Asymptotic behavior of a nonlocal advection system with two populations

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Abstract

In this paper, we consider a nonlocal advection model for two populations on a bounded domain. The first part of the paper is devoted to the existence and uniqueness of solutions and the associated semi-flow properties. By employing the notion of solution integrated along the characteristics, we rigorously prove the segregation property of solutions. Furthermore, we construct an energy functional to investigate the asymptotic behavior of solutions. To resolve the lack of compactness of the positive orbits, we obtain a description of the asymptotic behavior of solutions by using the narrow convergence in the space of Young measures. The last section of the paper is devoted to numerical simulations, which confirm and complement our theoretical results.

1 Introduction

In this work, we study a two-species model with nonlocal advection

$$\begin{cases}\partial_t u_1(t,x) + \operatorname{div}\,\big(u_1(t,x)\mathbf{v}(t,x)\big) = u_1(t,x)h_1(u_1(t,x),u_2(t,x))\\ \partial_t u_2(t,x) + \operatorname{div}\,\big(u_2(t,x)\mathbf{v}(t,x)\big) = u_2(t,x)h_2(u_1(t,x),u_2(t,x))\end{cases} \quad t>0,\, x\in\mathbb{R}^N. \tag{1.1}$$

The velocity field $\mathbf{v} = -\nabla P$ is derived from pressure P

$$P(t,x) := \left(\rho * (u_1+u_2)(t,\cdot)\right)(x),$$

where $*$ is the convolution in $\mathbb{R}^N$. Suppose system (1.1) is supplemented with a periodic initial distribution

$$\mathbf{u}_0(x) := \begin{pmatrix} u_1(0,x)\\ u_2(0,x)\end{pmatrix} \in \mathbb{R}^2_+ \text{ where } \mathbf{u}_0 \text{ is a } 2\pi\text{–periodic function in each direction.} \tag{1.2}$$

We consider the solutions of system (1.1) which are *periodic in space*. Here a function $u(x)$ is said to be 2π-*periodic in each direction* (or for simplicity *periodic*) if

$$u(x+2k\pi) = u(x), \text{ for any } k\in\mathbb{Z}^N, x\in\mathbb{R}^N.$$

When $u(x)$ is periodic, we can reduce the convolution to the N–dimensional torus $\mathbb{T}^N := \mathbb{R}^N/2\pi\mathbb{Z}^N$ by the following observation

$$\begin{aligned}(\rho * u)(x) &= \int_{\mathbb{R}^N}\rho(x-y)u(y)\,\mathrm{d}y\\ &= \sum_{k\in\mathbb{Z}^N}\int_{[0,2\pi]^N}\rho(x-(y+2k\pi))u(y+2k\pi)\,\mathrm{d}y\\ &= \sum_{k\in\mathbb{Z}^N}\int_{[0,2\pi]^N}\rho(x-y-2k\pi)u(y)\,\mathrm{d}y.\end{aligned}$$

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Hence, we can reformulate as

$$(\rho * u)(x) = \frac{1}{(2\pi)^N} \int_{[0,2\pi]^N} K(x-y)u(y)\,\mathrm{d}y,$$

where K is again 2π–periodic in each direction and defined by

$$K(x) = (2\pi)^N \sum_{k\in\mathbb{Z}^N} \rho(x+2\pi k),\ x \in \mathbb{R}^N.$$

The fast decay of ρ is necessary to ensure the convergence of the above series (see Remark 1.3 for details). We can rewrite the velocity field $\mathbf{v}$ as follows:

$$\mathbf{v}(t,x) = -\nabla\left[K \circ (u_1+u_2)(t,\cdot)\right](x), \tag{1.3}$$

where $\circ$ denotes the convolution on the N–dimensional torus $\mathbb{T}^N := \mathbb{R}^N/2\pi\mathbb{Z}^N \simeq [0,2\pi]^N$. For any 2π-periodic and measurable function φ and ψ, it is defined by

$$(\varphi \circ \psi)(x) = |\mathbb{T}^N|^{-1} \int_{\mathbb{T}^N} \varphi(x-y)\psi(y)\,\mathrm{d}y.$$

Our motivation for this problem comes from a cell monolayer co-culture experiment in the study of human breast cancer cells. In [26, Figure 1], two types of cells grow into segregated islets over 7 days and the cell growth stops when they are locally saturated.

In this work, we model this mechanism by using a nonlocal advection system with contact inhibition. As we will see, our model captures the finite propagating speed in cell co-culture. In the context of cell sorting, the impact of cell adhesion and repulsion on pattern formation has been studied by many authors. We refer to the work of Armstrong, Painter and Sherratt [1] and Painter et al. [25]. From a more general perspective, our study is connected to cell segregation and border formation. Taylor et al. [31] concluded that the heterotypic repulsion and homotypic cohesion account for cell segregation and border formation. We also refer the readers to Dahmann et al. [10] and the references therein for more about boundary formation with its application. These observations and results in biological experiments lead us to a nonlocal advection system which is able to explain the phenomena such as cell propagation and segregation. The segregation property was brought up in the 80's by Shigesada, Kawasaki and Teramoto [30] and Mimura and Kawasaki [23] through the models with cross-diffusion. Since then, the cross-diffusion models have been widely studied and we refer to Lou and Ni [18, 19] for more results about this subject.

The well-posedness of nonlocal advection models with nonlinear diffusion has been considered by Bertozzi and Slepcev [6] and Bedrossian et al. [3] on a bounded domain $\Omega \subset \mathbb{R}^N$ with non-flux boundary condition. Bertozzi et al. [5, 4] studied the finite time blowup property and the well-posednees in L^p spaces of such nonlocal advection system in high dimensional space. For the studies of the asymptotic behavior of nonlocal equations, we refer to Bodnar and Velazquez [8] and Raoul [28]. The traveling wave solutions of such nonlocal system with or without linear diffusion were also considered by many authors. We refer the readers to [2, 21, 22] for models concerning swarms. Hamel and Henderson [16] investigated the existence of traveling fronts under a general assumption on the kernel with logistic source $f(u) = u(1-u)$. We also mention that system (1.1) is also related to the hyperbolic Keller-Segel equations (see Perthame and Dalibard [27]).

A single-species version of system (1.1) has been studied by Ducrot and Magal [13] (see the derivation of the model therein). Compared to [13], one of the technical difficulties in this work is that, a priori L^2-uniform boundedness of solutions is missing. This is because the nonlinear function h is more general (see Assumptions 1.1 and 4.1). This difficulty obliges us to find another method to prove the L^∞ uniform boundedness of solutions (see Lemma 4.9, Remark 4.11 and Theorem 4.10). Moreover, we prove the segregation property of the two species by employing the notion of solutions integrated along the characteristics. In addition, the positivity of Fourier coefficients in Assumption 4.4 enables to construct a decreasing energy functional, this condition has also been considered in [2] and [13]. With the help of this property, we can prove the L^∞ convergence of the sum of two species when the initial distribution is strictly positive (see Corollary 4.12). Furthermore, the segregation property preserves when t tends to infinity in the sense of narrow convergence (see Lemma 5.15).

We first specify the assumption on the reaction terms $h_i, i = 1, 2$, in system (1.1).

Assumption 1.1 *For $i = 1, 2$, suppose $h_i : \mathbb{R}^2_+ \to \mathbb{R}$ are of class C^1 satisfying*

$$\sup_{u_1,u_2\geq 0} h_i(u_1,u_2) < \infty, \qquad \sup_{u_1,u_2\geq 0} \partial_{u_j} h_i(u_1,u_2) < \infty, \quad j = 1,2.$$

An example of function h_i is

$$h_i(u_1,u_2) = \lambda_i(1-(u_1+u_2)).$$

Therefore, $u_i\, h_i(u_1,u_2)$ is of Lotka-Volterra type. Another example of function h_i which fits Assumption 1.1 is

$$h_i(u_1,u_2) = \frac{b_i}{1+\gamma_i\,(u_1+u_2)} - \mu_i.$$

Such a choice is motivated by Ducrot et al. [11] where h_i is used to describe the contact inhibition phenomenon (i.e. cells stop growing when they are locally saturated). The parameter $b_i > 0$ represents the division rate, $\mu_i > 0$ is the mortality rate and $\gamma_i > 0$ is the coefficient relating to the dormant phase of cells (see [11] for details). Notice that h_i is bounded from below. Therefore, we cannot apply the same arguments as in [13] to obtain an L^∞ bound of solutions. Hence, we extend the results in [13] to a more general class of nonlinear functions.

Assumption 1.2 *The kernel $K : \mathbb{R}^N \to \mathbb{R}$ is a $\mathbb{T}^N$–periodic function of class C^m on $\mathbb{R}^N$ for some integer $m \geq \frac{N+5}{2}$.*

Remark 1.3 *The above regularity Assumption 1.2 can be reduced to $m \geq 3$ in the proof of the existence and uniqueness of solutions. The higher regularity is mainly for Lemma 4.9. For the dimension $N \leq 3$, the regularity condition in Assumption 1.2 is always satisfied when $K \in C^4$. As for the choice of ρ in (1.1), it suffices to choose $\rho \in C^m(\mathbb{R}^N)$ satisfying for any $\varepsilon > 0$ and multi-index α with $|\alpha| \leq m$, there exists $M > 0$ such that for any $|x| \geq M$*

$$|D^\alpha \rho(x)| \leq C/|x|^{N+\varepsilon},$$

where C is a positive constant. For each multi-index α with $|\alpha| \leq m$, the series

$$x \longmapsto \sum_{k\in\mathbb{Z}^N} D^\alpha \rho(x+2\pi k)$$

is uniformly converging on $\mathbb{T}^N$. Hence, K satisfies Assumption 1.2.

The paper is organized as follows. In Section 2, we investigate the existence and uniqueness of solutions integrated along the characteristics. In Section 3, we prove the segregation property. In Sections 4 and 5, the asymptotic behavior of solutions will be studied using Young measures (a generalization of L^∞ weak $*$–convergence). Section 6 is devoted to numerical simulations and these numerical simulations complement our analysis.

2 Solutions integrated along the characteristics

In this section, we study the existence and uniqueness of solution for (1.1)-(1.3) with initial data $\mathbf{u}_0 \in L^\infty_{per}\left(\mathbb{R}^N\right)^2$. Before going further, let us introduce some notations. For each $k \in \mathbb{N}$, $C^k_{per}\left(\mathbb{R}^N\right)$ denotes the Banach space of functions of class C^k from $\mathbb{R}^N$ into $\mathbb{R}$ and $[0,2\pi]^N$–periodic endowed with the usual supremum norm

$$\|\varphi\|_{C^k} = \sum_{p=0}^{k} \sup_{x\in\mathbb{R}^N} |D^p\varphi(x)|\,.$$

For each $p \in [1,+\infty]$, $L^p_{per}\left(\mathbb{R}^N\right)$ denotes the space of measurable and $[0,2\pi]^N$–periodic functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that

$$\|\varphi\|_{L^p} := \|\varphi\|_{L^p\left((0,2\pi)^N\right)} < +\infty.$$

Then $L^p_{per}\left(\mathbb{R}^N\right)$ endowed with the norm $\|\varphi\|_{L^p}$ is a Banach space. We also introduce its positive cone $L^p_{per,+}\left(\mathbb{R}^N\right)$ consisting of all the functions in $L^p_{per}\left(\mathbb{R}^N\right)$ that are almost everywhere positive.

Remark 2.1 *When we study the product space* $C^k_{per}\left(\mathbb{R}^N\right)^n, L^p_{per}\left(\mathbb{R}^N\right)^n$ *with* $n\in\mathbb{N}$*, for simplicity, we use the same notation* $\|\cdot\|_{C^k}$ *and* $\|\cdot\|_{L^p}$ *for the norm in product space.*

Lemma 2.2 *Let Assumption 1.2 be satisfied. Let* $u_i\in C\left([0,\tau], L^1_{per}\left(\mathbb{R}^N\right)\right), i=1,2$ *be given. Then for each* $s\in[0,\tau]$ *and each* $z\in\mathbb{R}^N$*, setting* $\mathbf{v}(t,x)=-\nabla\left[K\circ(u_1+u_2)(t,\cdot)\right](x)$*, the following non-autonomous system*

$$\begin{cases}\partial_t\Pi_{\mathbf{v}}(t,s;z)=\mathbf{v}(t,\Pi_{\mathbf{v}}(t,s;z)), \text{ for each } t\in[0,\tau]\\ \Pi_{\mathbf{v}}(s,s;z)=z,\end{cases}\tag{2.1}$$

generates a unique non-autonomous continuous flow $\{\Pi_{\mathbf{v}}(t,s)\}_{t,s\in[0,\tau]}$*, i.e.,*

$$\Pi_{\mathbf{v}}(t,r;\Pi_{\mathbf{v}}(r,s;z))=\Pi_{\mathbf{v}}(t,s;z), \text{ for any } t,s,r\in[0,\tau], \text{ and } \Pi_{\mathbf{v}}(s,s;.)=I$$

and the map $(t,s,z)\to\Pi_{\mathbf{v}}(t,s;z)$ *is continuous. Moreover for each* $t,s\in[0,\tau]$*, we have*

$$\Pi_{\mathbf{v}}(t,s;z+2\pi k)=\Pi_{\mathbf{v}}(t,s;z)+2\pi k, \text{for any } z\in\mathbb{R}^{\mathbb{N}},\ k\in\mathbb{Z}^N,$$

the map $z\to\Pi_{\mathbf{v}}(t,s;z)$ *is continuously differentiable and furthermore, for the determinant of the Jacobian matrix*

$$\det(\partial_z\Pi_{\mathbf{v}}(t,s;z))=\exp\left(\int_s^t \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,s;z))dl\right).\tag{2.2}$$

Proof. By Assumption 1.2, one has $\mathbf{v}(t,x)\in C\left([0,\tau],C^1_{per}\left(\mathbb{R}^N\right)^N\right)$ which implies the following estimates

$$\|\mathbf{v}(t,\cdot)\|_{C^0}\le\|\nabla K\|_{C^0}\|(u_1+u_2)(t,\cdot)\|_{L^1},$$
$$\|\operatorname{div}\mathbf{v}(t,\cdot)\|_{C^0}\le\|\Delta K\|_{C^0}\|(u_1+u_2)(t,\cdot)\|_{L^1}.$$

Therefore, the first part of the results follows by using classical arguments in ordinary differential equations. For the proof of (2.2), note that

$$\begin{cases}\partial_t\partial_z\Pi_{\mathbf{v}}(t,s;z)=\partial_x\mathbf{v}(t,\Pi_{\mathbf{v}}(t,s;z))\partial_z\Pi_{\mathbf{v}}(t,s;z) \quad t\in[0,\tau],\\ \partial_z\Pi_{\mathbf{v}}(s,s;z)=I.\end{cases}$$

For any matrix-valued C^1 function $A:t\mapsto A(t)$, the Jacobi's formula reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t)=\det A(t)\operatorname{tr}\left(A^{-1}(t)\frac{d}{dt}A(t)\right).$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\det\partial_z\Pi_{\mathbf{v}}(t,s;z)=\det\partial_z\Pi_{\mathbf{v}}(t,s;z)\times\operatorname{tr}\left(\partial_x\mathbf{v}(t,\Pi_{\mathbf{v}}(t,s;z))\right).$$

Note that $\operatorname{tr}\left(\partial_x\mathbf{v}(t,\Pi_{\mathbf{v}}(t,s;z))\right)=\operatorname{div}\ \mathbf{v}(t,\Pi_{\mathbf{v}}(t,s;z))$, the result follows. ∎

To precise the notion of solution in this work, we first assume that

$$\mathbf{u}=(u_1,u_2)\in C^1\left([0,\tau]\times\mathbb{R}^N,\mathbb{R}\right)^2\cap C\left([0,\tau],C^0_{per,+}(\mathbb{R}^N)\right)^2$$

is a classical solution of (1.1)-(1.3). We consider the solution with each component $u_i(t,\cdot)$ along the characteristic $\Pi_{\mathbf{v}}(t,0;x)$ respectively, we obtain for $i=1,2$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(u_i(t,\Pi_{\mathbf{v}}(t,0;z)\Big)&=\partial_t u_i(t,\Pi_{\mathbf{v}}(t,0;z))+\nabla u_i(t,\Pi_{\mathbf{v}}(t,0;z))\cdot\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;z))\\&=u_i(t,\Pi_{\mathbf{v}}(t,0;z))\Big[-\operatorname{div}\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;z))+h_i\big(\mathbf{u}(t,\Pi_{\mathbf{v}}(t,0;z))\big)\Big],\end{aligned}$$

where $h_i(\mathbf{u}(t,\Pi_{\mathbf{v}}(t,0;z))=h_i(u_1(t,\Pi_{\mathbf{v}}(t,0;z),u_2(t,\Pi_{\mathbf{v}}(t,0;z))$. Hence a classical solution of (1.1)-(1.3) (i.e. C^1 in time and space) must satisfy

$$u_i(t,\Pi_{\mathbf{v}}(t,0;z))=\exp\left(\int_0^t h_i\big(\mathbf{u}(l,\Pi_{\mathbf{v}}(l,0;z))-\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;z))dl\right)u_i(0,z), i=1,2,\tag{2.3}$$

or equivalently

$$u_i(t,z) = \exp\left(\int_0^t h_i\big(\mathbf{u}(l,\Pi_{\mathbf{v}}(l,t;z))\big) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t;z))dl\right) u_i\left(0,\Pi_{\mathbf{v}}(0,t;z)\right), i=1,2, \tag{2.4}$$

where

$$\mathbf{v}(t,x) = -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \nabla K(x-y)(u_1+u_2)(t,y)\,\mathrm{d}y. \tag{2.5}$$

The above arguments yield the following definition of solutions.

Definition 2.3 (Solutions along the characteristics) *Let* $\mathbf{u}_0 \in L^\infty_{per,+}\left(\mathbb{R}^N\right)^2$, $\tau>0$ *be given. A function* $\mathbf{u}\in C\left([0,\tau], L^1_{per,+}\left(\mathbb{R}^N\right)\right)^2 \cap L^\infty\left((0,\tau), L^\infty_{per,+}\left(\mathbb{R}^N\right)\right)^2$ *is said to be a solution integrated along the characteristics of* (1.1)-(1.3) *if* u_i *satisfies* (2.4) *for* $i=1,2$, *with* $\mathbf{v}$ *defined in* (2.5).

We use a fixed point theorem to prove the existence and uniqueness of the solutions integrated along the characteristics. Consider

$$\mathbf{w} = (w_1,w_2), \quad w_i(t,x) := u_i(t,\Pi_{\mathbf{v}}(t,0;x)),\ i=1,2, \tag{2.6}$$

we will construct a fixed point problem for the pair $(\mathbf{w},\mathbf{v})$.

