

Functional differential equation with infinite delay in a space of exponentially bounded and uniformly continuous functions

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Abstract

In this article we study a class of delay differential equations with infinite delay in weighted spaces of uniformly continuous functions. We focus on the integrated semigroup formulation of the problem and so doing we provide a spectral theory. As a consequence we obtain a local stability result and a Hopf bifurcation theorem for the semiflow generated by such a problem.

Key words. Functional differential equations, infinite delay, integrated semigroup, stability, Hopf bifurcation

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1 Introduction

Functional differential equations with finite and infinite delay have been extensively studied in the literature. Finite delay differential equations have firstly been studied in the 1970s by the group of Hale's [19, 20, 22]. Since then people tried to extend some bifurcation results for ordinary differential equations to functional differential equations. In order to do so, one of the main difficulties is to understand the relationship between the spectral properties of the linearized system (around a given equilibrium) and the dynamical properties of nonlinear perturbed systems. This type of questions have been directly considered for delay differential equations by deriving a so called variation of constant formula. We also refer to Arino and Sanchez [4] and Kappel [28] for more results about this topic.

In the 1980s and 1990s variation of constant formula was reconsidered by using non classical perturbation idea coming from semigroup theory. Sun-star adjoint spaces and semigroup theory have been firstly successfully applied to delay differential equations. We refer to Diekmann et al. [11] for a nice survey about this topic. We refer to Kaashoek and Verduyn Lunel [27], Frasson and Verduyn Lunel [17] and Diekmann Getto and Gyllenberg [10] and references therein for more results in that direction.

Around the same period Adimy [1] and Thieme [46] observed that integrated semigroups can also be used to derive a variation of constant formula to describe functional differential equations. We also refer to Adimy [2], Adimy and Arino [3] and Ezzinbi and Adimy [16] for more results about this topic. In the present article, we will also use integrated semigroup theory and we will reconsider the formulation introduced by Liu, Magal and Ruan [30] to study infinite delay differential equations. More recently, infinite delay has also been considered by Walther [51] in the context of state dependent delay differential equations.

Consider the weighted space of uniformly continuous functions

$$BUC_\eta = \{ \varphi \in C((-\infty, 0], \mathbb{R}^n) : \theta \rightarrow e^{\eta\theta} \varphi(\theta) \text{ is bounded and uniformly continuous} \}$$

which is a Banach space endowed with the norm

$$\|\varphi\|_\eta := \sup_{\theta \leq 0} e^{\eta\theta} \|\varphi(\theta)\|.$$

In this article we consider the following class of functional differential

equations on the space BUC_η

$$(FDE) \quad \begin{cases} \frac{dx(t)}{dt} = f(x_t), \forall t \geq 0, \\ x_0 = \varphi \in BUC_\eta, \end{cases} \quad (1.1)$$

where $f : BUC_\eta \rightarrow \mathbb{R}^n$ is Lipschitz on bounded sets. Recall that for any given map $x \in C((-\infty, \tau], \mathbb{R}^n)$ (for some $\tau \geq 0$) and each $t \leq \tau$ the map $x_t \in C((-\infty, 0], \mathbb{R}^n)$ is defined by

$$x_t(\theta) = x(t + \theta), \forall \theta \leq 0.$$

Then it is easy to verify that if $x \in C((-\infty, \tau], \mathbb{R}^n)$ for some $\tau \geq 0$ then

$$x_0 \in BUC_\eta \Rightarrow x_t \in BUC_\eta, \forall t \in [0, \tau].$$

Recall the notion of solution for the FDE.

Definition 1.1 *A solution of the FDE (1.1) is a continuous map $x : (-\infty, \tau] \rightarrow \mathbb{R}^n$ (for some $\tau > 0$) satisfying*

$$x(t) = \begin{cases} \varphi(0) + \int_0^t f(x_l) dl, \forall t \geq 0, \\ \varphi(t), \forall t \leq 0. \end{cases} \quad (1.2)$$

Assume that

$$f(0_{BUC_\eta}) = 0.$$

Then 0 is an equilibrium solution of the system (1.1). Assume that f is differential at 0, and set

$$\widehat{L} := Df(0).$$

Then the linearized equation of (1.1) around 0 is

$$\begin{cases} \frac{dx(t)}{dt} = \widehat{L}(x_t), \forall t \geq 0, \\ x_0 = \varphi \in BUC_\eta. \end{cases} \quad (1.3)$$

The first main question addressed in this article is to understand the spectral properties of the linearized equation (1.3). Then by using the spectral properties of (1.3) we will derive a stability and Hopf bifurcation results for equation (1.1). Actually the equation (1.1) can be rewritten as

$$\begin{cases} \frac{dx(t)}{dt} = \widehat{L}(x_t) + g(x_t), \forall t \geq 0, \\ x_0 = \varphi \in BUC_\eta, \end{cases} \quad (1.4)$$

where $g := f - \widehat{L}$.

Assume that there exists $x \in C((-\infty, \tau], \mathbb{R}^n)$ (for some $\tau \geq 0$) a solution of (1.1). Set

$$u(t, \theta) := x_t(\theta) = x(t + \theta), \forall t \in [0, \tau] \text{ and } \forall \theta \leq 0. \quad (1.5)$$

Then u can be regarded as a solution of the following system

$$(PDE) \quad \begin{cases} \partial_t u(t, \theta) - \partial_\theta u(t, \theta) = 0, \text{ for } \theta \leq 0 \text{ and } t \geq 0, \\ \partial_\theta u(t, 0) = f(u(t, \cdot)), \text{ for } t \geq 0, \\ u(0, \cdot) = \varphi \in BUC_\eta. \end{cases} \quad (1.6)$$

The idea of using the PDE associated to the FDE, was successfully used by Travis and Webb [49, 50] and Webb [52] to apply nonlinear semigroup theory. We refer to Ruess [44] for more results and updated references on this topic. To our best knowledge no bifurcation results have been obtained by using this approach.

In order to reformulate the PDE problem as an abstract Cauchy problem, we will first incorporate the boundary condition into the state space, by considering the Banach space

$$X := \mathbb{R}^n \times BUC_\eta$$

endowed with the product norm

$$\|(\alpha, \varphi)\| := \|\alpha\|_{\mathbb{R}^n} + \|\varphi\|_\eta.$$

Define $A : D(A) \subset X \rightarrow X$ the linear operator by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} := \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \quad (1.7)$$

with

$$D(A) = \{0_{\mathbb{R}^n}\} \times BUC_\eta^1$$

where

$$BUC_\eta^1 := \{\varphi \in C^1((-\infty, 0], \mathbb{R}^n) : \varphi, \varphi' \in BUC_\eta((-\infty, 0], \mathbb{R}^n)\}.$$

It is important to note that the closure of the domain is

$$X_0 := \overline{D(A)} = \{0_{\mathbb{R}^n}\} \times BUC_\eta.$$

Therefore, A is non-densely defined. We also define $L : \overline{D(A)} \rightarrow X$ by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{L}(\varphi) \\ 0_{BUC_\eta} \end{pmatrix}.$$

We consider $F : \overline{D(A)} \rightarrow X$ and $G : \overline{D(A)} \rightarrow X$ the maps defined by

$$F \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} f(\varphi) \\ 0_{BUC_\eta} \end{pmatrix} \text{ and } G \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} g(\varphi) \\ 0_{BUC_\eta} \end{pmatrix}.$$

