Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

HOPF BIFURCATION FOR A SPATIALLY AND AGE STRUCTURED POPULATION DYNAMICS MODEL

Zhihua Liu and Hui Tang

School of Mathematical Sciences, Beijing Normal University
Beijing 100875, People's Republic of China

Pierre Magal

University of Bordeaux, IMB, UMR CNRS 5251,
33076 Bordeaux, France.

Abstract. This paper is devoted to the study of a spatially and age structured population dynamics model. We study the stability and Hopf bifurcation of the positive equilibrium of the model by using a bifurcation theory in the context of integrated semigroups. This problem is a first example for Hopf bifurcation for a spatially and age/size structured population dynamics model. Bifurcation analysis indicates that Hopf bifurcation occurs at a positive age/size dependent steady state of the model. The results are confirmed by some numerical simulations.

1. **Introduction.** In this article we consider a mathematical population dynamics model to describe the growth of trees. The main characteristic taken into account for the growth of such population are the size of trees and the spatial location. Since the small trees are growing in the shade of the bigger trees, we should take into account the competition for light between big and small trees. Therefore we introduce a competition for light between the small trees and the big trees. The model considered in this article is the following

$$\begin{cases} \dfrac{\partial u(t,s,x)}{\partial t}+\dfrac{\partial u(t,s,x)}{\partial s}=-\mu u(t,s,x) \text{ for } s\geq 0 \text{ and } x\in(0,1), \\ \left(I-d^2\Delta_x\right)u(t,0,x)=\alpha h\left(\int_0^{+\infty}\gamma(s)u(t,s,x)ds\right) \text{ for } x\in(0,1), \\ u(0,.,.)=u_0\in L^1\left((0,+\infty),L^1_+(0,1)\right), \end{cases} \tag{1}$$

where $\mu>0$ is the mortality rate, $\alpha>0$ is the birth rate in absence of birth limitations and the birth limitations (competition for food, space or light) are described by

$$h(x)=x\exp(-\beta x),$$

where $\beta>0$. This function is known as Ricker's [24, 25] type birth limitation. This type of birth function has been commonly used in the literature. One may observe that the Ricker's type birth limitation was introduced for fishes population in order to describe the cannibalism of adult fishes on lava during the reproduction season. Here we refer to Ducrot, Magal and Seydi [17] for a mathematical justification of the Ricker function by using a singular perturbation idea. For tree populations the

2010 *Mathematics Subject Classification.* 37L10, 37G15, 35B32, 35K55, 92D25.

Key words and phrases. Hopf bifurcation; spatially and age/size structure; population dynamics; integrated semigroups.

process is similar since the large tree takes most of the light and the young tree can not (or can almost not) grow in the shade of the adult trees.

Here the density of population $u(t,s,x)$ depends on t the time, s the size of individuals (which serves as a clock for the reproduction of trees), and x the spatial location of tree. The density of population here means that

$$\int_{s_1}^{s_2}\int_{x_1}^{x_2} u(t,s,x)dxds$$

is the number of trees with a size $s\in[s_1,s_2]$ and located in the region $[x_1,x_2]$ at time t.

The spatial displacement takes place here only at birth when the seeds are spreading around the mother tree. More precisely, the Laplacian operator $\Delta_x : W^{2,1}(0,1)\to L^1(0,1)$ is the diffusion operator with Neumann boundary conditions. Therefore we assume a no flux condition at the boundary of the domain $[0,1]$. The bounded linear operator $\left(I-d^2\Delta_x\right)^{-1} : L^1(0,1)\to L^1(0,1)$ is also defined by

$$\left(I-d^2\Delta_x\right)^{-1}\psi=\varphi \Leftrightarrow \begin{cases} \varphi-d^2\varphi''=\psi \\ \text{with} \\ \varphi'(0)=\varphi'(1)=0. \end{cases}$$

The map $\gamma\in L^\infty(0,+\infty)$ is defined by

$$\gamma(\theta)=(\theta-\tau)^n e^{-\varsigma(\theta-\tau)}1_{[\tau,+\infty)}(\theta)=\begin{cases} (\theta-\tau)^n e^{-\varsigma(\theta-\tau)}, \text{ if } a\geq\tau \\ 0, \text{ otherwise,} \end{cases}$$

where $\tau\geq 0$, $\varsigma\geq 0, n\in\mathbb{N}$. By using a convenient rescaling of α and β (defined above), we can always assume that

$$\int_0^{+\infty}\gamma(s)e^{-\mu s}ds=1.$$

In practice, the function $\gamma(s)$ should be understood as a probability to reproduce at size s.

In this article we explore the oscillating properties of the solutions around the positive equilibrium. One observe that similar model has been studied by Ducrot [10]. In [10], the spatial domain is the real line and the existence of oscillating traveling waves has been proved around the positive equilibrium. Here the domain is bounded and we will analyze the oscillation by using a Hopf bifurcation theorem for structured population dynamics models.

In the special case

$$\gamma(\theta)=1_{[\tau,+\infty)}(\theta),$$

by setting

$$U(t,x):=\int_\tau^{+\infty}u(t,s,x)ds,$$

we obtain

$$\partial_t U(t,.)=e^{-\mu\tau}\left(I-d^2\Delta_x\right)^{-1}\left(\alpha h\left(U(t-\tau,.)\right)\right)-\mu U(t,.), \text{ for } t\geq\tau. \tag{2}$$

Therefore the system (1) can also be regarded as a delay differential equation in some special cases. We refer to Wu [32] for more results on this subject.

Recently based on the center manifold theorem proved in Magal and Ruan [19], a Hopf bifurcation theorem has been presented for abstract non densely defined Cauchy problem in Liu Magal and Ruan [16]. These theorems have been successfully applied to study the existence of Hopf bifurcation for some age/size-structured models, see [2, 4, 5, 6, 19, 21, 26, 31]. More results can be found in the context of cell population dynamics in Doumic et al. [12], and in the context of structured neuron population in Pakdaman et al. [22]. We refer to Cushing [8, 9], Prüss [23], Swart [27], Kostova and Li [15], Bertoni [1] for more results on this subject. Early examples of periodic solutions suspected to appear by Hopf bifurcations in age/size-structured models are mentioned in the literature (Castillo-Chavez et al. [3], Inaba [13, 14], Zhang et al. [33]). In this article we consider a first example for Hopf bifurcation appearing in a spatially and age/size structured population dynamics model. As we will see, due to the spatial structure the bifurcation analysis is more complex than for a model without spatial structure.

The plan of the paper is the following. In section 2 we formulate the model (1) as a non-densely defined Cauchy problem and recall the Hopf bifurcation theorem for the abstract non-densely defined Cauchy problem obtained in [16]. In section 3 we investigate the existence and the uniqueness of the positive equilibrium and consider the linearized system around this positive equilibrium. In section 4, we derive a family of characteristic equations and the main result of this paper, that is, the existence of Hopf bifurcation is obtained by analyzing the spectrum property of the non-densely defined linear operator. In section 5 we present some numerical simulations of the model.

2. **Preliminaries.**

Consider the Banach space

$$X = L^1\left((0,1),\mathbb{R}\right) \times L^1\left((0,+\infty), L^1\left((0,1),\mathbb{R}\right)\right),$$

or equivalently

$$X = Y \times L^1\left((0,+\infty), Y\right)$$

with

$$Y := L^1\left((0,1),\mathbb{R}\right).$$

The space X is endowed with the product norm

$$\|x\| = \|\alpha\|_Y + \|\varphi\|_{L^1((0,+\infty),Y)}, \ \forall x = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in X.$$

We consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) = \{0_Y\} \times W^{1,1}\left((0,+\infty), Y\right).$$

Set

$$X_0 := \overline{D(A)}.$$

Define $H : X_0 \to X$ by

$$H\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} H_1\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \\ 0_{L^1((0,+\infty),Y)} \end{pmatrix},$$

where

$$H_1\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \left(I - d^2\Delta_x\right)^{-1}\left(\alpha h\left(\int_0^{+\infty} \gamma(\theta)\varphi(\theta,.)d\theta\right)\right)(x).$$

Then by identifying $u(t)$ to $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, the problem (1) can be considered as the following Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H(v(t)) \text{ for } t \geq 0,\ v(0) = x = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in X_0. \tag{3}$$

The resolvent of A is defined for each $\lambda > -\mu$ by

$$\begin{aligned} (\lambda I - A)^{-1} \begin{pmatrix} \chi \\ \psi \end{pmatrix} &= \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \Leftrightarrow \\ \varphi(a,x) &= e^{-(\lambda+\mu)a}\chi(x) + \int_0^a e^{-(\lambda+\mu)(a-\sigma)}\psi(\sigma,x)\, d\sigma. \end{aligned}$$

Therefore

$$(-\mu, +\infty) \subset \rho(A) \text{ (the resolvent set of } A)$$

and

$$\left\| (\lambda I - A)^{-n} \right\| \leq \frac{1}{(\lambda+\mu)^n}, \forall \lambda > -\mu, \forall n \geq 1$$

or equivalently A is a Hille-Yosida operator.

