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GLOBAL STABILITY FOR DIFFERENTIAL EQUATIONS WITH HOMOGENEOUS NONLINEARITY AND APPLICATION TO POPULATION DYNAMICS

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Abstract. In this paper we investigate global stability for a differential equation containing a positively homogeneous nonlinearity. We first consider perturbations of the infinitesimal generator of a strongly continuous semigroup which has a simple dominant eigenvalue. We prove that for "small" perturbation by a positively homogeneous nonlinearity the qualitative properties of the linear semigroup persist. From this result, we deduce a global stability result when one adds a certain type of saturation term. We conclude the paper by an application to a phenotype structured population dynamic model.

1. **Introduction.** The objective of this paper is to investigate the asymptotic behavior of solutions of abstract semilinear differential equations with homogeneous nonlinearities. The equation we consider has the form

$$\frac{du_x(t)}{dt} = Au_x(t) + g\left(u_x(t)\right) - F(u_x(t))u_x(t),\ t \geq 0,\ u_x(0) = x. \tag{1}$$

We suppose that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of positive linear operators in a Banach lattice X, g is a nonlinear operator in X_+ satisfying $g(cx) = cg(x)$, $x \in X_+$, $c \geq 0$, and F is a continuous linear functional satisfying $F(x) > 0, \forall x \in X_+ \setminus \{0\}$. Moreover, to assure the positivity of the solutions, we assume there exists $\rho_0\left(g\right) > 0$ such that $(g+\rho_0\left(g\right)Id)(X_+) \subset X_+$.

Let us now consider the positively homogeneous Cauchy problem

$$\frac{dv_x(t)}{dt} = Av_x(t) + g\left(v_x(t)\right),\ t \geq 0,\ v_x(0) = x. \tag{2}$$

In section 2, we will recall a global center manifold theorem that will be used in section 3. In section 3, we will investigate the behavior of equation (2). More precisely, we assume that A has a dominating simple real eigenvalue $\lambda_0 \in \mathbb{R}$ associated to some positive eigenvectors $\phi_0 \in X_+ \setminus \{0\}$ and $\phi_0^* \in X_+^* \setminus \{0\}$ with $\phi_0^*(\phi_0) = 1$, with $\phi_0^*(\phi) > 0, \forall \phi \in X_+ \setminus \{0\}$. Then (see Theorem 3.1) when the Lipschitz norm of g and $\rho_0\left(g\right)$ are small enough (A being fixed), there exists $\overline{v}_g \in X_+ \setminus \{0\}$ with $\|\overline{v}_g\| = 1$ and there exists $\mu_g \in \mathbb{R}$ such that

$$\overline{v}_g \in D(A),\ \text{and } \mu_g\overline{v}_g = A\overline{v}_g + g\left(\overline{v}_g\right).$$

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Moreover for each $x \in X_+ \setminus \{0\}$ there exists $\alpha_x > 0$ such that

$$\frac{v_x(t)}{e^{\mu_g t}} \to \alpha_x \overline{v}_g \text{ as } t \to +\infty.$$

A usual way to prove such a result is to use the same technics as in Wysocki [15] and Takac [9] (see also references there in). In order to apply such a technic, some compactness arguments are necessary. Here we have not such a compactness property. So we use a direct approach related with the contraction of the semiflow due to the second eigenvalue of the linear semigroup. We refer to Webb [14] for a result going in this direction.

Since g is positively homogeneous one has the following relation between the solutions of equations (1) and (2) (see Proposition 3.3)

$$u_x(t) = \frac{v_x(t)}{1 + \int_0^t F(v_x(s))ds}, \ t \geq 0, x \in X_+. \tag{3}$$

In section 4, we will use equation (3) and results of section 3, to derive a global stability result for equation (1). The global stability result proved here is a general version of Theorem 4.11 p:236 in Magal and Webb [3]. An important argument to apply the technics used in [3] is the existence of a compact global attractor. In [3] the existence of global attractor is due to the compactness of the linear semigroup. Here we avoid this problem and we allow some weaker compactness conditions on the linear semigroup. Finally, in section 5, we will apply the result to an example coming from population dynamics.

2. **A reduction result.** In this section we state some results proved in Magal [5]. These results are variations of some of the results proved in Vanderbauwhede [10][11], and in Vanderbauwhede and Iooss [12].

Let $(X, \|.\|)$ be a Banach space, let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \geq 0$, in X, and let $g : X \to X$ be a (nonlinear) operator from X into X, g is Lipschitz continuous. Then we want to reduce the following semi-linear problem:

$$S(t)u_0 = T(t)u_0 + \int_0^t T(t-s)g(S(s)u_0)ds, \forall t \geq 0. \tag{4}$$

Assumption 2.1: *a*) $X = X_s \oplus X_c$ where X_s and X_c are closed subspaces, satisfying $T(t)X_s \subset X_s$ and $T(t)X_c \subset X_c, \forall t \geq 0$. We denote $\Pi_s \in L(X)$, and $\Pi_c \in L(X)$ the projection operators satisfying $\Pi_s(X) = X_s$, $\Pi_c(X) = X_c$, and $Id - \Pi_c = \Pi_s$.
b) $\dim(X_c) < +\infty$, and $X_c \subset D(A)$ and $\sigma(A_c) \subset \mathbb{R}i$, where $A_c = A\Pi_c \in L(X)$.
c) There exist $\beta > 0$ and $M_s \geq 1$ such that

$$\|T(t)\Pi_s\|_{L(X)} \leq M_s e^{-\beta t}, \forall t \geq 0.$$

d) There exists $M_c \geq 1$ such that $\sup_{t\in\mathbb{R}} \left\|e^{A_c t}\right\|_{L(X)} \leq M_c$. The center manifold will be related with function $u \in C(\mathbb{R}, X)$ solution of

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^t T(t-s)g(u(s))ds, \forall t, t_0 \in \mathbb{R}, \text{ with } t \geq t_0. \tag{5}$$

We denote

$$Lip(X,X) = \left\{ g : X \to X : \|g\|_{Lip} = \sup_{x,y\in X: x\neq y} \frac{\|g(x)-g(y)\|}{\|x-y\|} < +\infty \right\},$$

and for all $\eta \in \mathbb{R}$,

$$BC^{\eta}(\mathbb{R}, X) = \left\{ w \in C(\mathbb{R}, X) : \|w\|_{\eta} = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|w(t)\| < +\infty \right\}.$$

The *center manifold* is defined for $\eta \in (0, \beta)$, as follows:

$$M_{\eta} = \{u_0 \in X : \text{ there exists } u \in BC^{\eta}(\mathbb{R}, X) \text{ a solution of (5), with } u(0) = u_0\}.$$

We refer to Magal [5] Theorem 1.3 for a proof of the following theorem.

Theorem 2.1. *: Under Assumption 2.1. Let $\eta \in (0, \beta)$ be fixed. Let $\delta_0 = \delta_0(A, \eta) > 0$ be such that for each $g \in Lip(X, X)$, with $\|g\|_{Lip} \leq \delta_0$, there exists a unique map $\Psi : X_c \to X_s$ which is Lipschitz continuous and such that*

$$M_{\eta} = \{x_c + \Psi(x_c) : x_c \in X_c\}.$$

We refer to Magal [5] Theorem 1.4 for a proof of the following theorem.

