 e^{\frac{z-y}{\sigma}} U(y)dy.$$

Since for any $z \le 0$, $\frac{2}{2+\hat{\chi}} \le U(z) \le 1$, we have the following estimate:

$$\begin{aligned}\sigma\big(P'(z) - P'(0)\big) &\ge \frac{1}{2\sigma}\int_{-\infty}^0 e^{\frac{y}{\sigma}} \times \frac{2}{2+\hat{\chi}} dy - \frac{1}{2\sigma}\int_{-\infty}^z e^{-\frac{z-y}{\sigma}} dy \\ &\quad + \frac{1}{2\sigma}\int_z^0 e^{\frac{z-y}{\sigma}} \frac{2}{2+\hat{\chi}} dy \\ &= \frac{1}{2+\hat{\chi}} - \frac{1}{2} + \frac{1}{2+\hat{\chi}}(1 - e^{\frac{z}{\sigma}}) \\ &= \frac{1}{2+\hat{\chi}}\left(2 - e^{\frac{z}{\sigma}} - \frac{1}{2}(2+\hat{\chi})\right) = \frac{1}{2+\hat{\chi}}\left(1 - \frac{\hat{\chi}}{2} - e^{\frac{z}{\sigma}}\right). \qquad (5.5)\end{aligned}$$

By our assumption $z < \sigma \ln(1 - \frac{\hat{\chi}}{2})$, we deduce that $P'(z) - P'(0) > 0$.

Notice that, if $\hat{\chi} < 1$, we have $\sigma \ln(\frac{\hat{\chi}}{2}) < \sigma \ln(1 - \frac{\hat{\chi}}{2})$ and the estimate is done. If $1 \le \hat{\chi} < 2$ we still need to fill a gap between the two bounds.

Step 4. We assume that $\hat{\chi} \ge 1$ and we prove that

$$\begin{aligned}P'(z) - P'(0) &\ge -\int_z^0 P''(y)dy \\ &\ge \frac{z}{2\sigma^2} - \frac{1}{2\sigma}\ln\left(\frac{\hat{\chi}}{2}\right) + \frac{1}{\sigma(2+\hat{\chi})}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right) > 0 \qquad (5.6)\end{aligned}$$

for any $z \in [\sigma \ln(\frac{\hat{\chi}}{2}) - \frac{\sigma}{2+\hat{\chi}}(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}), \sigma\ln(\frac{\hat{\chi}}{2})]$. Notice that

$$\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2} > 0,$$

because $x \mapsto x\ln(x)$ is strictly convex.

By Step 2 we have for all $z \le 0$:

$$P''(z) \le \frac{1}{2\sigma^2},$$

therefore if $z \in [\sigma \ln(\frac{\hat{\chi}}{2}) - \frac{\sigma}{2+\hat{\chi}}(\frac{\hat{\chi}}{2}\ln(\frac{\hat{\chi}}{2}) + 1 - \frac{\hat{\chi}}{2}), \sigma\ln(\frac{\hat{\chi}}{2})]$ we have

$$\begin{aligned}
P'(z) - P'(0) &= P'(z) - P'\left(\sigma\ln\left(\frac{\hat{\chi}}{2}\right)\right) + P'\left(\sigma\ln\left(\frac{\hat{\chi}}{2}\right)\right) - P'(0)\\
&\ge -\int_z^{\sigma\ln\left(\frac{\hat{\chi}}{2}\right)} P''(y)dy + \frac{1}{\sigma(2+\hat{\chi})}\left(\frac{\hat{\chi}}{2\sigma}\sigma\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right)\\
&\ge -\frac{1}{2\sigma^2}\left(\sigma\ln\left(\frac{\hat{\chi}}{2}\right) - z\right) + \frac{1}{\sigma(2+\hat{\chi})}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right)\\
&\ge \frac{z}{2\sigma^2} - \frac{\ln\left(\frac{\hat{\chi}}{2}\right)}{2\sigma} + \frac{1}{\sigma(2+\hat{\chi})}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right) > 0.
\end{aligned}$$

We have proved the desired estimate.

Step 5. We show the local uniformity. If $\hat{\chi} < 1$ the local uniformity follows from Steps 2 and 3 because $1 - \frac{\hat{\chi}}{2} < \frac{\hat{\chi}}{2}$. If $1 \le \hat{\chi} < 2$, then

$$\ln\left(\frac{\hat{\chi}}{2}\right) - \frac{2}{2+\hat{\chi}}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right) < \ln\left(1 - \frac{\hat{\chi}}{2}\right), \tag{5.7}$$

because of Assumption 2.3 and Lemma A.1 (notice that (5.7) is equivalent to $f(\hat{\chi}) < 0$, where f is as defined in Lemma A.1). By the estimates (5.4)–(5.6) from Steps 2–4, we find that $P'(z) - P'(0) > 0$ on every compact subset of $(-\infty, 0)$ and is bounded from below by a constant independent of U. This finishes the proof of Lemma 5.1. □

Before resuming to the proof, let us define the mapping $\mathcal{T}$ to which we want to apply a fixed-point theorem. Fix $U \in \mathcal{A}$, we define $\mathcal{T}(U)$ as

$$\mathcal{T}(U)(z) := \mathcal{U}(\tau^{-1}(z)) \quad \text{for all } z < 0 \tag{5.8}$$

and $\mathcal{T}(U)(z) \equiv 0$ for all $z \ge 0$, where $\tau : \mathbb{R} \mapsto (-\infty, 0)$ is the solution of the following scalar ordinary differential equation:

$$\begin{cases} \tau'(t) = \chi(P'(0) - P'(\tau(t))), \\ \tau(0) = -1 \end{cases} \tag{5.9}$$

and

$$\mathcal{U}(t) = \left[(1+\hat{\chi})\int_{-\infty}^{t} \exp\left(-\int_l^t 1 + \hat{\chi}P(\tau(s))ds\right)dl\right]^{-1}, \quad \text{for all } t \in \mathbb{R}.$$

Lemma 5.2. (Stability of $\mathcal{A}$) *Let Assumption* 2.3 *be satisfied, let* U *be admissible in the sense of Definition* 5.1 *and* $\mathcal{T}$ *be the map defined by* (5.8). *Then the image of* U *by* $\mathcal{T}$ *has the following properties*:

(i) $\frac{2}{2+\hat{\chi}} \leq \mathcal{T}(U)(z) \leq 1$ *for all* $z \leq 0$;
(ii) $\mathcal{T}(U)$ *is strictly decreasing on* $(-\infty, 0]$;
(iii) $\mathcal{T}(U) \in C^1((-\infty,0),\mathbb{R})$ *and* $\mathcal{T}(U)(0^-) = \lim_{z\to 0^-} \mathcal{T}(U)(z) = \frac{1+\hat{\chi}P(0)}{1+\hat{\chi}}$.