Consider $A_0 : D(A_0) \subset X_0 \to X_0$ the part of A in X_0, that is the linear operator on X_0 defined as follows

$$A_0 x = Ax,\ \forall x \in D(A_0) = \{x \in D(A) \cap X_0 : Ax \in X_0\}.$$

Since A is a Hille-Yosida operator, it follows that A_0 generates a strongly continuous semigroup of bounded linear operators $\{T_{A_0}(t)\}_{t\geq 0}$ on X_0. This semigroup is defined by

$$T_{A_0}(t)\begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \widehat{T}_{A_0}(t)(\varphi) \end{pmatrix}$$

where

$$\widehat{T}_{A_0}(t)\varphi = \begin{cases} e^{-\mu t}\varphi(s-t,x), & s > t \\ 0_{L^1((0,+\infty),Y)}, & s \leq t. \end{cases} \tag{4}$$

The global existence, uniqueness and positive of solution of the equation (3) follow from the results of Thieme [28], Magal [18] and Magal and Ruan [20]. Consider the positive cone

$$X_+ = Y_+ \times L^1_+((0,+\infty), Y)$$

where

$$Y_+ := L^1_+((0,1), \mathbb{R}),$$

and

$$L^1_+((0,+\infty), Y) := \left\{ \varphi \in L^1((0,+\infty), Y) : \varphi(a) \in Y_+ \ a.e. \text{ in } (0,+\infty) \right\}.$$

Define

$$X_{0+} = \{0_Y\} \times L^1_+((0,+\infty), Y).$$

Lemma 2.1. *There exists a unique continuous semiflow $\{V(t)\}_{t\geq 0}$ on X_{0+}, such that for each $x \in X_{0+}$ the map $t \to V(t)x$ is the unique mild solution of (3), that is to say that*

$$\int_0^t V(s)x ds \in D(A), \forall t \geq 0,$$

and

$$V(t)x = x + A\int_0^t V(s)x ds + \int_0^t H(V(s)x)\, ds, \forall t \geq 0.$$

In the following, we recall the Hopf bifurcation theorem obtained in [16] for the following abstract Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(\mu, u(t)), \forall t \geq 0, \ u(0) = x \in \overline{D(A)}, \tag{5}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space X, $F : \mathbb{R} \times \overline{D(A)} \to X$ is a C^k map with $k \geq 4$, and $\mu \in \mathbb{R}$ is the bifurcation parameter. Set

$$X_0 := \overline{D(A)} \text{ and } A_0 := A_{X_0},$$

where A_0 is the part of A in X_0, which is defined by

$$A_0 x = Ax, \ \forall x \in D(A_0) = \{x \in D(A) \cap X_0 : Ax \in X_0\}.$$

We make the following assumptions on the linear operator A and F.

Assumption 2.2. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|.\|)$. We assume that A is a Hille-Yosida operator. That is to say that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and*

$$\| (\lambda I - A)^{-n} \|_{\mathcal{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \ \forall \lambda > \omega_A, \forall n \geq 1.$$

Assumption 2.2 implies that A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0 and A generates a uniquely determined integrated semigroup $\{S_A(t)\}_{t\geq 0}$.

Assumption 2.3. *Let $\varepsilon > 0$ and $F \in C^k((-\varepsilon, \varepsilon) \times B_{X_0}(0, \varepsilon); X)$ for some $k \geq 4$. Assume that the following conditions are satisfied:*

- ***(a)*** *$F(\mu, 0) = 0, \forall \mu \in (-\varepsilon, \varepsilon)$, and $\partial_x F(0, 0) = 0$.*
- ***(b)*** ***(Transversality condition)*** *For each $\mu \in (-\varepsilon, \varepsilon)$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_x F(\mu, 0))_0$, denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, such that the map $\mu \to \lambda(\mu)$ is continuously differentiable,*

$$Im(\lambda(0)) > 0, \ Re(\lambda(0)) = 0, \ \frac{dRe(\lambda(0))}{d\mu} \neq 0,$$

  and

$$\sigma(A_0) \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\}.$$

- ***(c)*** *The essential growth rate of $\{T_{A_0}(t)\}_{t\geq 0}$ is strictly negative, that is*

$$\omega_{0,ess}(A_0) < 0.$$

The following result has been proved in [16].

Theorem 2.4 (Hopf Bifurcation)**.** *Let Assumptions 2.2-2.3 be satisfied. Then there exist $\varepsilon^* > 0$, three C^{k-1} maps, $\varepsilon \to \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to x_\varepsilon$ from $(0, \varepsilon^*)$ into $\overline{D(A)}$, and $\varepsilon \to \gamma(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $\gamma(\varepsilon)$-periodic function $u_\varepsilon \in C^k(\mathbb{R}, X_0)$, which is an integrated solution of (5) with the parameter value equals $\mu(\varepsilon)$ and the initial value equals x_ε. So for each $t \geq 0$, u_ε satisfies*

$$u_\varepsilon(t) = x_\varepsilon + A \int_0^t u_\varepsilon(l) dl + \int_0^t F(\mu(\varepsilon), u_\varepsilon(l)) dl.$$

Moreover, we have the following properties

(i) *There exist a neighborhood* N *of* 0 *in* X_0 *and an open interval* I *in* $\mathbb{R}$ *containing* 0, *such that for* $\widehat{\mu} \in I$ *and any periodic solution* $\widehat{u}(t)$ *in* N *with minimal period* $\widehat{\gamma}$ *close to* $\frac{2\pi}{\omega(0)}$ *of* (5) *for the parameter value* $\widehat{\mu}$, *there exists* $\varepsilon \in (0, \varepsilon^*)$ *such that* $\widehat{u}(t) = u_\varepsilon(t+\theta)$ *(for some* $\theta \in [0, \gamma(\varepsilon))$*),* $\mu(\varepsilon) = \widehat{\mu}$, *and* $\gamma(\varepsilon) = \widehat{\gamma}$.

(ii) *The map* $\varepsilon \to \mu(\varepsilon)$ *is a* C^{k-1} *function and we have the Taylor expansion*

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0, \varepsilon^*),$$

where $[\frac{k-2}{2}]$ *is the integer part of* $\frac{k-2}{2}$.

(iii) *The period* $\gamma(\varepsilon)$ *of* $t \to u_\varepsilon(t)$ *is a* C^{k-1} *function and*

$$\gamma(\varepsilon) = \frac{2\pi}{\omega(0)}[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \gamma_{2n}\varepsilon^{2n}] + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where $\omega(0)$ *is the imaginary part of* $\lambda(0)$ *defined in Assumption* 2.3.

Remark 1. The Crandall and Rabinowitz's [7] condition applys here. If we only assume that $k \geq 2$, and

$$\sigma(A_0) \cap i\omega(0)\mathbb{Z} = \left\{\lambda(0), \overline{\lambda(0)}\right\}$$

in Assumption 2.3, that is to say that the spectrum of A_0 does not contain a multiple of $\lambda(0)$. Then the first part Theorem 2.4 hold (excepted (ii) and (iii)).

Since $\omega_{0,ess}(A_0) < 0$ implies that A_0 has at most a finite number of eigenvalues on the imaginary axis. Therefore the Crandall and Rabinowitz's condition applies if the Assumption 2.3-(b) is replaced by the following condition:

(b1) For each $\mu \in (-\varepsilon, \varepsilon)$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_x F(\mu, 0))_0$, denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, such that the map $\mu \to \lambda(\mu)$ is continuously differentiable,

$$Im(\lambda(0)) > 0,\ Re(\lambda(0)) = 0,\ \frac{dRe(\lambda(0))}{d\mu} \neq 0,$$

and

$$Im(\lambda(0)) > Im\left(\widehat{\lambda}\right), \forall\ \widehat{\lambda} \in \sigma(A_0) \cap i\mathbb{R} \text{ with } \widehat{\lambda} \neq \lambda(0).$$

3. **Existence of equilibrium and linearized equation.** If $\overline{v} = \begin{pmatrix} 0_Y \\ \overline{u} \end{pmatrix} \in X_0$ is a equilibrium of system (3), we must have

$$\overline{v} \in D(A) \text{ and } A\overline{v} + H(\overline{v}) = 0$$

which is equivalent to the following equations

$$\begin{aligned} &\frac{\partial \overline{u}}{\partial s} + \mu\overline{u} = 0, \text{ for } s \geq 0, \text{ and } x \in (0,1), \\ &\overline{u}(0,x) = \left(I - d^2\Delta_x\right)^{-1}\left(\alpha h\left(\int_0^{+\infty}\gamma(\theta)\overline{u}(\theta,\cdot)d\theta\right)\right)(x). \end{aligned}$$

Hence we can obtain

$$\overline{v} = \begin{pmatrix} 0_Y \\ \overline{u} \end{pmatrix} \text{ with } \overline{u}(s,x) = \chi(x)e^{-\mu s},$$

where $\chi \in L^1_+((0,1),\mathbb{R})$ is a solution of

$$\chi = \left(I - d^2\Delta_x\right)^{-1}\left(\alpha h\left(\chi(.)\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right)\right).$$

Note that

$$\int_0^{+\infty}\gamma(a)e^{-\mu a}da = 1.$$

Then

$$\chi(x) = \left(I - d^2\Delta_x\right)^{-1}\left(\alpha h\left(\chi(.)\right)\right)(x),$$

or equivalently χ satisfies the non-linear boundary value problem

$$\left\{\begin{array}{l} \chi(x) - d^2\chi''(x) = \alpha h\left(\chi(x)\right),\ \forall x \in (0,1) \\ \chi'(0) = \chi'(1) = 0. \end{array}\right. \tag{6}$$

By using some boot strapping we deduce that $\chi \in C^2([0,1],\mathbb{R}) \cap C_+([0,1],\mathbb{R})$ and the following Lemma holds.