Then

$$F = L + G.$$

Set

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}.$$

As we will see in section 2, the FDE (1.1) or the PDE (1.6) can be reformulated as the following abstract non-densely defined Cauchy problem

$$(ACP) \begin{cases} \frac{dv(t)}{dt} = Av(t) + L(v(t)) + G(v(t)), t \geq 0, \\ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}. \end{cases} \quad (1.8)$$

Several examples of infinite delay differential equations have been considered in the literature. Here we present two examples. One is the following chemostat model with a distributed delay considered in Ruan and Wolkowicz [42]

$$\begin{cases} \frac{dS}{dt} = (S^0 - S(t))D - ax(t)p(S(t)), \\ \frac{dx}{dt} = x(t) \left[-D_1 + \int_{-\infty}^t F(t - \tau)p(S(\tau))d\tau \right], \\ S(s) = \phi(s) > 0, -\infty < s \leq 0, x(0) = x_0 > 0, \\ \phi(s) \text{ is a continuous function on } (-\infty, 0], \end{cases} \quad (1.9)$$

where $S(t)$ and $x(t)$ denote the concentration of the nutrient and the populations of microorganisms at time t . Another example is the following model of a fishery with fish stock involving delay equations investigated by Auger and Ducrot [7]

$$\begin{cases} \frac{dn}{dt} = rn(1 - n) - \varphi(n, E), \\ \frac{dE}{dt} = p(1 - \eta)\varphi(n, E) + \eta p \int_{-\infty}^0 \delta e^{\delta\theta} \varphi(n(t + \theta), E(t + \theta))d\theta - cE. \end{cases} \quad (1.10)$$

where $n(t), E(t)$ denote the density of the resource and the fishing effort, respectively. We refer to McCluskey [41], Rost and Wu [43], and Gourley, Rost and Thieme [18] for more examples in the context of population dynamics. We would like to mention that we can derive a stability and Hopf bifurcation result for the models (1.9) and (1.10) by using the results presented in this article.

The last example is a model describing the interaction between floating structures in shallow water. More precisely, in order to describe the movement of the surface of the water Bocchi [8] derives the following equation

$$f(\delta(t))\ddot{\delta}(t) = \int_0^t F(s)\dot{\delta}(t-s)ds + g(\delta(t), \dot{\delta}(t))$$

where f and g are smooth functions and the range of f does not contain 0. The map $t \rightarrow F(t)$ is a continuous function which converges exponentially fast to 0 when t goes to infinity. Our results apply to this class of equation and we refer to [8] for more results.

The main tool of this article is to apply integrated semigroup theory. We use essentially the results in Thieme [46, 47, 48], Magal and Ruan [34, 36] and Liu, Magal and Ruan [30]. We will first prove that A is a Hille-Yosida operator in order to define the mild solution of ACP. By using the uniqueness of the mild solution of ACP, we will prove that each mild solution of ACP corresponds to a solution of the FDE and the reverse is also true. Then in order to obtain a spectral theory, we study the essential growth rate of the semigroup generated by A_0 and by $(A+L)_0$. The operators A_0 (respectively $(A+L)_0$) is the part of A (respectively $A+L$) in $\overline{D(A)}$ (see section 4). This part is crucial to understand the spectral properties of these semigroup. Actually the fact that $\eta > 0$ is crucial to obtain a stability results, as well as a center manifold theorem (see Magal and Ruan [35]) and a Hopf bifurcation theorem. We refer to the book of Magal and Ruan [37] for a nice survey on this topic.

By transforming the system it is also possible to study the problem

$$BUC := BUC_0. \tag{1.11}$$

Consider the isometry form $\Psi : BUC_\eta \rightarrow BUC$ defined by

$$\Psi(u)(\theta) := e^{\eta\theta}u(\theta).$$

By setting $\widehat{u}(t, \theta) := e^{\eta\theta}u(t, \theta)$, the PDE (1.6) gives

$$(PDE) \quad \begin{cases} \partial_t \widehat{u}(t, \theta) - \partial_\theta \widehat{u}(t, \theta) = -\eta \widehat{u}(t, \theta), & \text{for } \theta \leq 0 \text{ and } t \geq 0, \\ \partial_\theta \widehat{u}(t, 0) - \eta \widehat{u}(t, 0) = f(e^{-\eta \cdot} \widehat{u}(t, \cdot)), & \text{for } t \geq 0, \\ \widehat{u}(0, \cdot) = \varphi \in BUC. \end{cases} \quad (1.12)$$

Identifying $\widehat{u}(t, \cdot)$ and $\widehat{v}(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{u}(t, \cdot) \end{pmatrix}$ the last PDE can be rewritten as an abstract Cauchy problem

$$\frac{d\widehat{v}(t)}{dt} = B\widehat{v}(t) + H(\widehat{v}(t)), t \geq 0, \widehat{v}(0) = \widehat{v}_0 \in \overline{D(B)} \quad (1.13)$$

where $B : D(B) \subset Y \rightarrow Y$ (where the Banach $Y := \mathbb{R}^n \times BUC$ is endowed with the usual product norm) is the linear operator defined by

$$B \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} := \begin{pmatrix} -\varphi'(0) + \eta\varphi(0) \\ \varphi' - \eta\varphi \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(B),$$

with

$$D(B) = \{0_{\mathbb{R}^n}\} \times BUC^1$$

and $BUC^1 := BUC_0^1$. $H : \overline{D(B)} \rightarrow X$ is the map defined by

$$H \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} f(e^{-\eta \cdot} \varphi(\cdot)) \\ 0_{BUC} \end{pmatrix}.$$

For infinite delay differential equations various class of semi-normed spaces have been considered firstly by Hale and Kato [23]. We refer to the book Hino, Murakami and Naito [25] for more results and a nice survey on this subject. Along this line a variation of constant formula has been obtained by Hino, Murakami, Naito and Minh [26]. We also refer to Diekmann and Gyllenberg [12] for infinite delay differential equations in weighted L^1 space. Along this line Matsunaga, Murakami, Nagabuchi and Van Minh [40] recently proved a center manifold theorem for difference equation in L^1 space. We shall also mention that a Hopf bifurcation theorem has been obtained by Hassard, Kazarinoff and Wan [24, Chapter 4 Section 5] in L^2 function space.

This article is entirely devoted to the first class of problem in BUC_η . The paper is organized as follows. In section 2 we study the properties of the linear operator A . In order to obtain an explicit formula for the integrated solution of the abstract Cauchy problem (1.8) we firstly consider

a special case of (1.8) in section 3. In section 4, explicit formulas for some mild solutions are given and the properties of the linear operator $A + L$ are investigated. In section 5, we obtain an explicit formula for the projectors on the generalized eigenspaces associated to some eigenvalues. The projector for a simple eigenvalue is considered in section 6. Sections 7 and 8 deal with the nonlinear semiflow and local stability of equilibria respectively. In Section 9, we show a few comments and remarks on the center manifold theorem, Hopf bifurcation theory and normal form theory for infinite delay differential equations.

2 Preliminary results

In order to apply integrated semigroup theory we need to verify the Hille-Yosida properties for the linear operator A .