In particular, $\mathcal{A}$ *is left stable by* $\mathcal{T}$

$$\mathcal{T}(\mathcal{A}) \subset \mathcal{A}.$$

Proof. We divide the proof in three steps.

Step 1. We prove that $\frac{2}{2+\hat{\chi}} \leq \mathcal{T}(U)(z) \leq 1$ for all $z < 0$. For any $z \in \mathbb{R}$ we have

$$P(z) = \int_{-\infty}^{\infty} \rho(y)U(z-y)dy \leq \int_{-\infty}^{+\infty} \rho(y)dy = 1,$$
$$P(z) = \int_{-\infty}^{\infty} \rho(y)U(z-y)dy \geq 0.$$

Since $\frac{2}{2+\hat{\chi}} \leq U(z) \leq 1$ for all $z < 0$, we have for $z < 0$,

$$P(z) \geq \frac{1}{2\sigma}\int_{z}^{+\infty} \exp\left(-\frac{|y|}{\sigma}\right) \times \frac{2}{2+\hat{\chi}} dy = \frac{2}{2+\hat{\chi}}\left(1 - \frac{e^{\frac{z}{\sigma}}}{2}\right) \geq \frac{1}{2+\hat{\chi}}.$$

Thus, for any $z \leq 0$, we have $\frac{1}{2+\hat{\chi}} \leq P(z) \leq 1$. Since $\tau(t)$ is the solution of

$$\begin{cases} \tau'(t) = \chi(P'(0) - P'(\tau(t))), \\ \tau(0) = -1 \end{cases}$$

and due to Lemma 5.1, $t \to \tau(t)$ is strictly decreasing, continuous and

$$\lim_{t\to-\infty} \tau(t) = 0, \quad \lim_{t\to+\infty} \tau(t) = -\infty.$$

Therefore,

$$\frac{1}{2+\hat{\chi}} \leq P(\tau(t)) \leq 1, \quad t \in \mathbb{R}.$$

Since by definition $\mathcal{U}(t) = [(1+\hat{\chi})\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}dl]^{-1}$, $\mathcal{U}$ is monotone with respect to P, and we compute on the one hand

$$\begin{aligned}\mathcal{U}(t) &\leq \left[(1+\hat{\chi})\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}ds}dl\right]^{-1}\\ &= \left[(1+\hat{\chi})\int_{-\infty}^{t} e^{-(1+\hat{\chi})(t-l)}dl\right]^{-1} = 1.\end{aligned}$$

On the other hand, we can see that

$$\begin{aligned}\mathcal{U}(t) &\geq \left[(1+\hat{\chi})\int_{-\infty}^{t} \exp\left(-\int_l^t 1+\frac{\hat{\chi}}{2+\hat{\chi}}ds\right)dl\right]^{-1}\\ &= \left[(1+\hat{\chi})\int_{-\infty}^{t} \exp\left(-\left(1+\frac{\hat{\chi}}{2+\hat{\chi}}\right)(t-l)\right)dl\right]^{-1} = \frac{2}{2+\hat{\chi}}.\end{aligned}$$

This implies $\frac{2}{2+\hat{\chi}} \leq \mathcal{U}(t) \leq 1$, for all $t \in \mathbb{R}$. Since τ^{-1} maps $(-\infty,0)$ to $\mathbb{R}$, for any $z<0$ we have indeed

$$\frac{2}{2+\hat{\chi}} \leq \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z)) \leq 1.$$

Item 5.2 is proved.

Step 2. We prove that $z \mapsto \mathcal{T}(U)(z)$ is strictly decreasing on $(-\infty,0)$. First, we prove that $t \mapsto \mathcal{U}(t)$ is strictly increasing. Indeed $\mathcal{U}$ is differentiable and we have

$$\mathcal{U}'(t) = \frac{-1}{1+\hat{\chi}} \times \frac{1+\int_{-\infty}^{t} -\big(1+\hat{\chi}P(\tau(t))\big)e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}dl}{\left[\int_{-\infty}^{t}\exp\big(-\int_l^t 1+P(\tau(s))ds\big)dl\right]^2}. \tag{5.10}$$

Moreover, for any $l<t$, we have $\tau(t)<\tau(l)$. Since P is strictly decreasing, $P(\tau(l)) < P(\tau(t))$. We deduce

$$\begin{aligned}\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}(1+\hat{\chi}P(\tau(t)))dl &> \int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}\big(1+\hat{\chi}P(\tau(l))\big)dl\\ &= \int_{-\infty}^{t} \frac{d}{dl}\left(e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}\right) = 1.\end{aligned}$$

This implies $\mathcal{U}'(t) > 0$ and $t \mapsto \mathcal{U}(t)$ is strictly increasing. Note that the inverse map $z \mapsto \tau^{-1}(z)$ is strictly decreasing, therefore the composition of two mappings

$$z \longmapsto \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z))$$

is also strictly decreasing on $(-\infty,0)$. Item 5.2 is proved.

Step 3. We prove that $\mathcal{T}(U) \in C^1((-\infty,0),\mathbb{R})$ and compute the limit of $\mathcal{T}(U)$ as $z \to 0^-$.

Since for any $z<0$

$$\sigma^2 P''(z) = -U(z) + P(z) \in C((-\infty,0),\mathbb{R}),$$

P belongs to $C^2((-\infty,0),\mathbb{R})$, which implies that $t \mapsto \tau(t)$ belongs to $C^1(\mathbb{R},(-\infty,0))$. By (5.10), the function $t \mapsto \mathcal{U}'(t)$ is continuous and the inverse map $z \to \tau^{-1}(z)$ is also of class C^1 from $(-\infty,0)$ to $\mathbb{R}$. Thus, the function

$$z \longmapsto \mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z))$$

is of class C^1 from $(-\infty,0)$ to $\mathbb{R}$. Moreover, the map $t \mapsto \mathcal{U}(t)$ is strictly decreasing and is bounded from below by $\frac{2}{2+\hat{\chi}} > 0$, thus $\lim_{t\to-\infty}\mathcal{U}(t)$ exists. In particular

$$\mathcal{T}(U)(0^-) := \lim_{z\to 0^-} \mathcal{U}(\tau^{-1}(z)) = \lim_{t\to-\infty} \mathcal{U}(t).$$

By the definition of $\mathcal{U}$

$$\begin{aligned}\mathcal{T}(U)(0^-) &= \lim_{t\to-\infty} \mathcal{U}(t)\\ &= \lim_{t\to-\infty} \left[(1+\hat{\chi})\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds} dl\right]^{-1}\\ &= \lim_{t\to-\infty} \frac{e^{\int_0^t 1+\hat{\chi}P(\tau(s))ds}}{(1+\hat{\chi})\int_{-\infty}^t e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds} dl}.\end{aligned}$$