Lemma 3.1. *If $\chi \in C^2([0,1],\mathbb{R}) \cap C_+([0,1],\mathbb{R})$ is a solution of (6), then either $\chi(x) > 0, \forall x \in [0,1]$ or $\chi(x) = 0, \forall x \in [0,1]$.*

It is easy to observe that the constant fixed point of (6) are

$$\chi(x) = \frac{\ln\alpha}{\beta}, \forall x \in [0,1],$$

and

$$\chi \geq 0 \text{ and } \chi \neq 0 \Leftrightarrow \alpha > 1.$$

Lemma 3.2. *Assume that $\alpha \leq 1$. Then system (3) has a unique positive equilibrium $\overline{v} = 0$.*

Proof. By multiplying the first equation of (6) by $\chi(x)$ and integrating on $(0,1)$, we obtain

$$\int_0^1 (\chi(x))^2\,dx - d^2\int_0^1 \chi(x)\chi''(x)dx = \alpha\int_0^1 \chi(x)h\left(\chi(x)\right)dx.$$

By integrating by parts, we get

$$\int_0^1 \chi(x)\chi''(x)dx = \left[\chi(x)\chi'(x)\right]_0^1 - \int_0^1 \left(\chi'(x)\right)^2 dx.$$

Notice that $h(x) = x\exp(-\beta x)$. By using the boundary values $\chi'(0) = \chi'(1) = 0$, we have

$$\int_0^1 (\chi(x))^2\,dx + d^2\int_0^1 \left(\chi'(x)\right)^2 dx = \alpha\int_0^1 (\chi(x))^2\exp(-\beta\chi(x))dx.$$

Since $\alpha \leq 1$, we obtain

$$d^2\int_0^1 \left(\chi'(x)\right)^2 dx \leq 0.$$

Thus

$$\chi'(x) = 0, \forall x \in [0,1].$$

□

Lemma 3.3. *Assume that $\alpha > 1$. If $\chi \in C^2([0,1],\mathbb{R})$ and $\chi \geq 0$ is a solution of (6), then $\chi(x)$ is a constant.*

Proof. It is sufficient to consider the case $\chi \neq 0$. Define

$$G(x) = \max(x,0)^2 = \begin{cases} x^2 \text{ if } x \geq 0 \\ 0, \text{ if } x \leq 0. \end{cases}$$

Then G is C^1 and

$$G'(x) = 2x^+ = \begin{cases} 2x \text{ if } x \geq 0 \\ 0, \text{ if } x \leq 0, \end{cases}$$

where

$$x^+ = \max(x,0).$$

By multiplying the first equation of (6) by $G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)$ and integrating on $(0,1)$, we obtain

$$\begin{aligned} &\int_0^1 G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\chi(x)dx - d^2\int_0^1 G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\chi''(x)dx \\ = &\int_0^1 G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\alpha h\left(\chi(x)\right)dx. \end{aligned}$$

By integrating by parts, we obtain

$$\begin{aligned} &\int_0^1 G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\chi''(x)dx \\ = &\left[G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\chi'(x)\right]_0^1 - \int_0^1 G'\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\left(\chi'(x)\right)^2 dx. \end{aligned}$$

Now by using the boundary values $\chi'(0) = \chi'(1) = 0$, and the fact that $\alpha h(x) \leq x, \forall x \geq \frac{\ln\alpha}{\beta}$, we deduce that

$$\begin{aligned} &2d^2\int_0^1 \left(\chi(x) - \frac{\ln\alpha}{\beta}\right)^+ \chi'(x)^2 dx \\ = &\int_0^1 G\left(\chi(x) - \frac{\ln\alpha}{\beta}\right)\left[\alpha h\left(\chi(x)\right) - \chi(x)\right]dx \leq 0. \end{aligned}$$

Thus we obtain

$$\int_0^1 \left(\chi(x) - \frac{\ln\alpha}{\beta}\right)^+ \left(\chi'(x)\right)^2 dx = 0. \tag{7}$$

From this equality it follows that

$$\chi(x) \leq \frac{\ln\alpha}{\beta}, \forall x \in [0,1]. \tag{8}$$

Indeed, assume that there exists $x_0 \in [0,1]$ such that $\chi(x_0) > \frac{\ln\alpha}{\beta}$. Let

$$a = \min\left\{0 \leq x \leq x_0 : \chi(y) > \frac{\ln\alpha}{\beta}, \forall y \in [x, x_0]\right\}$$

and

$$b = \max\left\{1 \geq x \geq x_0 : \chi(y) > \frac{\ln\alpha}{\beta}, \forall y \in [x_0, x]\right\}.$$

Then

$$\chi(x) > \frac{\ln\alpha}{\beta}, \forall x \in [a,b],$$

and by using (7) we deduce that

$$\chi(x) = \chi(x_0), \forall x \in [a,b].$$

Thus by the continuity of χ we deduce that

$$\chi(x) = \chi(x_0) > \frac{\ln\alpha}{\beta}, \forall x \in [0,1],$$

which is impossible.

By multiplying the first equation of (6) by $G\left(\frac{\ln\alpha}{\beta} - \chi(x)\right)$ and integrating on $(0,1)$, we obtain

$$\begin{aligned} & \int_0^1 G\left(\frac{\ln\alpha}{\beta} - \chi(x)\right)\chi(x)dx - d^2\int_0^1 G\left(\frac{\ln\alpha}{\beta} - \chi(x)\right)\chi''(x)dx \\ = & \int_0^1 G\left(\frac{\ln\alpha}{\beta} - \chi(x)\right)\alpha h\left(\chi(x)\right)dx. \end{aligned}$$

By using the same arguments as above and the fact that $\alpha h(x) \geq x, \forall x \in \left[0, \frac{\ln\alpha}{\beta}\right]$, we deduce that

$$\begin{aligned} & -2d^2\int_0^1 \left(\frac{\ln\alpha}{\beta} - \chi(x)\right)^+ \left(\chi'(x)\right)^2 dx \\ = & \int_0^1 G\left(\frac{\ln\alpha}{\beta} - \chi(x)\right)\left[\alpha h\left(\chi(x)\right) - \chi(x)\right]dx \geq 0. \end{aligned}$$

Thus

$$\int_0^1 \left(\frac{\ln\alpha}{\beta} - \chi(x)\right)^+ \left(\chi'(x)\right)^2 dx = 0 \tag{9}$$

and the result follows. □

Theorem 3.4. *Assume that $\alpha > 1$. Then system (3) has two equilibria*

$$0_X \text{ and } \overline{v}_\alpha = \begin{pmatrix} 0_Y \\ \overline{u}_\alpha \end{pmatrix},$$

with $\overline{u}_\alpha(s) := \overline{C}(\alpha)e^{-\mu s}$ and $\overline{C}(\alpha) := \frac{\ln\alpha}{\beta}$.

Let's compute the linearized equation around the positive equilibrium $\overline{v}_\alpha$. We can get that

$$\begin{aligned} & DH\begin{pmatrix} 0 \\ \psi \end{pmatrix}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ = & \begin{pmatrix} \left(I - d^2\Delta_x\right)^{-1}\left(\alpha Dh\left(\int_0^{+\infty}\gamma(\theta)\psi(\theta)d\theta\right)\left(\int_0^{+\infty}\gamma(\theta)\varphi(\theta)d\theta\right)\right)(x) \\ 0 \end{pmatrix}. \end{aligned}$$

Thus we can get that

$$DH\left(\overline{v}_\alpha\right)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \left(I - d^2\Delta_x\right)^{-1}\left(\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\varphi(\theta)d\theta\right)(x) \\ 0 \end{pmatrix},$$

where

$$\eta(\alpha) = \alpha\left(1 - \beta\overline{C}(\alpha)\right)\exp\left(-\beta\overline{C}(\alpha)\right) = 1 - \ln(\alpha).$$

The linearized equation of (3) around the equilibrium $\overline{v}_\alpha$ is

$$\frac{dv(t)}{dt} = Av(t) + DH(\overline{v}_\alpha)v(t) \text{ for } t \geq 0, \;\; v(0) = y = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in X_0. \tag{10}$$

Define

$$\Omega := \{\lambda \in \mathbb{C} : Re(\lambda) > -\mu\},$$

and $B_\alpha : D(A) \subset X \to X$ the linear operator defined by $B_\alpha := A + DH\left(\overline{v}_\alpha\right)$, that is to say that

$$B_\alpha \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) + \left(I - d^2\Delta_x\right)^{-1}\left(\eta\left(\alpha\right)\int_0^{+\infty}\gamma(s)\varphi(s)ds\right) \\ -\varphi' - \mu\varphi \end{pmatrix}.$$

We observe that $\Omega \subset \rho\left(A\right)$ the resolvent set of A, and for each $\lambda \in \Omega$,

$$(\lambda I - A)^{-1}\begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \Leftrightarrow \varphi(s) = e^{-(\lambda+\mu)s}\alpha + \int_0^s e^{-(\lambda+\mu)(s-l)}\psi(l)dl.$$