Lemma 2.1 *We have $(0, +\infty) \subset \rho(A)$ (where $\rho(A)$ is the resolvent set of A), and we have for each $\lambda > 0$ and each $\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in X$*

$$\begin{aligned} (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} &= \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\ \Leftrightarrow \psi(\theta) &= \frac{1}{\lambda} e^{\lambda\theta} [\alpha + \varphi(0)] + \int_{\theta}^0 e^{\lambda(\theta-l)} \varphi(l) dl. \end{aligned} \quad (2.1)$$

Proof. Let $\lambda \in (0, \infty)$. For $\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in X$ and $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \in D(A)$, we have

$$\begin{aligned} (\lambda I - A) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} &= \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \Leftrightarrow \begin{cases} \psi'(0) = \alpha \\ \lambda\psi - \psi' = \varphi \end{cases} \\ \Leftrightarrow \begin{cases} \lambda\psi(0) = \alpha + \varphi(0) \\ \lambda\psi - \psi' = \varphi \end{cases} \\ \Leftrightarrow \begin{cases} \lambda\psi(0) = \alpha + \varphi(0) \\ \psi(\theta) = e^{\lambda(\theta-\hat{\theta})} \psi(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} e^{\lambda(\theta-l)} \varphi(l) dl, \forall \theta \geq \hat{\theta} \end{cases} \\ \Leftrightarrow \begin{cases} \lambda\psi(0) = \alpha + \varphi(0) \\ \psi(\hat{\theta}) = e^{\lambda\hat{\theta}} \psi(0) - \int_0^{\hat{\theta}} e^{\lambda(\hat{\theta}-l)} \varphi(l) dl, \forall \hat{\theta} \leq 0 \end{cases} \\ \Leftrightarrow \psi(\hat{\theta}) = \frac{1}{\lambda} e^{\lambda\hat{\theta}} [\alpha + \varphi(0)] - \int_0^{\hat{\theta}} e^{\lambda(\hat{\theta}-l)} \varphi(l) dl, \forall \hat{\theta} \leq 0. \end{aligned}$$

■

Lemma 2.2 *The linear operator $A : D(A) \subset X \rightarrow X$ is a Hille-Yosida operator. More precisely, we have*

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda^n}, \forall n \geq 1, \forall \lambda > 0. \quad (2.2)$$

Proof. Using (2.1), we obtain

$$\begin{aligned} & \left\| (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| \\ & \leq \sup_{\theta \leq 0} \left[e^{\eta\theta} e^{\lambda\theta} \left| \frac{1}{\lambda} [\varphi(0) + \alpha] \right| + e^{\eta\theta} \int_{\theta}^0 e^{\lambda(\theta-s)} |\varphi(s)| ds \right] \\ & \leq \frac{1}{\lambda} |\alpha| + \sup_{\theta \leq 0} \left[\left(\frac{e^{\eta\theta} e^{\lambda\theta}}{\lambda} + \frac{(1 - e^{\eta\theta} e^{\lambda\theta})}{\lambda} \right) \|\varphi\|_{\eta} \right] \\ & \leq \frac{1}{\lambda} [|\alpha| + \|\varphi\|_{\eta}] \\ & = \frac{1}{\lambda} \left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\|. \end{aligned}$$

Therefore, (2.2) holds and the proof is completed. ■

Lemma 2.3

$$\overline{D(A)} = \{0\} \times BUC_{\eta}.$$

Proof. Let $\psi \in BUC_{\eta}((-\infty, 0], \mathbb{R}^n)$. Define for each $\varepsilon > 0$ and each $\theta \leq 0$

$$\psi_{\varepsilon}(\theta) = e^{-\eta\theta} \frac{1}{\varepsilon} \int_{\theta-\varepsilon}^{\theta} e^{\eta l} \psi(l) dl.$$

Then since $l \rightarrow e^{\eta l} \psi(l)$ is bounded and uniformly continuous we deduce that for each $\varepsilon > 0$

$$\psi_{\varepsilon} \in BUC_{\eta}^1$$

and

$$\lim_{\varepsilon \rightarrow 0} \|\psi_{\varepsilon} - \psi\|_{\eta} = 0,$$

the proof is complete. ■

3 A special class of mild solutions

In this section in order to obtain an explicit formula for the integrated solution of the abstract Cauchy problem (1.8) we firstly consider the following PDE

$$\begin{cases} \partial_t u(t, \theta) - \partial_\theta u(t, \theta) = 0, & \text{for } \theta \leq 0 \text{ and } t \geq 0 \\ \partial_\theta u(t, 0) = h(t), & \text{for } t \geq 0 \\ u(0, \cdot) = \varphi \in BUC_\eta \end{cases} \quad (3.1)$$

where the map $h \in L^1((0, \tau); \mathbb{R}^n)$ is a given perturbation at the boundary.

In that case the system (1.8) becomes

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} h(t) \\ 0 \end{pmatrix}, t \geq 0, v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}, \quad (3.2)$$

where $h \in L^1((0, \tau), \mathbb{R}^n)$.

Recall that $v \in C([0, \tau], X)$ is an integrated solution of (3.2) if and only if

$$\int_0^t v(s) ds \in D(A), \forall t \in [0, \tau] \quad (3.3)$$

and

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A \int_0^t v(s) ds + \int_0^t \begin{pmatrix} h(s) \\ 0 \end{pmatrix} ds. \quad (3.4)$$

We refer to Arendt [5], Thieme [47], Kellermann and Hieber [29], and the book by Arendt et al. [6] for a nice overview on this subject. We also refer to Magal and Ruan [37] for more results and updated references.

From (3.2) we note that if v is an integrated solution we must have

$$v(t) = \lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} v(s) ds \in \overline{D(A)}.$$

Hence

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}$$

with

$$u \in C([0, \tau], BUC_\eta).$$

In order to obtain the uniqueness of the integrated solutions of (3.2) we want to prove that A generates an integrated semigroup. So firstly we need to study the resolvent of A . Since A is a Hille-Yosida operator, A generates

a non-degenerated integrated semigroup $\{S_A(t)\}_{t \geq 0}$ on X . It follows from Thieme [47], and Kellerman and Hieber that the abstract Cauchy problem (3.2) has at most one integrated solution.

Lemma 3.1 *Let $h \in L^1((0, \tau), \mathbb{R}^n)$ and $\varphi \in BUC_\eta$. Then there exists $t \rightarrow v(t)$ a unique integrated solution of the Cauchy problem (3.2). Moreover $v(t)$ is explicitly given by the following formula*

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}$$

with

$$u(t)(\theta) = x(t + \theta), \forall t \in [0, \tau], \forall \theta \leq 0, \quad (3.5)$$

where

$$x(t) = \begin{cases} \varphi(0) + \int_0^t h(s) ds, & \text{if } t \in [0, \tau], \\ \varphi(t), & \text{if } t \leq 0. \end{cases}$$

Proof. Since A is a Hille-Yosida operator, there is at most one integrated solution of the Cauchy problem (3.2). So it is sufficient to prove that u defined by (3.5) satisfies for each $t \in [0, \tau]$ the following

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_0^t u(l) dl \end{pmatrix} \in D(A) \quad (3.6)$$

and

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_0^t u(l) dl \end{pmatrix} + \begin{pmatrix} \int_0^t h(l) dl \\ 0 \end{pmatrix}. \quad (3.7)$$

Since

$$\int_0^t u(l)(\theta) dl = \int_0^t x(l + \theta) dl = \int_\theta^{t+\theta} x(s) ds$$

and therefore $\int_0^t u(l) dl \in BUC_\eta^1$ and (3.6) follows. Moreover

$$\begin{aligned} A \begin{pmatrix} 0 \\ \int_0^t u(l) dl \end{pmatrix} &= \begin{pmatrix} -(x(t) - x(0)) \\ (x(t + \cdot) - x(\cdot)) \end{pmatrix} \\ &= - \begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \begin{pmatrix} -(x(t) - \varphi(0)) \\ x(t + \cdot) \end{pmatrix}. \end{aligned}$$