By employing L'Hôpital rule

$$\begin{aligned}\mathcal{T}(U)(0^-) &= \lim_{t\to-\infty} \frac{e^{\int_0^t 1+\hat{\chi}P(\tau(s))ds}}{(1+\hat{\chi})\int_{-\infty}^t e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds} dl}\\ &= \lim_{t\to-\infty} \frac{\big(1+\hat{\chi}P(\tau(t))\big)e^{\int_0^t 1+P(\tau(s))ds}}{(1+\hat{\chi})\, e^{\int_0^t 1+P(\tau(s))ds}}\\ &= \frac{1+\hat{\chi}P(0)}{1+\hat{\chi}}.\end{aligned}$$

Therefore, $\mathcal{T}(U) \in C^1((-\infty,0),\mathbb{R}) \cap C((-\infty,0],\mathbb{R})$ and $\mathcal{T}(U)(0) = (1 + \hat{\chi}P(0))/(1+\hat{\chi})$. This proves item (iii) and concludes the proof of Lemma 5.2. □

Next, we focus on the continuity of $\mathcal{T}$ for a particular topology.

Lemma 5.3. (Continuity of $\mathcal{T}$) *Define the weighted norm*

$$\|U\|_\eta := \sup_{z\in(-\infty,0)} \alpha(z)|U(z)|, \tag{5.11}$$

where

$$\alpha(z) := \sqrt{-z}\, e^{\eta z} \le \frac{1}{\sqrt{2e\eta}}, \quad \text{for all } z \le 0,$$

with $0 < \eta < \sigma^{-1}$. If Assumption 2.3 is satisfied, then the map $\mathcal{T}$ is continuous on $\mathcal{A}$ for the distance induced by $\|\cdot\|_\eta$.

Proof. Let $U \in \mathcal{A}$ and $\varepsilon \in (0, 2\sqrt{2/\eta e})$ be given. Let $\tilde{U} \in \mathcal{A}$ be given and define the corresponding pressure and rescaled variable $\tilde{P} := \rho \star \tilde{U}$ and $\tilde{\tau}$ as the solution to (5.9) with U replaced by $\tilde{U}$. We remark that

$$
\begin{aligned}
&|\mathcal{T}(U)(z) - \mathcal{T}(\tilde{U})(z)| \\
&\quad = |\mathcal{T}(U)(z)\mathcal{T}(\tilde{U})(z)| \left| \int_{-\infty}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} dl \right. \\
&\qquad \left. - \int_{-\infty}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right| \\
&\quad \le \left| \int_{-\infty}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} dl - \int_{-\infty}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right|,
\end{aligned}
$$

by Lemma 5.2. Define $T_{-L}(U) := \int_{-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl$. We have $\mathcal{T}(U) = T_{-\infty}(U)$ and

$$
\begin{aligned}
&|T_{-\infty}(U) - T_{-\infty}(\tilde{U})| \\
&\le \left| \int_{-\infty}^{\tilde{\tau}^{-1}(z)-L} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} dl - \int_{-\infty}^{\tau^{-1}(z)-L} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right| \\
&\quad + \left| \int_{\tilde{\tau}^{-1}(z)-L}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} dl - \int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right| \\
&\le e^{-L} + e^{-L} \\
&\quad + \left| \int_{\tilde{\tau}^{-1}(z)-L}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}P(\tilde{\tau}(s))ds} dl - \int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right| \\
&\le \frac{\varepsilon}{2}\sqrt{2\eta e} \\
&\quad + \left| \int_{\tilde{\tau}^{-1}(z)-L}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} dl - \int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds} dl \right| \\
&= \frac{\varepsilon}{2}\sqrt{2\eta e} + |T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)|,
\end{aligned}
$$

for $L := -\ln(\frac{\varepsilon}{2}\sqrt{\frac{e\eta}{2}}) > 0$.

Let z_0 and z_1 be respectively the smallest and the biggest negative root of the equation

$$
\eta z + \frac{1}{2}\ln(-z) = \ln\left(\frac{\varepsilon}{4}\right).
$$

The choice of ε ensures that z_0 and z_1 exist. Then if $z \notin [z_0, z_1]$ we have $\sqrt{-z}e^{\eta z} \leq \frac{\varepsilon}{4}$ and, since $|T_{-L}(U)| \leq 1$ we have

$$\sqrt{-z}e^{\eta z}|T_{-L}(U)(z)| = \sqrt{-z}e^{\eta z}\left|\int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}dl\right|$$
$$\leq \frac{\varepsilon}{4}\int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1ds}dl = \frac{\varepsilon}{4}(1-e^{-L}) \leq \frac{\varepsilon}{4}.$$

Similarly, we have

$$\sqrt{-z}e^{\eta z}|T_{-L}(\tilde{U})(z)| \leq \frac{\varepsilon}{4}.$$

We have shown

$$\sup_{z\notin[z_0,z_1]} \sqrt{-z}e^{\eta z}|\mathcal{T}(U)(z) - \mathcal{T}(\tilde{U})(z)| \leq \varepsilon.$$

There remains to estimate $\sqrt{-z}e^{\eta z}|T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)|$ when $z \in [z_0, z_1]$. We have

$$\begin{aligned}
&|T_{-L}(U)(z) - T_{-L}(\tilde{U})(z)| \\
&\quad = \left|\int_{\tilde{\tau}^{-1}(z)-L}^{\tilde{\tau}^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds}dl - \int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}dl\right| \\
&\quad \leq 2|\tilde{\tau}^{-1}(z) - \tau^{-1}(z)| \\
&\qquad + \left|\int_{\tau^{-1}(z)-L}^{\tau^{-1}(z)} e^{-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} - e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}dl\right| \\
&\quad \leq 2|\tilde{\tau}^{-1}(z) - \tau^{-1}(z)| \\
&\qquad + L\sup_{l\in(\tau^{-1}(z)-L,\tau^{-1}(z))}\left|e^{\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds-\int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds} - 1\right|
\end{aligned}$$

and we remark that

$$\begin{aligned}
&\left|\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds - \int_l^{\tilde{\tau}^{-1}(z)} 1+\hat{\chi}\tilde{P}(\tilde{\tau}(s))ds\right| \\
&\quad \leq 2|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| + \hat{\chi}\left|\int_l^{\tau^{-1}(z)} P(\tau(s)) - \tilde{P}(\tilde{\tau}(s))ds\right| \\
&\quad \leq 2|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| + \hat{\chi}L\sup_{s\in(\tau^{-1}(z)-L,\tau^{-1}(z))}|P(\tau(s)) - P(\tilde{\tau}(s))| \\
&\qquad + \hat{\chi}L\sup_{s\in(\tau^{-1}(z)-L,\tau^{-1}(z))}|P(\tilde{\tau}(s)) - \tilde{P}(\tilde{\tau}(s))|.
\end{aligned}$$