Theorem 2.2. *: Under Assumption 2.1. Let $\eta \in (0, \beta)$ be fixed. Then there exists $\delta_1 = \delta_1(A, \eta) > 0$, such that for each $g \in Lip(X, X)$, with $\|g\|_{Lip} \leq \delta_1$, there exists a map $H : X \to M_{\eta}$ such that for each $x \in X$,*

$$M_{\eta} \cap \widetilde{M}_{\eta}(x) = \{H(x)\},$$

where

$$\widetilde{M}_{\eta}(x) = \left\{ y \in X : \sup_{t \geq 0} e^{\eta t} \|S(t)x - S(t)y\| < +\infty \right\}.$$

3. **Homogeneous problem.** Let X_+ be a positive cone of a Banach space X, i.e. X_+ closed convex subset of X such that *i)* $\lambda X_+ \subset X_+, \forall \lambda \geq 0$, and *ii)* $X_+ \cap -X_+ = \{0\}$. Such a cone X_+ induces a partial order on X, denoted $\leq$ and defined by

$$x \leq y \Leftrightarrow y - x \in X_+.$$

In the sequel, we denote X^* the topological dual space of X (i.e. the space of continuous linear forms on X), and we denote X_+^* the dual cone defined by

$$X_+^* = \{\varphi \in X^* : \varphi(x) \geq 0, \forall x \in X_+\}.$$

We recall that a bounded linear operator $L \in L(X)$ is said to be positive if $L(X_+) \subset X_+$.

Let $(X, \|.\|)$ be a Banach lattice with positive cone X_+ (see Schaefer [8]). In this section, we will make the following assumptions.

Assumption 3.1: *a)* $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of positive bounded linear operators $T(t)$, $t \geq 0$ in X.
b) There exist $\phi_0 \in X_+ \setminus \{0\}$ and $\phi_0^* \in X_+^* \setminus \{0\}$ with $\phi_0^*(\phi_0) = 1$, such that

$$T(t)P_0 = P_0, P_1T(t) = T(t)P_1, \forall t \geq 0,$$

and there exist $\beta > 0$ and $M \geq 1$, such that

$$\|T(t)P_1\phi\| \leq Me^{-\beta t} \|P_1\phi\|, \forall \phi \in X,$$

where $P_0(\phi) = \phi_0^*(\phi)\phi_0, \forall \phi \in X$, and $P_1 = Id - P_0$.
c) ϕ_0^* is strictly positive i.e. $\phi_0^*(\phi) > 0, \forall \phi \in X_+ \setminus \{0\}$.
d) g is a nonlinear operator from X_+ to X, g is Lipschitz continuous on X_+, and there exists $\rho_0(g) > 0$ such that $(g + \rho_0(g) Id)(X_+) \subset X_+$.
e) g is positively homogeneous, i.e. $g(\lambda x) = \lambda g(x), \forall \lambda \geq 0, \forall x \in X_+$.

Remark: 1) Assumption 3.1 *b*) implies that the spectral bound of A satisfies $s(A) = 0$.
2) Assume that g is only Lipschitz continuous on bounded sets. Denote $k > 0$ the Lipschitz constant of g on $B(0,1) \cap X_+$. Then if $x, y \in B(0,M) \cap X_+$, by using Assumption 3.1 *e*) we have

$$\|g(x) - g(y)\| = M \left\| g(\frac{x}{M}) - g(\frac{y}{M}) \right\| \leq Mk \left\| \frac{x}{M} - \frac{y}{M} \right\| \leq k \|x - y\|.$$

So, g is $k-$Lipschitz on X_+.
3) Assumption 3.1 *b*) and *c*) will be verified if A is the infinitesimal generator of a strongly continuous semigroup of positive operator $T(t)$, which is quasi-compact and irreducible i.e. $\forall x \in X_+ \setminus \{0\}$, and $\forall \varphi \in X_+^* \setminus \{0\}, \exists t > 0$ such that $\varphi(T(t)x) > 0$, and the spectral bound of A satisfies

$$s(A) = 0.$$

This result can be found in the book of Nagel [6] see Theorem 2.1 p:343, and remark d) p:344. But here we don't assume that ϕ_0 is a quasi-interior point. So the situation that we consider here is more general than the case of irreducible semigroups. We also refer to Webb [13] (see Proposition 2.3 and Proposition 2.5 and Remark 2.2) for a generalized version of this result, in which the compactness conditions on the linear semigroup are weaker.

We denote $\{S_g(t)\}_{t\geq 0}$ the nonlinear semigroup solution of

$$S_g(t)x = T(t)x + \int_0^t T(t-s)g(S_g(s)x)ds, \forall t \geq 0, \forall x \in X_+. \tag{6}$$

We extend equation (6) by setting

$$\widetilde{g}(x) = g(|x|), \forall x \in X.$$

One can note (since $|\lambda x| = |\lambda| |x|, \forall \lambda \in \mathbb{R}, \forall x \in X$, see Schaefer [8] p:207) the map $\widetilde{g}$ is also positively homogeneous, namely

$$\widetilde{g}(\lambda x) = \lambda \widetilde{g}(x), \forall x \in X, \forall \lambda \geq 0.$$

Moreover, let be $x, y \in X$, (since X is a Banach lattice, and $||x| - |y|| \leq |x-y|, \forall x, y \in X$, see Schaefer [8] p:207) we deduce that

$$\begin{aligned} \|\widetilde{g}(x) - \widetilde{g}(y)\| &\leq \|g(|x|) - g(|y|)\| \leq \|g\|_{Lip} \||x| - |y|\| = \|g\|_{Lip} \|||x| - |y|||\| \\ &\leq \|g\|_{Lip} \||x-y|\| = \|g\|_{Lip} \|x-y\| \end{aligned}$$

so we obtain

$$\|\widetilde{g}(x) - \widetilde{g}(y)\| \leq \|g\|_{Lip} \|x - y\|. \tag{7}$$

So

$$\|\widetilde{g}\|_{Lip} \leq \|g\|_{Lip}. \tag{8}$$

We denote $\{S_{\widetilde{g}}(t)\}_{t\geq 0}$ the nonlinear semigroup solution of

$$S_{\widetilde{g}}(t)x = T(t)x + \int_0^t T(t-s)\widetilde{g}(S_{\widetilde{g}}(s)x)ds, \forall t \geq 0, \forall x \in X, \tag{9}$$

We start with a preliminary result.

Proposition 3.1. *: Under Assumption 3.1. For each $x \in X_+$, there exists a unique global continuous solution $t \to S_g(t)x$ of equation (6), and we have the following:*
i) $S_g(t)(0) = 0, \forall t \geq 0$.
ii) $S_g(t)(X_+ \setminus \{0\}) = X_+ \setminus \{0\}, \forall t \geq 0$.
iii) $S_g(t)(\lambda x) = \lambda S_g(t)(x), \forall t \geq 0, \forall x \in X_+, \forall \lambda \geq 0$.

For each $x \in X$, there exists a unique global solution of (9) and
iv) $S_{\widetilde{g}}(t)x = S_g(t)x, \forall t \geq 0, \forall x \in X_+$.
v) $S_{\widetilde{g}}(t)(\lambda x) = \lambda S_{\widetilde{g}}(t)(x), \forall t \geq 0, \forall x \in X, \forall \lambda \geq 0$.
We denote $S_{\varepsilon,\widetilde{g}}(t)x =^{def} e^{\varepsilon t} S_{\widetilde{g}}(t)x, \forall t \geq 0, \forall x \in X, \forall \varepsilon \in \mathbb{R}$, *then* $\{S_{\varepsilon,\widetilde{g}}(t)\}_{t \geq 0}$ *is a strongly continuous nonlinear semigroup, and*

$$\begin{aligned} S_{\varepsilon,\widetilde{g}}(t)x &= e^{\varepsilon t} T(t)x + \int_0^t e^{\varepsilon(t-s)} T(t-s) \widetilde{g}(S_{\varepsilon,\widetilde{g}}(s)x) ds, \\ &= T(t)x + \int_0^t T(t-s)(\widetilde{g} + \varepsilon Id)(S_{\varepsilon,\widetilde{g}}(s)x) ds. \end{aligned}$$

vi) $\forall \varepsilon > \rho_0(g), \forall x \in X_+ \setminus \{0\}$, *we have*

$$\|S_{\varepsilon,\widetilde{g}}(t)x\| \to +\infty, \text{ as } t \to +\infty, \tag{10}$$

where $\rho_0(g) \geq 0$ *is the constant introduced in Assumption 3.1 d).*

Proof: $i) - v)$ use classical arguments. We now prove $vi)$. Let be $x \in X_+ \setminus \{0\}$. We have

$$\begin{aligned} S_{\varepsilon,\widetilde{g}}(t)x = & e^{-(\rho_0(g)+\varepsilon)t} T(t)x \\ & + \int_0^t e^{(-\rho_0(g)+\varepsilon)(t-s)} T(t-s) \left[\widetilde{g}(S_{\varepsilon,\widetilde{g}}(s)x) + \rho_0(g) S_{\varepsilon,\widetilde{g}}(s)x\right] ds, \end{aligned}$$

so by using Assumption 3.1 $d)$ we have

$$S_{\varepsilon,\widetilde{g}}(t)x \geq e^{-\rho_0(g)t} e^{\varepsilon t} T(t)x$$

and since X is a Banach lattice we have

$$\|S_{\varepsilon,\widetilde{g}}(t)x\| \geq e^{(\varepsilon - \rho_0(g))t} \|T(t)x\|$$

and from Assumption 3.1 $b)$ we have

$$\lim_{t \to +\infty} \|T(t)x\| = \|P_0(x)\| = \phi_0^*(x) \|\phi_0\|,$$

and from Assumption 3.1 $c)$ we have $\phi_0^*(x) > 0$, and the result follows. □