Therefore, (3.7) is satisfied if and only if

$$x(t) = \varphi(0) + \int_0^t h(s) ds. \quad (3.8)$$

The proof is completed. ■

4 Linear abstract Cauchy problem

For a given bounded linear operator $L \in \mathcal{L}(X)$, $\|L\|_{ess}$ is the essential norm of L defined by

$$\|L\|_{ess} = \kappa(L(B_X(0,1))),$$

here $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$, and for each bounded set $B \subset X$, $\kappa(B) = \inf \{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$ is the Kuratovsky measure of non-compactness. Let $L : D(L) \subset X \rightarrow X$ be the infinitesimal generator of a linear C_0 -semigroup $\{T_L(t)\}_{t \geq 0}$ on a Banach space X . Define the growth bound $\omega_0(L) \in [-\infty, +\infty)$ of L by

$$\omega_0(L) := \lim_{t \rightarrow +\infty} \frac{\ln \left(\|T_L(t)\|_{\mathcal{L}(X)} \right)}{t}.$$

The essential growth bound $\omega_{0,ess}(L) \in [-\infty, +\infty)$ of L is defined by

$$\omega_{0,ess}(L) := \lim_{t \rightarrow +\infty} \frac{\ln \left(\|T_L(t)\|_{ess} \right)}{t}.$$

Recall that $A_0 : D(A_0) \subset \overline{D(A)} \rightarrow \overline{D(A)}$ the part of A in $\overline{D(A)}$ is defined by

$$A_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times BUC_\eta^1 : \varphi'(0) = 0 \right\}.$$

Now by using the fact that A is a Hille-Yosida operator, we deduce that A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ and $v(t) = T_{A_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t), t \geq 0, v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}.$$

Using Lemma 3.1 with $h = 0$, we obtain the following result.

Lemma 4.1 *The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ of bounded linear operators on $\overline{D(A)}$ which is defined by*

$$T_{A_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{T}_{A_0}(t)\varphi \end{pmatrix}, \quad (4.1)$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} \varphi(0), & \text{if } t + \theta \geq 0, \\ \varphi(t + \theta), & \text{if } t + \theta \leq 0. \end{cases}$$

The semigroup $\{T_{A_0}(t)\}_{t \geq 0}$ can be rewritten as follows

$$T_{A_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ T\varphi + S(t)(\varphi) \end{pmatrix}, \quad (4.2)$$

where $T\varphi = \varphi(0)$ and

$$S(t)(\varphi)(\theta) = \begin{cases} 0, & \text{if } t + \theta \geq 0, \\ \varphi(t + \theta) - \varphi(0), & \text{if } t + \theta \leq 0. \end{cases}$$

Note that T is a finite-rank operator and thus compact. For each $t \geq 0$, we have

$$\sup_{\theta \leq -t} e^{\eta\theta} \|S(t)(\varphi)(\theta)\| \leq 2e^{-\eta t} \|\varphi\|_{\eta}.$$

Thus we obtain the following lemma.

Lemma 4.2 *The essential growth bound of A_0 satisfies*

$$\omega_{0,ess}(A_0) \leq -\eta.$$

The above lemma is crucial in order to apply some compact perturbation results (see Ducrot et al. [13]).

Since A is a Hille-Yosida operator, we know that A generates an integrated semigroup $\{S_A(t)\}_{t \geq 0}$ on X , and $t \rightarrow S_A(t) \begin{pmatrix} x \\ \varphi \end{pmatrix}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} x \\ \varphi \end{pmatrix}, t \geq 0, v(0) = 0.$$

Since $S_A(t)$ is linear we have

$$S_A(t) \begin{pmatrix} x \\ \varphi \end{pmatrix} = S_A(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + S_A(t) \begin{pmatrix} x \\ 0 \end{pmatrix},$$

where

$$S_A(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \int_0^t T_{A_0}(l) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} dl$$

and $S_A(t) \begin{pmatrix} x \\ 0 \end{pmatrix}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} x \\ 0 \end{pmatrix}, t \geq 0, v(0) = 0.$$

Therefore, by using Lemma 3.1 with $h(t) = x$ and the above results, we obtain the following result.

Lemma 4.3 *The linear operator A generates an integrated semigroup $\{S_A(t)\}_{t \geq 0}$ on X . Moreover, we have the following explicit formula*

$$S_A(t) \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{S}_A(t)(x, \varphi) \end{pmatrix}, \begin{pmatrix} x \\ \varphi \end{pmatrix} \in X,$$

where $\widehat{S}_A(t)$ is the linear operator defined by

$$\widehat{S}_A(t)(x, \varphi) = \widehat{S}_A(t)(0, \varphi) + \widehat{S}_A(t)(x, 0)$$

with

$$\widehat{S}_A(t)(0, \varphi)(\theta) = \int_0^t T\varphi + S(l)(\varphi)dl$$

and

$$\widehat{S}_A(t)(x, 0)(\theta) = \begin{cases} (t + \theta)x, & \text{if } t + \theta \geq 0, \\ 0, & \text{if } t + \theta \leq 0. \end{cases}$$

Now we focus on the spectrums of A and $A + L$. Since A is a Hille-Yosida operator and L is a bounded linear operator, we can get that $A + L$ is a Hille-Yosida operator. Moreover $(A + L)_0 : D((A + L)_0) \subset \overline{D(A)} \rightarrow \overline{D(A)}$ the part of $A + L$ in $\overline{D(A)}$ is the linear operator defined by

$$(A + L)_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D((A + L)_0),$$

where

$$D((A + L)_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times BUC_\eta^1 : \varphi'(0) = \widehat{L}(\varphi) \right\}.$$

By Lemma 2.1 in this article and Lemma 2.1 in Magal and Ruan [35], we know that

$$\sigma(A) = \sigma(A_0) \text{ and } \sigma(A + L) = \sigma((A + L)_0)$$

since $LT_{A_0}(t)$ is compact for any $t > 0$.

By using the main result by Thieme in [48] or by Ducrot et al. [13], one obtain the following lemma.

Lemma 4.4 *The essential growth bound of $(A + L)_0$ satisfies*

$$\omega_{0,ess}((A + L)_0) \leq \omega_{0,ess}(A_0) \leq -\eta.$$

In the following lemma, we start specifying the point spectrum of $(A + L)_0$. Let

$$\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\eta\}.$$

We now apply some results taken from Engel and Nagel [15] and Webb [53, 54].

Lemma 4.5 *The point spectrum of $(A + L)_0$ is the set*

$$\sigma(A + L) \cap \Omega = \sigma_P((A + L)_0) \cap \Omega = \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\},$$

where

$$\Delta(\lambda) = \lambda I - \widehat{L}(e^{\lambda \cdot} I) \in M_n(\mathbb{C}). \quad (4.3)$$

Proof. Let $\lambda \in \Omega$. Since $\omega_{0,ess}((A + L)_0) \leq -\eta$ it follows that

$$\sigma(A + L)_0 \cap \Omega = \sigma_P((A + L)_0) \cap \Omega$$

(see [15, 53, 54]). But $\lambda \in \sigma_P((A + L)_0)$ if and only if there exists $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D((A + L)_0) \setminus \{0\}$ such that

$$(A + L)_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \lambda \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}.$$

That is to say that $\lambda \in \sigma_P((A + L)_0)$ if and only if there exists $\varphi \in BUC_\eta^1((-\infty, 0], \mathbb{C}^n) \setminus \{0\}$ such that

$$\varphi'(\theta) = \lambda \varphi(\theta), \forall \theta \leq 0 \quad (4.4)$$

and

$$\varphi'(0) = \widehat{L}(\varphi). \quad (4.5)$$

Equation (4.4) is equivalent to

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \forall \theta \leq 0. \quad (4.6)$$

Therefore,

$$\varphi \neq 0 \Leftrightarrow \varphi(0) \neq 0.$$

By combining (4.5) and (4.6), we obtain

$$\lambda\varphi(0) = \widehat{L}(e^{\lambda\cdot}\varphi(0)).$$

The proof is completed. ■

From the discussion in this section, we obtain the following proposition.