To conclude the proof of the continuity of $\mathcal{T}$, we show that each of those three terms can be made arbitrarily small (uniformly on $[z_0, z_1]$) by choosing $\tilde{U}$ sufficiently close to U in the $\|\cdot\|_\eta$ norm. We start with the second one. We have for all $z \le 0$:

$$\begin{aligned}
|P(z) - \tilde{P}(z)| &= \frac{1}{2\sigma}\left|\int_{-\infty}^{0} e^{-\frac{|z-y|}{\sigma}}(U(y) - \tilde{U}(y))dy\right| \\
&\le \frac{1}{2\sigma}\int_{-\infty}^{z} e^{\frac{y-z}{\sigma}}|U(y) - \tilde{U}(y)|dy + \frac{1}{2\sigma}\int_{z}^{0} e^{\frac{z-y}{\sigma}}|U(y) - \tilde{U}(y)|dy \\
&\le \frac{1}{2\sigma}\sqrt{\frac{2\eta}{e}}e^{-\frac{z}{\sigma}}\int_{-\infty}^{z}\frac{e^{(1-\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}\|U - \tilde{U}\|_\eta dy \\
&\quad + \frac{1}{2}\sqrt{\frac{2\eta}{e}}e^{\frac{z}{\sigma}}\int_{z}^{0}\frac{e^{-(1+\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}\|U - \tilde{U}\|_\eta dy \\
&= \sigma^{-1}\sqrt{\frac{\eta}{2e}}\left[e^{-\frac{z}{\sigma}}\int_{-\infty}^{z}\frac{e^{(1-\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}dy + e^{\frac{z}{\sigma}}\int_{z}^{0}\frac{e^{-(1+\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}dy\right]\|U - \tilde{U}\|_\eta \\
&=: C_P(z)\|U - \tilde{U}\|_\eta.
\end{aligned}$$

A similar computation shows that, for all $z \le 0$,

$$\begin{aligned}
|P'(z) - \tilde{P}'(z)| &\le \sigma^{-2}\sqrt{\frac{\eta}{2e}}\left[e^{-\frac{z}{\sigma}}\int_{-\infty}^{z}\frac{e^{(1-\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}dy + e^{\frac{z}{\sigma}}\int_{z}^{0}\frac{e^{-(1+\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}dy\right]\|U - \tilde{U}\|_\eta \\
&= \frac{1}{\sigma}C_P(z)\|U - \tilde{U}\|_\eta.
\end{aligned}$$

In particular for $z = 0$ we have

$$|P'(0) - \tilde{P}'(0)| \le \sigma^{-2}\sqrt{\frac{\eta}{2e}}\int_{-\infty}^{0}\frac{e^{(1-\sigma\eta)\frac{y}{\sigma}}}{\sqrt{-y}}dy\|U - \tilde{U}\|_\eta$$

and therefore $P'(0)$ and $\tilde{P}'(0)$ can be chosen arbitrarily small. Next, we show that $\tau(t)$ and $\tilde{\tau}(t)$ are uniformly close for $t \in [\tau^{-1}(z_0) - L, \tau^{-1}(z_1)]$. Indeed, we compute:

$$\begin{aligned}
|(\tau - \tilde{\tau})(t)| &= \chi\left|\int_0^t P'(0) - P'(\tau(s))ds - \int_0^t \tilde{P}'(0) - \tilde{P}'(\tilde{\tau}(s))ds\right| \\
&\le \chi\left|t(P'(0) - \tilde{P}'(0)) + \int_0^t \tilde{P}'(\tau(s)) - P'(\tau(s))ds\right| \\
&\quad + \chi\left|\int_0^t \tilde{P}'(\tilde{\tau}(s)) - \tilde{P}'(\tau(s))ds\right| \\
&\le \chi t[C_P(0) + \max_{0\le s\le t} C_P(\tau(s))]\|U - \tilde{U}\|_\eta + \hat{\chi}\int_0^t |\tilde{\tau}(s) - \tau(s)|ds,
\end{aligned}$$

where we have used the fact that $\sigma^2|P''(z)| = |P(z) - U(z)| \le 1$. By Grönwall's lemma, we have therefore

$$|\tau(t) - \tilde{\tau}(t)| \le \chi t\big[C_P(0) + \max_{0\le s\le t} C_P(\tau(s))\big]\|U - \tilde{U}\|_\eta e^{\hat{\chi}t}$$

and we have shown that τ and $\tilde{\tau}$ can be made arbitrarily close by choosing $\|U-\tilde{U}\|_\eta$ sufficiently small. This gives an arbitrary control on the term

$$\sup_{s\in(\tau^{-1}(z)-L,\tau^{-1}(z))} |P(\tau(s)) - P(\tilde{\tau}(s))| \le |P'(0)||\tau(s) - \tilde{\tau}(s)|,$$

since $P'(0) < P'(z) \le 0$ by Lemma 5.1, and on the term

$$\sup_{s\in(\tau^{-1}(z)-L,\tau^{-1}(z))} |P(\tilde{\tau}(s)) - \tilde{P}(\tilde{\tau}(s))| \le \left[\sup_{s\in(\tau^{-1}(z)-L,\tau^{-1}(z))} C_P(\tilde{\tau}(s))\right] \|U - \tilde{U}\|_\eta.$$

Finally, we estimate $\tau^{-1}(z) - \tilde{\tau}^{-1}(z)$ by the remark:

$$\begin{aligned}
|\tau^{-1}(z) - \tilde{\tau}^{-1}(z)| &= \left|\int_{-1}^{z} \frac{1}{\tau'(\tau^{-1}(y))}dy - \int_{-1}^{z} \frac{1}{\tilde{\tau}'(\tilde{\tau}^{-1}(y))}dy\right| \\
&= \frac{1}{\chi}\left|\int_{-1}^{z} \frac{1}{P'(0) - P'(y)} - \frac{1}{\tilde{P}'(0) - \tilde{P}'(y)}dy\right| \\
&\le \frac{1}{\chi}\int_{-1}^{z} \frac{|P'(0) - \tilde{P}'(0)| + |P'(y) - \tilde{P}'(y)|}{|P'(0) - P'(y)||\tilde{P}'(0) - \tilde{P}'(y)|}dy,
\end{aligned}$$

recalling that we have a uniform lower bound for $|P'(0)-P'(y)|$ and $|\tilde{P}'(0)-\tilde{P}'(y)|$ by Lemma 5.1.