Lemma 3.5. *The linear operator $B_\alpha : D(A) \subset X \to X$ is a Hille-Yosida operator and its part $\left(B_\alpha\right)_0$ in X_0 satisfies*

$$\omega_{0,ess}\left(\left(B_\alpha\right)_0\right) \leq -\mu. \tag{11}$$

Proof. Since $DH(\overline{v}_\alpha)$ is a bounded linear operator and A is a Hille-Yosida operator, it follows that $B_\alpha = A + DH(\overline{v}_\alpha)$ is a Hille-Yosida operator. From (4), we deduce that $\omega_{0,ess}(A_0) \leq \omega_0(A_0) \leq -\mu$. Moreover, $DH(\overline{v}_\alpha)$ is compact. By using the perturbation results in Thieme [30] or Ducrot, Liu and Magal [11], we obtain

$$\omega_{0,ess}((B_\alpha)_0) \leq \omega_0(A_0) \leq -\mu.$$

$\square$

4. **Hopf bifurcation.** Recall that $d^2\Delta_x : D(d^2\Delta_x) \subset L^1\left(0,1\right) \to L^1\left(0,1\right)$ is considered as the linear operator

$$d^2\Delta_x\varphi = d^2\varphi''$$

with

$$D(d^2\Delta_x) = \{v \in W^{2,1}(0,1) : v'(x) = 0 \text{ at } x = 0, 1\}.$$

Then it is well known that the spectrum of $d^2\Delta_x$

$$\sigma\left(d^2\Delta_x\right) = \left\{-d^2(j\pi)^2 : j \in \mathbb{N}\right\}.$$

Let $\lambda \in \Omega$. Since $\lambda I - A$ is invertible, it follows that $\lambda I - B_\alpha$ is invertible if and only if $I - DH\left(\overline{v}_\alpha\right)(\lambda I - A)^{-1}$ is invertible. Moreover, when $I - DH\left(\overline{v}_\alpha\right)(\lambda I - A)^{-1}$ is invertible we have

$$(\lambda I - B_\alpha)^{-1} = (\lambda I - A)^{-1}(I - DH\left(\overline{v}_\alpha\right)(\lambda I - A)^{-1})^{-1}.$$

Consider the system

$$(I - DH\left(\overline{v}_\alpha\right)(\lambda I - A)^{-1})\begin{pmatrix} \widehat{\delta} \\ \widehat{\psi}(s) \end{pmatrix} = \begin{pmatrix} \delta \\ \psi(s) \end{pmatrix}. \tag{12}$$

Since

$$
\begin{aligned}
& \left(I-DH\left(\overline{v}_{\alpha}\right)(\lambda I-A)^{-1}\right)\begin{pmatrix} \widehat{\delta} \\ \widehat{\psi}(s) \end{pmatrix} \\
= & \begin{pmatrix} \widehat{\delta} \\ \widehat{\psi}(s) \end{pmatrix} - DH\left(\overline{v}_{\alpha}\right)\begin{pmatrix} 0 \\ e^{-(\lambda+\mu)s}\widehat{\delta}+\int_0^s e^{-(\lambda+\mu)(s-l)}\widehat{\psi}(l)dl \end{pmatrix} \\
= & \begin{pmatrix} \widehat{\delta} \\ \widehat{\psi}(s) \end{pmatrix} - \begin{pmatrix} \left(I-d^2\Delta_x\right)^{-1}\left(\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\begin{pmatrix} e^{-(\lambda+\mu)\theta}\widehat{\delta} \\ +\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\widehat{\psi}(l)dl \end{pmatrix} d\theta\right) \\ 0 \end{pmatrix} \\
= & \begin{pmatrix} \widehat{\delta}-\left(I-d^2\Delta_x\right)^{-1}\left(\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\begin{pmatrix} e^{-(\lambda+\mu)\theta}\widehat{\delta} \\ +\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\widehat{\psi}(l)dl \end{pmatrix} d\theta\right) \\ \widehat{\psi}(s) \end{pmatrix},
\end{aligned}
$$

system (12) can be written as the following

$$
\begin{cases}
\delta=\widehat{\delta}-\left(I-d^2\Delta_x\right)^{-1}\left(\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\left(e^{-(\lambda+\mu)\theta}\widehat{\delta}+\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\widehat{\psi}(l)dl\right)d\theta\right) \\
\psi(s)=\widehat{\psi}(s).
\end{cases}
$$

The first equation of the above system is rewritten as

$$
\begin{array}{l}
\delta=\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]\widehat{\delta} \\
-\eta(\alpha)\left(I-d^2\Delta_x\right)^{-1}\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\widehat{\psi}(l)dld\theta
\end{array} \tag{13}
$$

with

$$
\Theta=\Theta(\lambda,\alpha):=\eta(\alpha)\int_0^{+\infty}\gamma(\theta)e^{-(\lambda+\mu)\theta}d\theta.
$$

We have

$$
I-\Theta\left(I-d^2\Delta_x\right)^{-1}=\left[I-\Theta I-d^2\Delta_x\right]\left(I-d^2\Delta_x\right)^{-1}.
$$

Therefore $I-\Theta\left(I-d^2\Delta_x\right)^{-1}$ is invertible if and only if $I-\Theta I-d^2\Delta_x$ is invertible and

$$
\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]^{-1}=\left(I-d^2\Delta_x\right)\left[I-\Theta I-d^2\Delta_x\right]^{-1}.
$$

Hence

$$
\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]^{-1}=I+\Theta\left[I-\Theta I-d^2\Delta_x\right]^{-1} \tag{14}
$$

and

$$
\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]^{-1}\left(I-d^2\Delta_x\right)^{-1}=\left[I-\Theta I-d^2\Delta_x\right]^{-1}. \tag{15}
$$

By using (13), (14) and (15), we have

$$
\begin{array}{l}
\widehat{\delta}=\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]^{-1}\delta \\
+\eta(\alpha)\left[I-\Theta\left(I-d^2\Delta_x\right)^{-1}\right]^{-1}\left(I-d^2\Delta_x\right)^{-1}\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\psi(l)dld\theta \\
=\left[I+\Theta\left[I-\Theta I-d^2\Delta_x\right]^{-1}\right]\delta \\
+\eta(\alpha)\left[I-\Theta I-d^2\Delta_x\right]^{-1}\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\psi(l)dld\theta.
\end{array}
$$

Set

$$
\Lambda(\lambda,\alpha):=1-\Theta(\lambda,\alpha)=1-\eta(\alpha)\int_0^{+\infty}\gamma(\theta)e^{-(\lambda+\mu)\theta}d\theta.
$$

It makes sense to consider the **characteristic operator** $\Delta(\lambda,\alpha): D(\Delta(\lambda,\alpha)) \subset L^1(0,1) \to L^1(0,1)$

$$\Delta(\lambda,\alpha)=\Lambda(\lambda,\alpha)I-d^2\Delta_x \text{ with } D(\Delta(\lambda,\alpha))=D\left(d^2\Delta_x\right).$$

From the above discussion, we obtain the following lemma.

Lemma 4.1. *For the linear operator $B_\alpha : D(A) \subset X \to X$, we have the following properties:*

(i)

$$\sigma\left(B_\alpha\right)\cap\Omega=\sigma_p\left(B_\alpha\right)\cap\Omega=\left\{\lambda\in\Omega:\Delta_j(\lambda,\alpha)=0 \textit{ for some } j\in\mathbb{N}\right\},$$

where

$$\Delta_j(\lambda,\alpha)=1+d^2\left(j\pi\right)^2-\eta\left(\alpha\right)\int_0^{+\infty}\gamma(\theta)e^{-(\lambda+\mu)\theta}d\theta. \tag{16}$$

(ii) If $\lambda\in\Omega\cap\rho\left(B_\alpha\right)$, we have the following explicit formula for the resolvent

$$\begin{array}{l}\left(\lambda I-B_\alpha\right)^{-1}\left(\begin{array}{c}\delta\\ \psi\end{array}\right)=\left(\begin{array}{c}0\\ \varphi\end{array}\right)\\ \Leftrightarrow \varphi(s)=\int_0^s e^{-(\lambda+\mu)(s-l)}\psi(l)dl\\ +e^{-(\lambda+\mu)s}\left[\begin{array}{c}\left[I+\Theta(\lambda,\alpha)\Delta(\lambda,\alpha)^{-1}\right]\delta\\ +\eta\left(\alpha\right)\Delta(\lambda,\alpha)^{-1}\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\psi(l)dld\theta\end{array}\right].\end{array}$$

Proof. We already proved that

$$\sigma\left(B_\alpha\right)\cap\Omega\subset\left\{\lambda\in\Omega:\Delta_j(\lambda,\alpha)=0 \text{ for some } j\in\mathbb{N}\right\}.$$

To prove the converse inclusion, let

$$\lambda_0\in\left\{\lambda\in\Omega:\Delta_j(\lambda,\alpha)=0 \text{ for some } j\in\mathbb{N}\right\}.$$

Then there exists $j_0\in\mathbb{N}$ such that

$$\Lambda(\lambda_0,\alpha)=1-\eta\left(\alpha\right)\int_0^{+\infty}\gamma(\theta)e^{-(\lambda_0+\mu)\theta}d\theta=-d^2(j_0\pi)^2,$$

or equivalently

$$\Lambda(\lambda_0,\alpha)\in\sigma\left(d^2\Delta_x\right).$$

So there exists

$$\chi_{j_0}=\cos(j_0\pi x)\in D(d^2\Delta_x)\backslash\{0\}$$

such that

$$d^2\Delta_x\chi_{j_0}=-d^2(j_0\pi)^2\chi_{j_0}=\Lambda(\lambda_0,\alpha)\chi_{j_0}.$$