If there exists a solution integrated along the characteristics, then by (2.3) we have

$$w_i(t,x) = \exp\left(\int_0^t h_i\big(\mathbf{w}(l,x)\big) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right) u_i(0,x),\ i=1,2, \tag{2.7}$$

where $h_i(\mathbf{w}(t,x)) = h_i(w_1(t,x),w_2(t,x))$ for $i=1,2$. From the definition of $\mathbf{v}$

$$\begin{aligned}
\mathbf{v}(t,x) &= -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \nabla K(x-y)(u_1+u_2)(t,y)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^N} \nabla\rho(x-y)(u_1+u_2)(t,y)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^N} \nabla\rho\left(x-\Pi_{\mathbf{v}}(t,0;z)\right)\sum_{i=1,2} u_i(t,\Pi_{\mathbf{v}}(t,0;z))\,\det\partial_z(\Pi_{\mathbf{v}}(t,0;z))\,\mathrm{d}z \\
&= -\int_{\mathbb{R}^N} \nabla\rho\left(x-\Pi_{\mathbf{v}}(t,0;z)\right)\sum_{i=1,2} w_i(t,z)\,\det\partial_z(\Pi_{\mathbf{v}}(t,0;z))\,\mathrm{d}z,
\end{aligned} \tag{2.8}$$

where we used the change of variables $y=\Pi_{\mathbf{v}}(t,0;z)$. Replacing the determinant of Jacobian matrix by (2.2) and using (2.7), we deduce that

$$w_i(t,z)\det\,\partial_z(\Pi_{\mathbf{v}}(t,0;z)) = e^{\int_0^t h_i(\mathbf{w}(l,z))\,\mathrm{d}l} u_i\left(0,z\right),\ i=1,2.$$

Thus, equation (2.8) writes

$$\begin{aligned}
\mathbf{v}(t,x) &= -\int_{\mathbb{R}^N} \nabla\rho\left(x-\Pi_{\mathbf{v}}(t,0;z)\right)\sum_{i=1,2} e^{\int_0^t h_i(\mathbf{w}(l,z))dl} u_i\left(0,z\right)dz \\
&= -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \nabla K(x-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2} e^{\int_0^t h_i(\mathbf{w}(l,z))dl} u_i\left(0,z\right)dz.
\end{aligned} \tag{2.9}$$

Therefore, incorporating equations (2.7) and (2.9), the fixed point problem can be formulated as follows

$$\begin{cases}
w_i(t,x) = \exp\left(\int_0^t h_i\big(\mathbf{w}(l,x)\big) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right) u_i\left(0,x\right) \quad i=1,2, \\
\mathbf{v}(t,x) = -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \nabla K(x-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2} e^{\int_0^t h_i\left(\mathbf{w}(l,z)\right)dl} u_i\left(0,z\right)dz.
\end{cases} \tag{2.10}$$

We observe the following estimation

$$\left\|\int_0^t h_i\big(\mathbf{w}(l,x)\big) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right\|_{L^\infty} \le t\left(\bar{h} + \|\mathbf{v}\|_{C^1}\right),\ i=1,2,$$

where $\bar{h} := \sup_{u_1,u_2\geq 0}\sum_{i=1,2} h_i(u_1,u_2)$. Hence we can choose a proper space for $(\mathbf{w},\mathbf{v})$

$$\mathbf{w}=(w_1,w_2)\in C\left([0,\tau], L^{\infty}_{per,+}\left(\mathbb{R}^N\right)\right)^2,\quad \mathbf{v}\in C([0,\tau], C^1_{per}(\mathbb{R}^N)^N).$$

We reformulate our fixed point problem as follows

$$\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}\in\begin{pmatrix}C\left([0,\tau], L^{\infty}_{per,+}\left(\mathbb{R}^N\right)\right)^2\\ C([0,\tau], C^1_{per}(\mathbb{R}^N)^N)\end{pmatrix}\quad \text{and}\quad \mathcal{T}\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}=\begin{pmatrix}\mathbf{w^1}\\ \mathbf{v^1}\end{pmatrix},$$

wherein $\mathbf{w^1}$ and $\mathbf{v^1}$ are defined by

$$\begin{aligned}\mathbf{w^1}(t,x)&=\begin{pmatrix}\exp\left(\int_0^t h_1\big(\mathbf{w}(l,x)\big)-\operatorname{div}\mathbf{v^1}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right)u_1(0,x)\\ \exp\left(\int_0^t h_2\big(\mathbf{w}(l,x)\big)-\operatorname{div}\mathbf{v^1}(l,\Pi_{\mathbf{v}}(l,0;x))dl\right)u_2(0,x)\end{pmatrix},\\ \mathbf{v^1}(t,x)&=-\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N}\nabla K(x-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2}e^{\int_0^t h_i\left(\mathbf{w}(l,z)\right)dl}u_i(0,z)\,\mathrm{d}z.\end{aligned}\tag{2.11}$$

Theorem 2.4 *Let Assumption 1.1 and Assumption 1.2 be satisfied. For each $\mathbf{u}_0\in L^{\infty}_{per,+}\left(\mathbb{R}^N\right)^2$, system (1.1)-(1.3) has a unique solution integrated along the characteristics*

$$t\mapsto U(t)\mathbf{u}_0 \text{ in } C\left([0,+\infty), L^1_{per,+}\left(\mathbb{R}^N\right)\right)^2\cap L^{\infty}_{loc}\left([0,\infty), L^{\infty}_{per,+}\left(\mathbb{R}^N\right)\right)^2.$$

Moreover $\{U(t)\}_{t\geq 0}$ is a continuous semiflow on $L^1_{per,+}\left(\mathbb{R}^N\right)^2$, i.e.,

(i) *$U(t)U(s)=U(t+s)$, for any $t,s\geq 0$ and $U(0)=I$;*

(ii) *The map $(t,\mathbf{u}_0)\to U(t)\mathbf{u}_0$ maps every bounded set of $[0,+\infty)\times L^{\infty}_{per,+}\left(\mathbb{R}^N\right)^2$ into a bounded set of $L^{\infty}_{per,+}\left(\mathbb{R}^N\right)^2$;*

(iii) *If a sequence $\{t_n\}_{n\in\mathbb{N}}(\subset[0,+\infty))$ converges to a finite time t and $\{\mathbf{u}_0^n\}_{n\in\mathbb{N}}$ is bounded sequence in $L^{\infty}_{per,+}\left(\mathbb{R}^N\right)^2$ such that $\|\mathbf{u}_0^n-\mathbf{u}_0\|_{L^1}\to 0$ as $n\to+\infty$, then*

$$\|U(t_n)\mathbf{u}_0^n-U(t)\mathbf{u}_0\|_{L^1}\to 0 \text{ as } n\to+\infty,$$

where the norm is the product norm of $L^1_{per,+}\left(\mathbb{R}^N\right)^2$ (see Remark 2.1).

The semiflow U also satisfies the two following two properties

$$U(t)\mathbf{u}_0\geq 0,\ \text{for any } \mathbf{u}_0\geq 0, t\geq 0,\tag{2.12}$$

$$\|U(t)\mathbf{u}_0\|_{L^1}\leq e^{t\bar{h}}\|\mathbf{u}_0\|_{L^1},\ \text{for any } t\geq 0,\tag{2.13}$$

where we define

$$\bar{h} := \sup_{u_1,u_2\geq 0}\sum_{i=1,2} h_i(u_1,u_2).\tag{2.14}$$

We need the following lemma before we prove Theorem 2.4.

Lemma 2.5 *Suppose $\mathbf{v},\tilde{\mathbf{v}}\in C([0,\tau], C^1_{per}(\mathbb{R}^N)^N)$. Then for any $\tau>0$, we have*

$$\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty}\leq\tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)-\tilde{\mathbf{v}}(t,\cdot)\|_{L^\infty}e^{\tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)\|_{C^1}}.$$

Proof. For any fixed $t\in[0,\tau]$, from (2.1)

$$\partial_t\left(\Pi_{\mathbf{v}}(t,0;x)-\Pi_{\tilde{\mathbf{v}}}(t,0;x)\right)=\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x))-\tilde{\mathbf{v}}(t,\Pi_{\tilde{\mathbf{v}}}(t,0;x)),$$

which is equivalent to

$$\Pi_{\mathbf{v}}(t,0;x)-\Pi_{\tilde{\mathbf{v}}}(t,0;x)=\int_0^t\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))-\tilde{\mathbf{v}}(l,\Pi_{\tilde{\mathbf{v}}}(l,0;x))\,\mathrm{d}l.$$

We have the following estimate

$$
\begin{aligned}
&\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty}\\
&=\left\|\int_0^t \mathbf{v}(t,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))-\tilde{\mathbf{v}}(t,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))+\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\cdot))-\mathbf{v}(t,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))\,\mathrm{d}l\right\|_{L^\infty}\\
&\le t\|\mathbf{v}(t,\cdot)-\tilde{\mathbf{v}}(t,\cdot)\|_{L^\infty}+\int_0^t \|\mathbf{v}(t,\cdot)\|_{C^1}\|\Pi_{\mathbf{v}}(l,0;\cdot)-\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot)\|_{L^\infty}\,\mathrm{d}l.
\end{aligned}
$$

By Gronwall inequality, we obtain

$$
\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty}\le \tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)-\tilde{\mathbf{v}}(t,\cdot)\|_{L^\infty}e^{\tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)\|_{C^1}}.
$$

The result follows. ■

Proof of Theorem 2.4. We prove this theorem by showing that the contraction mapping theorem applies for $\mathcal{T}$ as long as $\tau>0$ is small enough. This ensures that the local existence and uniqueness of solutions. To that aim, we fix $\tau>0$ which will be chosen later and we define Banach space Z by $Z:=X\times Y$ where

$$
X:=C\left([0,\tau],L^\infty_{per}\left(\mathbb{R}^N\right)\right)^2,\quad Y:=C([0,\tau],C^1_{per}(\mathbb{R}^N)^N)
$$

endowed with the norm:

$$
\left\|\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}\right\|_Z=\|\mathbf{w}\|_X+\|\mathbf{v}\|_Y,
$$

where

$$
\|\mathbf{w}\|_X=\|w_1\|_{C\left([0,\tau],L^\infty_{per}(\mathbb{R}^N)\right)}+\|w_2\|_{C\left([0,\tau],L^\infty_{per}(\mathbb{R}^N)\right)}.
$$

We also introduce the closed subset $X_+\subset X$ defined by:

$$
X_+=C\left([0,\tau],L^\infty_{per,+}\left(\mathbb{R}^N\right)\right)^2,
$$

and define $Z_+=X_+\times Y$. Note that due to (2.11) one has

$$
\mathcal{T}(Z_+)\subset Z_+. \tag{2.15}
$$

For each given $\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}\in X$ and $\kappa>0$, let $\overline{B}_Z\left(\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix},\kappa\right)$ be the closed ball in Z centered at $\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}$ with radius κ. Now for any $\kappa>0$ and any initial distribution

$$
\mathbf{u}_0=(u_1(0,\cdot),u_2(0,\cdot))\in X_+,\quad \mathbf{v}_0=-\nabla K\circ((u_1+u_2)(0,\cdot)),
$$

we claim that there exists $\widehat{\tau}>0$ such that for each $\tau\in(0,\widehat{\tau})$

$$
\mathcal{T}\left(Z_+\cap\overline{B}_Z\left(\begin{pmatrix}\mathbf{u}_0\\ \mathbf{v}_0\end{pmatrix},\kappa\right)\right)\subset Z_+\cap\overline{B}_Z\left(\begin{pmatrix}\mathbf{u}_0\\ \mathbf{v}_0\end{pmatrix},\kappa\right). \tag{2.16}
$$

To prove this claim, for any $\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}\in Z_+\cap\overline{B}_Z\left(\begin{pmatrix}\mathbf{u}_0\\ \mathbf{v}_0\end{pmatrix},\kappa\right)$, we estimate component $\mathbf{w},\mathbf{v}$ separately. Recalling the definition of $\mathbf{w}$ in (2.11), one obtains

$$
\begin{aligned}
&\|\mathbf{w}^1(t,\cdot)-\mathbf{u}_0(\cdot)\|_{L^\infty}\\
&=\left\|\exp\left(\int_0^t h_1(\mathbf{w}(l,\cdot))-\operatorname{div}\mathbf{v}^{\mathbf{1}}(l,\Pi_{\mathbf{v}}(l,0;\cdot))dl\right)u_1(0,\cdot)-u_1(0,\cdot)\right\|_{L^\infty}\\
&\quad+\left\|\exp\left(\int_0^t h_2(\mathbf{w}(l,\cdot))-\operatorname{div}\mathbf{v}^{\mathbf{1}}(l,\Pi_{\mathbf{v}}(l,0;\cdot))dl\right)u_2(0,\cdot)-u_2(0,\cdot)\right\|_{L^\infty}\\
&\le\|\mathbf{u}_0\|_{L^\infty}\sum_{i=1,2}\left\|\exp\left(\int_0^t h_i(\mathbf{w}(l,\cdot))-\operatorname{div}\mathbf{v}^{\mathbf{1}}(l,\Pi_{\mathbf{v}}(l,0;\cdot))dl\right)-1\right\|_{L^\infty}.
\end{aligned}
$$

Note that by the classic inequality $|e^x - 1| \leq |x|e^{|x|}$ for any $x \in \mathbb{R}$, we can deduce that

$$\sup_{t\in[0,\tau]} \|\mathbf{w^1}(t,\cdot) - \mathbf{u}_0(\cdot)\|_{L^\infty} \leq \|\mathbf{u}_0\|_{L^\infty}\theta(\tau)e^{\theta(\tau)}, \tag{2.17}$$

where

$$\begin{aligned}\sum_{i=1}^{2}\int_0^\tau \|h_i\big(\mathbf{w}(l,x)\big) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\cdot))\|_{L^\infty}\,\mathrm{d}l &\leq \tau\left(h_\kappa + \|\mathbf{v}\|_Y\right)\\ &\leq \tau(h_\kappa + \kappa + \|\mathbf{v}_0\|_Y) := \theta(\tau),\end{aligned}$$

and we set

$$h_\kappa := \sup_{0\leq u_1,u_2\leq \kappa+\|\mathbf{u}_0\|_{L^\infty}} \sum_{i=1,2}|h_i(u_1,u_2)|. \tag{2.18}$$

On the other hand,

$$\begin{aligned}&\sup_{t\in[0,\tau]}\|\mathbf{v^1}(t,\cdot)-\mathbf{v}_0(\cdot)\|_{C^1}\\ &\leq \|\mathbf{u}_0\|_{L^\infty}\frac{1}{|\mathbb{T}^N|}\sup_{t\in[0,\tau]}\left\|\int_{\mathbb{T}^N}\nabla K(\cdot-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2}e^{\int_0^t h_i(\mathbf{w}(l,z))\,\mathrm{d}l} - \nabla K(\cdot - z)dz\right\|_{C^1}\\ &\leq \|\mathbf{u}_0\|_{L^\infty}\frac{1}{|\mathbb{T}^N|}\sup_{t\in[0,\tau]}\left\|\int_{\mathbb{T}^N}\nabla K(\cdot-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2}e^{\int_0^t h_i(\mathbf{w}(l,z))\,\mathrm{d}l} - \nabla K(\cdot-\Pi_{\mathbf{v}}(t,0;z))\right.\\ &\quad\left.+\nabla K(\cdot-\Pi_{\mathbf{v}}(t,0;z)) - \nabla K(\cdot - z)dz\right\|_{C^1}\\ &\leq \|\mathbf{u}_0\|_{L^\infty}\left\{\left(\|K\|_{C^1}+\|K\|_{C^2}\right)|e^{\tau h_\kappa}-1| + \left(\|K\|_{C^2}+\|K\|_{C^3}\right)\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\cdot\|_{L^\infty}\right\}\\ &\leq 2\|\mathbf{u}_0\|_{L^\infty}\|K\|_{C^3}\left\{|e^{\tau h_\kappa}-1| + \sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\mathbf{v}_0}(t,0;\cdot)\|_{L^\infty} + \sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}_0}(t,0;\cdot)-\cdot\|_{L^\infty}\right\}.\end{aligned} \tag{2.19}$$

Recalling Lemma 2.5, we have

$$\begin{aligned}\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\mathbf{v}_0}(t,0;\cdot)\|_{L^\infty} &\leq \tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)-\mathbf{v}_0(t,\cdot)\|_{L^\infty}e^{\tau\sup_{t\in[0,\tau]}\|\mathbf{v}(t,\cdot)\|_{C^1}}\\ &\leq \tau\kappa e^{\tau(\kappa+\|\mathbf{v}_0\|_Y)}.\end{aligned}$$

Therefore, we rewrite equation (2.19)

$$\begin{aligned}&\sup_{t\in[0,\tau]}\|\mathbf{v^1}(t,\cdot)-\mathbf{v}_0(\cdot)\|_{C^1}\\ &\leq 2\|\mathbf{u}_0\|_{L^\infty}\|K\|_{C^3}\left\{|e^{\tau h_\kappa}-1| + \tau\kappa e^{\tau(\kappa+\|\mathbf{v}_0\|_Y)} + \sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}_0}(t,0;\cdot)-\cdot\|_{L^\infty}\right\}.\end{aligned}$$

Since we have

$$\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}_0}(t,0;\cdot)-\cdot\|_{L^\infty} \leq \int_0^\tau \|\mathbf{v}_0(l,\Pi_{\mathbf{v}_0}(l,0;\cdot))\|_{L^\infty}\,\mathrm{d}l \to 0, \quad \text{as } \tau\to 0.$$

Incorporating (2.15), (2.17) and (2.19), the above estimations implies (2.16) by choosing a $\hat{\tau}$ small enough.

We now claim that for any

$$\begin{pmatrix}\mathbf{w}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\tilde{\mathbf{w}}\\ \tilde{\mathbf{v}}\end{pmatrix} \in Z_+ \cap \overline{B}_Z\left(\begin{pmatrix}\mathbf{u}_0\\ \mathbf{v}_0\end{pmatrix},\kappa\right),$$

where

$$\mathbf{w}(t,x) = \mathbf{u}(t,\Pi_{\mathbf{v}}(t,0;x)),\ \tilde{\mathbf{w}}(t,x) = \tilde{\mathbf{u}}(t,\Pi_{\tilde{\mathbf{v}}}(t,0;x)),$$

there exists $\tau^* \in (0, \widehat{\tau})$ such that for each $\tau \in (0, \tau^*)$ we can find some $L(\tau) \in (0,1)$ such that

$$\left\| \mathcal{T}\begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix} - \mathcal{T}\begin{pmatrix} \tilde{\mathbf{w}} \\ \tilde{\mathbf{v}} \end{pmatrix} \right\|_Z \leq L(\tau) \left\| \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix} - \begin{pmatrix} \tilde{\mathbf{w}} \\ \tilde{\mathbf{v}} \end{pmatrix} \right\|_Z . \tag{2.20}$$

To prove this claim, as before we estimate each component separately. For any given $\tau \in (0, \tau^*)$

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \left\| \mathbf{w^1}(t,\cdot) - \widetilde{\mathbf{w}}^{\mathbf{1}}(t,\cdot) \right\|_{L^\infty} \\
&= \sum_{i=1}^{2} \|\mathbf{u}_0\|_{L^\infty} \sup_{t\in[0,\tau]} \| e^{\int_0^t h_i(\mathbf{w}(l,\cdot)) - \operatorname{div} \mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\cdot))\,\mathrm{d}l} - e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,\cdot)) - \operatorname{div} \tilde{\mathbf{v}}(l,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))\,\mathrm{d}l} \|_{L^\infty} \\
&\leq \|\mathbf{u}_0\|_{L^\infty} \Big(e^{\tau(\kappa + \|\mathbf{v}_0\|_Y)} \underbrace{\sum_{i=1}^{2} \sup_{t\in[0,\tau]} \| e^{\int_0^t h_i(\mathbf{w}(l,\cdot))\,\mathrm{d}l} - e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,\cdot))\,\mathrm{d}l} \|_{L^\infty}}_{\mathrm{I}} \\
&\quad + \underbrace{e^{\tau h_\kappa} \sup_{t\in[0,\tau]} \| e^{-\int_0^t \operatorname{div} \mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\cdot))\,\mathrm{d}l} - e^{-\int_0^t \operatorname{div} \tilde{\mathbf{v}}(l,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))\,\mathrm{d}l} \|_{L^\infty}}_{\mathrm{II}} \Big).
\end{aligned}$$

Estimation for I: Since for any $x, y \in \mathbb{R}$, we have $|e^x - e^y| \leq e^{\max\{|x|,|y|\}}|x-y|$. Thus

$$\begin{aligned}
&\sum_{i=1}^{2} \sup_{t\in[0,\tau]} \left\| e^{\int_0^t h_i(\mathbf{w}(l,\cdot))\,\mathrm{d}l} - e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,\cdot))\,\mathrm{d}l} \right\|_{L^\infty} \\
&\leq e^{\tau h_\kappa} \sum_{i=1}^{2} \left\| \int_0^t h_i\big(\mathbf{w}(l,\cdot)\big) - h_i\big(\tilde{\mathbf{w}}(l,\cdot)\big)\,\mathrm{d}l \right\|_{L^\infty} \\
&\leq \tau e^{\tau h_\kappa} |\nabla h_\kappa| \|\mathbf{w} - \tilde{\mathbf{w}}\|_X,
\end{aligned} \tag{2.21}$$

where $|\nabla h_\kappa| = \sum_{i=1}^{2} \sup_{u_1,u_2\in[0,\|\mathbf{u}_0\|_{L^\infty}+\kappa]} |\nabla h_i(u_1,u_2)|$ and h_κ is defined in (2.18).