This finishes the proof of Lemma 5.3. □

Lemma 5.4. *Suppose U is admissible in the sense of Definition* 5.1 *and that Assumption* 2.3 *holds. Then $\mathcal{T}(U) \in C^1((-\infty,0],\mathbb{R})$ and*

$$\mathcal{T}(U)'(z) = \mathcal{T}(U)(z)\frac{1+\hat{\chi}P(z) - (1+\hat{\chi})\mathcal{T}(U)(z)}{\chi\big(P'(0) - P'(z)\big)}, \quad \textit{for all } z<0. \tag{5.12}$$

Moreover

$$\lim_{z\to 0^-} \mathcal{T}(U)'(z) = \frac{P'(0)}{1+\hat{\chi}}\frac{1+\hat{\chi}P(0)}{1+\hat{\chi}U(0^-)}.$$

Proof. We divide the proof in two steps.

Step 1. We prove (5.12).

We observe that

$$\tau'(\tau^{-1}(z)) := \chi\big(P'(0) - P'(z)\big),$$

therefore $\mathcal{T}(U)$ is differentiable for each $z<0$ and

$$\mathcal{T}(U)'(z) = \mathcal{U}'(\tau^{-1}(z))\frac{1}{\tau'(\tau^{-1}(z))} = \mathcal{U}'(\tau^{-1}(z))\frac{1}{\chi\big(P'(0) - P'(z)\big)}.$$

By Eq. (5.10) in Lemma 5.2 we have

$$\begin{aligned}\mathcal{U}'(t) &= \frac{1}{1+\hat{\chi}}\left[\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}dl\right]^{-2} \\ &\quad\times\left(\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}\big(1+\hat{\chi}P(\tau(t))\big)dl-1\right) \\ &= \left[(1+\hat{\chi})\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}dl\right]^{-2} \\ &\quad\times\left((1+\hat{\chi})\int_{-\infty}^{t} e^{-\int_l^t 1+\hat{\chi}P(\tau(s))ds}dl(1+\hat{\chi}P(\tau(t)))-(1+\hat{\chi})\right) \\ &= \mathcal{U}^2(t)(\mathcal{U}^{-1}(t)(1+\hat{\chi}P(\tau(t)))-(1+\hat{\chi})) \\ &= \mathcal{U}(t)(1+\hat{\chi}P(\tau(t))-(1+\hat{\chi})\mathcal{U}(t)).\end{aligned}$$

Therefore, we can rewrite $\mathcal{T}(U)'(z)$ as

$$\begin{aligned}\mathcal{T}(U)'(z) &= \frac{\mathcal{U}'(\tau^{-1}(z))}{\chi(P'(0)-P'(z))} \\ &= \mathcal{U}(\tau^{-1}(z))\frac{1+\hat{\chi}P(z)-(1+\hat{\chi})\mathcal{U}(\tau^{-1}(z))}{\chi(P'(0)-P'(z))} \\ &= \mathcal{T}(U)(z)\frac{1+\hat{\chi}P(z)-(1+\hat{\chi})\mathcal{T}(U)(z)}{\chi(P'(0)-P'(z))}.\end{aligned}$$

Equation (5.12) follows.

Step 2. Next, we prove

$$\lim_{z\to 0^-}\mathcal{T}(U)'(z) = \frac{P'(0)}{1+\hat{\chi}}\frac{1+\hat{\chi}P(0)}{1+\hat{\chi}U(0)}.$$

Recall that

$$\begin{aligned}\mathcal{T}(U)(z) = \mathcal{U}(\tau^{-1}(z)) &= \frac{1}{(1+\hat{\chi})\int_{-\infty}^{\tau^{-1}(z)} e^{-\int_l^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}dl} \\ &= \frac{e^{\int_0^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}}{(1+\hat{\chi})\int_{-\infty}^{\tau^{-1}(z)} e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds}dl}.\end{aligned}$$

We have shown in Step 1 that for any $z<0$

$$\mathcal{T}(U)'(z) = \mathcal{T}(U)(z)\frac{1+\hat{\chi}P(z)-(1+\hat{\chi})\mathcal{T}(U)(z)}{\chi(P'(0)-P'(z))} \tag{5.13}$$

and by Lemma 5.2 we have

$$\lim_{z\to 0^-}\mathcal{T}(U)(z) = \frac{1+\hat{\chi}P(0)}{1+\hat{\chi}}.$$

Moreover,

$$\frac{1+\hat{\chi}P(z)-(1+\hat{\chi})\mathcal{T}(U)(z)}{\chi(P'(0)-P'(z))}$$
$$= \frac{(1+\hat{\chi}P(z))\int_{-\infty}^{\tau^{-1}(z)} e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds}dl - e^{\int_0^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}}{\chi(P'(0)-P'(z))\int_{-\infty}^{\tau^{-1}(z)} e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds}} =: \frac{N(z)}{D(z)}$$

and

$$\frac{N'(z)}{D'(z)} = \frac{\hat{\chi}P'(z)\int_{-\infty}^{\tau^{-1}(z)} e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds}dl}{-\chi P''(z)\int_{-\infty}^{\tau^{-1}(z)} e^{\int_0^l 1+\hat{\chi}P(\tau(s))ds}ds + \chi\big(P'(0)-P'(z)\big)(\tau^{-1})'(z)e^{\int_0^{\tau^{-1}(z)} 1+\hat{\chi}P(\tau(s))ds}}$$
$$= \frac{P'(z)}{\hat{\chi}(U(z)-P(z))+(1+\hat{\chi})\mathcal{T}(U)(z)} \xrightarrow[z\to 0^-]{} \frac{P'(0)}{\hat{\chi}U(0^-)+1}.$$

Therefore, by using L'Hôpital's rule, $\mathcal{T}(U)'(z)$ admits a limit when $z \to 0^-$ and

$$\lim_{z\to 0^-} \mathcal{T}(U)'(z) = \frac{P'(0)}{1+\hat{\chi}}\frac{1+\hat{\chi}P(0)}{1+\hat{\chi}U(0^-)}.$$ □

Lemma 5.5. (Compactness of $\mathcal{T}$) *Let Assumption* 2.3 *hold. The metric space $\mathcal{A}$ equipped with the distance induced by the $\|\cdot\|_\eta$ norm* (*defined in* (5.11)) *is a complete metric space on which the map $\mathcal{T}:\mathcal{A}\to\mathcal{A}$ is compact.*

Proof. Let us first briefly recall that the space $\mathcal{A}$ is complete. Let B_η be the set of all continuous functions defined on $(-\infty,0)$ with finite $\|\cdot\|_\eta$ norm:

$$B_\eta := \{u \in C^0((-\infty,0)) \mid \|u\|_\eta < +\infty\}.$$

It is classical that B_η equipped with the norm $\|\cdot\|_\eta$ is a Banach space. Therefore, in order to prove the completeness of $\mathcal{A}$, it suffices to show that $\mathcal{A}$ is closed in B_η. Let $U_n \in \mathcal{A}$, $U \in B_\eta$ be such that $\lim \|U_n - U\|_\eta = 0$. Then U_n converges to U locally uniformly on $(-\infty,0)$, and in particular we have

$$U(z) \in \left[\frac{2}{2+\hat{\chi}}, 1\right] \quad \text{for all } z \le 0,$$
$$U \text{ is nonincreasing.}$$

Therefore, $u \in \mathcal{A}$ and the completeness is proved.