Fix

$$\varphi(s)=e^{-(\lambda_0+\mu)s}\chi_{j_0}.$$

By recalling that

$$B_\alpha\left(\begin{array}{c}0\\ \varphi\end{array}\right)=\left(\begin{array}{c}-\varphi(0)+\left(I-d^2\Delta_x\right)^{-1}\left(\eta\left(\alpha\right)\int_0^{+\infty}\gamma(s)\varphi(s)ds\right)\\ -\varphi'-\mu\varphi\end{array}\right)$$

and observing that

$$\begin{aligned}
& -\varphi(0)+\left(I-d^2\Delta_x\right)^{-1}\left(\eta(\alpha)\int_0^{+\infty}\gamma(s)\varphi(s)ds\right) \\
= & \left(I-d^2\Delta_x\right)^{-1}\left[\left(I-d^2\Delta_x\right)\chi_{j_0}+\eta(\alpha)\int_0^{+\infty}\gamma(s)e^{-(\lambda_0+\mu)s}ds\chi_{j_0}\right] \\
= & \left(I-d^2\Delta_x\right)^{-1}\left[\Lambda(\lambda_0,\alpha)\chi_{j_0}-d^2\Delta_x\chi_{j_0}\right] \\
= & \left(I-d^2\Delta_x\right)^{-1}0,
\end{aligned}$$

we have

$$B_\alpha\begin{pmatrix}0\\ \varphi\end{pmatrix}=\lambda_0\begin{pmatrix}0\\ \varphi\end{pmatrix}.$$

Thus

$$\{\lambda\in\Omega:\Delta_j(\lambda,\alpha)=0 \text{ for some } j\in\mathbb{N}\}\subset\sigma(B_\alpha)\cap\Omega.$$

□

For each $j\in\mathbb{N}$,

$$\lambda_j:=-d^2(j\pi)^2$$

is a simple eigenvalue of $d^2\Delta_x$ associated to the eigenfunction

$$\chi_j(x):=\cos(j\pi x).$$

That is to say that

$$\lim_{\lambda\to\lambda_j}(\lambda-\lambda_j)\left(\lambda I-d^2\Delta_x\right)^{-1}=\Pi_j,$$

where the convergence is understood in norm of operator and Π_j is the projector defined by

$$\Pi_j(\varphi)(x)=\frac{\int_0^1\cos(j\pi y)\varphi(y)dy\cos(j\pi x)}{\int_0^1\cos(j\pi\sigma)^2d\sigma}.$$

Lemma 4.2. *Assume that there exists $\widehat{\lambda}_j\in\Omega$ such that*

$$\Lambda(\widehat{\lambda}_j,\alpha)=\lambda_j \text{ and } \partial_\lambda\Lambda(\widehat{\lambda}_j,\alpha)\neq 0.$$

Then $\widehat{\lambda}_j$ is a simple and isolate eigenvalue of B_α associated to the eigenfunction

$$\varphi(s)=e^{-(\lambda_j+\mu)s}\cos(j\pi x)$$

and the projector of the generalized eigenspace of B_α is defined by

$$\begin{aligned}
\widehat{\Pi}_j\begin{pmatrix}\delta\\ \psi\end{pmatrix} &= \begin{pmatrix}0\\ \varphi\end{pmatrix}\Leftrightarrow \\
\varphi &= e^{-(\widehat{\lambda}_j+\mu)s}\left(\partial_\lambda\Lambda(\widehat{\lambda}_j,\alpha)\right)^{-1} \\
&\quad\times\Pi_j\left[\Theta(\widehat{\lambda}_j,\alpha)\delta+\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\widehat{\lambda}_j+\mu)(\theta-l)}\psi(l)dld\theta\right].
\end{aligned}$$

Proof. Since $\partial_\lambda\Lambda(\widehat{\lambda}_j,\alpha)\neq 0$, by the inverse function theorem there exists at most one root λ

$$\Lambda(\lambda,\alpha)=\lambda_j$$

in some neighborhood of $B\left(\widehat{\lambda}_j,\varepsilon\right)$. Therefore for each $\lambda \in B\left(\widehat{\lambda}_j,\varepsilon\right)\setminus\left\{\widehat{\lambda}_j\right\}$ the resolvent $\left[\Lambda(\lambda,\alpha)I-d^2\Delta_x\right]^{-1}$ is well defined. Moreover

$$\begin{aligned}
&\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)\left[\Lambda(\lambda,\alpha)I-d^2\Delta_x\right]^{-1}\\
=&\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\frac{\left(\lambda-\widehat{\lambda}_j\right)}{\left(\Lambda(\lambda,\alpha)-\Lambda(\widehat{\lambda}_j,\alpha)\right)}\\
&\times\left(\Lambda(\lambda,\alpha)-\Lambda(\widehat{\lambda}_j,\alpha)\right)\left[\Lambda(\lambda,\alpha)I-d^2\Delta_x\right]^{-1}.
\end{aligned}$$

Thus

$$\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)\left[\Lambda(\lambda,\alpha)I-d^2\Delta_x\right]^{-1}=\left(\partial_\lambda\Lambda(\widehat{\lambda}_j,\alpha)\right)^{-1}\Pi_j.$$

Now by using the explicite formula for the resolvent B_α we obtain

$$\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)(\lambda I-B_\alpha)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}=\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)\begin{pmatrix}0\\ \varphi_\lambda\end{pmatrix},$$

where

$$\begin{aligned}
&\left(\lambda-\widehat{\lambda}_j\right)\varphi_\lambda\\
=&\left(\lambda-\widehat{\lambda}_j\right)\left[\int_0^s e^{-(\lambda+\mu)(s-l)}\psi(l)dl\right]\\
&+e^{-(\lambda+\mu)s}\left(\lambda-\widehat{\lambda}_j\right)\left[\begin{array}{c}\left[I+\Theta(\lambda,\alpha)\Delta(\lambda,\alpha)^{-1}\right]\delta\\ +\eta(\alpha)\,\Delta(\lambda,\alpha)^{-1}\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\psi(l)dld\theta\end{array}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)\varphi_\lambda\\
=&\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left[\begin{array}{c}e^{-(\lambda+\mu)s}\left(\lambda-\widehat{\lambda}_j\right)\\ \times\Delta(\lambda,\alpha)^{-1}\left(\Theta(\lambda,\alpha)\delta+\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\lambda+\mu)(\theta-l)}\psi(l)dld\theta\right)\end{array}\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{\substack{\lambda\to\widehat{\lambda}_j\\ \lambda\neq\widehat{\lambda}_j}}\left(\lambda-\widehat{\lambda}_j\right)\varphi_\lambda\\
=&\ e^{-(\widehat{\lambda}_j+\mu)s}\left(\partial_\lambda\Lambda(\widehat{\lambda}_j,\alpha)\right)^{-1}\times\Pi_j\left[\begin{array}{c}\Theta(\widehat{\lambda}_j,\alpha)\delta\\ +\eta(\alpha)\int_0^{+\infty}\gamma(\theta)\int_0^\theta e^{-(\widehat{\lambda}_j+\mu)(\theta-l)}\psi(l)dld\theta\end{array}\right]
\end{aligned}$$

and the result follows. □

4.1. **Existence of purely imaginary eigenvalues.** One may first observe that
$$\sigma\left(B_{\alpha}\right)\cap i\mathbb{R}=\{i\omega\in i\mathbb{R}:\Delta_{j}(i\omega,\alpha)=0\text{ for some }j\in\mathbb{N}\}$$
is finite. By using the formula
$$\Delta_{j}(0,\alpha)=1+d^{2}(j\pi)^{2}-\eta\left(\alpha\right)\int_{0}^{+\infty}\gamma(\theta)e^{-\mu\theta}d\theta,$$
it is easy to observe that
$$j\neq k\Rightarrow\Delta_{j}(i\omega,\alpha)\neq\Delta_{k}(i\omega,\alpha).$$
Next

Lemma 4.3. *$\lambda=0$ is not a root of the characteristic equation $\Delta_{j}(\lambda,\alpha)=0$, where $\Delta_{j}(\lambda,\alpha)$ is explicitly defined in (16).*

Proof. We have
$$\Delta_{j}(0,\alpha)=1+d^{2}(j\pi)^{2}-1+\ln(\alpha).$$
Since $\alpha>1$ and $1+d^{2}(j\pi)^{2}\geq 1$ we obtain that
$$\Delta_{j}(0,\alpha)>0$$
and the result follows. □

Theorem 4.4. *If $1<\alpha\leq e$, then the positive equilibrium $\overline{v}_{\alpha}$ of the system (3) is locally asymptotically stable.*

Proof. We consider the characteristic equation
$$1+d^{2}(j\pi)^{2}=\eta\left(\alpha\right)\int_{0}^{+\infty}\gamma(\theta)e^{-(\lambda+\mu)\theta}d\theta.$$
Then we have
$$\begin{aligned}\left|1+d^{2}(j\pi)^{2}\right| &= \left|\eta\left(\alpha\right)\int_{0}^{+\infty}\gamma(\theta)e^{-(\lambda+\mu)\theta}d\theta\right|\\ &\leq \left|\eta\left(\alpha\right)\right|\int_{0}^{+\infty}\gamma(\theta)e^{-(Re\lambda+\mu)\theta}d\theta.\end{aligned}$$
If $Re(\lambda)\geq 0$, we must have
$$\left|1+d^{2}(j\pi)^{2}\right|\leq\left|\eta\left(\alpha\right)\right|\int_{0}^{+\infty}\gamma(\theta)e^{-\mu\theta}d\theta=\left|\eta\left(\alpha\right)\right|.$$
By Lemma 4.2, we have
$$1-\ln(\alpha)<\left|\eta\left(\alpha\right)\right|.$$
So if
$$\eta\left(\alpha\right)=1-\ln(\alpha)\geq 0,$$
then there will be no roots of the characteristic equations in (16) with non-negative real part, and the result follows. □

Next we will study the existence of Hopf bifurcation when α is regarded as the bifurcation parameter of the system. By Theorem 4.4 we already knew that the positive equilibrium $\overline{v}_{\alpha}$ of the system (3) is locally asymptoticlly stable if
$$1<\alpha\leq e.$$
So we will study the existence of a bifurcation value $\alpha>e$.