Estimation for II: For the second term, we obtain

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \| e^{-\int_0^t \operatorname{div} \mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;\cdot))\,\mathrm{d}l} - e^{-\int_0^t \operatorname{div} \tilde{\mathbf{v}}(l,\Pi_{\tilde{\mathbf{v}}}(l,0;\cdot))\,\mathrm{d}l} \|_{L^\infty} \\
&\leq \tau e^{\tau(\kappa+\|\mathbf{v}_0\|_Y)} \sup_{t\in[0,\tau]} \| \operatorname{div} \mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;\cdot)) - \operatorname{div} \tilde{\mathbf{v}}(t,\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)) \|_{L^\infty}.
\end{aligned}$$

While due to the form of $\mathbf{v}$ in (2.9) we can estimate the last term

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \| \operatorname{div} \mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;\cdot)) - \operatorname{div} \tilde{\mathbf{v}}(t,\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)) \|_{L^\infty} \\
&\leq \frac{1}{|\mathbb{T}^N|} \sum_{i=1}^{2} \sup_{t\in[0,\tau]} \Big\| \int_{\mathbb{T}^N} \Delta K(\Pi_{\mathbf{v}}(t,0;\cdot) - \Pi_{\mathbf{v}}(t,0;z)) e^{\int_0^t h_i(\mathbf{w}(l,z))\,\mathrm{d}l} \\
&\quad - \Delta K(\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot) - \Pi_{\tilde{\mathbf{v}}}(t,0;z)) e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,z))\,\mathrm{d}l}\,\mathrm{d}z \Big\|_{L^\infty} \|\mathbf{u}_0\|_{L^\infty} \\
&\leq \|\mathbf{u}_0\|_{L^\infty} \Big\{ \|K\|_{C^2} \sum_{i=1}^{2} \sup_{t\in[0,\tau]} \left\| e^{\int_0^t h_i(\mathbf{w}(l,\cdot))\,\mathrm{d}l} - e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,\cdot))\,\mathrm{d}l} \right\|_{L^\infty} \\
&\quad + 2e^{\tau h_\kappa} \|K\|_{C^3} \sup_{t\in[0,\tau]} \|\Pi_{\mathbf{v}}(t,0;\cdot) - \Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty} \Big\},
\end{aligned}$$

where the first part can be estimated by (2.21). As for the second part, recalling Lemma 2.5 and $\mathbf{v}, \tilde{\mathbf{v}} \in \overline{B}_Y(\mathbf{v}_0, \kappa)$ we have

$$\sup_{t\in[0,\tau]} \|\Pi_{\mathbf{v}}(t,0;\cdot) - \Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty} \leq \tau \sup_{t\in[0,\tau]} \|\mathbf{v}(t,\cdot) - \tilde{\mathbf{v}}(t,\cdot)\|_{L^\infty} e^{\tau(\kappa+\|\mathbf{v}_0\|_Y)}. \tag{2.22}$$

Incorporating the estimation in (2.21), we can find some $L_1(\tau)$ with $\lim_{\tau\to 0} L_1(\tau) = 0$ satisfying the following estimation

$$\left\| \mathbf{w^1} - \widetilde{\mathbf{w}}^{\mathbf{1}} \right\|_X \leq L_1(\tau)\big(\|\mathbf{w} - \tilde{\mathbf{w}}\|_X + \|\mathbf{v} - \tilde{\mathbf{v}}\|_Y \big). \tag{2.23}$$

To complete the proof, notice that

$$
\begin{aligned}
\left\|\mathbf{v^1}-\widetilde{\mathbf{v}}^{\mathbf{1}}\right\|_Y &= \sup_{t\in[0,\tau]}\left\|\mathbf{v^1}(t,\cdot)-\widetilde{\mathbf{v}}^{\mathbf{1}}(t,\cdot)\right\|_{C^1}\\
&= \frac{1}{|\mathbb{T}^N|}\sup_{t\in[0,\tau]}\| \int_{\mathbb{T}^N}\nabla K(\cdot-\Pi_{\mathbf{v}}(t,0;z))\sum_{i=1,2}e^{\int_0^t h_i(\mathbf{w}(l,z))\,\mathrm{d}l}u_i(0,z)\,\mathrm{d}z\\
&\quad -\int_{\mathbb{T}^N}\nabla K(\cdot-\Pi_{\tilde{\mathbf{v}}}(t,0;z))\sum_{i=1,2}e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,z))\,\mathrm{d}l}u_i(0,z)dz\|_{C^1}\\
&\le \|\mathbf{u}_0\|_{L^\infty}\Bigg\{2e^{\tau h_\kappa}\left(\|K\|_{C^2}+\|K\|_{C^3}\right)\sup_{t\in[0,\tau]}\|\Pi_{\mathbf{v}}(t,0;\cdot)-\Pi_{\tilde{\mathbf{v}}}(t,0;\cdot)\|_{L^\infty}\\
&\quad +\left(\|K\|_{C^1}+\|K\|_{C^2}\right)\sum_{i=1}^{2}\sup_{t\in[0,\tau]}\|e^{\int_0^t h_i(\mathbf{w}(l,\cdot))\,\mathrm{d}l}-e^{\int_0^t h_i(\tilde{\mathbf{w}}(l,\cdot))\,\mathrm{d}l}\|_{L^\infty}\Bigg\}.
\end{aligned}
\tag{2.24}
$$

Using (2.21) and (2.22), we can find some $L_2(\tau)$ with $\lim_{\tau\to 0}L_2(\tau)=0$ satisfying

$$\left\|\mathbf{v^1}-\widetilde{\mathbf{v}}^{\mathbf{1}}\right\|_Y \le L_2(\tau)\big(\|\mathbf{w}-\tilde{\mathbf{w}}\|_X+\|\mathbf{v}-\tilde{\mathbf{v}}\|_Y\big)$$

Let $L(\tau):=L_1(\tau)+L_2(\tau)$ and together with (2.23) and (2.24) we complete the proof of (2.20).

Finally, one concludes from (2.16) and (2.20) that for τ small enough, the contraction mapping theorem applies to operator $\mathcal{T}$. Hence the operator $\mathcal{T}$ has a unique fixed point in $Z_+\cap\overline{B}_Z\left(\begin{pmatrix}\mathbf{u}_0\\ \mathbf{v}_0\end{pmatrix},\kappa\right)$. Recalling (2.6), this ensures the existence and uniqueness of a local solution integrated along the characteristic of (1.1). The positivity property (2.12) follows from the property (2.16). The semiflow property in Theorem 2.4-(i) follows by a standard uniqueness argument. Next we show that the semiflow is globally defined and the properties (ii) and (iii) of the semiflow. In fact, one can see that

$$u_i(t,x)=\exp\left(\int_0^t h_i\big(\mathbf{u}(l,\Pi_{\mathbf{v}}(l,t;x))\big)-\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t;x))\,\mathrm{d}l\right)u_i\left(0,\Pi_{\mathbf{v}}(0,t;x)\right). \tag{2.25}$$

Therefore, one has

$$u_i(t,x)\le\exp\left(t\bar{h}\right)\exp\left(\int_0^t -\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t;x))\,\mathrm{d}l\right)u_i\left(0,\Pi_{\mathbf{v}}(0,t;x)\right), i=1,2,$$

then integrating over $\mathbb{T}^N$ and using the change of variable $x=\Pi_{\mathbf{v}}(t,0,z)$ to right hand side, which completes the estimation of u in L^1 norm (2.13), i.e.,

$$\|u_i(t,\cdot)\|_{L^1}\le e^{t\bar{h}}\|u_i(0,\cdot)\|_{L^1},\ i=1,2,\text{for any } t\ge 0. \tag{2.26}$$

Moreover, recall the definition $\bar{h}$ in (2.14) we have

$$\sup_{t\in[0,\tau]}\|\mathbf{u}(t,\cdot)\|_{L^\infty}\le e^{\tau\left(\bar{h}+\|\Delta K\|_{L^\infty}e^{\tau\bar{h}}\|\mathbf{u}_0\|_{L^\infty}\right)}\|\mathbf{u}_0\|_\infty,\text{ for any }\tau\ge 0. \tag{2.27}$$

The result (ii) follows. Lastly, we study the L^1 continuity of the semiflow. For any $0\le s\le t$,

$$
\begin{aligned}
&\|U(t)\mathbf{u}_0-U(s)\mathbf{u}_0\|_{L^1}\le e^{s\bar{h}}\|U(t-s)\mathbf{u}_0-\mathbf{u}_0\|_{L^1}\\
&=e^{s\bar{h}}\sum_{i=1}^{2}\|e^{\int_0^{t-s}h_i(\mathbf{u}(l,\Pi_{\mathbf{v}}(l,t-s;\cdot)))-\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t-s;\cdot))\,\mathrm{d}l}u_i(0,\Pi_{\mathbf{v}}(0,t-s;\cdot))-u_i(0,\cdot)\|_{L^1}.
\end{aligned}
\tag{2.28}
$$

Since

$$\sum_{i=1}^{2}\|\int_0^{t-s}h_i(\mathbf{u}(l,\Pi_{\mathbf{v}}(l,t-s;\cdot)))-\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,t-s;\cdot))\,\mathrm{d}l\|_{L^\infty}\le J(t-s),$$

where

$$J(\tau):=\tau\left(\bar{h}+\|\Delta K\|_{C^0}e^{\tau\bar{h}}\|\mathbf{u}_0\|_{L^\infty}\right),$$

we can rewrite (2.28) as

$$\begin{aligned}&\|U(t)\mathbf{u}_0 - U(s)\mathbf{u}_0\|_{L^1}\\ &\leq e^{s\bar{h}}\|\mathbf{u}_0(\Pi_{\mathbf{v}}(0,t-s;\cdot)) - \mathbf{u}_0\|_{L^1} e^{J(t-s)} + e^{s\bar{h}}\|\mathbf{u}_0\|_{L^1}\left|e^{J(t-s)} - 1\right| \to 0, \quad s \to t.\end{aligned} \tag{2.29}$$

If $\{\mathbf{u}_0^n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty_{per,+}\left(\mathbb{R}^N\right)$ such that $\|\mathbf{u}_0^n - \mathbf{u}_0\|_1 \to 0$ as $n \to +\infty$, then by (2.26), we have

$$\|U(t)\mathbf{u}_0^n - U(t)\mathbf{u}_0\|_{L^1} \to 0, \quad n \to +\infty,$$

together with (2.29), we have proved the continuity of the semiflow in (iii). ∎

Proposition 2.6 *Let Assumption 1.1 and Assumption 1.2 be satisfied. In addition,* $\mathbf{u}_0 \in W^1_{per}\left(\mathbb{R}^N\right)^2$*, then* $U(\cdot)\mathbf{u}_0 \in C^1\left([0,+\infty), L^1_{per}\left(\mathbb{R}^N\right)\right)^2$*. Moreover, if* $\mathbf{u}_0 \in C^1_{per}\left(\mathbb{R}^N\right)^2$ *then* $\mathbf{u}(t,x) = U(t)\mathbf{u}_0(x)$ *belongs to* $C^1\left([0,+\infty)\times\mathbb{R}^N\right)^2$ *and* $u(t,x)$ *is a classical solution of system (1.1)-(1.3).*

Sketch of the proof. If $\mathbf{u}_0 \in W^1_{per}(\mathbb{R}^N)^2$, we claim $U(\cdot)\mathbf{u}_0 \in C^1([0,\infty), L^1_{per}(\mathbb{R}^N))^2$. In fact, we define for $i = 1,2$,

$$w_i(t,x) = e^{\int_0^t h_i(\mathbf{w}(l,x)) - \operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))\,\mathrm{d}l} u_i(0,x) =: e^{\int_0^t h_i(\mathbf{w}(l,x))\,\mathrm{d}l} B_i(t,x), \tag{2.30}$$

where $B_i(t,x) := e^{\int_0^t -\operatorname{div}\mathbf{v}(l,\Pi_{\mathbf{v}}(l,0;x))\,\mathrm{d}l} u_i(0,x)$ is $C([0,\tau], W^1_{per}(\mathbb{R}^N))$ by our assumption. Define the formal derivative $\tilde{w}_i(t,\cdot) = \nabla_x w_i(t,\cdot)$, solving the following fixed point problem

$$\mathcal{T}\begin{pmatrix}\tilde{w}_1(t,x)\\ \tilde{w}_2(t,x)\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\left(\int_0^t \sum_{j=1}^2 \partial_{u_j} h_1(\mathbf{w}(l,x))\tilde{w}_j(l,x)\,\mathrm{d}l\, B_1(t,x) + \nabla_x B_1(t,x)\right) e^{\int_0^t h_1(\mathbf{w}(l,x))\,\mathrm{d}l}\\ \left(\int_0^t \sum_{j=1}^2 \partial_{u_j} h_2(\mathbf{w}(l,x))\tilde{w}_j(l,x)\,\mathrm{d}l\, B_2(t,x) + \nabla_x B_2(t,x)\right) e^{\int_0^t h_2(\mathbf{w}(l,x))\,\mathrm{d}l}\\ -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \nabla K(x - \Pi_{\mathbf{v}}(t,0;z)) \sum_{i=1,2} e^{\int_0^t h_i\left(\mathbf{w}(l,z)\right)\,\mathrm{d}l} u_i(0,z)\,\mathrm{d}z\end{pmatrix},$$

on space $C([0,\tau], L^\infty_{per}(\mathbb{R}^N)^N)^2 \times C([0,\tau], C^1_{per}(\mathbb{R}^N)^N)$ where $\partial_{u_j} h_i(u_1,u_2)$ is the partial derivative of h_i. Similarly, one can show that the mapping $\mathcal{T}$ is from $C([0,\tau], L^\infty_{per}(\mathbb{R}^N)^N)^2 \times C([0,\tau], C^1_{per}(\mathbb{R}^N)^N)$ to itself and is a contraction if τ is small. Therefore,

$$\tilde{w}_i(t,x) = \left(\int_0^t \sum_{j=1}^2 \partial_{u_j} h_i(\mathbf{w}(l,x))\tilde{w}_j(l,x)\,\mathrm{d}l\, B_i(t,x) + \nabla_x B_i(t,x)\right) e^{\int_0^t h_i(\mathbf{w}(l,x))\,\mathrm{d}l}, i = 1,2,$$

on $[0,\tau]$. Since by our assumption

$$\sup_{u_1,u_2\geq 0} \partial_{u_j} h_i(u_1,u_2) < \infty, \quad i = 1,2,\ j = 1,2,$$

applying Gronwall inequality, we have $\tilde{\mathbf{w}} \in C([0,\infty), L^1_{per}(\mathbb{R}^N)^N)^2$ for any positive time.

By definition we have for $i = 1,2$, $w_i(t, \Pi_{\mathbf{v}}(0,t;x)) = u_i(t,x)$, and

$$\partial_t u_i(t,x) = \partial_t w_i(t,\Pi_{\mathbf{v}}(0,t;x)) + \tilde{w}_i(t,x)\cdot\partial_t \Pi_{\mathbf{v}}(0,t;x) \in C([0,\infty); L^1_{per}(\mathbb{R}^N)).$$

If $\mathbf{u}_0 \in C^1(\mathbb{R}^N)^2$, then $B_i(t,x) \in C^1\left([0,+\infty)\times\mathbb{R}^N\right)$ and by (2.30) we have $\mathbf{w} \in C^1([0,\infty)\times\mathbb{R}^N)^2$. Therefore, u is a classical solution. ∎

Remark 2.7 (Conservation law) *The above computations imply the following conservation law: for each Borel set* $A \subset \mathbb{T}^N$ *and each* $0 \leq s \leq t$*:*

$$\int_{\Pi_{\mathbf{v}}(t,s;A)} u_i(t,x)\,\mathrm{d}x = \int_A \exp\left[\int_s^t h_i\left(\mathbf{u}\left(l,\Pi_{\mathbf{v}}(l,s;z)\right)\right)\,\mathrm{d}l\right] u_i(s,z)\,\mathrm{d}z, i = 1,2.$$

3 Segregation property

Our next theorem will show that the solutions along the characteristics can easily prove the segregation property.

Theorem 3.1 *Suppose $\mathbf{u} = \mathbf{u}(t, x)$ is the solution of (1.1)-(1.3) given by Theorem 2.4. For any initial distribution with $u_1(0, x)u_2(0, x) = 0$ for all $x \in \mathbb{T}^N$. Then $u_1(t, x)u_2(t, x) = 0$ for any $t > 0$ and $x \in \mathbb{T}^N$.*

Proof. We argue by contradiction. Assuming that there exist $t_1 > 0, x_1 \in \mathbb{T}^N$ such that

$$u_1(t_1, x_1)u_2(t_1, x_1) > 0.$$

Since $z \to \Pi_{\mathbf{v}}(t, s; z)$ is invertible from $\mathbb{R}^N \to \mathbb{R}^N$, then there exists some $x_0 \in \mathbb{R}^N$ such that $\Pi_{\mathbf{v}}(t_1, 0; x_0) = x_1$. Denote $x_0 = \tilde{x}_0 + 2\pi k_0$ for some $\tilde{x}_0 \in \mathbb{T}^N$ and $k_0 \in \mathbb{Z}^N$, thus by Lemma 2.2 we have

$$0 < u_i(t_1, \Pi_{\mathbf{v}}(t_1, 0; x_0)) = u_i(t_1, \Pi_{\mathbf{v}}(t_1, 0; \tilde{x}_0) + 2\pi k_0) = u_i(t_1, \Pi_{\mathbf{v}}(t_1, 0; \tilde{x}_0)).$$

Thus, for any $i = 1, 2$,

$$u_i(t_1, \Pi_{\mathbf{v}}(t_1, 0; \tilde{x}_0)) = \exp\left(\int_0^{t_1} h_i\big(\mathbf{u}(l, \Pi_{\mathbf{v}}(l, 0; \tilde{x}_0)) - \operatorname{div} \mathbf{v}(l, \Pi_{\mathbf{v}}(l, 0; \tilde{x}_0)\big)\, \mathrm{d}l\right) u_i\left(0, \tilde{x}_0\right) > 0,$$

which implies $u_i\left(0, \tilde{x}_0\right) > 0$. This is a contradiction. ■

Remark 3.2 *We give an illustration (see Figure 1) of the segregation of solutions integrated along the characteristics $u_i(t, \Pi_{\mathbf{v}}(t, 0; x))$ for $i = 1, 2$ when the dimension $N = 1$. In fact, if there exists for some x_0 such that $u_i(0, x_0) = 0$ for $i = 1, 2$. Then from equation (2.3) we obtain*

$$u_1(t, \Pi_{\mathbf{v}}(t, 0; x_0)) = 0 = u_2(t, \Pi_{\mathbf{v}}(t, 0; x_0)), \text{ for any } t > 0.$$

Therefore, the characteristics $t \mapsto \Pi_{\mathbf{v}}(t, 0; x_0)$ forms a segregation barrier for the two cell populations.

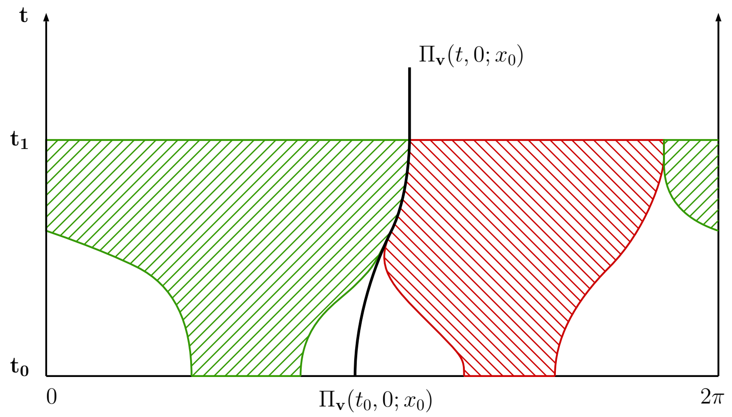

Figure 1: *The shaded areas represent the supports of two populations (red and green) evolving along time. Notice that if one starts with two separated supports and choose x_0 where $u_i(0, x_0) = 0$ for $i = 1, 2$, then the characteristic $t \mapsto \Pi_{\mathbf{v}}(t, 0; x_0)$ forms a segregation "wall" between the two cell populations, which indicates no matter how close they are, they stay separated.*

4 Asymptotic behavior

In the rest of the work, we always assume that the initial distributions for the two populations are separated.

Assumption 4.1 *For initial value $\mathbf{u}_0 \in L^\infty_{per,+}\left(\mathbb{R}^N\right)^2$, we assume that*

$$u_1(0, x)u_2(0, x) = 0, \text{for any } x \in \mathbb{T}^N.$$

Furthermore, we suppose that h_i in equation (1.1) has the following form

$$h_i(u_1, u_2) = h_i(u_1 + u_2),\ i = 1, 2,$$

with $h_i(r_i) = 0$ for some $r_i > 0$, $i = 1, 2$, and

$$h_i(u) > 0, \text{ for any } u \in [0, r_i), \quad h_i(u) < 0, \text{ for any } u > r_i, \quad \limsup_{u\to\infty} h_i(u) < 0,\ i = 1, 2.$$

Moreover, $u \longmapsto uh_i(u)$ is a concave function for $i = 1, 2$.