Proposition 4.6 *The linear operator $A + L : \overline{D(A)} \rightarrow X$ is a Hille-Yosida operator, and $(A + L)_0$ the part of $A + L$ in $\overline{D(A)}$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L)_0}(t)\}_{t \geq 0}$ of bounded linear operators on $\overline{D(A)}$.*

5 Projectors on the eigenspaces

Since $\omega_{0,ess}((A + L)_0) \leq -\eta$, we obtain that $\sigma(A + L) \cap \Omega$ is nonempty and finite and each $\lambda_0 \in \sigma(A + L) \cap \Omega$ is a pole of $(\lambda I - (A + L))^{-1}$ of finite order $k_0 \geq 1$. This means that λ_0 is isolated in $\sigma(A + L)$ and the Laurent's expansion of the resolvent around λ_0 takes the following form

$$(\lambda I - (A + L))^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda - \lambda_0)^n B_n^{\lambda_0}. \quad (5.1)$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $(A + L)$ associated to λ_0 . The goal of this section is to provide a method to compute $B_{-1}^{\lambda_0}$.

We remark that

$$(\lambda - \lambda_0)^{k_0} (\lambda I - (A + L))^{-1} = \sum_{m=0}^{+\infty} (\lambda - \lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0} = \lim_{\lambda \rightarrow \lambda_0} \frac{1}{(k_0 - 1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left((\lambda - \lambda_0)^{k_0} (\lambda I - (A + L))^{-1} \right). \quad (5.2)$$

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results. The proof of the lemma is similar to the proof of Lemma 4.1 in Liu Magal and Ruan [30].

Lemma 5.1 *For each $\lambda \in \rho(A + L)$, we have the following explicit formula for the resolvent of $A + L$*

$$\begin{aligned} (\lambda I - (A + L))^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} &= \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\ \Leftrightarrow \psi(\theta) &= \int_{\theta}^0 e^{\lambda(\theta-s)} \varphi(s) ds + e^{\lambda\theta} \Delta(\lambda)^{-1} \left[\widehat{L} \left(\int_{\theta}^0 e^{\lambda(\cdot-s)} \varphi(s) ds \right) \right]. \end{aligned} \quad (5.3)$$

Furthermore, we have that

$$\begin{aligned} \sigma(A + L) \cap \Omega &= \sigma((A + L)_0) \cap \Omega \\ &= \sigma_P((A + L)_0) \cap \Omega \\ &= \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\}. \end{aligned}$$

Now we introduce the following linear operators $F_1 : X_0 \rightarrow \mathbb{R}^n$ and $F_2 : \mathbb{R}^n \rightarrow X_0$ defined by

$$F_1 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \varphi(0), \quad F_2 \alpha = \begin{pmatrix} \alpha_{\mathbb{R}^n} \\ 0 \end{pmatrix}.$$

From Lemma 5.1, we have

$$\begin{aligned} F_1 (\lambda I - (A + L))^{-1} F_2 \alpha &= \Delta(\lambda)^{-1} \alpha, \\ \forall \lambda &\in \{\lambda \in \rho((A + L)) : \operatorname{Re}(\lambda) \geq \omega_{0,ess}((A + L)_0)\}, \\ \forall \alpha &\in \mathbb{R}^n. \end{aligned}$$

Since $\lambda \rightarrow (\lambda I - (A + L))^{-1}$ is holomorphic from Ω into $\mathcal{L}(X)$, we deduce from the above formula that the map $\lambda \rightarrow \Delta(\lambda)^{-1}$ is holomorphic in Ω . We

know that $\Delta(\cdot)^{-1}$ has only finite order poles with order $\widehat{k}_0 \geq 1$. Therefore, $\Delta(\lambda)^{-1}$ has the Laurent's expansion around λ_0 and takes the following form

$$\Delta(\lambda)^{-1} = \sum_{n=-\widehat{k}_0}^{+\infty} (\lambda - \lambda_0)^n \Delta_n, \quad \Delta_n \in \mathcal{L}(\mathbb{R}^n).$$

From the following lemma we know that $\widehat{k}_0 = k_0$.

Lemma 5.2 *Let $\lambda_0 \in \sigma(A + L) \cap \Omega$. Then the following are equivalent*

- (i) λ_0 is a pole of order k_0 of $(\lambda I - (A + L))^{-1}$;
- (ii) λ_0 is a pole of order k_0 of $\Delta(\lambda)^{-1}$;
- (iii) $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{k_0} \Delta(\lambda)^{-1} \neq 0$, and $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{k_0+1} \Delta(\lambda)^{-1} = 0$.

Proof. The proof follows trivially from the explicit formula of the resolvent of $A + L$ obtained in Lemma 5.1. ■

Lemma 5.3 *The matrices $\Delta_{-1}, \dots, \Delta_{-k_0}$ must satisfy*

$$\Delta_{k_0}(\lambda_0) \begin{pmatrix} \Delta_{-1} \\ \Delta_{-2} \\ \vdots \\ \Delta_{-k_0+1} \\ \Delta_{-k_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\left(\Delta_{-k_0} \quad \Delta_{-k_0+1} \quad \cdots \quad \Delta_{-2} \quad \Delta_{-1} \right) \Delta_{k_0}(\lambda_0) = \left(0 \quad \cdots \quad 0 \right),$$

where

$$\Delta_{k_0}(\lambda_0) := \begin{pmatrix} \Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)! \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \Delta^{(2)}(\lambda_0)/2! \\ \vdots & & \ddots & \ddots & \Delta^{(1)}(\lambda_0) \\ 0 & \cdots & \cdots & 0 & \Delta(\lambda_0) \end{pmatrix}.$$

From the above results we can obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0 , which is given in the following proposition.

Proposition 5.4 *Each $\lambda_0 \in \sigma((A + L))$ with $\text{Re}(\lambda_0) \geq \omega_{0,ess}((A + L)_0)$ is a pole of $(\lambda I - (A + L))^{-1}$ of order $k_0 \geq 1$. Moreover k_0 is the only integer such that there exists $\Delta_{-k_0} \in M_n(\mathbb{R})$ with $\Delta_{-k_0} \neq 0$, such that*

$$\Delta_{-k_0} = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^{k_0} \Delta(\lambda)^{-1}.$$

Furthermore the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace of $(A + L)$ associated λ_0 is defined by the following formula

$$B_{-1}^{\lambda_0} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{bmatrix} \mathbf{0}_{\mathbb{R}^n} \\ \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^2(\lambda_0) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \end{bmatrix}, \quad (5.4)$$

where

$$\Delta_{-j} = \lim_{\lambda \rightarrow \lambda_0} \frac{1}{(k_0 - j)!} \frac{d^{k_0-j}}{d\lambda^{k_0-j}} \left((\lambda - \lambda_0)^{k_0} \Delta(\lambda)^{-1} \right), j = 1, \dots, k_0,$$

$$L_0^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = e^{\lambda\theta} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda(\cdot-s)} \varphi(s) ds \right) \right],$$

and

$$\begin{aligned} L_j^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} &= \frac{d^j}{d\lambda^j} \left[L_0^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right] \\ &= \sum_{k=0}^j C_j^k \theta^k e^{\lambda\theta} \frac{d^{j-k}}{d\lambda^{j-k}} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda(\cdot-s)} \varphi(s) ds \right) \right], j \geq 1, \end{aligned}$$

here

$$\begin{aligned} &\frac{d^i}{d\lambda^i} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda(\cdot-s)} \varphi(s) ds \right) \right] \\ &= \widehat{L} \left(\int_{\cdot}^0 (\cdot - s)^i e^{\lambda(\cdot-s)} \varphi(s) ds \right), i \geq 1. \end{aligned}$$

6 Projector for a simple eigenvalue

For Hopf bifurcation it is useful to get the projector for a simple eigenvalue. In this section we study this case, i.e. λ_0 is a simple eigenvalue of $(A + L)$. That is to say that λ_0 is pole of order 1 of the resolvent of $(A + L)$, and the dimension of the eigenspace of $(A + L)$ associated to the eigenvalue λ_0 is 1.