In the following we will consider $\gamma\left(a\right)$ as a special case

$$\gamma(a) = 1_{[\tau,+\infty)}(a) \text{ for some } \tau > 0.$$

Then we have

$$\int_0^{+\infty} \gamma(a)\, e^{-(\lambda+\mu)a} da = \frac{e^{-(\lambda+\mu)\tau}}{\lambda+\mu},$$

and we obtain a family of **characteristic equations**

$$\Delta_j(\lambda,\alpha) = 1 + d^2(j\pi)^2 - \eta(\alpha)\frac{e^{-(\lambda+\mu)\tau}}{\lambda+\mu} = 0, \tag{17}$$

for $\lambda \in \Omega$, and $j = 0, 1, 2, ...$

Now we will consider the characteristic equations in (17). First we give the following lemma to show that for any given $\alpha > e$ there exists at most one pair of purely imaginary solutions of the characteristic equation in (17).

Lemma 4.5. *For each real number δ_1, there exists at most one $\delta_2 \in (0, +\infty)$ such that if*

$$\lambda \in \Omega,\ \operatorname{Re}(\lambda) = \delta_1 \text{ and } \Delta_j(\alpha,\lambda) = 0,$$

then

$$\operatorname{Im}(\lambda) = \pm\delta_2.$$

Proof. Since $\Delta_j(\alpha,\lambda) = 0$, we obtain

$$1 + d^2(j\pi)^2 - \eta(\alpha)\frac{e^{-(\lambda+\mu)\tau}}{\lambda+\mu} = 0,\ \lambda \in \Omega.$$

Then

$$|\lambda+\mu|\left(1 + d^2(j\pi)^2\right) = \left|\eta(\alpha)e^{-(\lambda+\mu)\tau}\right|,$$

i.e.,

$$\left((\operatorname{Re}(\lambda)+\mu)^2 + (\operatorname{Im}(\lambda))^2\right)\left(1 + d^2(j\pi)^2\right)^2 = |\eta(\alpha)|^2 e^{2(-\operatorname{Re}(\lambda)-\mu)\tau}.$$

Now set

$$f(\operatorname{Im}(\lambda)) := \left((\operatorname{Re}(\lambda)+\mu)^2 + (\operatorname{Im}(\lambda))^2\right)\left(1 + d^2(j\pi)^2\right)^2 - |\eta(\alpha)|^2 e^{2(-\operatorname{Re}(\lambda)-\mu)\tau}.$$

If $\operatorname{Im}(\lambda) > 0$, then

$$f'(\operatorname{Im}(\lambda)) = 2(\operatorname{Im}(\lambda))\left(1 + d^2(j\pi)^2\right)^2 > 0.$$

Thus for any fixed $\operatorname{Re}(\lambda)$, we can find at most one $\operatorname{Im}(\lambda) > 0$ satisfying the characteristic equation in (17). The proof is complete. □

In the following, we will consider the existence of the purely imaginary solutions of the characteristic equations in (17).

Proposition 1. *Let $\tau > 0$, $\mu > 0$ be fixed. Then the characteristic equation in (17) has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $k \in \mathbb{Z}$ such that $\omega > 0$ is a solution of equation*

$$\arctan\left(\frac{\omega}{\mu}\right) + \pi + \omega\tau = 2k\pi \tag{18}$$

and

$$\eta(\alpha) = -\left(1 + d^2(j\pi)^2\right)\sqrt{\mu^2+\omega^2}e^{\mu\tau}.$$

Moreover, for each $k \in \mathbb{N}^+$, there exists a unique $\omega_k > 0$ (which is a function of τ, μ) satisfying equation (18).

Proof. Fix j and let $\lambda = i\omega$ with $\omega > 0$ be a purely imaginary root of the characteristic equation

$$\Delta_j(\lambda, \alpha) = 1 + d^2(j\pi)^2 - \eta(\alpha) \frac{e^{-(\lambda+\mu)\tau}}{\lambda + \mu} = 0 \tag{19}$$

in (17). Then

$$1 + d^2(j\pi)^2 = \eta(\alpha) \frac{e^{-(i\omega+\mu)\tau}}{i\omega + \mu}$$

or

$$i\omega + \mu = \frac{\eta(\alpha)e^{-(i\omega+\mu)\tau}}{1 + d^2(j\pi)^2}. \tag{20}$$

Since

$$i\omega + \mu = \sqrt{\mu^2 + \omega^2} e^{\left(i \arctan\left(\frac{\omega}{\mu}\right)\right)}, \ \eta(\alpha) = 1 - \ln(\alpha) \text{ and } \alpha > e,$$

(20) becomes

$$\sqrt{\mu^2 + \omega^2} e^{\left(i \arctan\left(\frac{\omega}{\mu}\right)\right)} = \frac{\eta(\alpha)e^{-\mu\tau}}{1 + d^2(j\pi)^2} e^{-i\omega\tau} \tag{21}$$

with $\eta(\alpha) < 0$. (21) is equivalent to

$$\begin{aligned} &-\sqrt{\mu^2 + \omega^2} \left(\cos\left(\arctan\left(\frac{\omega}{\mu}\right) + \pi \right) + i \sin\left(\arctan\left(\frac{\omega}{\mu}\right) + \pi \right) \right) \\ = &\ \frac{\eta(\alpha)e^{-\mu\tau}}{1 + d^2(j\pi)^2} \left(\cos(-\omega\tau) + i \sin(-\omega\tau) \right). \end{aligned}$$

Therefore, the characteristic equation (19) has a purely imaginary solution $\lambda = i\omega$ with $\omega > 0$ if and only if there exists $k \in \mathbb{Z}$ such that ω is a solution of equation

$$\arctan\left(\frac{\omega}{\mu}\right) + \pi + \omega\tau = 2k\pi$$

and

$$-\sqrt{\mu^2 + \omega^2} = \frac{\eta(\alpha)e^{-\mu\tau}}{1 + d^2(j\pi)^2}.$$

The proof is complete. $\square$

Since

$$\eta(\alpha) = 1 - \ln\alpha,$$

we obtain that the bifurcation curves are

$$\alpha_k^j = \exp\left(1 + \left(1 + d^2(j\pi)^2\right) \sqrt{\mu^2 + (\omega_k)^2} e^{\mu\tau} \right), \ k \in \mathbb{N}^+, \tag{22}$$

and ω_k is described in Proposition 1. Now we fix $\alpha = \alpha_k^j$, $k \in \mathbb{N}^+, j \in \mathbb{N}$. From the discussion above we have

$$1 + d^2(j\pi)^2 = -\eta\left(\alpha_k^j\right) \frac{e^{-\mu\tau}}{\sqrt{\mu^2 + (\omega_k)^2}},$$

$$\begin{aligned} 1 + d^2(\widehat{j}\pi)^2 &\neq -\eta\left(\alpha_k^j\right) \frac{e^{-\mu\tau}}{\sqrt{\mu^2 + \omega_i^2}}, \\ \text{for } i &= k, k+1, \cdots, \ \widehat{j} > j \text{ and } \widehat{j} \in \mathbb{N}, \end{aligned}$$

and

$$
\begin{aligned}
1+d^2(\widehat{j}\pi)^2 &\neq -\eta\left(\alpha_k^j\right)\frac{e^{-\mu\tau}}{\sqrt{\mu^2+\omega_i^2}}, \\
\text{for } i &= 1,2,\cdots,k,\ \widehat{j}<j \text{ and } \widehat{j}\in\mathbb{N}.
\end{aligned}
$$

Thus we can obtain that at $\alpha=\alpha_k^j,\ k\in\mathbb{N}^+, j\in\mathbb{N}$ the characteristic equations $\Delta_j(\lambda,\alpha)=0,\ j\in\mathbb{N}$ have at most a finite number of pair of purely imaginary eigenvalues. We denote the maximum of these purely imaginary eigenvalues by $i\omega_k^j$. Then

$$
\sigma\left(B_\alpha\right)\cap i\omega_k^j\mathbb{Z}=\left\{i\omega_k^j,-i\omega_k^j\right\}.
$$

4.2. **Transversality condition.** In this subsection we will prove the transversality condition for the model.