The following theorem is the main result of this section.

Theorem 3.1. *: Under Assumption 3.1. Let* $\eta \in (0, \beta)$ *be fixed. Then there exists* $\delta^* = \delta^*(A, \eta) > 0$, *such that if* $\|g\|_{Lip} < \delta^*$ *and* $\rho_0(g) < \eta$ *(where* $\rho_0(g)$ *is the constant introduced in Assumption 3.1 d)), there exists* $\overline{v}_g \in X_+ \setminus \{0\}$ *with* $\|\overline{v}_g\| = 1$, *and there exists* $\mu_g \in \mathbb{R}$ *such that*

$$\overline{v}_g \in D(A), \text{ and } \mu_g \overline{v}_g = A\overline{v}_g + g(\overline{v}_g). \tag{11}$$

Moreover for each $x \in X_+ \setminus \{0\}$ *there exists* $\alpha_x > 0$ *and* $M_x \geq 0$ *such that*

$$\left\| \frac{S_g(t)x}{e^{\mu_g t}} - \alpha_x \overline{v}_g \right\| \leq M_x e^{-(\eta + \mu_g)t}, \forall t \geq 0, \tag{12}$$

and

$$\eta + \mu_g > 0. \tag{13}$$

Proof: Let be $\delta^* > 0$, such that if $\|g\|_{Lip} < \delta^*$, then Theorem 2.1 and Theorem 2.2 apply to equation (9) (which is possible because $\|\widetilde{g}\|_{Lip} \leq \|g\|_{Lip}$). We start by describing the center manifold. Let us start by noting that if $u \in BC^\eta(\mathbb{R}, X) = \left\{ w \in C(\mathbb{R}, X) : \|w\|_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|w(t)\| < +\infty \right\}$ is a solution of

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^t T(t-s)\widetilde{g}(u(s)) ds, \forall t, t_0 \in \mathbb{R}, \text{ with } t \geq t_0. \tag{14}$$

then since $\widetilde{g}$ is positively homogeneous, $\forall\lambda \geq 0$, $\lambda u(.) \in BC^{\eta}(\mathbb{R}, X)$, and $t \to \lambda u(t)$ is a solution of (14). By definition the center manifold is

$$M_\eta = \{u_0 \in X : \exists u \in BC^{\eta}(\mathbb{R}, X) \text{ a solution of (14), and } u(0) = u_0\}, \quad (15)$$

so we deduce that

$$\forall x \in M_\eta,\ \forall \lambda \geq 0,\ \lambda x \in M_\eta. \quad (16)$$

Furthermore by Theorem 2.1 there exists a map $\Psi : P_0(X) \to (Id - P_0)(X)$ such that

$$M_\eta = \{x_0 + \Psi(x_0) : x_0 \in P_0(X)\}. \quad (17)$$

In particular from (16) we deduce that $0 \in M_\eta$, so $\Psi(0) = 0$. Moreover, if $y \in M_\eta \setminus \{0\}$, then $P_0(y) \neq 0$ (because Ψ is a map). So from (16) we deduce that there exist $y_1, y_2 \in X$ with $\phi_0^*(y_1) > 0$, and $\phi_0^*(y_2) < 0$, such that

$$M_\eta = \mathbb{R}_+ y_1 \cup \mathbb{R}_+ y_2. \quad (18)$$

Furthermore, $S_{\widetilde{g}}(t)M_\eta \subset M_\eta, \forall t \geq 0$, we deduce that for each $y \in M_\eta \setminus \{0\}$, there exists a maximal open interval $I \subset \mathbb{R}$ with $0 \in I$, and there exists a continuous map $\alpha \in C(I, R)$ with $\alpha(0) = 1$ such that

$$t \to \alpha(t)y \text{ is a solution of (14), and } \alpha(t) > 0, \forall t \in I. \quad (19)$$

We first note that

$$\mathbb{R}_- \subset I. \quad (20)$$

Otherwise there exists $t_0 < 0$ such that $\alpha(t_0) = 0$, and $\alpha(t) = 0, \forall t \geq t_0$, a contradiction. Moreover, we have

$$\alpha(t)y = T(t - t_0)\alpha(t_0)y + \int_{t_0}^{t} T(t - s)\widetilde{g}(\alpha(s)y)ds, \forall t, t_0 \in I, \text{ with } t \geq t_0,$$

and since $\widetilde{g}$ is positively homogeneous, we have

$$\alpha(t)y = \alpha(t_0)T(t - t_0)y + \int_{t_0}^{t} \alpha(s)T(t - s)\widetilde{g}(y)ds, \forall t, t_0 \in I, \text{ with } t \geq t_0. \quad (21)$$

Let be $\lambda > 0$. By Assumption 3.1 *b*), λ is in the resolvent set of A, and we denote $R(\lambda, A) := (\lambda I - A)^{-1}$ the resolvent operator of A. We have for all $t, t_0 \in I$, with $t \geq t_0$,

$$\alpha(t)R(\lambda, A)y = \alpha(t_0)T(t - t_0)R(\lambda, A)y + \int_{t_0}^{t} \alpha(s)T(t - s)R(\lambda, A)\widetilde{g}(y)ds. \quad (22)$$

As $R(\lambda, A)x \in D(A), \forall x \in X$, the maps from $\mathbb{R}_+$ into X defined by

$$t \to T(t)R(\lambda, A)y \text{ and } t \to T(t)R(\lambda, A)\widetilde{g}(y),$$

are continuously differentiable. So from (22) we deduce that

$$\alpha \in C^1(I, \mathbb{R}). \quad (23)$$

Furthermore, from (21) with $t_0 = 0$,

$$T(t)y = \alpha(t)y - \int_0^t \alpha(s)T(t - s)\widetilde{g}(y)ds, \forall t \in I \cap \mathbb{R}_+,$$

so

$$T(t)y = \alpha(t)y - \int_0^t \alpha(t - l)T(l)\widetilde{g}(y)ds, \forall t \in I \cap \mathbb{R}_+,$$

and from (23), we deduce that the map $t \to T(t)y$ is continuously differentiable, so

$$y \in D(A). \quad (24)$$

Finally by using (22), we have $\forall t, t_0 \in I$, with $t \geq t_0$,

$$\alpha'(t)R(\lambda, A)y = \alpha(t_0)AT(t-t_0)R(\lambda, A)y + \int_{t_0}^{t} \alpha(s)AT(t-s)R(\lambda, A)\widetilde{g}(y)ds + \alpha(t)R(\lambda, A)\widetilde{g}(y)$$

so

$$\alpha'(t)R(\lambda, A)y = A\alpha(t)R(\lambda, A)y + \alpha(t)R(\lambda, A)\widetilde{g}(y), \forall t \in I,$$

and since $y \in D(A)$, we obtain

$$\alpha'(t)y = \alpha(t)Ay + \alpha(t)\widetilde{g}(y), \forall t \in I,$$

and since $y \neq 0$, we deduce that there exists a certain constant $\mu \in \mathbb{R}$ such that

$$\frac{\alpha'(t)}{\alpha(t)} = \mu \in \mathbb{R}, \forall t \in I.$$

So

$$I = \mathbb{R}, \text{ and } \alpha(t) = e^{\mu t}, \forall t \in \mathbb{R}, \tag{25}$$

and

$$\mu y = Ay + \widetilde{g}(y). \tag{26}$$

By using (18) we deduce that there exist $\mu^+, \mu^- \in \mathbb{R}$, and $v^+, v^- \in X \setminus \{0\}$, with $\phi_0^*(v^+) > 0$, and $\phi_0^*(v^-) < 0$, such that

$$\begin{cases} v^+, v^- \in D(A), \\ M_\eta = \mathbb{R}_+ v^+ \cup \mathbb{R}_+ v^-, \\ \text{and} \\ \mu^+ v^+ = Av^+ + \widetilde{g}(v^+), \ \mu^- v^- = Av^- + \widetilde{g}(v^-). \end{cases} \tag{27}$$