Remark 4.2 *Notice that the segregation property in Theorem 3.1 implies the following equality:*

$$u_i(t, x)h_i(u_1(t, x) + u_2(t, x)) = u_i(t, x)h_i(u_i(t, x)),\ i = 1, 2, \text{ for any } (t, x) \in [0, \infty) \times \mathbb{T}^N. \tag{4.1}$$

Lemma 4.3 *Let Assumptions 1.1, 1.2 and 4.1 be satisfied. Suppose $\mathbf{u} = \mathbf{u}(t, x)$ is a solution of (1.1)-(1.3). Then we have*

(i) $\sup_{t\geq 0} \|u_i(t, \cdot)\|_{L^1} \leq \max\{\|u_i(0, \cdot)\|_{L^1}, |\mathbb{T}^N|\}$, $i = 1, 2$.

(ii) $\mathbf{v}(t, x) := (\nabla K \circ (u_1 + u_2)(t, \cdot))(x)$ *satisfies* $\mathbf{v} \in L^\infty((0, \infty), W^{1,\infty}_{per}(\mathbb{R}^N))^N$ *and*

$$\|\mathbf{v}(t, \cdot)\|_{C^1} \leq 2\|K\|_{C^2} \max\{\|u_1(0, \cdot)\|_{L^1}, \|u_2(0, \cdot)\|_{L^1}, |\mathbb{T}^N|\}.$$

Proof. To prove above estimates (i) and (ii), we first assume $\mathbf{u}$ is a classical solutions. Due to segregation property in (4.1), equation (1.1) can be rewritten as

$$\partial_t u_i + \operatorname{div}\left(u_i \mathbf{v}\right) = u_i h_i(u_i),\ i = 1, 2. \tag{4.2}$$

By Assumption 4.1 the function $f_i(u) = uh_i(u)$ is concave for each i, integrating (4.2) over $\mathbb{T}^N$ and using Jensen's inequality, we have for classical solution

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_i(t, \cdot)\|_1 = \|f(u_i(t, \cdot))\|_1 \leq f\left(\|u_i(t, \cdot)\|_{L^1}\right).$$

Then the result follows using the usual ODE arguments with Assumption 4.1, where we can prove

$$\sup_{t\geq 0} \|u_i(t, \cdot)\|_{L^1} \leq \max\{\|u_i(0, \cdot)\|_{L^1}, |\mathbb{T}^N|\},\ i = 1, 2.$$

Let $\mathbf{u}_0 \in L^\infty_{per,+}\left(\mathbb{R}^N\right)^2$ be given and $\mathbf{u}$ be the corresponding solution integrated along the characteristics. Consider a sequence $\{\mathbf{u}_0^n\}_{n\geq 0}$ in $C^1_{per,+}(\mathbb{R}^N)^2$ such that $\|\mathbf{u}_0^n - \mathbf{u}_0\|_{L^1} \to 0$ as $n \to +\infty$. Then denote $\mathbf{u}^n$ the solutions corresponding to $\mathbf{u}_0^n$, from Theorem 2.4 we have $\|\mathbf{u}^n(t, \cdot) - \mathbf{u}(t, \cdot)\|_{L^1} \to 0$ and $\mathbf{u}(t, \cdot) \in L^\infty_{per,+}\left(\mathbb{R}^N\right)^2$. Therefore, by using Lebesgue convergence theorem, result (i) follows. Then result (ii) is a direct consequence of (i). ■

4.1 Energy functional

Assumption 4.4 *The Fourier's coefficients of function K on $\mathbb{T}^N$ denoted by $\{c_n[K]\}_{n\in\mathbb{Z}^N}$ satisfy $c_n[K] > 0$, for any $n \in \mathbb{Z}^N \setminus \{0\}$. The Fourier coefficients are defined by*

$$c_n[K] = |\mathbb{T}^N|^{-1} \int_{\mathbb{T}^N} e^{-in\cdot x} K(x)\,\mathrm{d}x, \quad \text{for any } n \in \mathbb{Z}^N.$$

Remark 4.5 *If Fourier transformation $\widehat{\rho}(\xi) > 0$ for all $\xi \in \mathbb{R}^N$, then for kernel K in system (1.1), we have $c_n[K] > 0$ for all $n \in \mathbb{Z}^N$. This implies Assumption 4.4.*

We construct the functional for $u_i, i = 1, 2$, as

$$E_i[u_i(t, \cdot)] = \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} G_i(u_i(t, x))\,\mathrm{d}x,$$

where $G_i : [0, \infty) \to [0, \infty)$ is defined by

$$G_i(u) := u \ln\left(\frac{u}{r_i}\right) - u + r_i. \tag{4.3}$$

Notice that $G_i'(u) = \ln(u/r_i)$ for $u > 0$ and we define the energy functional as

$$E[(u_1, u_2)(t, \cdot)] := \sum_{i=1,2} E_i[u_i(t, \cdot)]. \tag{4.4}$$

Theorem 4.6 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose* $\mathbf{u} = \mathbf{u}(t, x)$ *is a solution of* (1.1)-(1.3)*. Then for any* $t, \tau > 0$ *set* $u := u_1 + u_2$ *we have*

$$\begin{aligned} &E[(u_1, u_2)(t + \tau, \cdot)] - E[(u_1, u_2)(t, \cdot)] \\ &= -\int_t^{t+\tau} \sum_{k \in \mathbb{Z}^N} |k|^2 c_k[K] \left| c_k[u(s, \cdot)] \right|^2 \mathrm{d}s - \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} \sum_{i=1,2} u_i \left| h_i(u_i) \ln\left(\frac{u_i}{r_i}\right) \right| \mathrm{d}x\, \mathrm{d}s. \end{aligned} \tag{4.5}$$

Proof. For any $\delta > 0$, as before we first suppose $\mathbf{u} = (u_1, u_2)$ to be the classical solution. Setting $u = u_1 + u_2 \geq 0$, recalling the segregation property in (4.1) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_i[(u_i + \delta)(t, \cdot)] &= \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \ln\left(\frac{u_i + \delta}{r_i}\right) \partial_t u_i \,\mathrm{d}x \\ &= \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \ln\left(\frac{u_i + \delta}{r_i}\right) \left[\mathrm{div}\, [u_i \nabla(K \circ u)] + u_i h_i(u_i)\right] \mathrm{d}x \\ &= \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \frac{u_i^2}{u_i + \delta} \Delta(K \circ u) + u_i \nabla K \circ u \cdot \nabla\left(\frac{u_i}{u_i + \delta}\right) \mathrm{d}x \\ &\quad + \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} u_i h_i(u_i) \ln\left(\frac{u_i + \delta}{r_i}\right) \mathrm{d}x. \end{aligned}$$

Therefore, for any $t, \tau > 0$ we obtain

$$\begin{aligned} &E_i[(u_i + \delta)(t + \tau, \cdot)] - E_i[(u_i + \delta)(t, \cdot)] \\ &= \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} \frac{u_i^2}{u_i + \delta} \Delta(K \circ u) + u_i \nabla K \circ u \cdot \nabla\left(\frac{u_i}{u_i + \delta}\right) \mathrm{d}x\, \mathrm{d}s \\ &\quad + \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} u_i h_i(u_i) \ln\left(\frac{u_i + \delta}{r_i}\right) \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

Now by letting $\delta \to 0$ we can see that

$$\begin{aligned} &E_i[u_i(t + \tau, \cdot)] - E_i[u_i(t, \cdot)] \\ &= \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} u_i \Delta(K \circ u) \,\mathrm{d}x\, \mathrm{d}s + \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} u_i h_i(u_i) \ln\left(\frac{u_i}{r_i}\right) \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

Summing up the two functionals $E_i, i = 1, 2$, we obtain

$$\begin{aligned} &E[(u_1, u_2)(t + \tau, \cdot)] - E[(u_1, u_2)(t, \cdot)] \\ &= \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} u \Delta(K \circ u) \,\mathrm{d}x\, \mathrm{d}s + \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} \sum_{i=1,2} u_i h_i(u_i) \ln\left(\frac{u_i}{r_i}\right) \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

On the other hand, for each $\phi \in L^2_{per}(\mathbb{R}^N)$, one has $\phi(x) = \sum_{k \in \mathbb{Z}^N} \overline{c_k[\phi]} e^{in \cdot x}$ almost everywhere which implies

$$\begin{aligned} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \phi \Delta(K \circ \phi) \,\mathrm{d}x &= \sum_{k \in \mathbb{Z}^N} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \overline{c_k[\phi]} e^{in \cdot x} \Delta(K \circ \phi) \,\mathrm{d}x \\ &= \sum_{k \in \mathbb{Z}^N} \overline{c_k[\phi]} c_k[\Delta K \circ \phi] \\ &= -\sum_{k \in \mathbb{Z}^N} |k|^2 c_k[K] c_k[\phi]^2. \end{aligned}$$

Therefore, by the above calculation and by the fact that $h_i(u) \ln(u/r_i) < 0, i = 1, 2$, we have

$$\begin{aligned} &E[u(t + \tau, \cdot)] - E[u(t, \cdot)] \\ &= -\int_t^{t+\tau} \sum_{k \in \mathbb{Z}^N} |k|^2 c_k[K] \left| c_k[u(s, \cdot)] \right|^2 \mathrm{d}s - \frac{1}{|\mathbb{T}^N|} \int_t^{t+\tau} \int_{\mathbb{T}^N} \sum_{i=1,2} u_i \left| h_i(u_i) \ln\left(\frac{u_i}{r_i}\right) \right| \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

The usual limiting procedure as in Lemma 4.3 allows us to extend the estimation to the solutions integrated along the characteristics. ∎

Remark 4.7 *By Theorem 4.6, we can see that the energy functional E is non-negative and is decreasing, by letting $t \to +\infty$ we deduce from (4.5) that*

$$\lim_{t\to+\infty}\int_t^{t+\tau}\sum_{k\in\mathbb{Z}^N}|k|^2 c_k[K]\,|c_k[u(s,\cdot)]|^2\,\mathrm{d}s = 0, \tag{4.6}$$

and

$$\lim_{t\to+\infty}\int_t^{t+\tau}\int_{\mathbb{T}^N} u_i\left|h_i(u_i)\ln\left(\frac{u_i}{r_i}\right)\right|\,\mathrm{d}x\,\mathrm{d}s = 0,\ i=1,2. \tag{4.7}$$

We need Lemmas 4.8 and 4.9 to prove the L^∞ boundedness of the solution for all $t \geq 0$, i.e.,

$$\sup_{t\geq 0}\|u_i(t,\cdot)\|_{L^\infty} < \infty, \quad i=1,2.$$

Lemma 4.8 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose $\mathbf{u}=\mathbf{u}(t,x)$ is a solution of (1.1)-(1.3). Then for any $k\in\mathbb{Z}^N$ and for each $i=1,2$, the mapping*

$$t \longmapsto c_k[u_i(t,\cdot)]$$

is a C^1 function. Here $c_k[u_i(t,\cdot)]$, $k\in\mathbb{Z}^N$ are the Fourier coefficients. Moreover,

$$\sup_{t\geq 0}\left|\frac{\mathrm{d}}{\mathrm{d}t}c_k[u_i(t,\cdot)]\right| < \infty.$$

Proof. For any $k\in\mathbb{Z}^N$, suppose $\mathbf{u}=(u_1,u_2)$ is a classical solution. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}c_k[u_i(t,\cdot)] &= \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} e^{-ik\cdot x}\left[-\mathrm{div}\,(u_i\mathbf{v}) + u_i h_i(u_i)\right]\mathrm{d}x\\ &= \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} u_i\nabla\left(e^{-ik\cdot x}\right)\cdot\mathbf{v} + e^{-ik\cdot x}u_i h_i(u_i)\,\mathrm{d}x.\end{aligned}$$

Therefore, applying Jensen's inequality to $f_i(u) = uh_i(u)$, we derive

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}c_k[u_i(t,\cdot)]\right| \leq |k|\|u_i(t,\cdot)\|_1\|\mathbf{v}(t,\cdot)\|_{C^0} + f(\|u_i(t,\cdot)\|_{L^1}).$$

The result follows by using Lemma 4.3. The case for the solutions integrated along the characteristics can be proved by applying a classical regularization procedure. ∎

The regularity condition for kernel K defined in Assumption 1.2 serves mainly for the following result.

Lemma 4.9 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose $\mathbf{u}=\mathbf{u}(t,x)$ is a solution of (1.1)-(1.3) and define $u := u_1+u_2$. Then for $\mathbf{v}(t,x) = (\nabla K\circ u(t,\cdot))(x)$ we have*

$$\lim_{t\to+\infty}\|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} = 0.$$

Proof. By Assumption 1.2, $K\in C^m_{per}(\mathbb{R}^N)$ with $m \geq \frac{N+5}{2}$. Thus, from Temam [32, page. 50] one has

$$\sum_{k\in\mathbb{Z}^N}(1+|k|^2)^{\frac{N+5}{2}}c_k[K]^2 < \infty. \tag{4.8}$$

Moreover, we can deduce from (4.6) that for each $k\in\mathbb{Z}^N\backslash\{0\}$

$$\lim_{t\to+\infty}\int_t^{t+\tau}|c_k[u(s,\cdot)]|^2\,\mathrm{d}s = \lim_{t\to+\infty}\int_0^{\tau}|c_k[u(s+t,\cdot)]|^2\,\mathrm{d}s = 0.$$

By the last equality together with the results in Lemma 4.8, we can deduce

$$\lim_{t\to+\infty} c_k[u(t,\cdot)] = 0, \quad k \in \mathbb{Z}^N\backslash\{0\}. \tag{4.9}$$

We can compute that

$$\begin{aligned}
\operatorname{div} \mathbf{v}(t,x) &= -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \Delta K(x-y)u(t,y)\,\mathrm{d}y \\
&= -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \Delta K(x-y)\sum_{k\in\mathbb{Z}^N} e^{-ik\cdot y}c_k[u(t,\cdot)]\,\mathrm{d}y \\
&= -\frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} \sum_{k\in\mathbb{Z}^N} \Delta K(z)e^{ik\cdot(z-x)}c_k[u(t,\cdot)]\,\mathrm{d}z \\
&= \sum_{k\in\mathbb{Z}^N} |k|^2 c_k[K]c_k[u(t,\cdot)]e^{-ik\cdot x}.
\end{aligned}$$

By Lemma 4.3, we can find a constant $M > 0$ such that for each $k \in \mathbb{Z}^N$ we have

$$|c_k[u(t,\cdot)]| < \|u(t,\cdot)\|_{L^1} \le M, \text{ for any } t \ge 0.$$

Therefore,

$$\begin{aligned}
\|\operatorname{div} \mathbf{v}(t,x)\|_{C^0} &= \left\| \sum_{k\in\mathbb{Z}^N} |k|^2 c_k[K]c_k[u(t,\cdot)]e^{-ik\cdot x} \right\|_{C^0} \\
&\le M \sum_{k\in\mathbb{Z}^N} |k|^2 c_k[K] = M \sum_{k\in\mathbb{Z}^N\backslash\{0\}} |k|^{-\frac{N+1}{2}}|k|^{2+\frac{N+1}{2}}c_k[K] \\
&\le M\left(\sum_{k\in\mathbb{Z}^N\backslash\{0\}} \frac{1}{|k|^{N+1}}\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}^N\backslash\{0\}} |k|^{N+5}c_k[K]^2\right)^{\frac{1}{2}},
\end{aligned}$$

and due to (4.8), this last series converges. Hence, by Lebesgue dominated convergence theorem and (4.9) we have

$$\limsup_{t\to+\infty} \|\operatorname{div} \mathbf{v}(t,x)\|_{C^0} \le \limsup_{t\to+\infty} \sum_{k\in\mathbb{Z}^N} |k|^2 c_k[K]|c_k[u(t,\cdot)]| = 0.$$

The result follows. ∎

As a consequence of Lemma 4.9, we obtain Theorem 4.10 and Corollary 4.12 which are the main results of this section.

Theorem 4.10 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose* $\mathbf{u} = \mathbf{u}(t,x)$ *is a solution of* (1.1)-(1.3)*. Then we have for each* $i = 1, 2$,

$$\sup_{t\ge 0} \|u_i(t,\cdot)\|_{L^\infty} < +\infty,$$

and more precisely we have

$$\limsup_{t\to+\infty} \|u_i(t,\cdot)\|_{L^\infty} \le r_i.$$

Moreover, for any $x \in \mathbb{R}^N$ *such that* $u_i(0,x) > 0$*, the solution integrated along the characteristics converges point-wisely to the positive equilibrium* r_i *for* $i = 1, 2$*. That is, for any* $x \in \mathcal{U}_i$ *where* $\mathcal{U}_i = \{x \in \mathbb{R}^N : u_i(0,x) > 0\}$

$$\lim_{t\to\infty} u_i(t, \Pi_{\mathbf{v}}(t,0;x)) = r_i.$$

Or equivalently, for any $x \in \mathbb{R}^N$ *we have*

$$u_i(t,\Pi_{\mathbf{v}}(t,0;x)) \xrightarrow{p.w.} r_i \mathbb{1}_{\mathcal{U}_i}(x), \quad \textit{as } t \to \infty. \tag{4.10}$$

Remark 4.11 *Notice from the Theorem 4.10, we automatically obtain the following L^2 uniform boundedness of the solution $u = u_1 + u_2$, that is*

$$\sup_{t\geq 0} \|u(t,\cdot)\|_{L^2} < \infty.$$

Moreover, for any sequence $\{t_n\}_{n\geq 0}$ which tends to infinity, one has

$$\lim_{n\to\infty} c_k[u(t_n,\cdot)] = 0, \quad \text{for any } k \in \mathbb{Z}^N \backslash \{0\}.$$

Therefore, by Banach-Alaoglu-Bourbaki theorem, we deduce that there exists a subsequence $\{t_{n_l}\}_{l\geq 0}$ such that

$$u(t_{n_l},\cdot) \rightharpoonup c \text{ in } L^2,$$

where c is a constant which depends on the choice of the subsequence. With the above argument we can deduce

$$\lim_{t\to\infty} \|\mathbf{v}(t,\cdot)\|_{C^0} = 0. \tag{4.11}$$

In fact, for any sequence $\{t_n\}_{n\geq 0}$ with $t_n \to \infty$ as $n \to \infty$, we can find a subsequence such that

$$\mathbf{v}(t_{n_l},x) = \int_{\mathbb{T}^N} \nabla K(x-y)u(t_{n_l},y)dy \to c\int_{\mathbb{T}^N} \nabla K(x-y)dy = 0,$$

where the last equation is follows since K is periodic. Thus, equation (4.11) follows.