Let us show that $\mathcal{T}$ is a compact map of the metric space $\mathcal{A}$. We have shown in Lemma 5.2 that $\mathcal{T}$ is continuous on $\mathcal{A}$ and leaves $\mathcal{A}$ stable. Let $U_n \in \mathcal{A}$, then combining Eq. (5.12) and the local uniform lower bound of $P'(z)-P'(0)$ from Lemma 5.1, the family $\mathcal{T}(U_n)'|_{[-k,-1/k]}$ is uniformly Lipschitz continuous on $[-k,-1/k]$ for each $k \in \mathbb{N}$. Therefore, the Ascoli–Arzelà applies and the set $\{\mathcal{T}(U_n)|_{[-k,-1/k]}\}_{n\ge 0}$ is relatively compact for the uniform topology on $[-k,-1/k]$ for each $k\in\mathbb{N}$. Using a diagonal extraction process, there exists a subsequence $\varphi(n)$ and a continuous function U such that $U_{\varphi(n)} \to U$ uniformly on every compact subset of $(-\infty, 0)$. Let us show that $\|U_{\varphi(n)} - U\|_\eta \to 0$ as $n \to +\infty$. Let $\varepsilon > 0$ be given, and let z_0, z_1 be respectively the smallest and largest root of the equation:

$$\eta z + \frac{1}{2}\ln(-z) = \ln\left(\frac{\varepsilon}{2}\right).$$

Then, on the one hand, for any $z \notin [z_0, z_1]$, we have $\sqrt{-z}e^{\eta z} \leq \frac{\varepsilon}{2}$ and therefore

$$\sqrt{-z}e^{\eta z}|\mathcal{T}(U_{\varphi(n)})(z) - \mathcal{T}(U)(z)| \leq \sqrt{-z}e^{\eta z}(|U_{\varphi(n)}(z)| + |U(z)|) \leq \varepsilon.$$

On the other hand, since $\mathcal{T}(U_{\varphi(n)})$ converges locally uniformly to $\mathcal{T}(U)$, there is $n_0 \geq 0$ such that

$$\sup_{z\in[z_0,z_1]} \sqrt{-z}e^{\eta z}|\mathcal{T}(U_{\varphi(n)})(z) - \mathcal{T}(U)(z)| \leq \varepsilon, \quad \text{for all } n \geq n_0.$$

We conclude that

$$\|\mathcal{T}(U_{\varphi(n)}) - \mathcal{T}(U)\|_\eta \leq \varepsilon,$$

for all $n \geq n_0$. The convergence is proved. This ends the proof of Lemma 5.5. □

We are now in the position to prove Theorem 2.4.

Proof of Theorem 2.4. We remark that the set of admissible functions $\mathcal{A}$ is a nonempty, closed, convex, bounded subset of the Banach space B_η and $\mathcal{T}$ is a continuous compact operator on $\mathcal{A}$ (Lemma 5.5). Therefore, a direct application of the Schauder fixed-point Theorem (see e.g. Theorem 2.A on p. 57 in Ref. 51) shows that $\mathcal{T}$ admits a fixed point U in $\mathcal{A}$:

$$\mathcal{T}(U) = U.$$

Applying Lemmas 5.2 and 5.4, U is strictly decreasing on $(-\infty, 0)$, $U((-\infty, 0)) \subset [\frac{2}{2+\hat{\chi}}, 1]$, U is C^1 on $(-\infty, 0]$ and

$$\lim_{z\to 0^-} U(z) = \frac{1+\hat{\chi}P(0)}{1+\hat{\chi}} \quad \text{and} \quad \lim_{z\to 0^-} U'(z) = \frac{P'(0)}{1+\hat{\chi}}\frac{1+\hat{\chi}P(0)}{1+\hat{\chi}U(0)}.$$

Finally

$$U'(z) = U(z)\frac{1+\hat{\chi}P(z) - (1+\hat{\chi})U(z)}{\chi(P'(0) - P'(z))}, \quad \text{for all } z < 0, \tag{5.14}$$

therefore

$$\chi P'(0)U'(z) - \chi P'(z)U'(z) - \chi U(z)P''(z) = U(z)(1 - U(z)), \quad \text{for all } z < 0$$

and finally

$$\chi P'(0)U'(z) - \chi(P'(z)U(z))' = U(z)(1 - U(z)), \quad \text{for all } z < 0.$$

We now prove that $U(-\infty) := \lim_{z\to\infty} U(z) = 1$. Since U is monotone decreasing on $(-\infty, 0)$ and is bounded by 1 from above, $U(-\infty)$ exists and, by a direct application of Lebesgue's dominated convergence theorem, P also converges to a limit near $-\infty$, $P(-\infty) = U(-\infty)$. Therefore, $U'(z) \to 0$, $P'(z) \to 0$ and $P''(z) \to 0$ as $z \to -\infty$. We conclude that

$$\lim_{z\to-\infty} U(z)(1 - U(z)) = 0,$$

which implies that $U(-\infty) = 1$.