Now, by Theorem 2.2 we know that $\forall x \in X, \exists x_\eta \in M_\eta$, and there exists $M_x \geq 0$ such that

$$\|S_{\widetilde{g}}(t)x - S_{\widetilde{g}}(t)x_\eta\| \leq M_x e^{-\eta t}, \forall t \geq 0. \tag{28}$$

Let be $x \in X_+ \setminus \{0\}$. From (27) and (28) we deduce that there exist $\mu \in \mathbb{R}$, and $x_\eta \in M_\eta$, such that

$$\left\|S_{\widetilde{g}}(t)x - e^{\mu t}x_\eta\right\| \leq M_x e^{-\eta t}, \forall t \geq 0. \tag{29}$$

But since $\rho_0(g) < \eta$, we can choose $\varepsilon \in (\rho_0(g), \eta)$, and by Proposition 3.1 *vi*), we have

$$\left\|e^{\varepsilon t} S_{\widetilde{g}}(t)x\right\| \to +\infty, \text{ as } t \to +\infty. \tag{30}$$

But for all $t \geq 0$, we have

$$\begin{aligned} \left\|e^{(\mu+\varepsilon)t}x_\eta\right\| &\geq \|e^{\varepsilon t}S_{\widetilde{g}}(t)x\| - \left\|e^{\varepsilon t}S_{\widetilde{g}}(t)x - e^{(\mu+\varepsilon)t}x_\eta\right\| \\ &\geq \|e^{\varepsilon t}S_{\widetilde{g}}(t)x\| - M_x e^{(\varepsilon-\eta)t}, \end{aligned}$$

So we deduce that

$$e^{(\mu+\varepsilon)t}\|x_\eta\| \to +\infty, \text{ as } t \to +\infty. \tag{31}$$

Thus

$$x_\eta \neq 0, \text{ and } (\mu+\varepsilon) > 0, \tag{32}$$

but since $\varepsilon < \eta$, we have

$$\eta + \mu > 0. \tag{33}$$

Now by using (29) we have

$$\left\|\frac{S_{\widetilde{g}}(t)x}{e^{\mu t}} - x_\eta\right\| \leq M_x e^{-(\eta+\mu)t}, \forall t \geq 0. \tag{34}$$

So by Proposition 3.1 *ii*) and *iv*), and since $x \in X_+ \setminus \{0\}$, we know that

$$\frac{S_{\widetilde{g}}(t)x}{e^{\mu t}} \in X_+, \forall t \geq 0, \tag{35}$$

and by using (32) (33) (34) (35) and the fact that X_+ is closed, we deduce that

$$x_\eta \in X_+ \setminus \{0\}.$$

So $\phi_0^*(x_\eta) > 0$, and $x_\eta = \alpha_x v^+$ for some $\alpha_x > 0$, and $\mu = \mu^+ \in \mathbb{R}$. □

Corollary 3.1. *: Under Assumptions 3.1 a)-c). Let $g : X_+ \to X$ be a Lipschitz continuous map, such that $g(\lambda x) = \lambda g(x), \lambda \geq 0, x \in X_+$, and assume that there exists $\rho_0 > 0$ such that*

$$g(x) + \rho_0 x \in X_+, \forall x \in X_+.$$

Consider the following semilinear Cauchy problem

$$\frac{dv_x(t)}{dt} = Av_x(t) + \tau g(v_x(t)), \ t \geq 0, \ v_x(0) = x.$$

Then there exists $\tau^* > 0$ such that for all $\tau \in [0, \tau^*]$ the conclusions of Theorem 3.1 hold.

Proof: This result is a direct consequence of Theorem 3.1. □

We now prove some additional properties for the nonlinear eigenvalue μ_g and the nonlinear eigenvector $\overline{v}_g$.

Proposition 3.2. *: Under Assumption 3.1. Let $\eta \in (0, \beta)$ be fixed, and assume that $\|g\|_{Lip} < \delta^*$ and $\rho_0(g) < \eta$. Then*

$$-\eta \leq \mu_g \leq \|T\|_\infty \|g\|_{Lip},$$

with $\|T\|_\infty = \sup_{t\geq 0} \|T(t)\|_{L(X)}$ (which is finite because of Assumption 3.1 b)). Moreover

$$\mu_g \to 0, \text{ and } \overline{v}_g \to \frac{\phi_0}{\|\phi_0\|} \text{ as } \|g\|_{Lip} \to 0 \text{ (with } \rho_0(g) < \eta).$$

Proof: We first note that from Theorem 3.1 we have

$$\mu_g \geq -\eta. \tag{36}$$

Moreover, by Theorem 3.1, we also know that

$$\|S_g(t)\overline{v}_g\| = e^{\mu_g t} \|\overline{v}_g\|, \forall t \geq 0. \tag{37}$$

Furthermore, we have

$$S_g(t)\overline{v}_g = T(t)\overline{v}_g + \int_0^t T(t-s)g(S_g(s)\overline{v}_g)ds, \forall t \geq 0.$$

thus

$$\|S_g(t)\overline{v}_g\| \leq \|T\|_\infty \|\overline{v}_g\| + \|T\|_\infty \|g\|_{Lip} \int_0^t \|S_g(s)\overline{v}_g\| ds, \forall t \geq 0,$$

and by Gronwall's lemma we obtain

$$\|S_g(t)\overline{v}_g\| \leq \|T\|_\infty \|\overline{v}_g\| e^{\|T\|_\infty \|g\|_{Lip} t}, \forall t \geq 0. \tag{38}$$

Finally from equations (36) (37) and (38) we obtain

$$-\eta < \mu_g \leq \|T\|_\infty \|g\|_{Lip}. \tag{39}$$

So there exists $\delta_1^* > 0$ such that

$$\|g\|_{Lip} \leq \delta_1^* \text{ and } \rho_0(g) < \eta \Rightarrow |\mu_g| \leq \eta. \tag{40}$$

We denote

$$\widetilde{v}_g = \frac{\overline{v}_g}{\phi_0^*(\overline{v}_g)}.$$

Then one has

$$P_0(\widetilde{v}_g) = P_0(\phi_0),$$

so

$$P_0(\widetilde{v}_g - \phi_0) = 0, \text{ and } \widetilde{v}_g - \phi_0 = P_1(\widetilde{v}_g - \phi_0). \tag{41}$$

Moreover, since g is positively homogeneous, we also have

$$\mu_g \widetilde{v}_g = A\widetilde{v}_g + g(\widetilde{v}_g), \tag{42}$$

but $A\phi_0 = 0$, so

$$\mu_g \widetilde{v}_g = A(\widetilde{v}_g - \phi_0) + (g(\widetilde{v}_g) - g(\phi_0)) + g(\phi_0),$$

thus $\forall \lambda > 0$,

$$\mu_g R(\lambda, A)\widetilde{v}_g = AR(\lambda, A)(\widetilde{v}_g - \phi_0) + R(\lambda, A)(g(\widetilde{v}_g) - g(\phi_0)) + R(\lambda, A)g(\phi_0).$$