Proof of Theorem 4.10. Suppose that $\mathbf{u} = (u_1,u_2)$ is a classical solution. The usual limiting procedure allows us to extend the estimation to solutions integrated along the characteristics. We recall the notation in (2.6) where $w_i(t,x) := u_i(t,\Pi_{\mathbf{v}}(t,0;x))$, $i = 1,2$, and for any $x \in \mathbb{R}^N$ we have

$$\begin{aligned}\frac{\mathrm{d}w_i(t,x)}{\mathrm{d}t} &= w_i(t,x)\left[-\mathrm{div}\,\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x)) + h_i((w_1+w_2)(t,x))\right] \\ &= w_i(t,x)\left[-\mathrm{div}\,\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x)) + h_i(w_i(t,x))\right],\end{aligned}$$

where the second equation results from the segregation property. We compare the solution along the characteristics with the solution of the following ordinary differential equation. For any $\tau > 0$, let $\overline{w}_i(t)$ to be the solution of the following Cauchy problem

$$\begin{cases}\dfrac{\mathrm{d}\overline{w}_i(t)}{\mathrm{d}t} &= \overline{w}_i(t)\left[\sup\limits_{t\geq\tau}\|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} + h_i(\overline{w}_i(t))\right] \quad t > \tau, \\ \overline{w}_i(\tau) &= \|w_i(\tau,\cdot)\|_{L^\infty}.\end{cases}$$

Then we note that

$$\limsup_{t\to+\infty} \overline{w}_i(t) \leq \overline{\Phi}_i(\tau) := \inf\{z > r_i : \sup_{t\geq\tau}\|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} + h_i(y) \leq 0, \text{ for any } y \geq z\}.$$

If the set is empty, then $\overline{\Phi}_i(\tau) = +\infty$. By comparison principle, for any $\tau > 0$ we have

$$\limsup_{t\to+\infty} \|w_i(t,\cdot)\|_{L^\infty} \leq \limsup_{t\to+\infty} \overline{w}_i(t) \leq \overline{\Phi}_i(\tau),$$

while due to Assumption 4.1 where for any $u > r_i$, $h_i(u) < 0$, $\limsup_{u\to\infty} h_i(u) < 0$ and

$$\lim_{t\to+\infty} \|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} = 0$$

in Lemma 4.9, the limit $\lim_{\tau\to+\infty} \overline{\Phi}_i(\tau) = r_i$. Thus, we have

$$\limsup_{t\to+\infty} \|u_i(t,\Pi_{\mathbf{v}}(t,0;\cdot))\|_{L^\infty} \leq r_i. \tag{4.12}$$

Since $x \mapsto \Pi_{\mathbf{v}}(t,0;x)$ is invertible on $\mathbb{R}^N$, we have

$$\limsup_{t\to+\infty} \|u_i(t,\cdot)\|_{L^\infty} \leq r_i.$$

Together with the L^∞ estimation of $\mathbf{u}$ in finite time in (2.27) , we can see that

$$\sup_{t\geq 0}\|u_i(t,\cdot)\|_{L^\infty}<\infty.$$

Now we prove the second part of the theorem. For any fixed $x\in\mathbb{R}^N$ with $u_i(0,x)>0$, from the definition of solutions integrated along the characteristics (2.10)

$$w_i(t,x)=u_i(t,\Pi_{\mathbf{v}}(t,0;x))>0,\text{ for any } t>0.$$

For any $\tau>0$, define $\underline{w}_i(t)$ to be the solution of the following Cauchy problem

$$\begin{cases}\dfrac{\mathrm{d}\underline{w}_i(t)}{\mathrm{d}t} &= \underline{w}_i(t)\left[-\sup\limits_{t\geq\tau}\|\operatorname{div}\mathbf{v}(t,\cdot)\|_{C^0}+h_i(\underline{w}_i(t))\right]\\ \underline{w}_i(\tau) &= w_i(\tau,x)>0.\end{cases}$$

Then we note that

$$\liminf_{t\to+\infty}\underline{w}_i(t)\geq\underline{\Phi}_i(\tau):=\sup\{z>0:-\sup_{t\geq\tau}\|\operatorname{div}\mathbf{v}(t,\cdot)\|_{C^0}+h_i(y)\geq 0,\text{for any } y\leq z\}.$$

If the set is empty, then $\underline{\Phi}_i(\tau)=-\infty$. As before we use the comparison principle, for any $\tau>0$ and any $x\in\{x\in\mathbb{R}^N:u_i(0,x)>0\}$ we have

$$\liminf_{t\to+\infty}w_i(t,x)\geq\liminf_{t\to+\infty}\underline{w}_i(t)\geq\underline{\Phi}_i(\tau).$$

Due to Assumption 4.1 where $h_i(u)>0$ for any $u\in[0,r_i)$, one has $\lim_{\tau\to+\infty}\underline{\Phi}_i(\tau)=r_i$ thus we have for any $x\in\{x\in\mathbb{R}^N:u_i(0,x)>0\}$,

$$\liminf_{t\to+\infty}u_i(t,\Pi_{\mathbf{v}}(t,0;x))\geq r_i,$$

together with (4.12) the result (4.10) follows. ∎

Next corollary is a consequence of Theorem 4.10.

Corollary 4.12 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose $\mathbf{u}=\mathbf{u}(t,x)$ is a solution of (1.1)-(1.3). If for some constant $\delta>0$ and $u(0,x)=\sum_{i=1,2}u_i(0,x)\geq\delta>0$ for a.e. $x\in\mathbb{T}^N$. Moreover, we assume $r_1=r_2=:r$ in Assumption 4.1. Then*

$$\lim_{t\to\infty}\|u(t,\cdot)-r\|_{L^\infty}=0.$$

Proof. Here again we only prove the convergence when $\mathbf{u}=(u_1,u_2)$ is a classical solution. We use the same notations as in Theorem 4.10 and define

$$w(t,x):=w_1(t,x)+w_2(t,x).$$

Due to estimation (4.12) in Theorem 4.10 and segregation property, we have

$$\limsup_{t\to+\infty}\sup_{x\in\mathbb{R}^N}w(t,x)\leq r. \tag{4.13}$$

Moreover, we can obtain

$$\frac{\mathrm{d}w(t,x)}{\mathrm{d}t}=-w(t,x)\operatorname{div}\mathbf{v}(t,\Pi_{\mathbf{v}}(t,0;x))+\sum_{i=1}^{2}w_ih_i(w_i).$$

In order to use comparison principle, we set $\underline{h}(u)=\min_{u\geq 0}\{h_1(u),h_2(u)\}$ and by the separation property in Theorem 3.1 we have

$$w_1h_1(w_1)+w_2h(w_2)\geq w_1\underline{h}(w_1)+w_2\underline{h}(w_2)=(w_1+w_2)\underline{h}(w_1+w_2).$$

Hence,

$$\frac{\mathrm{d}w(t,x)}{\mathrm{d}t} \geq w(t,x)\left[-\sup_{t\geq\tau}\|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} + \underline{h}(w(t,x))\right],\ t\geq\tau.$$

For any $\tau > 0$, we have $\inf_{x\in\mathbb{R}^N} w(\tau,x) > 0$. In fact, by our assumption, $u(0,x) \geq \delta > 0$ on $\mathbb{T}^N$, thus $u(0,x) \geq \delta > 0$ on $\mathbb{R}^N$ and by equation (2.10) we have $w(\tau,x) > 0$ for any $x \in \mathbb{R}^N$ and since $w(t,x+2\pi) = w(t,x)$ for any $x \in \mathbb{R}^N$, we have $\inf_{x\in\mathbb{R}^N} w(\tau,x) \geq \tilde{\delta} > 0$ for some positive $\tilde{\delta}$. Thus, for any $\tau > 0$, we define $\underline{w}(t)$ to be the solution of the following ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}\underline{w}(t)}{\mathrm{d}t} & = \underline{w}(t)\left[-\sup\limits_{t\geq\tau}\|\mathrm{div}\,\mathbf{v}(t,\cdot)\|_{C^0} + \underline{h}(\underline{w}(t))\right] \\ \underline{w}(\tau) & = \inf_{x\in\mathbb{R}^N} w(\tau,x) > 0. \end{cases}$$

By similar arguments as in Theorem 4.10, we can see that

$$\liminf_{t\to+\infty}\inf_{x\in\mathbb{R}^N} w(t,x) \geq \liminf_{t\to+\infty}\underline{w}(t) \geq r.$$

Together with (4.13), we have

$$\lim_{t\to\infty}\|w(t,\cdot)-r\|_{L^\infty} = 0.$$

Since for any $t > 0$, the mapping $t \mapsto \Pi_{\mathbf{v}}(t,0;\cdot)$ is a bijection, we have

$$\|w(t,\cdot)-r\|_{L^\infty} = \|u(t,\Pi_{\mathbf{v}}(t,0;\cdot))-r\|_{L^\infty} = \|u(t,\cdot)-r\|_{L^\infty}.$$

Thus, we obtain

$$\lim_{t\to\infty}\|u(t,\cdot)-r\|_{L^\infty} = 0.$$

The result follows. ∎

Remark 4.13 *Note that in Corollary 4.12, we only assume the roots of two different reaction functions h_1, h_2 to be the same to obtain the convergence in L^∞.*

5 Young measures

In Corollary 4.12, we have the L^∞ convergence of the solution $u(= u_1 + u_2)$ when the initial distribution is strictly positive. Then one would like to know about the convergence of the solution when the initial distribution admits zero values.

We first introduce the notion of Young measures. The basic idea of Young measures is to replace the map $(t,x) \to u(t,x) = u_1(t,x) + u_2(t,x)$ by the map

$$(t,x) \to \delta_{u(t,x)}$$

from $[0,\infty)\times\mathbb{T}^N$ into a probability space. Namely, for some fixed t and x, the Dirac mass $\delta_{u(t,x)}$ is regarded as an element of the dual space the continuous functions $C([0,\gamma],\mathbb{R})$ (where $\gamma := \|u\|_{L^\infty([0,\infty)\times\mathbb{T}^N)}$) by using the following mapping

$$f \longmapsto \int_{[0,\gamma]} f(\lambda)\delta_{u(t,x)}(\mathrm{d}\lambda) = f(u(t,x)).$$

This means that the map $(t,x) \to \delta_{u(t,x)}$ is identified to an element of

$$L^\infty\left([0,\infty)\times\mathbb{T}^N, C([0,\gamma],\mathbb{R})^\star\right).$$

The goal of this procedure is to use the weak $\star-$topology to regard Young measure as an element the dual space of

$$L^1\left([0,\infty)\times\mathbb{T}^N, C([0,\gamma],\mathbb{R})\right).$$

The space of Young measures in our specific context is nothing but $L^\infty\left([0,\infty)\times\mathbb{T}^N, \mathbb{P}([0,\gamma])\right)$ (where $\mathbb{P}([0,\gamma])$ is the space of probabilities on $[0,\gamma]$) endowed with the weak $\star-$topology.

Theorem 5.1 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied. Suppose* $\mathbf{u} = \mathbf{u}(t,x)$ *is a solution of* (1.1)-(1.3) *given by Theorem 2.4. Furthermore, suppose we have*

$$r_1 = r_2 = r$$

in Assumption 4.1 and define

$$E_\infty := \lim_{t\to\infty} E[(u_1,u_2)(t,\cdot)],$$

where $E[(u_1,u_2)(t,\cdot)]$ *is the energy functional defined in* (4.4).

Then for each $i = 1,2$ *and each* $t \geq 0$ *the Dirac measure* $\delta_{(u_1+u_2)(t,x)}$ *belongs to the space of Young measures* $Y\left(\mathbb{T}^N;[0,\gamma]\right)$ $(\gamma := \sum_{i=1,2} \|u_i\|_{L^\infty([0,\infty)\times\mathbb{T}^N)})$, *i.e.,*

$$(u_1+u_2)(t,x) \in [0,\gamma], \text{ for all } t \geq 0 \text{ and almost every } x \in \mathbb{R}^N,$$

$$\int_{A\times[0,\gamma]} \eta(\lambda)\delta_{(u_1+u_2)(t,x)}(\,\mathrm{d}\lambda)\,\mathrm{d}x = \int_A \eta((u_1+u_2)(t,x))\,\mathrm{d}x, \text{ for any } A \in \mathcal{B}(\mathbb{T}^N), \text{ for any } \eta \in C([0,\gamma],\mathbb{R}).$$

Moreover, we can prove

$$r \leq E_\infty \leq 2r$$

and

$$\lim_{t\to\infty} \delta_{(u_1+u_2)(t,x)} = (E_\infty/r - 1)\delta_0 + (2 - E_\infty/r)\delta_r,$$

in the sense of the narrow convergence topology of $Y(\mathbb{T}^N;[0,\gamma])$. *This means that for each continuous function* $\eta : [0,\gamma] \to \mathbb{R}$ *and for any* $A \in \mathcal{B}(\mathbb{T}^N)$

$$\lim_{t\to\infty} \int_A \eta((u_1+u_2)(t,x))\,\mathrm{d}x = \int_A (E_\infty/r - 1)\eta(0) + (2 - E_\infty/r)\eta(r)\,\mathrm{d}x.$$

Remark 5.2 *Under the same assumptions as in Theorem 5.1, let* $\{t_n\}_{n\geq 0}$ *be any sequence tending to* ∞ *as* $n \to \infty$. *Then the sequence* $\{(u_1+u_2)(t_n,\cdot)\}_{n\geq 0} \subset L^\infty_{per}(\mathbb{R}^N)$ *is relatively compact in* $L^1_{per}(\mathbb{R}^N)$ *if and only if*

$$E_\infty = r \quad \text{or} \quad E_\infty = 2r.$$

The above result is a direct consequence of Young measure properties (see [9, Corollary 3.1.5]), which says if the sequence of Young measures $\{\delta_{(u_1+u_2)(t_n,x)}\}_{n\geq 0}$ *converges in the narrow sense to a Young measure* $\nu(x,\cdot)$ *and* $\nu(x,\cdot)$ *is a single Dirac measure* $\delta_{\phi(x)}(\cdot)$ *for almost all* $x \in \mathbb{T}^N$. *Then we have*

$$(u_1+u_2)(t_n,x) \xrightarrow{L^1} \phi(x), \quad n \to \infty.$$

In our case, when $E_\infty = r$ *(resp.* $= 2r$*), then*

$$(u_1+u_2)(t_n,x) \xrightarrow{L^1} r \text{ (resp. } 0), \quad n \to \infty.$$

Remark 5.3 *When* E_∞ *lies strictly in the interval* $(r,2r)$, *then* $\delta_{(u_1+u_2)(t,x)}$ *converges to two Dirac measures as* $t \to \infty$. *To illustrate the notion of narrow convergence to two Dirac measures, one may consider the following example. For each* $n \in \mathbb{N}$,

$$u_n(x) = \begin{cases} 1 & x \in \Delta x\,[j, j+p), \\ 0 & x \in \Delta x\,[j+p, j+1). \end{cases}, \quad j = 0,1,\ldots,n, \quad p \in (0,1),\ \Delta x = \frac{2\pi}{n+1}.$$

Then one can prove that

$$\lim_{n\to\infty} \delta_{u_n(x)} = p\delta_1 + (1-p)\delta_0$$

in the sense of narrow convergence. Indeed, for any $\eta \in C_b([0,1])$ *and* $\varphi \in L^1(0,2\pi)$ *one has*

$$\begin{aligned} &\int_{[0,2\pi]} \varphi(x) \int_{[0,1]} \eta(\lambda)\delta_{u_n(x)}(d\lambda)dx = \int_{[0,2\pi]} \varphi(x)\eta(u_n(x))\,\mathrm{d}x \\ &= \sum_{j=0}^{n} \int_{\Delta x[j,j+p)} \varphi(x)\eta(1)dx + \int_{\Delta x[j+p,j+1)} \varphi(x)\eta(0)\,\mathrm{d}x, \end{aligned}$$

and the result follows when $n \to \infty$.

Next, we introduce the notion of Young measures and the notion of narrow convergence topology in a general case.

Definition 5.4 (Young measure) *Let $(\mathcal{S}, d)$ be a separable metric space and let $\mathbb{P}(\mathcal{S})$ be the set of probability measures on $(\mathcal{S}, d)$. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space endowed with $\sigma-$algebra $\mathcal{A}$ (in our case μ is a Lebesgue measure). A map $\nu : \Omega \to \mathbb{P}(\mathcal{S})$ (i.e. the map ν maps each $x \in \Omega$ to a probability $B \to \nu(x, B)$ on $\mathcal{S}$) is said to be a* ***Young measure*** *if for each Borel set $B \in \mathcal{B}(\mathcal{S})$ the function $x \mapsto \nu(x, B)$ is measurable from $(\Omega, \mathcal{A})$ into $[0, 1]$. The set of all Young measures from $(\Omega, \mathcal{A})$ into $\mathcal{S}$ is denoted by $Y(\Omega, \mathcal{A}; \mathcal{S})$.*

Definition 5.5 (Narrow convergence topology) *The set $Y(\Omega, \mathcal{A}; \mathcal{S})$ is endowed with narrow convergence topology which is the weakest topology on $Y(\Omega, \mathcal{A}; \mathcal{S})$ such that for each functional from $Y(\Omega, \mathcal{A}; \mathcal{S})$ into $\mathbb{R}$ defined by*

$$\nu \longmapsto \int_A \int_{\mathcal{S}} \eta(\lambda) \nu(x, \mathrm{d}\lambda) \mu(\mathrm{d}x)$$

is continuous whenever $A \in \mathcal{A}$ and $\eta \in C_b(\mathcal{S}; \mathbb{R})$.

Remark 5.6 *Note that a sequence $\{\nu^n\}_{n \in \mathbb{N}} \subset Y(\Omega, \mathcal{A}; \mathcal{S})$ narrowly converges to $\nu \in Y(\Omega, \mathcal{A}; \mathcal{S})$ if and only if for any $\eta \in C_b(\mathcal{S}; \mathbb{R})$ and $A \in \mathcal{A}$*

$$\lim_{n \to \infty} \int_A \int_{\mathcal{S}} \eta(\lambda) \nu^n(x, \mathrm{d}\lambda) \mu(\mathrm{d}x) = \int_A \int_{\mathcal{S}} \eta(\lambda) \nu(x, \mathrm{d}\lambda) \mu(\mathrm{d}x).$$

For the sake of simplicity, we use $Y(\Omega; \mathcal{S})$ to denote $Y(\Omega, \mathcal{A}; \mathcal{S})$ if $\mathcal{A} = \mathcal{B}(\Omega)$.

Since the time variable t is in a unbounded domain, we introduce the local narrow convergence topology.

Definition 5.7 (Local narrow convergence topology) *Let $(\mathcal{S}, d)$ be a separable metric space and let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space (in practice μ will be a Lebesgue measure in our case). The set $Y(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{A}; \mathcal{S})$ is endowed with the local narrow convergence topology denoted by $\mathcal{T}_{loc}$ which is defined as the weakest topology on $Y(\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{A}; \mathcal{S})$ such that for each functional from $Y(\Omega, \mathcal{A}; \mathcal{S})$ into $\mathbb{R}$ defined by*

$$\nu \longmapsto \int_{I \times A} \left(\int_{\mathcal{S}} \eta(\lambda) \nu(t, x, \mathrm{d}\lambda) \right) (\mathrm{d}t \otimes \mu(\mathrm{d}x)),$$

is continuous for each bounded interval $I \subset \mathbb{R}$, $A \in \mathcal{A}$ and $\eta \in C_b(\mathcal{S}; \mathbb{R})$.

For our case, we consider $\Omega = \mathbb{T}^N$, $\mathcal{A} = \mathcal{B}(\mathbb{T}^N)$ is the Borel $\sigma-$ algebra and μ is the Lebegues measure, the set $\mathcal{S} = [0, \gamma]$ endowed with Euclidean norm. To simplify the notations, we set

$$Y(\mathbb{T}^N; [0, \gamma]) := Y(\mathbb{T}^N, \mathcal{B}(\mathbb{T}^N); [0, \gamma]).$$

We define $Y_{loc}(\mathbb{R} \times \mathbb{T}^N; [0, \gamma])$ to be the topological space $Y(\mathbb{R} \times \mathbb{T}^N; [0, \gamma])$ endowed with the local narrow convergence topology $\mathcal{T}_{loc}$. Furthermore, let us consider the probability space $\mathbb{P}(\mathbb{T}^N \times [0, \gamma])$ and let us recall that the usual weak $*-$topology on $\mathbb{P}(\mathbb{T}^N \times [0, \gamma])$ is metrizable by using the so-called bounded dual Lipschitz metric (Wasserstein metric W_p when $p = 1$) defined for each $\mu, \nu \in \mathbb{P}(\mathbb{T}^N \times [0, \gamma])$ by

$$\Theta(\mu, \nu) = \sup \left\{ \left| \int_{\mathbb{T}^N \times [0, \gamma]} f(x, \lambda) (\mu - \nu)(\mathrm{d}x, \mathrm{d}\lambda) \right| f \in \mathrm{Lip}\left(\mathbb{T}^N \times [0, \gamma]\right), \|f\|_{\mathrm{Lip}} \le 1 \right\}.$$

Recall that the Lipschitz norm for metric space (X, d) is defined as follows

$$\|f\|_{\mathrm{Lip}} = \sup_{x \in X} |f(x)| + \sup_{(x,y) \in X^2, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}, \text{ for any } f \in \mathrm{Lip}(X).$$

We refer to Dudley [12, Theorem 18] for the equivalence between the weak $\star-$topology on $\mathbb{P}(\mathbb{T}^N \times [0, \gamma])$ and the topology induced by $\Theta(\cdot, \cdot)$. In the following, the probability space $\mathbb{P}(\mathbb{T}^N \times [0, \gamma])$ is always endowed with the metric topology induced by Θ without further precision. Let $\{t_n\}_{n \ge 0}$ be a given increasing sequence tending to ∞ as $n \to \infty$. Using the above definition, we can prove the following lemma.

Lemma 5.8 *Let Assumptions 1.1, 1.2, 4.1 and 4.4 be satisfied and $T > 0$. The sequence of maps $\left\{t \longmapsto \mu_{i,t}^n\right\}_{n\in\mathbb{N}}$ from $[-T,T]$ to $\mathbb{P}\left(\mathbb{T}^N \times [0,\gamma]\right)$ (endowed with the above metric Θ) and defined by*

$$\int_{\mathbb{T}^N\times[0,\gamma]} g(x,y)\mu_{i,t}^n(\,\mathrm{d}x,\,\mathrm{d}y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N} g\left(x,u_i(t+t_n,x)\right)\mathrm{d}x, \textit{for any } g \in C\left(\mathbb{T}^N\times[0,\gamma];\mathbb{R}\right),$$

is relatively compact in $C\left([-T,T];\mathbb{P}\left(\mathbb{T}^N\times[0,\gamma]\right)\right)$.