We know that λ_0 is a pole of order 1 of the resolvent of $(A + L)$ if and only if there exists $\Delta_{-1} \neq 0$, such that

$$\Delta_{-1} = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0) \Delta(\lambda)^{-1}.$$

From Lemma 5.3, we have $\Delta_{-1} \Delta(\lambda_0) = \Delta(\lambda_0) \Delta_{-1} = 0$. Hence

$$\Delta_{-1} \left[B + \widehat{L}(e^{\lambda_0 \cdot} I) \right] = \left[B + \widehat{L}(e^{\lambda_0 \cdot} I) \right] \Delta_{-1} = \lambda_0 \Delta_{-1}.$$

Therefore, if $\dim [N(\Delta(\lambda_0))] = 1$, the rank of Δ_{-1} is 1 and the dimension of the eigenspace of $(A+L)$ associated to λ_0 is 1. Conversely, if $\dim [N(\Delta(\lambda_0))] > 1$, it is readily checked that the eigenspace of $(A + L)$ associated to λ_0 is

$$\left\{ \begin{pmatrix} 0 \\ e^{\lambda_0 \theta} x \end{pmatrix} : x \in N(\Delta(\lambda_0)) \right\}.$$

and λ_0 is not simple.

In that case, there exist $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$, such that

$$W_{\lambda_0}^T \Delta(\lambda_0) = 0, \text{ and } \Delta(\lambda_0) V_{\lambda_0} = 0. \quad (6.1)$$

Hence

$$\Delta_{-1} = V_{\lambda_0} W_{\lambda_0}^T. \quad (6.2)$$

Moreover since $B_{-1}^{\lambda_0}$ is a projector, we should have $B_{-1}^{\lambda_0} B_{-1}^{\lambda_0} = B_{-1}^{\lambda_0}$, i.e.

$$\begin{aligned} & e^{\lambda_0 \theta} \Delta_{-1} \left[\begin{array}{c} \Delta_{-1} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_0^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right] \\ + \widehat{L} \left(\int_0^0 e^{\lambda_0 \cdot} \Delta_{-1} \left[\begin{array}{c} \alpha + \varphi(0) \\ + \widehat{L} \left(\int_0^0 e^{\lambda_0(\cdot-l)} \varphi(l) dl \right) \end{array} \right] ds \right) \end{array} \right] \\ &= e^{\lambda_0 \theta} \Delta_{-1} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_0^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right]. \end{aligned} \quad (6.3)$$

Taking $\varphi = 0$ in (6.3), we obtain

$$\Delta_{-1} = \Delta_{-1} \left[I + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0 \cdot} ds \right) \right] \Delta_{-1}. \quad (6.4)$$

Taking $\alpha = 0$ in (6.3), we obtain

$$\begin{aligned} & \Delta_{-1} \left[\varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right] \\ = & \Delta_{-1} \Delta_{-1} \left[\varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right] \\ & + \Delta_{-1} \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0 \cdot} \Delta_{-1} \left[\varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right] ds \right) \\ = & \Delta_{-1} \left[I + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0 \cdot} ds \right) \right] \Delta_{-1} \left[\varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right]. \end{aligned}$$

Therefore, we obtain the following corollary.

Corollary 6.1 $\lambda_0 \in \sigma((A+L))$ is a simple eigenvalue of $(A+L)$ if and only if

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^2 \Delta(\lambda)^{-1} = 0$$

and

$$\dim [N(\Delta(\lambda_0))] = 1.$$

Moreover the projector on the eigenspace associated to λ_0 is

$$B_{-1}^{\lambda_0} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{bmatrix} 0_{\mathbb{R}^n} \\ e^{\lambda_0 \theta} \Delta_{-1} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0(\cdot-s)} \varphi(s) ds \right) \right] \end{bmatrix}, \quad (6.5)$$

where

$$\Delta_{-1} = V_{\lambda_0} W_{\lambda_0}^T$$

with $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$ are two vectors satisfying (6.1) and

$$\Delta_{-1} = \Delta_{-1} \left[I + \widehat{L} \left(\int_{\cdot}^0 e^{\lambda_0 \cdot} ds \right) \right] \Delta_{-1}.$$

7 Nonlinear semiflow

Remembering that $F := L + G$ one may rewrite the abstract Cauchy problem (1.8) as follows

$$\frac{dU(t)x}{dt} = AU(t)x + F(U(t)x), \quad t \geq 0, \quad U(0)x = x := \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_0. \quad (7.1)$$

We shall investigate the properties of the semiflow generated by the mild solution of (7.1). Namely the continuous function $U(\cdot)x : [0, \tau] \rightarrow X_0$ satisfies the fixed point problem

$$U(t)x = T_{A_0}(t)x + (S_A \diamond F(\cdot, U(\cdot, 0)x))(t), \quad t \geq 0, \quad (7.2)$$

or equivalently

$$U(t)x = x + A \int_0^t U(l)x dl + \int_0^t F(U(l)x) dl, \quad t \geq 0. \quad (7.3)$$

By using Lemma 3.1 with $h(t) = f(x_t)$ we obtain the following lemma.

Lemma 7.1 *$v \in C([0, \tau], X_0)$ is mild solution of the abstract Cauchy problem (7.1) is and only if*

$$v(t) = \begin{pmatrix} 0 \\ u(t, \cdot) \end{pmatrix}$$

with

$$u(t, \theta) = x(t + \theta), \quad \forall t \geq 0, \forall \theta \leq 0,$$

and

$$x(t) = \begin{cases} \varphi(0) + \int_0^t f(x_s) ds, & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \leq 0. \end{cases}$$

Lemma 7.1 is important since it shows that it is equivalent to consider the mild solutions of the abstract Cauchy problem (7.1) or the solutions of the functional differential equation (1.1). Therefore we can apply a certain number of results obtained for abstract Cauchy problems (see Thieme [46], Magal [33], Magal and Ruan [34, 35, 36, 37]).

Definition 7.2 *Consider two maps $\chi : X_0 \rightarrow (0, +\infty]$ and $U : D_\chi \rightarrow X_0$, where*

$$D_\chi = \{(t, x) \in [0, +\infty) \times X_0 : 0 \leq t < \chi(x)\}.$$

We will say that U is a maximal (autonomous) semiflow on X_0 if U satisfies the following properties:

- (i) $\chi(U(t)x) + t = \chi(x), \forall x \in X_0, \forall t \in [0, \chi(x)]$.
- (ii) $U(0)x = x, \forall x \in X_0$.
- (iii) $U(t-s)U(s)x = U(t)x, \forall x \in X_0, \forall t, s \in [0, \chi(x)]$ with $t \geq s$.
- (iv) If $\chi(x) < +\infty$, then

$$\lim_{t \rightarrow \chi(x)^-} \|U(t)x\| = +\infty.$$

Set

$$D = [0, +\infty) \times X_0.$$

In order to present a theorem on the existence and uniqueness of solutions to equation (7.1), we make the following definition.