But $AR(\lambda, A) = -Id + \lambda R(\lambda, A)$, so

$$\begin{aligned}(\widetilde{v}_g - \phi_0) = (\lambda - \mu_g)R(\lambda, A)(\widetilde{v}_g - \phi_0) + R(\lambda, A)(g(\widetilde{v}_g) - g(\phi_0)) \\ +R(\lambda, A)g(\phi_0) - \mu_g R(\lambda, A)\phi_0.\end{aligned}$$

But we have $R(\lambda, A)(\phi_0) = \int_0^{+\infty} e^{-\lambda t}T(t)\phi_0 dt = \frac{1}{\lambda}\phi_0$, and $P_1(\phi_0) = 0$. So, $\forall \lambda > 0$,

$$\begin{aligned}P_1(\widetilde{v}_g - \phi_0) = (\lambda - \mu_g)R(\lambda, A)P_1(\widetilde{v}_g - \phi_0) + R(\lambda, A)P_1(g(\widetilde{v}_g) - g(\phi_0)) \\ +R(\lambda, A)P_1 g(\phi_0).\end{aligned} \tag{43}$$

Consider now the equivalent norm

$$|x| = \max(\|P_0(x)\|, \sup_{t\geq 0} \frac{\|T(t)P_1 x\|}{e^{\beta t}}), \forall x \in X,$$

which is well defined by Assumption 3.1 b). Then we have

$$|T(t)P_1 x| \leq e^{-\beta t} |P_1 x|, \forall t \geq 0, \forall x \in X,$$

so

$$|R(\lambda, A)P_1 x| = \left|\int_0^{+\infty} e^{-\lambda t}T(t)P_1 x dt\right| \leq \frac{1}{\lambda + \beta} |x|, \forall x \in X, \forall \lambda > 0. \tag{44}$$

By using equations (40) (41)(43) and (44), we obtain $\forall \lambda > 0$,

$$\begin{aligned}|P_1(\widetilde{v}_g - \phi_0)| \leq \tfrac{\lambda+\eta}{\lambda+\beta} |P_1(\widetilde{v}_g - \phi_0)| + \tfrac{1}{\lambda+\beta} |g|_{Lip} |P_1|_{L(X)} |P_1(\widetilde{v}_g - \phi_0)| \\ + \tfrac{1}{\lambda+\beta} |g|_{Lip} |P_1|_{L(X)} |\phi_0|.\end{aligned}$$

By using the fact that $\eta < \beta$, we deduce that there exist $\delta_2^* > 0$ and $0 < C < 1$ such that

$$\|g\|_{Lip} \leq \delta_2^* \text{ and } \rho_0(g) < \eta \Rightarrow \frac{\lambda + \eta}{\lambda + \beta} + \frac{1}{\lambda + \beta} |g|_{Lip} |P_1|_{L(X)} \leq C.$$

We have

$$(1 - C) |P_1(\widetilde{v}_g - \phi_0)| \leq \frac{1}{\lambda + \beta} |g|_{Lip} |P_1|_{L(X)} |\phi_0|$$

By using (41) we obtain

$$|\widetilde{v}_g - \phi_0| \leq \frac{1}{1 - C} \frac{1}{\lambda + \beta} |g|_{Lip} |P_1|_{L(X)} |\phi_0|,$$

so $\widetilde{v}_g \to \phi_0$ as $|g|_{Lip} \to 0$. Thus

$$\overline{v}_g = \frac{\widetilde{v}_g}{\|\widetilde{v}_g\|} \to \frac{\phi_0}{\|\phi_0\|} \text{ as } |g|_{Lip} \to 0.$$

Furthermore, we have $\mu_g \widetilde{v}_g = A\widetilde{v}_g + g(\widetilde{v}_g)$ and since $s(A) = 0$, we have

$$\mu_g = \phi_0^* \left(g(\widetilde{v}_g) \right)$$

thus

$$|\mu_g| \leq \|\phi_0^*\|_{X^*} \|g\|_{Lip} \|\widetilde{v}_g\| \to 0 \text{ as } |g|_{Lip} \to 0. \qquad \square$$

We conclude this section with a result relating solution of equations (1) and (2).

Proposition 3.3. *: Under Assumptions 3.1 a), d), e), and assume that $F \in X_+^*$. Then for each $x \in X_+$, the semilinear problem*

$$u_x(t) = T(t)x + \int_0^t T(t-s) \left[g\left(u_x(s)\right) - F(u_x(s))u_x(s) \right] ds, \forall t \geq 0, \qquad (45)$$

admits a unique mild solution $u_x \in C\left([0, +\infty), X\right)$ which is given by

$$u_x(t) = \frac{v_x(t)}{1 + \int_0^t F(v_x(s))ds}, \forall t \geq 0,$$

where $v_x(t)$ is the unique solution of the semilinear problem

$$v_x(t) = T(t)x + \int_0^t T(t-s) g\left(v_x(s)\right) ds, \forall t \geq 0. \qquad (46)$$

Proof: We start by noting that the problem (46) has a unique $v_x(t)$. Also for $\lambda > 0$, we have

$$\begin{aligned} (\lambda Id - A)^{-1} v_x(t) &= (\lambda Id - A)^{-1} T(t)x + (\lambda Id - A)^{-1} \textstyle\int_0^t T(t-s) g\left(v_x(s)\right) ds \\ &= T(t) (\lambda Id - A)^{-1} x + \textstyle\int_0^t T(t-s) (\lambda Id - A)^{-1} g\left(v_x(s)\right) ds \end{aligned}$$

so by applying Theorem 2.4 p:107 in Pazy [7], we deduce that

$$\begin{cases} \frac{dv_{x,\lambda}(t)}{dt} = Av_{x,\lambda}(t) + (\lambda Id - A)^{-1} g\left(v_x(t)\right), \forall t \geq 0, \\ v_{x,\lambda}(0) = (\lambda Id - A)^{-1} x, \end{cases}$$

where

$$v_{x,\lambda}(t) = (\lambda Id - A)^{-1} v_x(t), \forall t \geq 0.$$

We denote

$$\widetilde{u}_{\lambda,x}(t) = \frac{v_{x,\lambda}(t)}{1 + \int_0^t F(v_x(s))ds}, \text{ and } \widetilde{u}_x(t) = \frac{v_x(t)}{1 + \int_0^t F(v_x(s))ds} \forall t \geq 0.$$

One can note that since $x \in X_+$, and $F \in X_+^*$, $\widetilde{u}_{\lambda,x}(t)$ and $\widetilde{u}_x(t)$ are defined for all $t \geq 0$. Moreover we have

$$\begin{cases} \frac{d\widetilde{u}_{\lambda,x}(t)}{dt} = A\widetilde{u}_{\lambda,x}(t) + \frac{1}{1+\int_0^t F(v_x(s))ds} (\lambda Id - A)^{-1} g\left(v_x(t)\right) \\ \qquad - \frac{v_{x,\lambda}(t)}{\left(1+\int_0^t F(v_x(s))ds\right)^2} F(v_x(s)), \forall t \geq 0, \\ \widetilde{u}_{\lambda,x}(0) = (\lambda Id - A)^{-1} x, \end{cases}$$

and since g is positively homogeneous,

$$\begin{cases} \frac{d\widetilde{u}_{\lambda,x}(t)}{dt} = A\widetilde{u}_{\lambda,x}(t) + (\lambda Id - A)^{-1} \left[g\left(\widetilde{u}_x(t)\right) - F(\widetilde{u}_x(t))\widetilde{u}_x(t) \right], \forall t \geq 0, \\ \widetilde{u}_{\lambda,x}(0) = (\lambda Id - A)^{-1} x, \end{cases}$$

So

$$\begin{aligned}\widetilde{u}_{\lambda,x}(t) &= T(t)\left(\lambda Id - A\right)^{-1} x \\ &+ \int_0^t T(t-s)\left(\lambda Id - A\right)^{-1}\left[g\left(\widetilde{u}_x(s)\right) - F(\widetilde{u}_x(s))\widetilde{u}_x(s)\right] ds\end{aligned}$$

we deduce that

$$\widetilde{u}_x(t) = T(t)x + \int_0^t T(t-s)\left[g\left(\widetilde{u}_x(s)\right) - F(\widetilde{u}_x(s))\widetilde{u}_x(s)\right] ds, \forall t \geq 0,$$

and by uniqueness of the solution of problem (45) (see Cazenave and Haraux [2] Lemma 4.3.2 p:56) the result follows. □

4. **Global stability.** In this section we investigate the global stability for the following Cauchy problem

$$\frac{du_x(t)}{dt} = Au_x(t) + \tau g(u_x(t)) - F(u_x(t))u_x(t),\ t \geq 0,\ u_x(0) = x. \tag{47}$$

Let $(X, \|.\|)$ be a Banach lattice with positive cone X_+. In this section, we will make the following assumptions.