Remark 5.9 *In the following, we use the notation*

$$\mu_{i,t}^n(\,\mathrm{d}x,\,\mathrm{d}y) = |\mathbb{T}^N|^{-1}\,\mathrm{d}x \otimes \delta_{u_i(t+t_n,x)}(\,\mathrm{d}y).$$

Proof. Let us first consider the classical solutions. For each $g \in C^1(\mathbb{T}^N\times\mathbb{R})$

$$\int_{\mathbb{T}^N} g(x,u_i(t,x))\,\mathrm{d}x - \int_{\mathbb{T}^N} g(x,u_i(s,x))\,\mathrm{d}x = \int_s^t \frac{\mathrm{d}}{\mathrm{d}l}\int_{\mathbb{T}^N} g(x,u_i(l,x))\,\mathrm{d}x\,\mathrm{d}l.$$

Since u_i is bounded, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N} g(x,u_i(t,x))\,\mathrm{d}x &= \int_{\mathbb{T}^N} \partial_u g(x,u_i(t,x))\partial_t u_i(t,x)\,\mathrm{d}x\\ &= \int_{\mathbb{T}^N} \partial_u g(x,u_i(t,x))\left(-\mathrm{div}(u_i\mathbf{v}) + u_i h_i(u_i)\right)\,\mathrm{d}x\\ &= \int_{\mathbb{T}^N} u_i\nabla_x\left[\partial_u g(x,u_i(t,x))\right]\cdot\mathbf{v} + \partial_u g(x,u_i(t,x))u_i h_i(u_i)\,\mathrm{d}x,\end{aligned} \tag{5.1}$$

where the last equality is obtained by applying Green's formula together with periodic boundary condition. We can see that

$$u_i(t,x)\nabla_x\left[\partial_u g(x,u_i(t,x))\right] = \nabla_x\left[u_i(t,x)\partial_u g(x,u_i(t,x)) - g(x,u_i(t,x))\right] + \mathbf{p}(x,u_i(t,x)),$$

where $\mathbf{p}(x,u) = \nabla_x g(x,u)$.

By substituting the last formula into (5.1) and by using again the periodicity we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N} g(x,u_i(t,x))\,\mathrm{d}x = &-\int_{\mathbb{T}^N}\left[u_i(t,x)\partial_u g(x,u_i(t,x)) - g(x,u_i(t,x))\right]\mathrm{div}\,\mathbf{v}(t,x)\,\mathrm{d}x\\ &+\int_{\mathbb{T}^N}\mathbf{p}(x,u_i(t,x))\cdot\mathbf{v}(t,x)\,\mathrm{d}x\\ &+\int_{\mathbb{T}^N}\partial_u g(x,u_i(t,x))u_i(t,x)h_i(u_i(t,x))\,\mathrm{d}x.\end{aligned} \tag{5.2}$$

The formula (5.1) extends to the solution integrated along the characteristics by usual density arguments. Incorporating the estimation of $\sup_{t\geq 0}\|u(t,\cdot)\|_{L^\infty}$ in Theorem 4.10, the estimation of $\mathbf{v}$ in Lemma 4.3 and the above equality (5.2), we deduce that there exists a constant $M > 0$ such that

$$\left|\int_{\mathbb{R}^N} g(x,u_i(t,x))\,\mathrm{d}x - \int_{\mathbb{R}^N} g(x,u_i(s,x))\,\mathrm{d}x\right| \leq M\|g\|_{\mathrm{Lip}(\mathbb{T}^N\times[0,\gamma])}|t-s|.$$

From the definition of the metric on $\Theta(\mu,\nu)$, we can see that

$$\Theta\left(\mu_{i,t}^n,\mu_{i,s}^n\right) \leq M|t-s|.$$

This implies that the mapping $t \to \mu_{i,t}^n$ is continuous from $[-T,T]$ to $\mathbb{P}\left(\mathbb{T}^N\times[0,\gamma]\right)$. By Prohorov's compactness theorem [7, Theorem 5.1], the space $\mathbb{P}\left(\mathbb{T}^N\times[0,\gamma]\right)$ endowed with the metric Θ is a compact metric space. Therefore, we can apply Arzela-Ascoli theorem and the result follows. ■

Since u is uniformly bounded, one can deduce the following compact result in the space of Young measures (see [29, Theorem 9.15]).

Lemma 5.10 *Suppose $\mathbf{u} = (u_1,u_2)$ is a solution of (1.1)-(1.3), the sequence $\left\{\delta_{u_i(t+t_n,x)}\right\}_{n\geq 0}$ is relatively compact in the local narrow convergence topology of $Y_{loc}\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right)$ for each $i = 1,2$.*

Using the above Lemma 5.8 and Lemma 5.10, up to a subsequence, one can assume that there exists a Young measure $\nu \equiv \nu_{i,t}(x,\cdot) \in Y\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right)$ such that

$$\lim_{n\to\infty} \delta_{u_i(t+t_n,x)} = \nu_{i,t}(x,\cdot) \text{ in the topology of } Y_{loc}\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right). \tag{5.3}$$

and

$$\lim_{n\to\infty} \mu^n_{i,t} = \mu^\infty_{i,t} \tag{5.4}$$

where the limit holds in the locally uniformly continuous topology of $C\left(\mathbb{R};\mathbb{P}\left(\mathbb{T}^N\times[0,\gamma]\right)\right)$. Note that the limits $\mu^\infty_{i,t}$ and $\nu_{i,t}(x,\cdot)$ depend on the choice of subsequence.

For each continuous function $f\in C\left(\mathbb{T}^N\times[0,\gamma];\mathbb{R}\right)$ and each $n\geq 0$, one has from definition that

$$\int_{\mathbb{T}^N\times[0,\gamma]} f(x,y)\mu^n_{i,t}(\mathrm{d}x,\,\mathrm{d}y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N}\int_{[0,\gamma]} f(x,y)\,\delta_{u_i(t+t_n,x)}(\mathrm{d}y)\,\mathrm{d}x.$$

From (5.3) and (5.4), passing to the limit $n\to\infty$ yields to

$$\int_{\mathbb{T}^N\times[0,\gamma]} f(x,y)\mu^\infty_{i,t}(\mathrm{d}x,\,\mathrm{d}y) = |\mathbb{T}^N|^{-1}\int_{\mathbb{T}^N}\int_{[0,\gamma]} f(x,y)\,\nu_{i,t}(x,\,\mathrm{d}y)\,\mathrm{d}x.$$

This can be rewrite as

$$\mu^\infty_{i,t}(\mathrm{d}x,\,\mathrm{d}y) = |\mathbb{T}^N|^{-1}\,\mathrm{d}x\otimes\nu_{i,t}(x,\,\mathrm{d}y). \tag{5.5}$$

The following Lemmas 5.11 and 5.12 show more properties about the family of measures $\nu_{i,t}(x,\cdot)$. Our next result describes the support of $\nu_{i,t}(x,\cdot)$.

Lemma 5.11 *Under the same assumptions of Lemma 5.8, for $i=1,2$, there exist measurable maps $a_i:\mathbb{R}\times\mathbb{T}^N\to\mathbb{R}$ such that $0\leq a_i(t,x)\leq 1$ a.e. $(t,x)\in\mathbb{R}\times\mathbb{T}^N$ and*

$$\nu_{i,t}(x,\cdot) = (1-a_i(t,x))\,\delta_0(.) + a_i(t,x)\delta_{r_i}(.),\ \ a.e.\,(t,x)\in\mathbb{R}\times\mathbb{T}^N.$$

Proof. Define $F_i(u) := u\,|h_i(u)\ln(u/r_i)|$ for $u\in[0,\infty)$ and recall that from equation (4.7), for any $\tau>0$ we have

$$\lim_{t\to+\infty}\int_t^{t+\tau}\int_{\mathbb{T}^N} F_i(u_i(s,x))\,\mathrm{d}x\,\mathrm{d}s = 0,\ i=1,2.$$

Therefore, for $i=1,2$ and from equations (5.4) and (5.5)

$$\begin{aligned} 0 &= \lim_{n\to\infty}\int_0^\tau\int_{\mathbb{T}^N} F_i(u_i(t+t_n,x))\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{n\to\infty}|\mathbb{T}^N|\int_0^\tau\int_{\mathbb{T}^N\times[0,\gamma]} F_i(\lambda)\mu^n_{i,t}(\mathrm{d}x,\,\mathrm{d}\lambda)\,\mathrm{d}t \\ &= \int_0^\tau\int_{\mathbb{T}^N\times[0,\gamma]} F_i(\lambda)\nu_{i,t}(x,\,\mathrm{d}\lambda)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Since the map $u\mapsto F_i(u)$ is non-negative and only vanishes at $u=0$ and $u=r_i$ one obtains that

$$\operatorname{supp}\nu_{i,t}(x,\cdot)\subset\{0\}\cup\{r_i\},\ \ a.e.\,(t,x)\in\mathbb{R}\times\mathbb{T}^N.$$

The above characterization of the support allows us to rewrite

$$\nu_{i,t}(x,\cdot) = \nu_{i,t}\left(x,\{0\}\right)\delta_0(.) + \nu_{i,t}\left(x,\{r_i\}\right)\delta_{r_i}(.),\ a.e.\,(t,x)\in\mathbb{R}\times\mathbb{T}^N.$$

Setting $a_i(t,x)\equiv\nu_{i,t}(x,\{r_i\})$ and recalling that $(t,x)\mapsto\nu_{i,t}(x,\cdot)$ is measurable with value as a probability measure, thus $\nu_{i,t}\left(x,\{0\}\right) = 1-\nu_{i,t}\left(x,\{r_i\}\right)$ and $(t,x)\mapsto a(t,x)$ is measurable, the result follows. ■

Our next result shows the measurable function $a_i(t,x)$ is independent of the time variable t.

Lemma 5.12 *Under the same assumptions of Lemma 5.8, there exists a measurable map* $c_i : \mathbb{T}^N \to \mathbb{T}^N$ *such that* $a_i \equiv a^i(t,x)$ *given by Lemma 5.11 satisfies*

$$a_i(t,x) \equiv c_i(x), \text{ a.e. } x \in \mathbb{T}^N, \text{ for any } t > 0, i = 1,2.$$

Moreover, for any $t \in \mathbb{R}$*,*

$$\nu_{i,t}(x,\cdot) = (1 - c_i(x))\delta_0(.) + c_i(x)\delta_{r_i}(.), \quad \text{a.e. } x \in \mathbb{T}^N,\ i = 1,2,$$

for some measurable functions $c_i : \mathbb{T}^N \to \mathbb{T}^N$*,* $i = 1,2$*.*

Furthermore, we have

$$\lim_{n\to\infty} \delta_{u_i(t+t_n,x)} = (1 - c_i(x))\delta_0 + c_i(x)\delta_{r_i}, \tag{5.6}$$

in the sense of the narrow convergence and where the limit depends on the choice of subsequence.

Proof. Suppose that $\mathbf{u} = (u_1,u_2)$ is a classical solution. For any $\{t_n\}_{n\geq 0}$ with $t_n \to \infty$ as $n \to 0$ and any $\phi \in C_c^1(\mathbb{T}^N)$,

$$\begin{aligned}&\int_{\mathbb{T}^N} \phi(x)\partial_t u_i(t+t_n,x)\,\mathrm{d}x + \int_{\mathbb{T}^N} \phi(x)\mathrm{div}(u(t+t_n,x)\mathbf{v}(t+t_n,x))\,\mathrm{d}x \\ &= \int_{\mathbb{T}^N} \phi(x)u_i(t+t_n,x)h_i(u_i(t+t_n,x))\,\mathrm{d}x.\end{aligned}$$

Since ϕ has compact support, we have

$$\begin{aligned}&\int_{\mathbb{T}^N} \phi(x)\partial_t u_i(t+t_n,x)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^N} \nabla\phi(x)\cdot\mathbf{v}(t+t_n,x)u(t+t_n,x)\,\mathrm{d}x + \int_{\mathbb{T}^N} \phi(x)u_i(t+t_n,x)h_i(u_i(t+t_n,x))\,\mathrm{d}x.\end{aligned}$$

Given any $T \in \mathbb{R}$ and $\delta > 0$, integrating both sides over $(T, T+\delta)$ leads to

$$\begin{aligned}&\int_{\mathbb{T}^N} \phi(x)\big(u_i(T+\delta+t_n,x) - u_i(T+t_n,x)\big)\,\mathrm{d}x \\ &= \int_T^{T+\delta}\int_{\mathbb{T}^N} \nabla\phi(x)\cdot\mathbf{v}(t+t_n,x)u(t+t_n,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{T+\delta}\int_{\mathbb{T}^N} \phi(x)u_i(t+t_n,x)h_i(u_i(t+t_n,x))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{5.7}$$

Now equation (5.7) is also true for any solution integrated along the characteristics. In fact, we can apply Theorem 2.4 (iii). Since the semiflow is continuous in L^1 norm, that is, for any $t \in [T, T+\delta]$,

$$\|u_i(t,x;\varphi_i^n) - u_i(t,x;\varphi_i)\|_{L^1} \to 0, \quad \text{as } n \to \infty,$$

where $\{\varphi_i^n\}_{n\geq 0} \subset C^1(\mathbb{T}^N)$ with

$$\varphi_i^n \xrightarrow{L^1(\mathbb{T}^N)} \varphi_i \in L^\infty(\mathbb{T}^N).$$

Hence, we can pass the limit to both sides of (5.7). For the right-hand-side of (5.7), due to

$$\lim_{t\to\infty} \|\mathbf{v}(t,\cdot)\|_{C^0} = 0$$

in Remark 4.11, we have for the first term

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_T^{T+\delta}\int_{\mathbb{T}^N} \nabla\phi(x)\cdot\mathbf{v}(t+t_n,x)u(t+t_n,x)\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \lim_{n\to\infty} \delta|\mathbb{T}^N|\|\phi\|_{C^1}\sup_{t\geq 0}\|u_i(t,\cdot)\|_{L^\infty}\|\mathbf{v}(t+t_n,\cdot)\|_{C^0} = 0.\end{aligned} \tag{5.8}$$

While the second term writes

$$\begin{aligned}&\int_T^{T+\delta}\int_{\mathbb{T}^N} \phi(x)u_i(t+t_n,x)h_i(u_i(t+t_n,x))\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_T^{T+\delta}\int_{\mathbb{T}^N} \phi(x)\left[\int_{[0,\gamma]} \lambda h_i(\lambda)\,\delta_{u_i(t+t_n,x)}(\mathrm{d}\lambda)\right]\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Letting $n \to \infty$ yields

$$\begin{aligned}
&\lim_{n\to\infty} \int_T^{T+\delta} \int_{\mathbb{T}^N} \phi(x) u_i(t+t_n,x) h_i(u_i(t+t_n,x))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_T^{T+\delta} \int_{\mathbb{T}^N} \phi(x) \left[\int_{[0,\gamma]} \lambda h_i(\lambda) \left[(1-a_i(t,x))\,\delta_0 + a_i(t,x)\delta_{r_i} \right] (\mathrm{d}\lambda) \right] \mathrm{d}x\,\mathrm{d}t \\
&= \int_T^{T+\delta} \int_{\mathbb{T}^N} \phi(x) \left[\int_{[0,\gamma]} r_i h_i(r_i)\, a_i(t,x) \right] \mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{5.9}$$

Therefore, by (5.8) and (5.9) we deduce

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{T}^N} \phi(x) \big(u_i(T+\delta+t_n,x) - u_i(T+t_n,x) \big)\,\mathrm{d}x \\
&= r_i \int_{\mathbb{T}^N} \phi(x) \big(a_i(T+\delta,x) - a_i(T,x) \big)\,\mathrm{d}x = 0.
\end{aligned}$$

Hence we have

$$\int_{\mathbb{T}^N} \phi(x) \big(a_i(T+\delta,x) - a_i(T,x) \big)\,\mathrm{d}x = 0, \text{ for any } \phi(x) \in C_c^1(\mathbb{T}^N).$$

Since $T \in \mathbb{R}$ and $\delta > 0$ is arbitrary, we deduce for any $t \in \mathbb{R}$

$$a_i(t,x) = c_i(x),\ a.e.\ x \in \mathbb{T}^N, \tag{5.10}$$

where $c_i : \mathbb{T}^N \to \mathbb{T}^N$ is a measurable function. The last part of the lemma now follows by the above equation (5.10) and Lemma 5.11. ■

Next, we study the narrow convergence of the measure $\delta_{(u_1+u_2)(t+t_n,x)}$ as $n \to \infty$.

Corollary 5.13 *Let $\{t_n\}_{n\geq 0}$ be a given increasing sequence tending to ∞ as $n \to \infty$. Then, up to a subsequence, we have two measurable functions $c_i(x) \in [0,1]$ for $i = 1,2$, such that for any $t \geq 0$,*

$$\lim_{n\to\infty} \delta_{(u_1+u_2)(t+t_n,x)} = \left(1 - \sum_{i=1,2} c_i(x) \right) \delta_0 + \sum_{i=1,2} c_i(x) \delta_{r_i}$$

in the sense of narrow convergence.

Proof. From segregation property in Theorem 3.1, for any $\eta \in C([0,\gamma])$ we have

$$\eta\left(u_1(t,x) + u_2(t,x)\right) + \eta(0) = \eta(u_1(t,x)) + \eta(u_2(t,x)), \quad \text{for any } (t,x) \in \mathbb{R}_+ \times \mathbb{T}^N,$$

which is equivalent to

$$\delta_0 + \delta_{(u_1+u_2)(t,x)} = \delta_{u_1(t,x)} + \delta_{u_2(t,x)}.$$

Therefore, for any $\varphi \in L^1(\mathbb{T}^N)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{T}^N} \varphi(x) \int_{[0,\gamma]} \eta(\lambda) \left(\delta_0 + \delta_{(u_1+u_2)(t+t_n,x)} \right) (\mathrm{d}\lambda)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_{\mathbb{T}^N} \varphi(x) \int_{[0,\gamma]} \eta(\lambda) \left(\delta_{u_1(t+t_n,x)} + \delta_{u_2(t+t_n,x)} \right) (\mathrm{d}\lambda)\,\mathrm{d}x \\
&= \int_{\mathbb{T}^N} \varphi(x) \int_{[0,\gamma]} \eta(\lambda) \left(\left(2 - \sum_{i=1,2} c_i(x) \right) \delta_0 + \sum_{i=1,2} c_i(x) \delta_{r_i} \right) (\mathrm{d}\lambda)\,\mathrm{d}x.
\end{aligned}$$

By subtracting the term δ_0 from each side, we deduce that

$$\lim_{n\to\infty} \delta_{(u_1+u_2)(t+t_n,x)} = \left(1 - \sum_{i=1,2} c_i(x) \right) \delta_0 + \sum_{i=1,2} c_i(x) \delta_{r_i} \tag{5.11}$$

in the sense of the narrow convergence topology of $Y(\mathbb{T}^N; [0,\gamma])$. ■

Lemma 5.14 *Under the same assumptions as in Lemma 5.8, the following equality holds*

$$r_1 c_1(x) + r_2 c_2(x) \equiv r_1 + r_2 - E_\infty, \text{ a.e. } x \in \mathbb{T}^N,$$

where $E_\infty := \lim_{t\to\infty} E\left[(u_1, u_2)(t, \cdot)\right]$ *in* (4.4).