Let us define $u(t,x) := U(x - ct)$, with $c := -\chi P'(0)$. The characteristics associated with $u(t,x)$ are

$$\frac{d}{dt}h(t,x) = -\chi(\rho_x \star u)(t, h(t,x)) = \chi(\rho \star U)(h(t,x) - ct) = -\chi P'(h(t,x) - ct),$$

and $u(t,x)$ satisfies for all x such that $h(t,x) - ct < 0$:

$$\begin{aligned}\partial_t u(t,h(t,x)) = \partial_t(U(h(t,x) - ct)) &= \left(\frac{d}{dt}(h(t,x) - ct)\right) U'(h(t,x) - ct)\\ &= \chi(-P'(h(t,x) - ct) + P'(0))U'(h(t,x) - ct)\\ &= u(t,h(t,x))(1 + \hat{\chi}(\rho \star u)(t,h(t,x)) - (1 + \hat{\chi})u(t,h(t,x))).\end{aligned}$$

If $h(t,x) - ct > 0$ then $u(t,h(t,x)) = U(h(t,x) - ct) = 0$ (locally in t) and therefore

$$\partial_t u(t,h(t,x)) = 0 = u(t,h(t,x))(1 + \hat{\chi}(\rho \star u)(t,h(t,x)) - (1 + \hat{\chi})u(t,h(t,x))).$$

Since $\{0\}$ is a negligible set for the Lebesgue measure, we conclude that $u(t,x)$ is a solution integrated along the characteristics to (1.1) and thus U is a traveling wave profile with speed $c = -P'(0) > 0$ as defined in Definition 2.2. Finally

$$c = -\chi P'(0) = \frac{\chi}{2\sigma}\int_{-\infty}^{0} e^y U(y)dy \in \left(\frac{\chi}{\sigma(2+\hat{\chi})}, \frac{\chi}{2\sigma}\right) = \left(\frac{\sigma\hat{\chi}}{2+\hat{\chi}}, \frac{\sigma\hat{\chi}}{2}\right).$$

This finishes the proof of Theorem 2.4 □

5.2. *Non-existence of continuous sharp traveling waves*

Remark 5.1. This result tells us if U is a sharp traveling wave solution to (1.1), then it must be discontinuous. This situation is very different from the porous medium case. However, it does not exclude the existence of *positive* continuous traveling wave solutions which decay to zero near $+\infty$. In fact, as we will show in the numerical simulations in the later section, we can observe numerically large speed traveling wave solutions that are smooth and strictly positive.

Proof of Proposition 2.4. We divide the proof in three steps.

Step 1. We show the estimate (2.9).

Assume by contradiction that there exists $x \in \mathbb{R}$ such that

$$-\chi\int_{\mathbb{R}} \rho_x(x - y)U(y)dy = c. \tag{5.15}$$

We let $P(x) := (\rho \star U)(x) = \int_{\mathbb{R}} \rho(x-y)U(y)dy$. Since $U \in C^0(\mathbb{R})$, we have that $P \in C^2(\mathbb{R})$. Differentiating, we find that

$$P'(x) = \int_{\mathbb{R}} \rho_x(x-y)U(y)dy = (\rho' \star U)(x),$$

$$\sigma^2 P''(x) = \int_{\mathbb{R}} \rho(x-y)U(y)dy - U(x) = P(x) - U(x).$$

Letting $Y(x) := -\chi(\rho_x \star U)(x) - c = -\chi P'(x) - c$, then $Y \in C^1(\mathbb{R})$ and we have

$$Y'(x) = -\chi P''(x) = \hat{\chi}(U(x) - (\rho \star U)(x)). \tag{5.16}$$

Since $\lim_{x\to+\infty} U(x) = 0$, we have $\lim_{x\to+\infty} Y(x) = -c < 0$. Remark that by our assumption (5.15), Y has at least one zero and therefore the largest root of Y is well-defined:

$$x_* := \inf\{x \,|\, \text{for all } y > x, Y(y) < 0\}.$$

We first remark that

$$\frac{d}{dt}\big(h(t,x) - ct\big) = \frac{d}{dt}h(t,x) - c = -\chi(\rho_x \star u)(t, h(t,x)) - c = Y(h(t,x) - ct), \tag{5.17}$$

where we recall that $u(t,x) := U(x-ct)$ is a solution to (1.1). In particular since $Y(x_*) = 0$ by the continuity of Y, we have $h(t,x_*) - ct = x_*$. Next by using (2.2) we have

$$\begin{aligned}\frac{d}{dt}u(t,h(t,x_*)) &= u(t,h(t,x_*))(1 + \hat{\chi}(\rho \star u)(t,h(t,x_*)) - (1+\hat{\chi})u(t,h(t,x_*))) \\ &= U(h(t,x_*) - ct)\big(1 + \hat{\chi}(\rho \star U)(h(t,x_*) - ct) \\ &\quad - (1+\hat{\chi})U(h(t,x_*) - ct)) \\ &= U(x_*)(1 + \hat{\chi}P(x_*) - (1+\hat{\chi})U(x_*))\end{aligned}$$

and since $u(t,h(t,x_*)) = U(h(t,x_*) - ct) = U(x_*)$ does not depend on t, this yields

$$0 = U(x_*)(1 + \hat{\chi}P(x_*) - (1+\hat{\chi})U(x_*)).$$

We conclude that either $U(x_*) = 0$ or $U(x_*) = \frac{1+\hat{\chi}P(x_*)}{1+\hat{\chi}} > 0$. In the remaining part of this step we will show that these two cases lead to contradiction.

Case 1. $U(x_*) = \frac{1+\hat{\chi}P(x_*)}{1+\hat{\chi}} > 0$**.** By (5.16) we have:

$$Y'(x_*) = \hat{\chi}(U(x_*) - P(x_*)) = (1 - P(x_*))\frac{\hat{\chi}}{1+\hat{\chi}},$$

however $U(x) \in [0,1]$, $U(x) \not\equiv 1$ and thus $P(x_*) = (\rho \star U)(x_*) < 1$ which shows $Y'(x_*) > 0$. Yet by definition of x_* we have $Y(x_*) = 0$ and $Y(x) < 0$ for all $x > x_*$, hence $Y'(x_*) \leq 0$, which is a contradiction.

Case 2. $U(x_*) = 0$**.** By (5.16) we have

$$Y'(x_*) = 0 - \hat{\chi}P(x_*) = -\hat{\chi}(\rho \star U)(x_*) < 0. \tag{5.18}$$

Hence by the continuity of Y, there exists a $x_0 < x_*$ such that

$$Y(x) > 0, \qquad \text{for all } x \in [x_0, x_*).$$

Recall that, by (5.17), we have for all $t > 0$

$$\frac{d}{dt}(h(t, x_0) - ct) = Y(h(t, x_0) - ct) > 0$$

as well as $h(0, x_0) - c \times 0 = x_0$, therefore the function $t \mapsto h(t, x_0) - ct$ is increasing and converges to x_* as $t \to +\infty$. In particular as $t \to +\infty$ we have $u(t, h(t, x_0)) = U(h(t, x_0) - ct) \to U(x_*) = 0$. Let $T > 0$ be such that $0 < u(t, h(t, x_0)) \leq \frac{1}{2(1+\hat{\chi})}$ for all $t \geq T$. We have

$$\begin{aligned}\frac{d}{dt}u(t, h(t, x_0)) &= u(t, h(t, x_0))\big(1 + \hat{\chi}(\rho \star u)(t, h(t, x_0)) - (1 + \hat{\chi})u(t, h(t, x_0))\big)\\ &\geq \frac{1}{2}u(t, h(t, x_0)),\end{aligned}$$

hence $u(t, h(t, x_0)) \geq u(T, h(T, x_0))e^{\frac{t-T}{2}}$. In particular letting

$$t^* := T - 2\ln\big(u(T, h(T, x_0))\big) > T,$$

we have

$$u(t^*, h(t^*, x_0)) \geq 1 > \frac{1}{2(1+\hat{\chi})},$$

which is a contradiction. Since both Cases 1 and 2 lead to contradiction, we have shown (2.9).