Assumption 4.1: *a*) $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of positive bounded linear operators $T(t)$, $t \geq 0$ in X.

b) There exist $\phi_0 \in X_+ \setminus \{0\}$ and $\phi_0^* \in X_+^* \setminus \{0\}$ with $\phi_0^*(\phi_0) = 1$, such that

$$T(t)P_0 = e^{s(A)t}P_0, P_1T(t) = T(t)P_1, \forall t \geq 0,$$

and there exist $\beta > 0$ and $M \geq 1$, such that

$$\|T(t)P_1\phi\| \leq Me^{(s(A)-\beta)t}\|P_1\phi\|, \forall \phi \in X,$$

where $P_0(\phi) = \phi_0^*(\phi)\phi_0, \forall \phi \in X$, and $P_1 = Id - P_0$.

c) ϕ_0^* is strictly positive i.e. $\phi_0^*(\phi) > 0, \forall \phi \in X_+ \setminus \{0\}$.

d) $g : X_+ \to X$ is Lipschitz continuous, and there exists $\rho_0 > 0$ such that

$$g(x) - \rho_0 x \in X_+, \forall x \in X_+.$$

e) $g(\lambda x) = \lambda g(x), \lambda \geq 0, x \in X_+$.

f) Let $F \in X^*$ be a linear functional which is strictly positive i.e. $F(x) > 0, \forall x \in X_+ \setminus \{0\}$.

We denote $\{T_0(t)\}_{t\geq 0}$ the strongly continuous semigroup of linear positive operators defined by

$$T_0(t) = e^{-s(A)t}T(t), t \geq 0,$$

we denote $\{U_\tau(t)\}_{t\geq 0}$ the strongly continuous semigroup of nonlinear operator from X_+ into X_+ solution of

$$U_\tau(t)x = T(t)x + \int_0^t T(t-s)\left[\tau g\left(U_\tau(s)x\right) - F(U_\tau(s)x)U_\tau(s)x\right] ds, \forall t \geq 0, \forall x \in X_+,$$

and we denote $\{S_\tau(t)\}_{t\geq 0}$ the strongly continuous semigroup of nonlinear operator from X_+ into X_+ solution of

$$S_\tau(t)x = T_0(t)x + \int_0^t T_0(t-s)\left[\tau g\left(S_\tau(s)x\right)\right] ds, \forall t \geq 0, \forall x \in X_+.$$

Theorem 4.1. *: Under Assumption 4.1. Then* $\forall \tau \geq 0$, *i*) $U_\tau(t)0 = 0, \forall t \geq 0;$ *ii*) $U_\tau(t)(X_+ \setminus \{0\}) \subset X_+ \setminus \{0\}, \forall t \geq 0$. *Moreover there exists* $\tau^* > 0$ *such that* $\forall \tau \in [0, \tau^*]$, *there exists* $v_\tau \in D(A) \cap X_+ \setminus \{0\}$, *with* $\|v_\tau\| = 1$, $\mu_\tau \in \mathbb{R}$ *and*

$$\left(\mu_\tau + s(A)\right)v_\tau = Av_\tau + \tau g\left(v_\tau\right),$$

and $\forall \tau \in [0, \tau^*]$, $\forall x \in X_+ \setminus \{0\}$,

$$\frac{U_\tau(t)x}{\|U_\tau(t)x\|} \to v_\tau \text{ as } t \to +\infty.$$

Furthermore $\forall \tau \in [0,\tau^*]$, $\forall x \in X_+ \setminus \{0\}$,
a) If $\mu_\tau + s(A) \leq 0$, *then*

$$U_\tau(t)x \to 0, \text{ as } t \to +\infty.$$

b) If $\mu_\tau + s(A) > 0$,

$$U_\tau(t)x \to \overline{u}_\tau := (s(A) + \mu_\tau)\frac{v_\tau}{F(v_\tau)}, \text{ as } t \to +\infty.$$

Assume in addition that $s(A) > 0$, *and that* g *can be extended locally around each point of* $X_+ \setminus \{0\}$ *by a continuously differentiable map. Then there exists* $\widehat{\tau} \in (0,\tau^*]$ *such that for all* $\tau \in [0,\widehat{\tau}]$, *we have*

$$s(A) + \mu_\tau > 0, \text{ and } \overline{u}_\tau \text{ is exponentially asymptotically stable.}$$

Proof: By Proposition 3.3 we have

$$U_\tau(t)x = \frac{e^{s(A)t}S_\tau(t)x}{1+\int_0^t F(e^{s(A)s}S_\tau(s)x)ds}, \forall t \geq 0, \forall x \in X_+. \tag{48}$$

By Corollary 3.1, there exists $\tau^* > 0$ such that $\forall \tau \in [0,\tau^*]$, $\forall x \in X_+ \setminus \{0\}$, $\exists \alpha_x > 0$, such that

$$\frac{S_\tau(t)x}{e^{\mu_\tau t}} \to \alpha_x v_\tau, \text{ as } t \to +\infty, \tag{49}$$

where $v_\tau \in D(A) \cap X_+ \setminus \{0\}$, with $\|v_\tau\| = 1$, $\mu_\tau \in \mathbb{R}$ and

$$\mu_\tau v_\tau = (A - s(A)Id)\, v_\tau + \tau g(v_\tau).$$

So in particular

$$\frac{U_\tau(t)x}{\|U_\tau(t)x\|} = \frac{S_\tau(t)x}{\|S_\tau(t)x\|} \to v_\tau, \text{ as } t \to +\infty.$$

We now prove a). Assume that $s(A) + \mu_\tau < 0$. Then by using (48) and (49) we deduce that

$$U_\tau(t)x = \frac{S_\tau(t)x}{e^{\mu_\tau t}} \frac{e^{(s(A)+\mu_\tau)t}}{1+\int_0^t F(e^{s(A)s}S_\tau(s)x)ds} \leq \frac{S_\tau(t)x}{e^{\mu_\tau t}} e^{(s(A)+\mu_\tau)t} \to 0, \text{ as } t \to +\infty.$$