Proof. Recall equation (4.3) where we have $G_i(0) = r_i, G(r_i) = 0$, we can see that

$$\begin{aligned}
\lim_{n\to\infty} E_i\left[u_i(t+t_n, \cdot)\right] &= \lim_{n\to\infty} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} G_i(u_i(t+t_n, x))\, \mathrm{d}x \\
&= \lim_{n\to\infty} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N\times[0,\gamma]} G_i(\lambda)\delta_{u_i(t+t_n,x)}(\,\mathrm{d}\lambda)\, \mathrm{d}x \\
&= \lim_{n\to\infty} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N\times[0,\gamma]} G_i(0)(1-c_i(x)) + G_i(r_i)c_i(x)\, \mathrm{d}x \\
&= r_i - \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} r_i c_i(x)\, \mathrm{d}x.
\end{aligned} \tag{5.12}$$

Meanwhile, from (4.9) the Fourier coefficients satisfy

$$\lim_{t\to\infty} c_k\left[(u_1+u_2)(t, \cdot)\right] = 0, \quad \text{for any } k \in \mathbb{Z}^N \backslash \{0\}.$$

On the other hand, we have for all $k \in \mathbb{Z}^N\backslash\{0\}$

$$\begin{aligned}
\lim_{n\to\infty} c_k\left[(u_1+u_2)(t+t_n, \cdot)\right] &= \lim_{n\to\infty} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} e^{-ikx}(u_1+u_2)(t+t_n, x)\, \mathrm{d}x \\
&= \lim_{n\to\infty} \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N\times[0\times\gamma]} e^{-ikx}\lambda\left(\delta_{u_1(t+t_n,x)} + \delta_{u_1(t+t_n,x)}\right)(\,\mathrm{d}\lambda)\, \mathrm{d}x \\
&= \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} e^{-ikx}(r_1 c_1(x) + r_2 c_2(x))\, \mathrm{d}x.
\end{aligned}$$

Since $c_1, c_2 \in L^\infty(\mathbb{T}^N) \subset L^2(\mathbb{T}^N)$ and $\{e^{-ikx}\}_{k\in\mathbb{Z}}$ is a basis of $L^2(\mathbb{T}^N)$. This implies that $r_1c_1(x)+r_2c_2(x)$ is a constant function. Recall that

$$E_\infty = \lim_{n\to\infty} \sum_{i=1,2} E_i\left[u_i(t+t_n, \cdot)\right] = r_1 + r_2 - \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} \sum_{i=1,2} r_i c_i(x) dx,$$

thus the result follows. ∎

Lemma 5.15 (Segregation at $t = \infty$) *Under the same assumptions as in Lemma 5.8, the following equation holds*

$$c_1(x)c_2(x) = 0, \quad a.e., \; x \in \mathbb{T}^N.$$

Moreover when $r_1 = r_2 = r$, *then*

$$r \le E_\infty \le 2r.$$

Proof. By using the segregation property in Theorem 3.1, for any $\eta \in C_b([0,\gamma])$ we can see that

$$\eta\big(\,(u_1(t,x) + u_2(t,x))^2\,\big) = \eta\big(u_1^2(t,x) + u_2^2(t,x)\big), \text{ for any } \; t \in \mathbb{R}_+, \; a.e.\; x \in \mathbb{T}^N.$$

Therefore, for any Borel set $A \in \mathcal{B}(\mathbb{T}^N)$, we deduce

$$\begin{aligned}
&\int_{A\times[0,\gamma]} \eta(\lambda^2)\delta_{(u_1+u_2)(t+t_n,x)}(\,\mathrm{d}\lambda)\, \mathrm{d}x \\
= &\int_{A\times[0,\gamma]^2} \eta(\lambda_1^2+\lambda_2^2)\delta_{u_1(t+t_n,x)}(\,\mathrm{d}\lambda_1)\delta_{u_2(t+t_n,x)}(\,\mathrm{d}\lambda_2)\, \mathrm{d}x.
\end{aligned} \tag{5.13}$$

By equation (5.6) and (5.11), we let $n \to \infty$, then for the left-hand-side (L.H.S.) of equation (5.13)

$$\begin{aligned}
\lim_{n\to\infty} \text{L.H.S. of (5.13)} &= \int_{A\times[0,\gamma]} \eta(\lambda^2)\left[\left(1 - \sum_{i=1,2} c_i(x)\right)\delta_0(\,\mathrm{d}\lambda) + \sum_{i=1,2} c_i(x)\delta_{r_i}(\,\mathrm{d}\lambda)\right] \\
&= \int_A \eta(0)\left(1 - \sum_{i=1,2} c_i(x)\right) + \sum_{i=1,2} \eta(r_i^2)c_i(x)\, \mathrm{d}x.
\end{aligned}$$

Then for the right-hand-side (R.H.S.) of equation (5.13)

$$\begin{aligned}\lim_{n\to\infty} \text{R.H.S. of (5.13)} &= \int_{A\times[0,\gamma]^2} \eta(\lambda_1^2+\lambda_2^2) \prod_{i=1,2} \left[(1-c_i(x))\,\delta_0(\,\mathrm{d}\lambda_i) + c_i(x)\delta_{r_i}(\,\mathrm{d}\lambda_i)\right] \mathrm{d}x \\ &= \int_A \Big(\eta(0) \prod_{i=1,2} (1-c_i(x)) + \eta(r_1^2)c_1(x)(1-c_2(x)) \\ &\quad + \eta(r_2^2)c_2(x)(1-c_1(x)) + \eta(r_1^2+r_2^2)c_1(x)c_2(x)\Big)\,\mathrm{d}x.\end{aligned}$$

Comparing the two limits and noticing that $A \in \mathcal{B}(\mathbb{T}^N)$ is arbitrary, we conclude that

$$c_1(x)c_2(x)\Big[\eta(0) + \eta(r_1^2+r_2^2) - \eta(r_1^2) - \eta(r_2^2)\Big] = 0, \text{ for a.e. } x \in \mathbb{T}^N.$$

Furthermore, since $\eta \in C_b([0,\gamma])$ is any given function, we can choose an η such that

$$\eta(0) + \eta(r_1^2+r_2^2) - \eta(r_1^2) - \eta(r_2^2) \neq 0,$$

thus

$$c_1(x)c_2(x) = 0, \quad a.e., \, x \in \mathbb{T}^N. \tag{5.14}$$

Since by Lemma 5.11 and 5.12, one has $0 \le c_i(x) \le 1$ for any $x \in \mathbb{T}^N$. Hence, one can deduce from Lemma 5.14

$$0 \le E_\infty \le r_1 + r_2.$$

Moreover, one can deduce from (5.14) that

$$\min\{r_1, r_2\} \le E_\infty \le r_1 + r_2.$$

If we assume $r_1 = r_2 = r$, then

$$r \le E_\infty \le 2r,$$

the result follows ■

Proof of Theorem 5.1. By Lemma 5.10, the sequence $\{\delta_{u_i(t+t_n,x)}\}_{n\ge 0}$ is relatively compact in $Y_{loc}\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right)$ with locally narrow topology, thus, up to a sequence, we have

$$\lim_{n\to\infty} \delta_{u_i(t+t_n,x)} = \nu_{i,t}(x,\cdot) \text{ in the topology of } Y_{loc}\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right).$$

The key arguments of the proof lies in the two consequences of the decreasing energy functional, namely, equation (4.6) and equation (4.7). Lemma 5.11 is a consequence of the first equation (4.6) by which we can determine the support of $\nu_{i,t}(x,\cdot)$, i.e., there exists measurable functions $a_i(t,x)$ such that

$$\nu_{i,t}(x,\cdot) = (1-a_i(t,x))\delta_0(.) + a_i(t,x)\delta_{r_i}(.), \;\; a.e. \; x \in \mathbb{T}^N, \; i = 1,2.$$

Moreover, Lemma 5.8 and Lemma 5.12 enable us to write $a_i(t,x) \equiv c_i(x)$, $i = 1,2$. Thus, we have

$$\lim_{n\to\infty} \delta_{u_i(t+t_n,x)} = (1-c_i(x))\delta_0 + c_i(x)\delta_{r_i} \text{ in the topology of } Y_{loc}\left(\mathbb{R}\times\mathbb{T}^N;[0,\gamma]\right)$$

Applying the segregation property, we have

$$\delta_0 + \delta_{(u_1+u_2)(t,x)} = \delta_{u_1(t,x)} + \delta_{u_2(t,x)}.$$

Hence by Corollary 4.12,

$$\lim_{n\to\infty} \delta_{(u_1+u_2)(t+t_n,x)} = \left(1 - \sum_{i=1,2} c_i(x)\right)\delta_0 + \sum_{i=1,2} c_i(x)\delta_{r_i} \tag{5.15}$$

If in addition, assume that $r_1 = r_2 = r$, applying Lemma 5.14 where we used the decay property of Fourier coefficients in equation (4.7), which yields

$$\sum_{i=1}^{2} c_i(x) = 2 - \frac{E_\infty}{r}.$$

Together with equation (5.15) we obtain

$$\lim_{n\to\infty} \delta_{(u_1+u_2)(t+t_n,x)} = (E_\infty/r - 1)\delta_0 + (2 - E_\infty/r)\delta_r,$$

in the sense of the narrow convergence topology of $Y(\mathbb{T}^N;[0,\gamma])$ and by Lemma 5.15 we have $E_\infty \in [r, 2r]$. Now the limit does not depend on t and the choice of the subsequence. Since $\{t_n\}_{n\geq 0}$ is any given sequence that tends to infinity and $\left(\mathbb{T}^N, \mathcal{B}(\mathbb{T}^N)\right)$ is a countably generated $\sigma-$algebra, then the topology $Y(\mathbb{T}^N;[0,\gamma])$ is metrizable (see for instance [34, Theorem 1] or the monograph [9]). Therefore, we conclude that

$$\lim_{t\to\infty} \delta_{(u_1+u_2)(t,x)} = (E_\infty/r - 1)\delta_0 + (2 - E_\infty/r)\delta_r.$$

As a result, Theorem 5.1 follows. ■

6 Discussion and numerical simulations

In this section we study system (1.1) for the one dimensional case with numerical simulations. Our original motivation is coming from two species of cells growing in a petri dish.

Here we will focus on the coexistence and the exclusion principle for these two species. From Theorem 5.1, we deduce that

$$\lim_{t\to\infty} \delta_{(u_1+u_2)(t,x)} = (E_\infty/r - 1)\delta_0 + (2 - E_\infty/r)\delta_r, \text{ in the sense of narrow convergence.}$$

Therefore, the limit $E_\infty := \lim_{t\to\infty} E[(u_1,u_2)(t,\cdot)]$ is an important index to determine whether the Dirac measure $\delta_{u_1+u_2}$ converges to a Young measure in the sense of narrow convergence or to a constant function in L^1 norm (see Remark 5.2). To that aim, we trace the curve $t \longmapsto E[(u_1,u_2)(t,\cdot)]$ in numerical simulations, which has been analytically proved decreasing in Theorem 4.6. Moreover, we also plot the curve $t \longmapsto E_i[u_i(t,\cdot)]$, $i = 1,2$, respectively. This will help us to understand the limit for each species u_i.

In the numerical simulations, we focus on the convergence of the energy functional which implies the convergence of the total number for each species. In fact, by using (5.6) we obtain

$$\lim_{t\to\infty} \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u_i(t,x)dx = \lim_{t\to\infty} \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}}\int_{[0,\gamma]} \lambda\delta_{u_i(t,x)}(d\lambda)dx = \frac{r_i}{|\mathbb{T}|}\int_{\mathbb{T}}\int_{[0,\gamma]} c_i(x)dx.$$

Hence by using (5.12) one has

$$\lim_{t\to\infty} E_i[u_i(t,\cdot)] = r_i\left(1 - \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} c_i(x)dx\right) = r_i - \lim_{t\to\infty}\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u_i(t,x)dx. \tag{6.1}$$

This means that the energy functional is related to the asymptotic total number of individuals for each species. We mainly investigate the following properties by numerical experiments.

Coexistence: If $r_1 = r_2 = r$, then $c_1(x), c_2(x) \in (0,1)$, *a.e.*, $x \in \mathbb{T}^N$. For each species, the following limits exist

$$\lim_{t\to\infty} \|u_i(t,\cdot)\|_{L^1} = r\int_{\mathbb{T}^N} c_i(x)dx \in (0,r),\ i = 1,2.$$

We will see that the relative location of each species has an impact on the asymptotic number in each species. Moreover, we have

$$(u_1+u_2)(t,x) \xrightarrow{L^1} r,\ t\to\infty.$$

Exclusion Principle: If $r_1 > r_2$ (resp. $r_1 < r_2$), then $c_1(x) = 1$, $c_2(x) = 0$ (resp. $c_1(x) = 0$, $c_2(x) = 1$) *a.e.*, $x \in \mathbb{T}^N$, which implies that

$$u_1(t,x) \xrightarrow{L^1} r_1,\quad u_2(t,x) \xrightarrow{L^1} 0,\ (resp.\, u_1(t,x) \xrightarrow{L^1} 0,\quad u_2(t,x) \xrightarrow{L^1} r_2),$$

and

$$(u_1+u_2)(t,x) \xrightarrow{L^1} \max\{r_1,r_2\},\ t\to\infty.$$

6.1 The case $r_1 = r_2$ implies the coexistence

Our first scenario is to present the results in Theorem 5.1. It is interesting to notice that in Theorem 5.1, we only assume the equilibrium of the corresponding ODE system for each species to be the same without imposing any other condition on h, which means that the dynamics for these two species can be different. Hence, we will use the following two different reaction functions for two species

$$u_1 h_1(u_1+u_2) = u_1\left(\frac{b_1}{1+\gamma(u_1+u_2)} - \mu\right), \quad u_2 h_2(u_1+u_2) = b_2 u_2\left(1 - \frac{u_1+u_2}{K}\right). \tag{6.2}$$

One can verify that h_i satisfies Assumption 1.1 and Assumption 4.1 with their roots (i.e., $h_i(r_i) = 0$, $i = 1,2$) as

$$r_1 := \frac{b_1 - \mu}{\gamma\mu}, \quad r_2 = K.$$

Our kernel ρ in the simulation is chosen as

$$\rho(x) = e^{-\pi|x|^2}, \quad x \in \mathbb{R}, \tag{6.3}$$

which is a Gaussian kernel. Therefore, due to Remark 1.3 and Remark 4.5, Assumption 1.2 and Assumption 4.4 are satisfied.

We set the initial distributions for two species to be of compact supports and separated. From Theorem 3.1, we can observe the segregation property of the two species as time evolves. Our parameters in system (1.1) are given as

$$b_1 = b_2 = 1.2,\ \mu = 1,\ \gamma = 1,\ K = 0.2. \tag{6.4}$$

Hence one can calculate that

$$r_1 = r_2 = 0.2.$$

Now we trace the curve $t \longmapsto E[(u_1,u_2)(t,\cdot)]$ in numerical simulations. We also plot the curve $t \longmapsto E_i[u_i(t,\cdot)]$, $i = 1,2$, respectively. Moreover, we plot the variation of the mean value of the total number of individuals for each species, that is

$$t \longmapsto \frac{1}{2\pi}\int_0^{2\pi} u_i(t,x)dx, \quad i = 1,2.$$

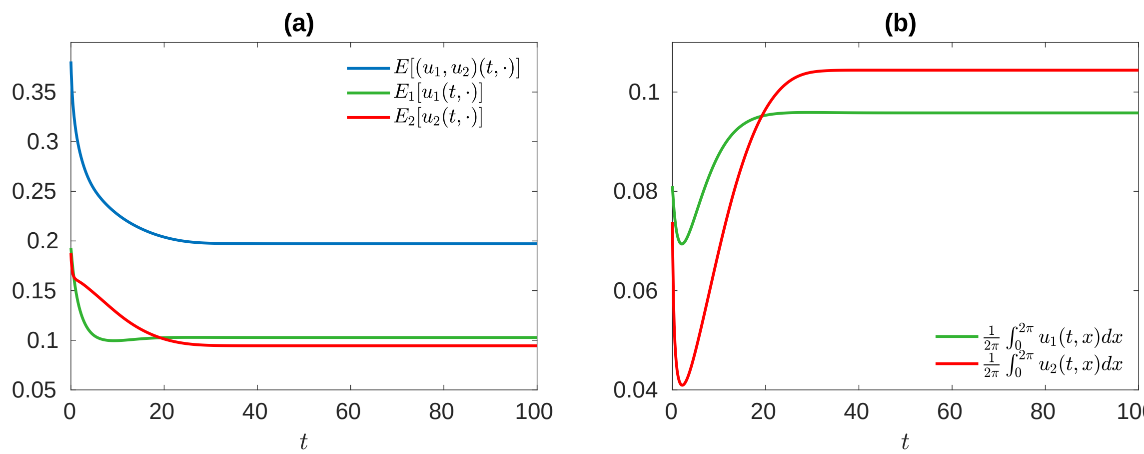

Figure 2: **(a).** *The energy functionals $t \longmapsto E_i[u_i(t,\cdot)]$, $i = 1,2$, and $t \longmapsto E[u_1,u_2)(t,\cdot)]$ under system (1.1). Parameters are set as in (6.4). Thus, one has $r_1 = r_2 = 0.2$.* **(b).** *Evolution of the mean value of individuals for each species.*

From Figure 2, we can see that the limit E_∞ exists and equals to $r = 0.2$. From Theorem 5.1 and Remark 5.2, the limit $E_\infty = r$ implies

$$(u_1+u_2)(t,x) \xrightarrow{L^1} r, \quad t \to \infty.$$

Moreover, from the simulation we note that each limit $E_{i,\infty} := \lim_{t\to\infty} E_i[u_i(t,\cdot)]$ exists for $i = 1, 2$. From (6.1) we have

$$E_{i,\infty} = r\left(1 - \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} c_i(x)dx\right), \quad i = 1, 2. \tag{6.5}$$

By our simulation, we can see that $E_{1,\infty}, E_{2,\infty} \in (0, r)$ while $E_{1,\infty} + E_{2,\infty} = r$, together with equation (6.5) we can deduce that $c_1(x), c_2(x) \in (0, 1)$, $c_1(x) + c_2(x) = 1$. Notice that $c_1(x), c_2(x) \in (0, 1)$ implies the limits

$$\lim_{n\to\infty} \delta_{u_i(t_n,x)} = (1 - c_i(x))\delta_0 + c_i(x)\delta_r, \quad i = 1, 2,$$

is not a single Dirac measure. Therefore, using Young measure and the weak compactness in $Y(\mathbb{T}; [0, \gamma])$ helps us to understand the limit of the solution.

Now we plot the evolution of two populations under system (1.1) in Figure 3.

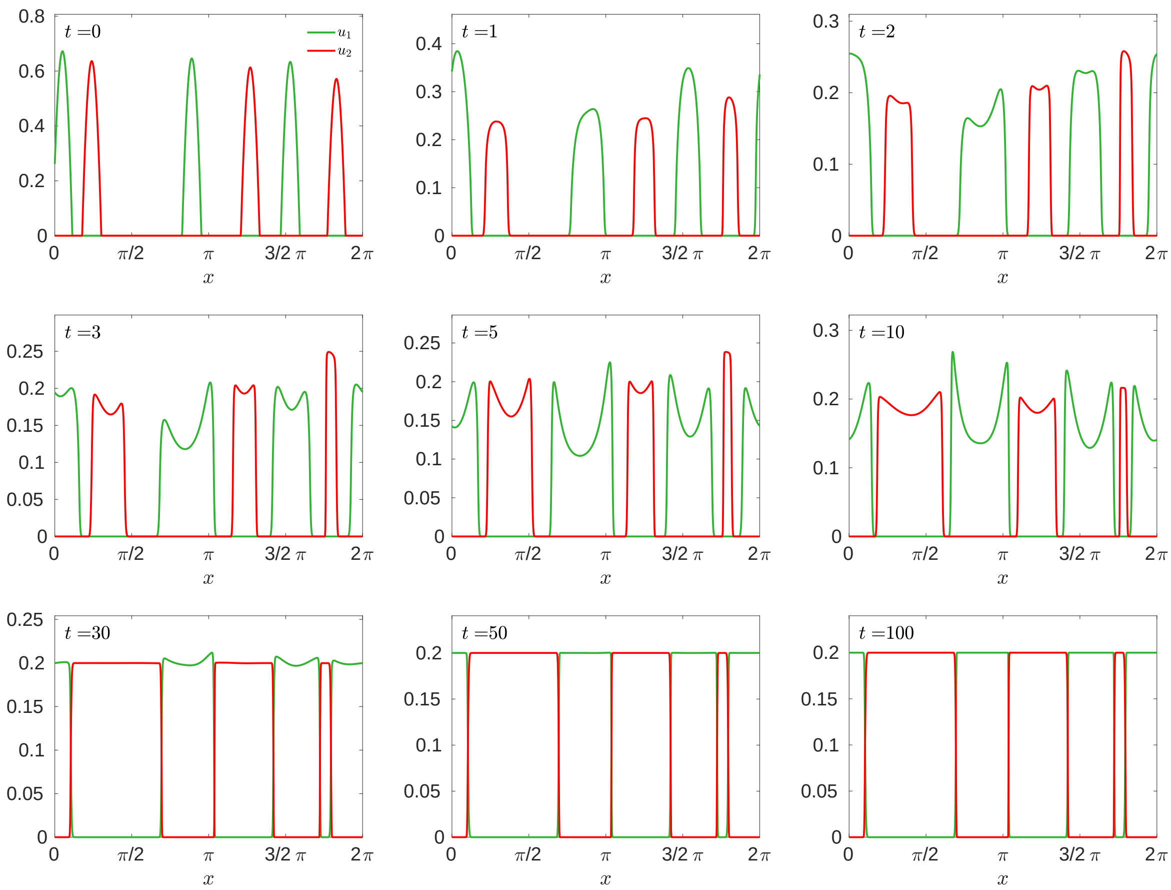

Figure 3: *The evolution of the two populations of system* (1.1)*. Parameters are set as in* (6.4)*. One has* $r_1 = r_2 = 0.2$ *which implies the coexistence of the two species. After* $t = 100$*, the distributions of the two species stay the same.*

For the asymptotic behavior of two populations, we can see from Figure 3 that the sum of two species $u_1 + u_2$ reaches a steady state at $t = 100$. From the pattern at each moment t, we can see two species keep segregated in stead of being mixed (as opposite to the case with linear diffusion).