Lemma 4.6. *If $\alpha>e,\ \lambda\in\Omega$ and $\Delta_j(\alpha,\lambda)=0$, then*

$$
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\alpha}<0.
$$

Proof. Since

$$
\Delta_j(\alpha,\lambda)=1+d^2(j\pi)^2-\eta\left(\alpha\right)\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}da
$$

with

$$
\eta(\alpha)=1-\ln\alpha,
$$

$$
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\alpha}=-\frac{d\eta(\alpha)}{d\alpha}\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}da=\frac{1}{\alpha}\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}da.
$$

If $\alpha>e,\ \lambda\in\Omega$ and $\Delta_j(\alpha,\lambda)=0$, then we obtain

$$
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\alpha}=\frac{1+d^2(j\pi)^2}{\alpha\eta(\alpha)}=\frac{1+d^2(j\pi)^2}{\alpha(1-\ln\alpha)}<0.
$$

□

Lemma 4.7. *If $\alpha>e,\ \lambda\in\Omega$ and $\Delta_j(\alpha,\lambda)=0$, then*

$$
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\lambda}=\left(1+d^2(j\pi)^2\right)\left(\tau+\frac{1}{\lambda+\mu}\right)\neq 0.
$$

Proof. We have

$$
\Delta_j(\alpha,\lambda)=1+d^2(j\pi)^2-\frac{\eta(\alpha)e^{-(\lambda+\mu)\tau}}{\lambda+\mu}
$$

and

$$
\begin{aligned}
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\lambda} &= \frac{\tau\eta(\alpha)e^{-(\lambda+\mu)\tau}\left(\lambda+\mu\right)+\eta(\alpha)e^{-(\lambda+\mu)\tau}}{\left(\lambda+\mu\right)^2} \\
&= \frac{\eta(\alpha)e^{-(\lambda+\mu)\tau}}{\lambda+\mu}\left(\tau+\frac{1}{\lambda+\mu}\right).
\end{aligned}
$$

If $\Delta_j(\alpha,\lambda)=0$, then we obtain

$$
\frac{\partial\Delta_j(\alpha,\lambda)}{\partial\lambda}=\left(1+d^2(j\pi)^2\right)\left(\tau+\frac{1}{\lambda+\mu}\right). \tag{23}
$$

Note that $\frac{\partial \Delta_j(\alpha,\lambda)}{\partial \lambda} = 0$ if and only if

$$\left(1 + d^2(j\pi)^2\right)\left(\tau + \frac{1}{\lambda+\mu}\right) = 0. \tag{24}$$

Since $\eta(\alpha) < 0$ for $\alpha > e$, we have for $\lambda \in \mathbb{R}$ and $\lambda > -\mu$ that

$$\Delta_j(\alpha,\lambda) = 1 + d^2(j\pi)^2 - \frac{\eta(\alpha)e^{-(\lambda+\mu)\tau}}{\lambda+\mu} > 0.$$

So the solutions of the characteristic equation in Ω can not be real numbers. If $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then $\tau + \frac{1}{\lambda+\mu} \in \mathbb{C}\setminus\mathbb{R}$ and we deduce that equation (24) can not be satisfied. The proof is complete. □

Theorem 4.8. *Let $i\omega_k^j$ be the maximum purely imaginary root of the characteristic equations in (17) associated to $\alpha_k^j > 0$ defined in (22). Then there exist $\rho_k^j > 0$ (small enough) and a C^1-map $\lambda_k^j : (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j) \to \mathbb{C}$ such that*

$$\begin{aligned}\lambda_k^j(\alpha_k^j) &= i\omega_k^j,\ \ \Delta_{\widehat{j}}(\alpha, \lambda_k^j(\alpha)) = 0,\ \textit{for some } \widehat{j} \in \mathbb{N} \textit{ and}\\ \forall\alpha &\in (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j),\ \textit{and } Re\left(\frac{d\lambda_k^j(\alpha_k^j)}{d\alpha}\right) > 0.\end{aligned}$$

Proof. By Lemma 4.6 we can use the implicit function theorem around each $\left(\alpha_k^j, i\omega_k^j\right)$ and obtain that there exists $\rho_k^j > 0$ and a C^1-map $\lambda_k^j : (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j) \to \mathbb{C}$ such that

$$\begin{aligned}\lambda_k^j(\alpha_k^j) &= i\omega_k^j,\ \ \Delta_{\widehat{j}}(\alpha, \lambda_k^j(\alpha)) = 0,\ \text{ for some } \widehat{j} \in \mathbb{N},\\ \text{and } \forall\alpha &\in (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j).\end{aligned}$$

Moreover, we have

$$\frac{\partial \Delta_{\widehat{j}}\left(\alpha, \lambda_k^j(\alpha)\right)}{\partial \alpha} + \frac{\partial \Delta_{\widehat{j}}\left(\alpha, \lambda_k^j(\alpha)\right)}{\partial \lambda}\frac{d\lambda_k^j(\alpha)}{d\alpha} = 0,\ \forall \alpha \in (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j).$$

So

$$\frac{d\lambda_k^j(\alpha)}{d\alpha} = -\frac{1}{\frac{\partial \Delta_{\widehat{j}}(\alpha,\lambda_k^j(\alpha))}{\partial\lambda}}\frac{\partial \Delta_{\widehat{j}}\left(\alpha, \lambda_k^j(\alpha)\right)}{\partial \alpha},\ \forall \alpha \in (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j).$$

By using Lemma 4.6, we deduce $\forall \alpha \in (\alpha_k^j - \rho_k^j, \alpha_k^j + \rho_k^j)$ that

$$\text{Re}\left(\frac{d\lambda_k^j(\alpha)}{d\alpha}\right) > 0 \Leftrightarrow \text{Re}\left(\frac{\partial \Delta_{\widehat{j}}\left(\alpha, \lambda_k^j(\alpha)\right)}{\partial \lambda}\right) > 0.$$

In particular, we have

$$\text{Re}\left(\frac{d\lambda_k^j(\alpha_k^j)}{d\alpha}\right) > 0 \Leftrightarrow \text{Re}\left(\frac{\partial \Delta_{\widehat{j}}\left(\alpha_k^j, i\omega_k^j\right)}{\partial \lambda}\right) > 0.$$

Using the equation (23), we get

$$\frac{\partial \Delta_{\widehat{j}}\left(\alpha_k^j, i\omega_k^j\right)}{\partial \lambda} = \left(1 + d^2(\widehat{j}\pi)^2\right)\left(\tau + \frac{1}{i\omega_k^j + \mu}\right)$$

and

$$\mathrm{Re}\left(\frac{\partial\Delta_{\widehat{j}}\left(\alpha_k^j, i\omega_k^j\right)}{\partial\lambda}\right) = \left(1+d^2(\widehat{j}\pi)^2\right)\left(\tau+\frac{\mu}{\mu^2+\left(\omega_k^j\right)^2}\right) > 0.$$

The result follows. □

Now we are in the position to present the main result of this paper. By using the Crandall and Rabinowitz's [7] condition for their Hopf bifurcation theorem (see Remark 1), we obtain the following result.

Theorem 4.9. *The spatially and age structured population model (1) undergoes a Hopf bifurcation at* $\alpha = \alpha_k^j$ *and* $\overline{u}_\alpha = \overline{u}_{\alpha_k^j}$ *given in theorem 3.4, where* $\alpha_k^j > 0$, $k = 1, 2, \ldots,$ $j \in \mathbb{N}$ *are defined in (22). In particular, a non-trivial periodic solution bifurcates from the equilibrium* $\overline{u}_\alpha = \overline{u}_{\alpha_k^j}$ *when* $\alpha = \alpha_k^j$.

5. **An example and numerical simulations.** In the following, we provide some numerical simulations to illustrate the Hopf bifurcation for the model (1). We consider system (1) with the parameters taking the values in the following

$$\mu = 0.2,\ d = 0.01,\ \beta = 0.5,\ \alpha = 10, 20 \text{ or } 30 \tag{25}$$

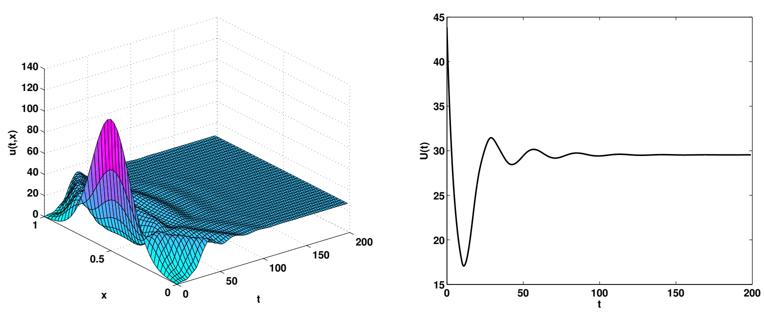
and

$$\gamma(\theta) = 1_{[10,+\infty)}(\theta). \tag{26}$$

We choose the following initial distribution

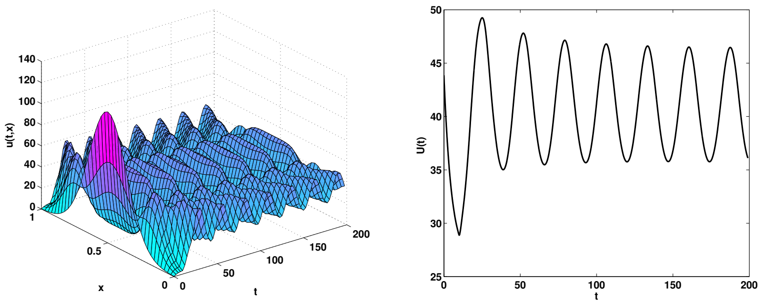
$$u(0, s, x) = 25e^{-0.2s}e^{-(5x-2.5)^2}. \tag{27}$$

In the figures 1-3 we only modify the parameter α. As predicted by the theory when the parameter α passes from $\alpha = 10$ (in Figure 1) to $\alpha = 20$ (in Figure 2) and $\alpha = 30$ (in Figure 3) we observe a Hopf bifurcation of the positive equilibrium. In figure 1 we observe the global stability of the positive equilibrium. In figures 2-3 the positive equilibrium is destabilized and the solutions converge to a periodic orbit.