Definition 7.3 We will say that $f : BUC_\eta \rightarrow \mathbb{R}^n$ is Lipschitz on bounded sets, if for each $\xi > 0$ there exists a constant $\kappa(\xi)$ satisfying

$$\|f(\varphi) - f(\psi)\| \leq \kappa(\xi) \|\varphi - \psi\|$$

whenever $\varphi, \psi \in BUC_\eta$ with $\|\varphi\| \leq \xi$ and $\|\psi\| \leq \xi$.

It is clear that if f is Lipschitz on bounded sets so is F . Therefore we can apply Theorem 5.2 in Magal and Ruan [34].

Theorem 7.4 Assume that $f : BUC_\eta \rightarrow \mathbb{R}^n$ is Lipschitz on bounded sets. Then there exist a map $\chi : X_0 \rightarrow (0, +\infty]$ and a maximal non-autonomous semiflow $U : D_\chi \rightarrow X_0$, such that for each $x \in X_0, U(\cdot)x \in C([s, s + \chi(s, x)], X_0)$ is a unique maximal solution of (7.2). Moreover, the subset D_χ is open in D and the map $(t, x) \rightarrow U(t)x$ is continuous from D_χ into X_0 . Furthermore

$$U(t) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{U}(t)(\varphi) \end{pmatrix} \quad (7.4)$$

where

$$\widehat{U}(t)(\varphi)(\theta) = x_\varphi(t + \theta) \quad (7.5)$$

and $x_\varphi : (-\infty, \tau_\varphi) \rightarrow \mathbb{R}^n$ (with $\tau_\varphi := \chi \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \leq +\infty$) is the unique continuous function satisfying the integral equation

$$x_\varphi(t) = \varphi(0) + \int_0^t f(x_{\varphi,s}) ds, \forall t \in [0, \chi \begin{pmatrix} 0 \\ \varphi \end{pmatrix}).$$

Moreover if $\tau(\varphi) := \chi \begin{pmatrix} 0 \\ \varphi \end{pmatrix} < +\infty$

$$\lim_{t \nearrow \tau(\varphi)^-} |x_\varphi(t)| = +\infty.$$

Remark 7.5 By using the identification $\varphi \rightarrow \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$, we deduce that the maps τ and \widehat{U} also define a maximal semiflow on BUC_η .

By using the results about (ACP) we can derive some extra result about the global existence of solutions, the positiveness of solution (see Martin and Smith [39]). The following result is based on Proposition 3.5 in Magal and Ruan [36] (see also Example 3.6 in [36]).

Proposition 7.6 Assume that $f : BUC_\eta \rightarrow \mathbb{R}^n$ is Lipschitz on bounded sets. Assume that for each constant $M > 0$ we can find $\lambda > 0$ such that

$$\lambda\varphi(0) + f(\varphi) \geq 0$$

whenever $\varphi \geq 0$ and $\|\varphi\| \leq M$. Then for each $\varphi \geq 0$ we have

$$x_\varphi(t) \geq 0, \forall t \in [0, \tau(\varphi)). \quad (7.6)$$

Next we focus on the property of asymptotic smoothness of the semiflow since some extra analysis is needed here. We now turn to the relative compactness of the positive orbits. This type of properties are needed in particular to talk about omega-limit sets of bounded positive orbit, or about global attractors (see Hale [21], Sell and You [45]).

Recall that Kuratovsky's measure of non-compactness is defined by

$$\kappa(B) = \inf \{ \varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon \}$$

whenever B is a bounded subset of X .

For various properties of Kuratowski's measure of noncompactness, we refer to Deimling [75], Martin [187], and Sell and You [233, Lemma 22.2]. Mention that the every bounded orbit is relatively compact (and precompact).

Lemma 7.7 Let $(X, \|\cdot\|)$ be a Banach space and $\kappa(\cdot)$ the measure of non-compactness defined as above. Then for any bounded subset B and \widehat{B} of X , we have the following properties:

- (i) $\kappa(B) = 0$ if and only if \overline{B} is compact;
- (ii) $\kappa(B) = \kappa(\overline{B})$;
- (iii) If $B \subset \widehat{B}$ then $\kappa(B) \leq \kappa(\widehat{B})$;
- (iv) $\kappa(B + \widehat{B}) \leq \kappa(B) + \kappa(\widehat{B})$, where $B + \widehat{B} = \{x + y : x \in B, y \in \widehat{B}\}$.

Moreover for each bounded linear operators $T \in \mathcal{L}(X)$ we have

$$\|T\|_{\text{ess}} := \kappa(T(B_X(0, 1))) \leq \|T\|_{\mathcal{L}(X)}. \quad (7.7)$$

Definition 7.8 We will say that a bounded subset $B \subset X_0$ is positively invariant by U if

$$U(t, x) \in B, \forall t \geq 0, \forall x \in B.$$

We say that U is asymptotically smooth if every positively invariant bounded set is attracted by a compact subset

Proposition 7.9 The semiflow U (or \widehat{U}) is asymptotically smooth.

Proof. Assume that $B \subset X_0$ is positive by U . The semiflow U can be rewritten as follows

$$U(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ T(t)(\varphi) + S(t)(\varphi) \end{pmatrix},$$

where

$$T(t)(\varphi)(\theta) = \begin{cases} \varphi(0) + \int_0^{t+\theta} f(x_{\varphi,s}) ds, & \text{if } t + \theta \geq 0, \\ \varphi(0), & \text{if } t + \theta \leq 0. \end{cases}$$

and

$$S(t)(\varphi)(\theta) = \begin{cases} 0, & \text{if } t + \theta \geq 0, \\ \varphi(t + \theta) - \varphi(0), & \text{if } t + \theta \leq 0. \end{cases}$$

By using Arzela-Ascoli theorem, we deduce that $T(t)B$ is a relatively compact subset. Therefore by Lemma 7.7 (d) and (a) we obtain

$$\kappa(U(t)B) \leq \kappa(T(t)B) + \kappa(S(t)B) = \kappa(S(t)B).$$

Assume that B is contained into a ball of radius $r > 0$. By using Lemma 7.7 (c) we deduce that

$$\kappa(S(t)B) \leq \kappa(S(t)B(0, r)) \leq r \|S(t)\|_{\mathcal{L}(X)} \leq 2re^{-\eta t}.$$

Therefore

$$\lim_{t \rightarrow +\infty} \kappa(U(t)B) = 0,$$

and the result follows by Lemma 2.1-(a) in Magal and Zhao [38]. ■

8 The local stability of equilibria

Assume that

$$f(0_{BUC_\eta}) = 0_{\mathbb{R}^n}.$$

Then

$$\bar{x}(t) = 0_{\mathbb{R}^n}, \forall t \in \mathbb{R}$$

is an equilibrium solution of the FDE (1.1). Similarly, 0_X is an equilibrium of the ACP (7.1). Assume that f is continuously differentiable locally around 0. Then the linearized equation of the FDE (1.1) around 0 is defined by (1.3). The linearized equation of the ACP (7.1) around 0 is defined by

$$\begin{cases} \frac{dU(t)x}{dt} = AU(t)x + L(U(t)x), & \text{for } t \geq 0, \\ U(0)x = x := \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_0, \end{cases} \quad (8.1)$$

where $L = DF(0)$.