Assume now that $s(A) + \mu_\tau = 0$, then

$$U_\tau(t)x = \frac{S_\tau(t)x}{e^{\mu_\tau t}} \frac{1}{1+\int_0^t F(\frac{S_\tau(s)x}{e^{\mu_\tau s}})ds},$$

and since

$$F(\frac{S_\tau(t)x}{e^{\mu_\tau t}}) \to \alpha_x F(v_\tau) > 0, \text{ as } t \to +\infty,$$

we deduce that

$$1 + \int_0^t F(\frac{S_\tau(s)x}{e^{\mu_\tau s}})ds \to +\infty, \text{ as } t \to +\infty,$$

so by using (48) and (49)

$$U_\tau(t)x \to 0, \text{ as } t \to +\infty.$$

We now prove b). Assume that $s(A) + \mu_\tau > 0$. Then

$$U_\tau(t)x = \frac{S_\tau(t)x}{e^{\mu_\tau t}} \frac{e^{(s(A)+\mu_\tau)t}}{1+\int_0^t F(e^{s(A)s}S_\tau(s)x)ds},$$

and $\frac{S_\tau(t)x}{e^{\mu_\tau t}} \to \alpha_x v_\tau$, as $t \to +\infty$, thus

$$\frac{(s(A)+\mu_\tau)e^{(s(A)+\mu_\tau)t}}{F(e^{s(A)s}S_\tau(t)x)} \to (s(A)+\mu_\tau)\frac{1}{F(\alpha_x v_\tau)}, \text{ as } t \to +\infty,$$

and

$$1+\int_0^t F(e^{s(A)s}S_\tau(s)x)ds \to +\infty, \text{ as } t \to +\infty,$$

so by applying l'Hospital's rule we deduce that

$$\frac{e^{(s(A)+\mu_\tau)t}}{1+\int_0^t F(e^{s(A)s}S_\tau(s)x)ds} \to (s(A)+\mu_\tau)\frac{1}{F(\alpha_x v_\tau)}, \text{ as } t \to +\infty.$$

Thus

$$U_\tau(t)x \to \overline{u}_\tau := (s(A)+\mu_\tau)\frac{v_\tau}{F(v_\tau)}, \text{ as } t \to +\infty.$$

We now assume that $s(A) > 0$, and that g can be extended locally around each point of $X_+ \setminus \{0\}$ by a continuously differentiable map. We now prove that for all $\tau > 0$ small enough $\overline{u}_\tau$ is locally stable. From Proposition 3.2 we know that

$$\mu_\tau \leq \tau \|T\|_\infty \|g\|_{Lip}, \forall \tau \in [0, \tau^*],$$

and since $s(A) > 0$, there exists $\tau_1^* > 0$, such that

$$s(A) + \mu_\tau \geq c > 0, \forall \tau \in [0, \tau_1^*].$$

The generator of the linearized equation of equation (47) at $\overline{u}_\tau$ is given by

$$A_\tau x := Ax + \tau Dg(\overline{u}_\tau)x - F(\overline{u}_\tau)x - F(x)\overline{u}_\tau, \forall x \in D(A).$$

and by definition of $\overline{u}_\tau$ we have

$$F(\overline{u}_\tau) = (s(A)+\mu_\tau),$$

so

$$A_\tau x = Ax - (s(A)+\mu_\tau)x + \tau Dg(\overline{u}_\tau)x - F(x)\overline{u}_\tau, \forall x \in D(A),$$

We denote $B_\tau \in L(X)$ the bounded linear operator defined by

$$B_\tau(x) = -\mu_\tau x + \tau Dg(\overline{u}_\tau)x - F(x)\left(\overline{u}_\tau - s(A)\frac{\phi_0}{F(\phi_0)}\right), \forall x \in X.$$

Then

$$A_\tau x = (A - s(A)Id)x - F(x)s(A)\frac{\phi_0}{F(\phi_0)} + B_\tau(x), \forall x \in D(A).$$

Since g is differentiable on X_+ and $X = X_+ - X_+$, we deduce that

$$\|Dg(x)\| \leq 2\|g\|_{Lip}, \forall x \in X_+ \setminus \{0\}. \tag{50}$$

Indeed for $h \in X_+$, $x \in X_+ \setminus \{0\}$, and $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon}\|g(\varepsilon h + x) - g(x)\| \leq \|g\|_{Lip}\|h\|$$

so

$$\|Dg(x)h\| \leq \|g\|_{Lip}\|h\|.$$

For $h \in X$, we have

$$\|Dg(x)h\| \leq \|Dg(x)h^+\| + \|Dg(x)h^-\| \leq \|g\|_{Lip}\left[\|h^+\| + \|h^-\|\right] \leq 2\|g\|_{Lip}\|h\|.$$

From (50) we deduce that

$$\begin{aligned}\|B_\tau\|_{L(X)} &\leq |\mu_\tau| + \tau 2\|g\|_{Lip} + \|F\|_{X^*}\left\|(s(A)+\mu_\tau)\tfrac{v_\tau}{F(v_\tau)} - s(A)\tfrac{\phi_0}{F(\phi_0)}\right\| \\ &\leq |\mu_\tau| + \tau 2\|g\|_{Lip} + \tfrac{\|F\|_{X^*}}{F(\phi_0)}\left\|(s(A)+\mu_\tau)\tfrac{v_\tau F(\phi_0)}{F(v_\tau)} - s(A)\phi_0\right\|\end{aligned}$$

and from Proposition 3.2, we deduce that for each $\varepsilon > 0$, there exists $\widehat{\tau} \in (0, \tau_1^*]$, such that

$$\|B_\tau\|_{L(X)} \leq \varepsilon, \forall \tau \in [0, \widehat{\tau}].$$

It now remains to investigate the exponential asymptotic stability of the linear semigroup generated by

$$A_0 x := (A - s(A)Id)\, x - F(x)s(A)\frac{\phi_0}{F(\phi_0)}, \forall x \in D(A).$$

But A_0 is the generator of $T_1(t)$ solution of

$$\begin{aligned} T_1(t)x &= T_0(t)x + \int_0^t T_0(t-s)\left(-F(T_1(s)x)s(A)\tfrac{\phi_0}{F(\phi_0)}\right)ds \\ &= T_0(t)x - s(A)\tfrac{\phi_0}{F(\phi_0)}\int_0^t F(T_1(s)x)ds \end{aligned}$$

Consider now the equivalent norm

$$|x| = \max(\|P_0(x)\|, \sup_{t\geq 0}\frac{\|T_0(t)P_1x\|}{e^{-\varepsilon t}}), \forall t \geq 0, \forall x \in X,$$

where

$$0 < \varepsilon < \min(\beta, s(A)). \tag{51}$$

Then we have

$$|P_1T_1(t)x| = |P_1T_0(t)x| \leq e^{-\varepsilon t}, \forall t \geq 0. \tag{52}$$

Moreover,

$$\phi_0^*(T_1(t)x) = \phi_0^*(x) - \int_0^t F(T_1(s)x)ds\; s(A)\frac{\phi_0^*(\phi_0)}{F(\phi_0)}, \forall t \geq 0.$$

So

$$\frac{d\phi_0^*(T_1(t)x)}{dt} = -s(A)\frac{1}{F(\phi_0)}F(T_1(t)x), \forall t \geq 0.$$

But

$$F(T_1(t)x) = F(P_0T_1(t)x + P_1T_1(t)x) = \phi_0^*(T_1(t)x)\,F(\phi_0) + F(P_1T_1(t)x),$$

thus

$$\frac{d\phi_0^*(T_1(t)x)}{dt} = -s(A)\phi_0^*(T_1(t)x) - s(A)\frac{1}{F(\phi_0)}F(P_1T_1(t)x), \forall t \geq 0.$$

thus by setting $\gamma(t) = e^{s(A)t}\phi_0^*(T_1(t)x)$, we have

$$\frac{d\gamma(t)}{dt} = -s(A)\frac{1}{F(\phi_0)}e^{s(A)t}F(P_1T_1(t)x),$$

thus

$$\gamma(t) = \gamma(0) - s(A)\frac{1}{F(\phi_0)}\int_0^t e^{s(A)s}F(P_1T_1(s)x)ds,$$

so, by using equations (51) and (52) we obtain

$$\begin{aligned} |\gamma(t)| &\leq |\phi_0^*(x)| + s(A)\tfrac{|F|_{X^*}}{F(\phi_0)}\int_0^t e^{(s(A)-\varepsilon)s}ds\,|x| \\ &\leq |\phi_0^*(x)| + s(A)\tfrac{|F|_{X^*}}{F(\phi_0)}\tfrac{1}{s(A)-\varepsilon}e^{(s(A)-\varepsilon)t}\,|x| \end{aligned}$$

thus

$$|\phi_0^*(T_1(t)x)| \leq e^{-s(A)t}\,|\phi_0^*(x)| + s(A)\frac{|F|_{X^*}}{F(\phi_0)}\frac{1}{s(A)-\varepsilon}e^{-\varepsilon t}\,|x|, \forall t \geq 0,$$

and the exponential stability follows. □

5. **Application to a population dynamics model.** In this section we investigate a model which was already considered in Magal [4]. In this model we consider the evolution of a population with a continuous varying phenotype.