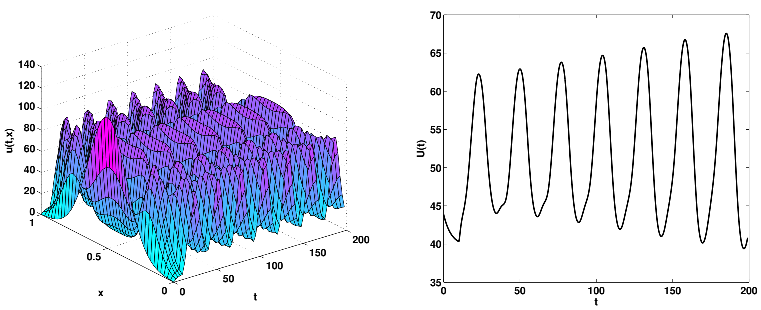
(a) (b)

FIGURE 1. *In this figure we fix* $\alpha = 10$. *In figure (a) we plot the distribution* $u(t,x) = \int_0^{+\infty} u(t,s,x)ds$ *and in figure (b) we plot the total number of individuals* $U(t) = \int_0^1\int_0^{+\infty} u(t,s,x)dsdx$. *We observe the global stability of the positive constant equilibrium distribution.*



(a) **(b)**

FIGURE 2. *In this figure we fix $\alpha = 20$. In figure (a) we plot the distribution $u(t,x) = \int_0^{+\infty} u(t,s,x)ds$ and in figure (b) we plot the total number of individuals $U(t) = \int_0^1 \int_0^{+\infty} u(t,s,x)dsdx$. The positive equilibrium admits a Hopf bifurcation and the distribution of population oscillates in time and converges to a periodic solution.*



(a) **(b)**

FIGURE 3. *In this figure we fix $\alpha = 30$. In figure (a) we plot the distribution $u(t,x) = \int_0^{+\infty} u(t,s,x)ds$ and in figure (b) we plot the total number of individuals $U(t) = \int_0^1 \int_0^{+\infty} u(t,s,x)dsdx$. As in Figure 2 the solution oscillates around positive equilibrium and converges to a periodic solution. Moreover we observe that the amplitude of the oscillations increases with α.*

REFERENCES

[1] S. Bertoni, Periodic solutions for non-linear equations of structured populations, *J. Math. Anal. Appl.,* **220** (1998), 250–267.

[2] P. Bi and X. Fu, Hopf bifurcation in an age-dependent population model with delayed birth process. *I. J. Bifurcation and Chaos*, **22** (2012), 1793–6551.

[3] C. Castillo-Chavez, H.W. Hethcote, V. Andreasen, S.A. Levin and W.M. Liu, Epidemiological models with age structure, proportionate mixing, and cross-immunity, *J. Math. Biol.,* **27** (1989), 233–258.

[4] J. Chu, A. Ducrot, P.Magal and S.Ruan, Hopf bifurcation in a size-structured population dynamic model with random growth, *Journal of Differential Equations,* **247** (2009), 956–1000.

[5] J. Chu and P. Magal, Hopf bifurcation for a size structured model with resting phase, *Discrete and Continuous Dynamical Systems*, **33** (2013), 4891–4921.
[6] J. Chu, P. Magal and R. Yuan, Hopf bifurcation for a maturity structured population dynamic model, *J. Nonlinear Sci.*, **21** (2011), 521–562.
[7] M. G. Crandall and P. H. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions, *Arch. Rational Mech. Anal.,* **67** (1977), 53–72.
[8] J.M. Cushing, Model stability and instability in age structured populations, *J. Theoret. Biol.,* **86** (1980), 709–730.
[9] J.M. Cushing, Bifurcation of time periodic solutions of the McKendrick equations with applications to population dynamics, *Comput. Math. Appl.,* **9** (1983), 459–478.
[10] A. Ducrot, Travelling waves for a size and space structured model in population dynamics: Point to sustained oscillating solution connections, *Journal of Differential Equations*, **250** (2011), 410–449.
[11] A. Ducrot, Z. Liu and P. Magal, Essential growth rate for bounded linear perturbation of non densely defined Cauchy problems, *J. Math. Anal. Appl.*, **341** (2008), 501–518.
[12] M. Doumic, A. Marciniak-Czochra, B. Perthame and J.P. Zubelli, A structured population model of cell differentiation, *SIAM J. Appl. Math.*, **71** (2011), 1918–1940.
[13] H. Inaba, Mathematical analysis for an evolutionary epidemic model, in: M.A. Horn, G. Simonett and G.F. Webb (Eds), Mathematical Models in Medical and Health Sciences, Vanderbilt Univ. Press, Nashville, TN, 1998, 213–236.
[14] H. Inaba, Endemic threshold and stability in an evolutionary epidemic model, in: C. Castillo-Chavez et al. (Eds), Mathematical Approaches for Emerging and Reemerging Infectious Diseases: Models, Methods, and Theory, Springer-Verlag, New York, 2002, 337–359.
[15] T. Kostova and J. Li, Oscillations and stability due to juvenile competitive effects on adult fertility, *Comput. Math. Appl.,* **32** (1996), 57–70.
[16] Z. Liu, P. Magal and S. Ruan, Hopf bifurcation for non-densely defined Cauchy problems, *Zeitschrift fur Angewandte Mathematik und Physik,* **62** (2011), 191–222.
[17] A. Ducrot, P. Magal and O. Seydi, Nonlinear boundary conditions derived by singular perturbation in age structured population dynamics model., *J. Appl. Anal. Comput.*, **1** (2011), 373–395.
[18] P. Magal, Compact attractors for time-periodic age structured population models, *Electron. J. Differential Equations,* **2001** (2001), 1–35.
[19] P. Magal and S. Ruan, Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models, *Mem. Amer. Math. Soc.,* **202** (951) (2009).
[20] P. Magal and S. Ruan, On semilinear Cauchy problems with non-dense domain, *Advances in Differential Equations,* **14** (2009), 1041–1084.
[21] P. Magal and S. Ruan, Sustained oscillations in an evolutionary epidemiological model of influenza A drift, *Proc. R. Soc. A,* **466** (2010), 965–992.
[22] K. Pakdaman, B. Perthame and D. Salort, Dynamics of a structured neuron population, *Nonlinearity,* **23** (2010), 55–75.
[23] J. Prüss, On the qualitative behaviour of populations with age-specific interactions, *Comput. Math. Appl.,* **9** (1983), 327–339.
[24] W.E. Ricker, Stock and recruitment, *J. Fish. Res. Board Canada*, **11** (1954), 559–623.
[25] W.E. Ricker, Computation and interpretation of biological studies of fish populations, *Bull. Fish. Res. Bd. Canada,* **191** (1975).
[26] Y. Su, S. Ruan and J. Wei, Periodicity and synchronization in blood-stage malaria infection, *Journal of Mathematical Biology*, **63** (2011), 557–574.
[27] J.H. Swart, Hopf bifurcation and the stability of nonlinear age-dependent population models, *Comput. Math. Appl.,* **15** (1988), 555–564.
[28] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations,* **3** (1990), 1035–1066.
[29] H. R. Thieme, Integrated semigroups and integrated solutions to abstract Cauchy problems, *J. Math. Anal. Appl.,* **152** (1990), 416–447.
[30] H. R. Thieme, Quasi-compact semigroups via bounded perturbation, in "Advances in Mathematical Population Dynamics-Molecules, Cells and Man," ed. by O. Arino, D. Axelrod and M. Kimmel, Ser. Math. Biol. Med., **6**, World Sci. Publ., River Edge, NJ, 1997, 691–711.
[31] Z. Wang and Z. Liu, Hopf bifurcation of an age-structured compartmental pest-pathogen model, *J. Math. Anal. Appl.,* **385** (2012), 1134–1150.

[32] J. Wu, *Theory and applications of partial functional-differential equations*, Springer-Verlag, New York, 1996.

[33] P. Zhang, Z. Feng and F. Milner, A schistosomiasis model with an age structure in human hosts and its application to treatment strategies, *Math. Biosci.,* **205** (2007), 83–107.

E-mail address: zhihualiu@bnu.edu.cn
E-mail address: pierre.magal@u-bordeaux2.fr
E-mail address: tanghui@mail.bnu.edu.cn