Step 2. Regularity of u.

We have shown in Step 1 that for all $x \in \mathbb{R}$ the strict inequality:

$$Y(x) = -\chi P'(x) - c < 0$$

holds. Let $x \in \mathbb{R}$ and $t_0 > 0$. Then, there exists $y \in \mathbb{R}$ such that $h(t_0, y) = x$, where h is the characteristic semiflow defined by (2.1). Since

$$\frac{d}{dt}(h(t, y) - ct) = -\chi(\rho_x \star u)(t, h(t, y)) - c = Y(h(t, y)) \neq 0,$$

the mapping $t \mapsto h(t, y) - ct$ has a C^1 inverse which we denote $\varphi(z)$, i.e.

$$\text{for all } z \,|\, \exists\, t > 0, \quad z = h(t, y) - ct, \quad h(\varphi(z), y) - c\varphi(z) = z.$$

Then we have

$$U(h(t, y) - ct) = u(t, h(t, y)) \Leftrightarrow U(z) = u(\varphi(z), h(\varphi(z), y)),$$

with $z = h(t, y)$ in a neighborhood of x. Since φ is C^1 and the function $t \mapsto u(t, h(t, y))$ is C^1, we conclude that U is C^1 in a neighborhood of x. The regularity is proved.

Step 3. We show that u is positive.

Combining Steps 1 and 2, we know that u is a classical solution to the equation:

$$\begin{aligned} -cU_x - \chi((\rho \star U)_x U)_x &= U(1-U) \\ (-c - \chi P')U_x &= U(1 + \hat{\chi}P - (1+\hat{\chi})U) \\ U_x &= \frac{U}{Y}(1 + \hat{\chi}P - (1+\hat{\chi})U) \end{aligned}$$

and since $Y < 0$, the right-hand side is a locally Lipschitz vector field in the variable U. In particular, the classical Cauchy–Lipschitz Theorem applies and the only solution with $U(x) = 0$ for some $x \in \mathbb{R}$ is $U \equiv 0$. Since U is nontrivial by assumption, U has to be positive. □

Appendix A. An Nonlinear Function

We study a function used in the proof of Lemma 5.1 and Assumption 2.3.

Lemma A.1. *The function*

$$f(x) := \ln\left(\frac{2-x}{x}\right) + \frac{2}{2+x}\left(\frac{x}{2}\ln\left(\frac{x}{2}\right) + 1 - \frac{x}{2}\right)$$

defined for $x \in (0,2)$ *is strictly decreasing and satisfies*

$$\lim_{x\to 0^+} f(x) = +\infty, \qquad \lim_{x \to 2^-} f(x) = -\infty.$$

In particular $f(x)$ *has a unique root in* $(0,2)$*.*

Finally, we have $f(1) > 0$*.*

Proof. The behavior of f at the boundary is standard. The strict monotony requires the computation of the derivative:

$$f'(x) = \left(\frac{-x-(2-x)}{x^2}\right) \times \frac{x}{2-x} + \frac{-2}{(2+x)^2}\left(\frac{x}{2}\ln\left(\frac{x}{2}\right) + 1 - \frac{x}{2}\right) + \frac{1}{2+x}\ln\left(\frac{x}{2}\right).$$

Recalling that

$$\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2} > 0, \tag{A.1}$$

for each $x \in (0,2)$ because $x \mapsto x\ln(x)$ is strictly convex, all three terms in the expression of $f'(x)$ are negative, therefore

$$f'(x) < 0$$

for all $x \in (0,2)$. The fact that $f(1) > 0$ can also be deduced from (A.1). Lemma A.1 is proved. □

Appendix B. Numerical Scheme

Our numerical scheme for the traveling waves in Sec. 3 reads

$$\begin{aligned}
&\frac{u_i^{n+1}-u_i^n}{\Delta t}+\frac{1}{\Delta x}(G(u_{i+1}^n,u_i^n)-G(u_i^n,u_{i-1}^n))=u_i^n(1-u_i^n),\\
&i=1,2,\dots,M,\quad n=0,1,2,\dots\\
&u_0=1,\quad u_{M+1}=0,
\end{aligned}$$

with $G(u_{i+1}^n,u_i^n)$ defined as

$$G(u_{i+1}^n,u_i^n)=(v_{i+\frac{1}{2}}^n)^+u_i^n-(v_{i+\frac{1}{2}}^n)^-u_{i+1}^n=\begin{cases}v_{i+\frac{1}{2}}^n u_i^n, & v_{i+\frac{1}{2}}^n\geq 0,\\ v_{i+\frac{1}{2}}^n u_{i+1}^n, & v_{i+\frac{1}{2}}^n<0,\end{cases}\qquad i=1,\dots,M.$$

Moreover, the velocity v is given by

$$v_{i+\frac{1}{2}}^n=-\frac{p_{i+1}^n-p_i^n}{\Delta x},\quad i=0,1,2,\dots,M,$$

where from (3.2) we define

$$P^n:=(I-A)^{-1}U^n,\quad P^n=\left(p_i^n\right)_{M\times 1}\quad U^n=\left(u_i^n\right)_{M\times 1},$$

where $A=(a_{i,j})_{M\times M}$ is the usual linear diffusion matrix with Neumann boundary condition. Therefore, by Neumann boundary condition $p_0=p_1$ and $p_{M+1}=p_M$, when $i=1,M$ we have

$$\begin{aligned}
G\left(u_1^n,u_0^n\right)&=0,\\
G\left(u_{M+1}^n,u_M^n\right)&=0,
\end{aligned}$$

which gives

$$\begin{aligned}
u_1^{n+1}&=u_1^n-d\,\frac{\Delta t}{\Delta x}G(u_2^n,u_1^n)+\Delta t\,f(u_1^n),\\
u_M^{n+1}&=u_M^n+d\,\frac{\Delta t}{\Delta x}G(u_M^n,u_{M-1}^n)+\Delta t\,f(u_M^n).
\end{aligned}$$

Owing to the boundary condition, we have the conservation of mass for Eq. (3.1) when the reaction term equals zero.

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