By combining Lemma 4.2, as a consequence of Proposition 7.1 in Magal and Ruan [36] (see also Thieme [46]) we can obtain the following stability theorem.

Theorem 8.1 *Assume that $f(0_{BUC_\eta}) = 0_{\mathbb{R}^n}$ and assume that f is continuously differentiable locally around 0_{BUC_η} .*

The equilibrium 0_X of the abstract Cauchy problem (7.1) is asymptotically stable if for each $\lambda \in \Omega$

$$\det(\Delta(\lambda)) = 0 \Rightarrow \operatorname{Re}(\lambda) < 0. \quad (8.2)$$

More precisely, if the above condition is satisfied, we can find three constants $M \geq 1$, $\delta > 0$ and $\varepsilon > 0$ such that

$$\|U(t)x\| \leq Me^{-\delta t} \|x\|, \forall t \geq 0, \quad (8.3)$$

whenever for each $x \in X_0$ with $\|x\| \leq \varepsilon$.

9 Hopf bifurcation

In this section we give a few comments and remarks concerning the results obtained in this paper. In order to apply the center manifold theorem to study Hopf bifurcation results for infinite delay differential equations with parameter

$$\begin{cases} \frac{dx(t)}{dt} = f(\mu, x_t), \forall t \geq 0, \\ x(\theta) = \varphi(\theta), \forall \theta \leq 0 \text{ with } \varphi \in BUC_\eta, \end{cases} \quad (9.1)$$

where $\mu \in \mathbb{R}$, and $f : \mathbb{R} \times BUC_\eta \rightarrow \mathbb{R}^n$ is a C^k map with $k \geq 4$. We assume that $f(\mu, 0) = 0, \forall \mu \in \mathbb{R}$. As before, by setting $v(t) = \begin{pmatrix} 0 \\ x_t \end{pmatrix}$ we can rewrite the delay differential equation (9.1) as the following abstract non-densely defined Cauchy problem on the Banach space $X = \mathbb{R}^n \times BUC_\eta$

$$\frac{dv(t)}{dt} = Av(t) + F(\mu, v(t)), \quad t \geq 0, \quad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}, \quad (9.2)$$

where $A : D(A) \subset X \rightarrow X$ is defined in (1.7) and $F : \mathbb{R} \times \overline{D(A)} \rightarrow X$ by

$$F\left(\mu, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f(\mu, \varphi) \\ 0_{BUC_\eta} \end{pmatrix}.$$

Set

$$L\left(\mu, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix}\right) = \partial_x F(\mu, 0) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_\varphi f(\mu, 0)\psi \\ 0_{BUC_\eta} \end{pmatrix} =: \begin{pmatrix} \widehat{L}(\mu, \psi) \\ 0_{BUC_\eta} \end{pmatrix}.$$

System (9.1) becomes

$$\frac{dv(t)}{dt} = Av(t) + L(\mu, v(t)) + G(\mu, v(t)), \quad t \geq 0, \quad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)} \quad (9.3)$$

with

$$G(\mu, v(t)) = F(\mu, v(t)) - L(\mu, v(t)).$$

By section 4, we know that the linear operator $A + L(\mu, \cdot) : D(A) \rightarrow X$ is a Hille-Yosida operator and $\omega_{0,ess}((A + L(\mu, \cdot))_0) \leq -\eta$. Let

$$\Omega := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\eta\}.$$

The point spectrum of $(A + L(\mu, \cdot))_0$ is the set

$$\sigma(A + L(\mu, \cdot)) \cap \Omega = \sigma_P((A + L(\mu, \cdot))_0) \cap \Omega = \{\lambda \in \mathbb{C} : \det(\Delta(\mu, \lambda)) = 0\},$$

where

$$\Delta(\mu, \lambda) := \lambda I - \widehat{L}(\mu, e^{\lambda \cdot} I).$$

Hence, $A + L(\mu, \cdot)$ satisfies Assumptions 1.1, 1.2 and 1.3(c) in [31]. In order to apply the Hopf bifurcation theorem obtained in [31], we need to make the following assumption.

Assumption 9.1 *Let $\varepsilon > 0$ and $f \in C^k((-\varepsilon, \varepsilon) \times B_{BUC_\eta}(0, \varepsilon); \mathbb{R}^n)$ for some $k \geq 4$. Assume that $\det(\Delta(0, \lambda)) = 0$ has a simple purely imaginary root $\lambda_0 = i\omega \neq 0$ and*

$$\{\lambda \in \mathbb{C} : \det(\Delta(0, \lambda)) = 0\} \cap i\mathbb{R} = \{i\omega, -i\omega\}. \quad (9.4)$$

Moreover, assume that $\frac{d\operatorname{Re}(\lambda(0))}{d\mu} \neq 0$, where $\lambda(\mu)$ is the branch of eigenvalues of $\det \Delta(\mu, \lambda) = 0$ through $i\omega$ at $\mu = 0$.

By combining the results presented in the previous sections and by using the same argument as in [31] one may extend the Hopf bifurcation theorem from finite to infinite delay differential equations.

Theorem 9.2 *Let Assumption 9.1 be satisfied. Then there exist $\varepsilon^* > 0$ and three C^{k-1} maps, $\varepsilon \rightarrow \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into \mathbb{R} , $\varepsilon \rightarrow \varphi_\varepsilon$ from $(0, \varepsilon^*)$ into BUC_η , and $\varepsilon \rightarrow \gamma(\varepsilon)$ from $(0, \varepsilon^*)$ into \mathbb{R} , such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $\gamma(\varepsilon)$ -periodic function $x_\varepsilon \in C^k(\mathbb{R}, \mathbb{R}^n)$, which is a solution of (9.1) for the parameter value equals $\mu(\varepsilon)$ and the initial value φ equals φ_ε . Moreover, we have the following properties*

- (i) *There exist a neighborhood N of 0 in \mathbb{R}^n and an open interval I in \mathbb{R} containing 0 such that for $\widehat{\mu} \in I$ and any periodic solution $\widehat{x}(t)$ in N with minimal period $\widehat{\gamma}$ close to $\frac{2\pi}{\omega}$ of (9.1) for the parameter value $\widehat{\mu}$, there exists $\varepsilon \in (0, \varepsilon^*)$ such that $\widehat{x}(t) = x_\varepsilon(t + \theta)$ (for some $\theta \in [0, \gamma(\varepsilon))$), $\mu(\varepsilon) = \widehat{\mu}$, and $\gamma(\varepsilon) = \widehat{\gamma}$.*
- (ii) *The map $\varepsilon \rightarrow \mu(\varepsilon)$ is a C^{k-1} function and*

$$\mu(\varepsilon) = \sum_{n=1}^{\lfloor \frac{k-2}{2} \rfloor} \mu_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where $\lfloor \frac{k-2}{2} \rfloor$ is the integer part of $\frac{k-2}{2}$.

(iii) The period $\gamma(\varepsilon)$ of $t \rightarrow u_\varepsilon(t)$ is a C^{k-1} function and

$$\gamma(\varepsilon) = \frac{2\pi}{\omega} \left[1 + \sum_{n=1}^{\lfloor \frac{k-2}{2} \rfloor} \gamma_{2n} \varepsilon^{2n} \right] + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where ω is the imaginary part of $\lambda(0)$ defined in Assumption 9.1.

Actually such a Hopf bifurcation result is based on the fact that such a system has a local center manifold. The existence of a local center manifold is a direct consequence of the center manifold theorem obtained in [35]. To conclude we would like to mention that it is also possible to apply the normal form theory presented in [32] to infinite delay differential equations.

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