6.2 Initial location matters

Consider two different initial distributions $\mathbf{u_0} = (u_1(0, x), u_2(0, x))$ and $\tilde{\mathbf{u}}_\mathbf{0} = (\tilde{u}_1(0, x), \tilde{u}_2(0, x))$ and assume that their L^1 norms are the same, that is

$$\int_{\mathbb{T}} u_i(0, x)dx = \int_{\mathbb{T}} \tilde{u}_i(0, x)dx, \quad i = 1, 2.$$

Under the same set of parameters, define

$$U_{i,\infty} := \lim_{t\to\infty} \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} u_i(t,x)dx, \quad \tilde{U}_{i,\infty} := \lim_{t\to\infty} \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \tilde{u}_i(t,x)dx, \quad i = 1,2, \tag{6.6}$$

we are interested in whether the limits $U_{i,\infty}$ and $\tilde{U}_{i,\infty}$ will be the same or not.

In the real biological experiments, this situation corresponds to the case where experimentalists use the same quantity of cells for each species for two separate petri dishes. Supposing the intrinsic mechanisms of cell populations for these two groups are the same, the only difference is the initial cell distributions in two petri dishes. We are interested in whether the final total mass for each population are the same. Before our simulation, we plot two different initial distributions as in Figure 4.

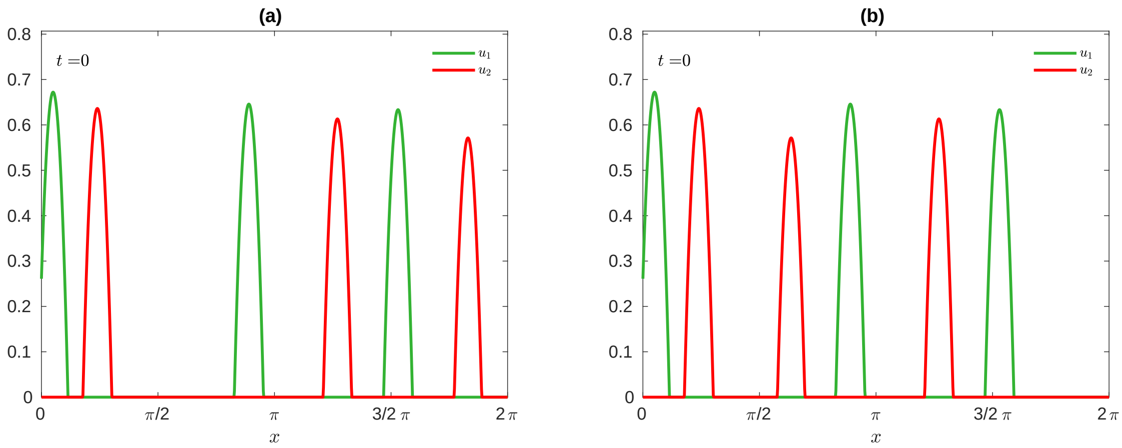

Figure 4: *(a) and (b) correspond to the initial distributions $\mathbf{u_0}$ and $\tilde{\mathbf{u}}_\mathbf{0}$ respectively. In (a), we shift a part of u_2 population at position in between $3/2\pi$ and 2π to the position in between $\pi/2$ to π. Hence, the number individuals for each species is conserved.*

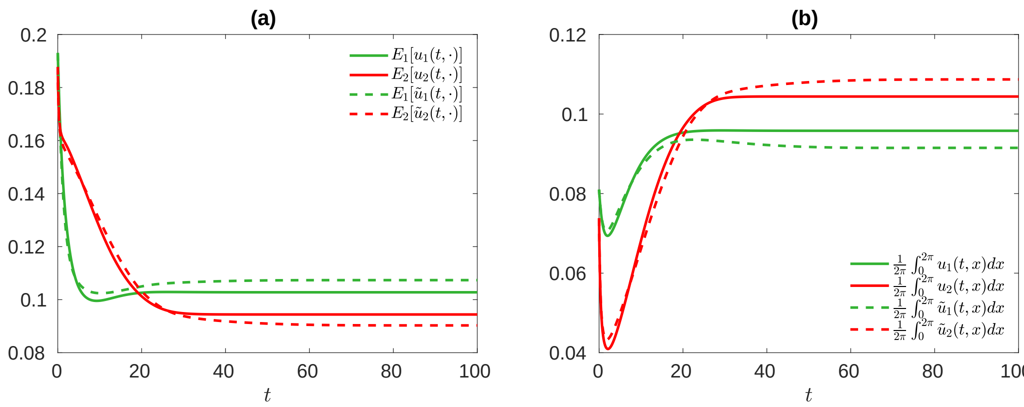

Figure 5: *The evolution of energy functional (a) and the mean value of individuals (b) corresponding to two sets of different initial distributions in Figure 4. The dashed lines correspond to the simulation with initial distribution as in Figure 4 (a) and solid lines correspond to initial distribution as in Figure 4 (b). The parameters are the same as in (6.4).*

In Figure 5, we plot the energy functionals and the number of individuals corresponding to each initial distribution in Figure 4. Since the limits $U_{i,\infty}$ and $\tilde{U}_{i,\infty}$ have a significant difference from Figure 5 (b), thus we conclude the final total mass depends on the position of the initial value.

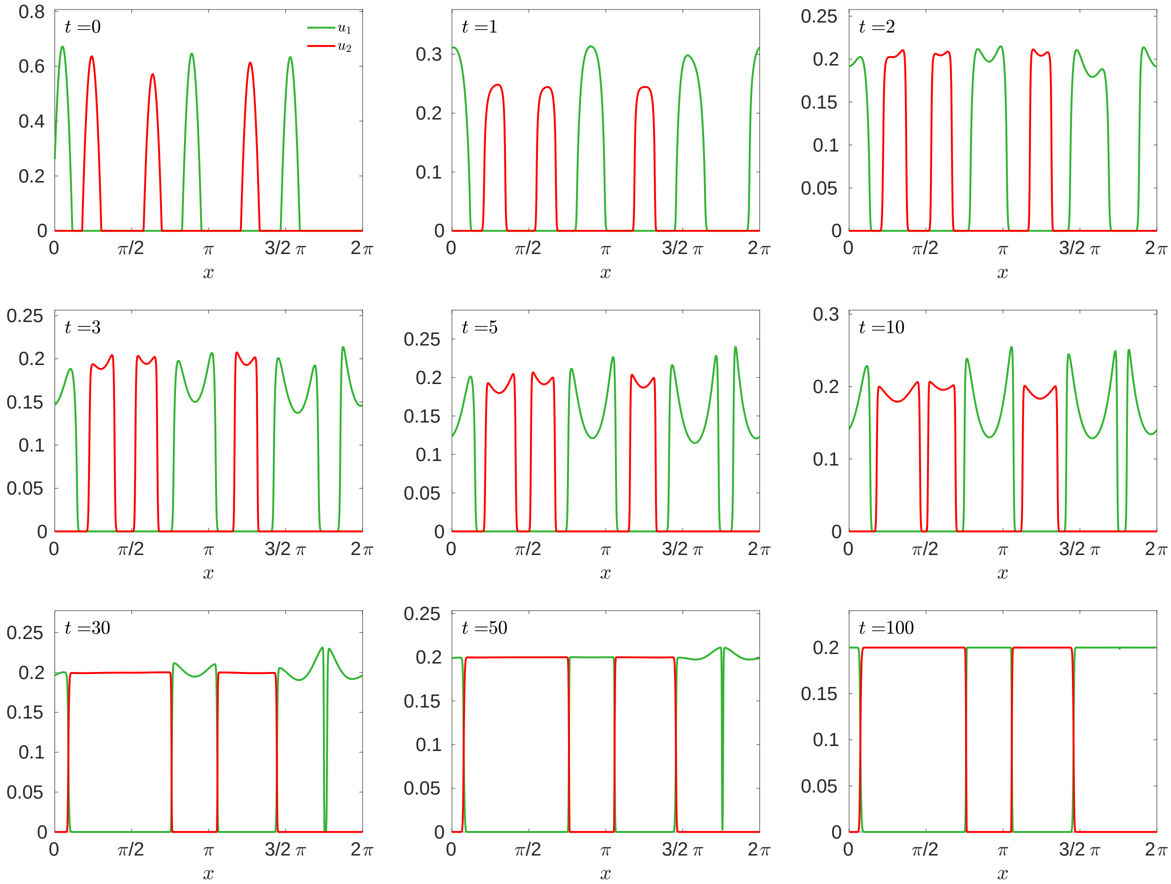

Figure 6: *The evolution of the two populations of system* (1.1)*. The initial condition is set as Figure 4* **(b)***. Parameters are set as in* (6.4)*. After $t = 100$, the distributions of the two species stay the same.*

Now we give the evolution of the two populations under system (1.1). As for the simulation in Figure 6, we can see that the same coexistence as in Figure 3 and the sum of the two populations

$$(u_1 + u_2)(t, x) \xrightarrow{L^1} r, \quad t \to \infty.$$

However, the final patterns of two species at $t = 100$ in Figure 6. (i) and Figure 3. (i) are evidently different.

6.3 The case $r_1 \neq r_2$ implies exclusion principle

Our second scenario complements the results in Theorem 5.1. Without loss of generality, we allow $r_1 > r_2$. This means species u_1 is favored in the environment. Our parameters for the reaction functions (6.2) are given as

$$b_1 = 1.5,\ b_2 = 1.2,\ \mu = 1,\ \gamma = 1,\ K = 0.2. \tag{6.7}$$

Hence we can calculate that

$$r_1 = 0.5 > r_2 = 0.2.$$

As before, we trace the curve $t \longmapsto E[(u_1, u_2)(t, \cdot)]$ in numerical simulation and we also plot the curve $t \longmapsto E_i[u_i(t, \cdot)]$, $i = 1, 2$, respectively. Moreover, we plot the variation of the mean value of the total number of individuals for each species.

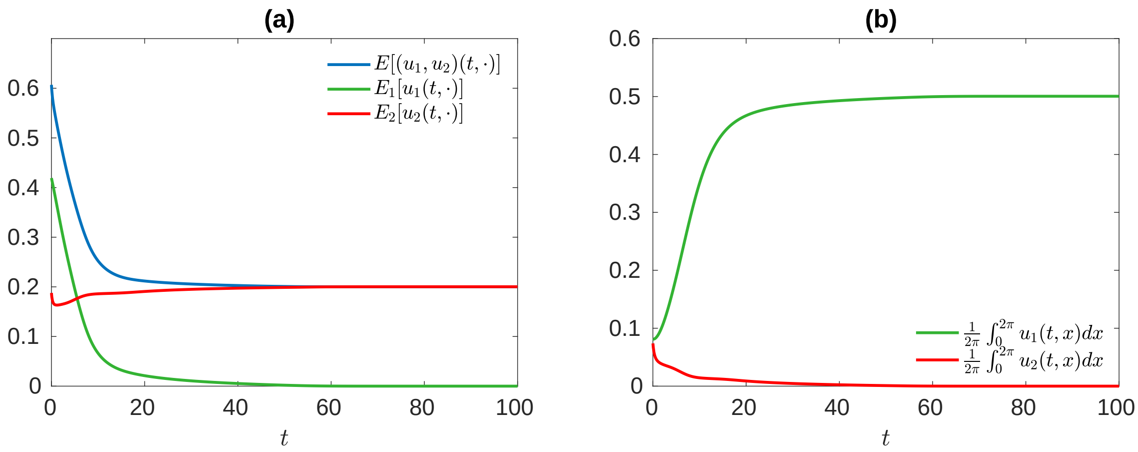

Figure 7: **(a).** *The energy functionals* $t \longmapsto E_i[u_i(t,\cdot)]$, $i = 1, 2$, *and* $t \longmapsto E[u_1, u_2)(t,\cdot)]$. *Parameters are set as in* (6.7)*. In such case, one has* $r_1 = 0.5 > r_2 = 0.2$. **(b).** *Evolution of the mean value of individuals for each species.*

By tracing the curve $t \longmapsto E[(u_1, u_2)(t,\cdot)]$, we can see from Figure 7 that it is strictly decreasing and it confirms again the result which has been proved in Theorem 4.6. We can also see that the curve $t \longmapsto E_1[u_1(t,\cdot)]$ is decreasing while $t \longmapsto E_2[u_2(t,\cdot)]$ is not monotone decreasing and their limits are

$$\lim_{t\to\infty} E_1[u_1(t,\cdot)] = 0, \quad \lim_{t\to\infty} E_2[u_1(t,\cdot)] = r_2.$$

If we have $E_{1,\infty} = 0, E_{2,\infty} = r_2$, since $c_i(x) \in [0, 1]$, $a.e.\, x \in \mathbb{T}$ for $i = 1, 2$ and by equation (6.5) one obtains $c_1(x) = 1$, $c_2(x) = 0$. Therefore, we have $c_1(x) + c_2(x) = 1$, $a.e.\, x \in \mathbb{T}$ and the convergence in Theorem 5.1 is in the sense of L^1 (see Remark 5.2)

$$u_1(t,x) \xrightarrow{L^1} r_1, \quad u_2(t,x) \xrightarrow{L^1} 0,\ t \to \infty,$$

and

$$(u_1 + u_2)(t,x) \xrightarrow{L^1} r_1,\ t \to \infty.$$

This means if $r_1 > r_2$ (resp. $r_2 > r_1$), the species u_1 will exclude u_2 (resp. u_2 will exclude u_1) when t tends to infinity. Therefore, we can conclude the exclusion principle as in the beginning of this section. We plot the evolution of the solution as follows.

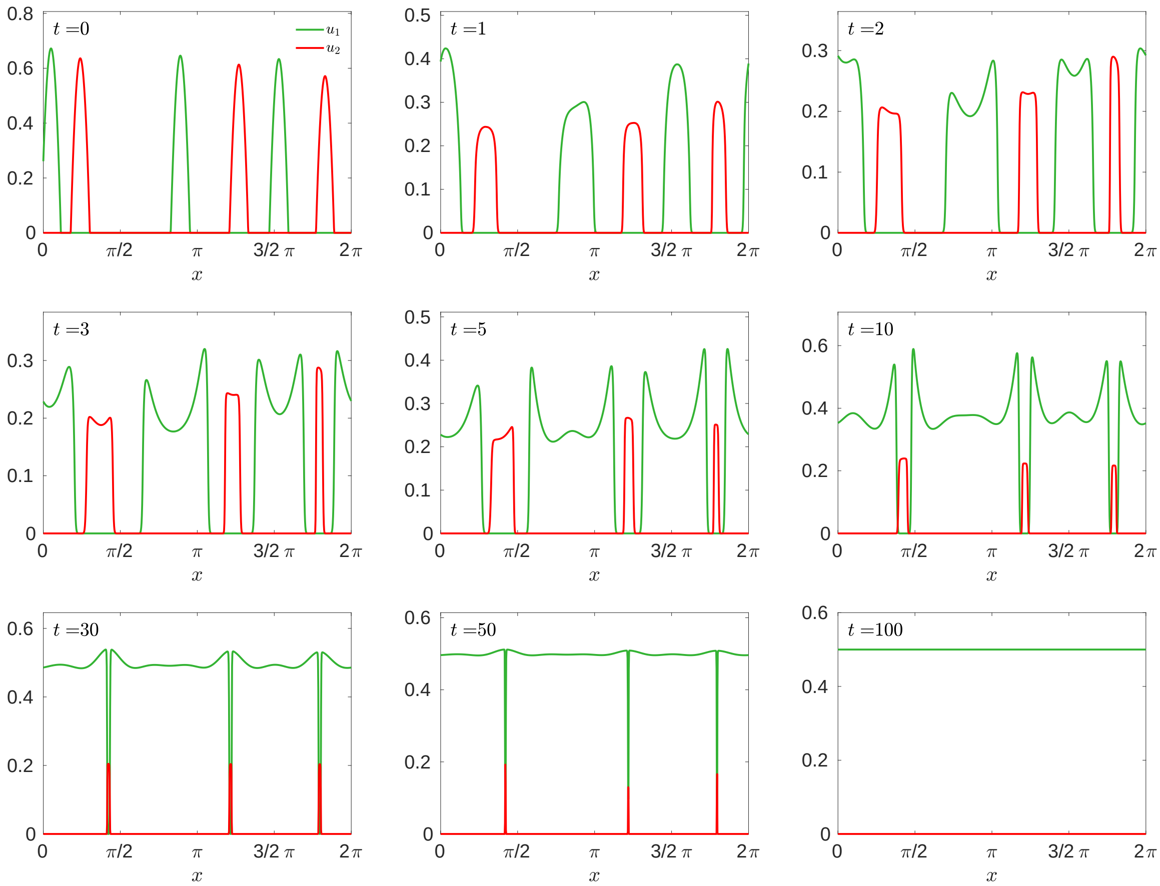

Figure 8: *The evolution of the two populations of system* (1.1) *with reaction functions as* (6.2) *and kernel* ρ *as Gaussian in* (6.3). *Parameters are set as in* (6.7). *In such case, one has* $r_1 = 0.5 > r_2 = 0.2$ *which implies the exclusion principle.*

In the simulations of Figure 8, species u_1 shows its dominance over u_2 when $t = 5$. As for the asymptotic behavior, in the last figure when $t = 100$, we can see that species u_1 crowds out species u_2 completely.

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7 Appendix

For simplicity, we give the numerical scheme for the following single-species and one dimensional model with periodic boundary condition

$$\begin{cases} \partial_t u + \partial_x (u\, v) &= \varepsilon \partial_x^2 u + u h(u) \quad t > 0,\, x \in \mathbb{T}, \\ v(t,x) &= -\partial_x (K \circ u(t,\cdot))(x) \\ u(0,x) &= u_0(x) \in L^1_{per}(\mathbb{T}). \end{cases}$$

The numerical method is based on finite volume scheme. We briefly illustrate our numerical scheme. The approximation of the convolution term is

$$(K \circ u(t,\cdot))(x) = \int_{\mathbb{T}} u(t,y) K(x-y) dy \approx \sum_j K(x - x_j) u(t, x_j) \Delta x.$$

In addition, we define

$$p_i^n := \sum_{j=1}^{M} K(x_i - x_j) u(t_n, x_j) \Delta x,$$

for $i=1,2,\ldots,M,\ n=0,1,2,\ldots,N$. We use the numerical scheme as illustrated in [33] to deal with the nonlocal convection and the scheme reads as follows

$$\begin{aligned} u_i^{n+1} &= u_i^n + \varepsilon \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) \\ &\quad - \frac{\Delta t}{\Delta x}\left(\phi(u_{i+1}^{n,-}, u_i^{n,+}) - \phi(u_i^{n,-}, u_{i-1}^{n,+})\right) + \Delta t\, u_i^n h(u_i^n), \\ & i = 1,2,\ldots,M,\ n=0,1,2,\ldots,N, \end{aligned}$$

with $\phi(u_{i+1}^n, u_i^n)$ defined as

$$\phi(u_{i+1}^{n,-}, u_i^{n,+}) = (v_{i+\frac12}^n)^+ u_i^{n,+} - (v_{i+\frac12}^n)^- u_{i+1}^{n,-} = \begin{cases} v_{i+\frac12}^n u_i^{n,+} & v_{i+\frac12}^n \geq 0, \\ v_{i+\frac12}^n u_{i+1}^{n,-} & v_{i+\frac12}^n < 0. \end{cases}$$

where

$$v_{i+\frac12}^n = -\frac{p_{i+1}^n - p_i^n}{\Delta x},\ i=1,2,\cdots,M-1,$$

and

$$\begin{aligned} u_i^{n,-} &= u_i^n - \frac12 \text{minmod}(u_{i+1}^n - u_i^n, u_i^n - u_{i-1}^n) \\ u_i^{n,+} &= u_i^n + \frac12 \text{minmod}(u_{i+1}^n - u_i^n, u_i^n - u_{i-1}^n) \end{aligned} \quad i=1,2,\cdots,M-1,$$

where the function minmod(a,b) is defined as

$$\text{minmod}(a,b) = \begin{cases} \text{sign}(a)\min\{a,b\} & \text{sign}(a)=\text{sign}(b), \\ 0 & \text{Otherwise}. \end{cases}$$

By the periodic boundary condition, let $v_{\frac12}^n = v_{M+\frac12}^n$ and $u_0^n = u_M^n$, $u_1^n = u_{M+1}^n$. Thus,

$$u_0^{n,\pm} = u_M^{n,\pm},\ u_1^{n,\pm} = u_{M+1}^{n,\pm},$$

the conservation law holds when the reaction term equals zero.

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