$$\begin{cases} \frac{du}{dt} = \gamma\left[L(u(t)) - u(t)\right] + \beta u(t) + \tau\left[R(u(t)) - u(t)\right] - \frac{1}{K}\int_0^1 \chi(\widehat{y})u(t)(\widehat{y})d\widehat{y}u(t) \\ u(0) = u_0 \in L^1_+(0,1), \end{cases} \tag{53}$$

where $u(t)(y)$ is the density of population, $\gamma \geq 0$ is the mutation rate, $\beta \in L^\infty((0,1),\mathbb{R})$ represents the fitness of individuals with respect to the phenotype, $\tau > 0$ is the recombination rate, $K > 0$ is the carrying capacity, and $\chi \in L^\infty_+((0,1),\mathbb{R})$ is positive almost everywhere. The bounded linear operator $L \in L(L^1(0,1), L^1(0,1))$ is defined by

$$L(\varphi)(y) = \int_0^1 K_0(y,\widehat{y})\varphi(\widehat{y})d\widehat{y},$$

the nonlinear operator $R : L^1_+(0,1) \to L^1_+(0,1)$ is defined by

$$R(\varphi)(y) = \begin{cases} \frac{\int_0^1 K_1(y,\widehat{y})\varphi(2y-\widehat{y})\varphi(\widehat{y})d\widehat{y}}{\int_0^1 \varphi(\widehat{y})d\widehat{y}}, \text{ if } \varphi \in L^1_+(0,1)\setminus\{0\}, \\ 0 \text{ if } \varphi = 0. \end{cases}$$

and the kernels K_0 and K_1 are defined by

$$K_0(y,\widehat{y}) = \begin{cases} \frac{1}{1-\alpha} \text{ if } 0 < \widehat{y} < 1, \text{ and } \alpha\widehat{y} < y < \alpha\widehat{y} + 1 - \alpha, \\ 0 \text{ elsewhere,} \end{cases}, \text{ with } 0 \leq \alpha < 1;$$

and

$$K_1(y,\widehat{y}) = \begin{cases} 2 \text{ if } 0 \leq y \leq \frac{1}{2} \text{ and } 0 \leq \widehat{y} \leq 2y \\ 2 \text{ if } \frac{1}{2} \leq y \leq 1 \text{ and } 2y-1 \leq \widehat{y} \leq 1 \\ 0 \text{ elsewhere.} \end{cases}$$

From now we denote

$$F(\varphi) = \frac{1}{K}\int_0^1 \chi(\widehat{y})\varphi(\widehat{y})d\widehat{y}, \forall\varphi \in L^1((0,1),\mathbb{R})$$

$$\underline{\beta} = \inf_{y\in(0,1)} ess\ \beta(y), \text{ and } \overline{\beta} = \sup_{y\in(0,1)} ess\ \beta(y).$$

In (53) $u = u(t,y)$ is the density of population with respect to a phenotype variable $y \in (0,1)$ at time t. The subpopulation of phenotype at time t in the range $[y_1, y_2] \subseteq (0,1)$ is given by $\int_{y_1}^{y_2} u(t,y)dy$. The population is viewed as evolving over time due to the three separate processes of mutation, selection, and recombination. In (53) the mutation process is represented by the kernel operator $\gamma\left[L(u) - u\right]$, where γ is the mutation rate, and $0 \leq \alpha < 1$ corresponds to a rate of movement in y per unit time for an individual which mutates. For example, if $\alpha = 1$, the result of the mutation of individuals during a unit of time will give a constant distribution. In (53) the selection process for the population depends on the fitness of individuals with respect to the phenotype represented by the function $\beta(y)$. Fitness is variable in y and the sign of $\beta(y)$ may be positive or negative. In (53) there is also a density dependent mortality independent of phenotype represented by the crowing term $\frac{1}{K}\int_0^1 u(t)(\widehat{y})d\widehat{y}$. The problem (53) also incorporates DNA exchange in phenotype evolution represented by the term $\tau\left[R(u(t)) - u(t)\right]$. The recombination operator R corresponds to the average rate at which two parent phenotypes y_1 and y_2 hybridize to yield offspring with phenotype $\frac{y_1+y_2}{2}$. This form of recombination inheritance is an idealization and other genetic recombination processes could be treated in similar way. Problem (53) thus models the evolution of phenotype structure from the initial phenotype distribution $u_0 \in L^1_+(0,1)$ at time 0 subject to these processes.

We also refer to Magal and Webb [3] for similar model where the mutation process is represented by a diffusion operator with Neumann boundary conditions. We refer to the book by Burger [1] for a comprehensive and update treatment of this topics.

In this section, we denote

$$X = L^1(0,1),\ X_+ = L^1_+(0,1).$$

We start with some properties of the operator R. The following result is proved in [3] (see Theorem 2.1).

Theorem 5.1. *R is a nonlinear operator from X_+ to X_+ satisfying the following properties:*
i) R is positive homogeneous, i.e. $R(c\phi) = cR(\phi), \forall \phi \in X_+, \forall c \geq 0$;
ii) R is Lipschitz continuous in X_+;

We now recall some properties of the operator L. The following theorem is proved in [4] (see Theorem 2.2).

Theorem 5.2. *The bounded linear operator $L \in L(X,X)$ satisfies the following properties:*
i) L is compact;
ii) L is irreducible;
iii) The spectrum of L is $\sigma(L) = \{\alpha^k : k = 0,1,2,...\} \cup \{0\}$, and $\forall k \geq 0$ the eigenvalue α^k is simple;

We are now interested in the linear part of the equation (53). So, we first consider the bounded linear operator

$$A_0 = \gamma (L - Id).$$

From Theorem 5.2, the spectrum of A_0 is

$$\sigma(A_0) = \{\gamma(\alpha^k - 1) : k = 0,1,2,...\} \cup \{-\gamma\}.$$

The following theorem can be found in [4] (see Theorem 2.3).

Theorem 5.3. *The bounded linear operator A_0 generates a uniformly continuous semigroup (see Pazy [7]) $T_0(t) = e^{A_0 t}$ which satisfies the following properties:*
i) $\int_0^1 T_0(t)(\phi)(y)dy = \int_0^1 \phi(y)dy, \forall \phi \in X_+$;
ii) $T_0(t) = e^{-\gamma t} Id + C(t)$, where $C(t) \in L(X)$ is compact $\forall t \geq 0$;
iii) $T_0(t)$ is irreducible, more precisely we have

$$x^*(T_0(t)x) > 0, \forall x \in X_+ \setminus \{0\}, \forall x^* \in X_+^* \setminus \{0\}, \forall t > 0;$$

iv) $T_0(t)$ has the property of asynchronous exponential growth, that is,

$$\lim_{t\to+\infty} T_0(t)\phi = P_0(\phi), \forall \phi \in X,$$
and
$$\|T_0(t)P_1\phi\| \leq Me^{-\gamma(1-\alpha)t} \|P_1\phi\|, \forall \phi \in X,$$

for some $M \geq 1$, with $P_0(\phi) = \int_0^1 \phi(y)dy\phi_0$, $\forall \phi \in X$, for some $\phi_0 \in X_+ \setminus \{0\}$ (with $\int_0^1 \phi_0(y)dy = 1$), and $P_1 = Id - P_0$.

We now recall some definitions taken from the book of Nagel [6]. We recall that given an operator $A \in L(X)$ then $s(A)$ the *spectral bound of A* is defined by

$$s(A) =^{def} \sup\{Re(\lambda) : \lambda \in \sigma(A)\},$$

and given a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ the *growth bound of* T is defined by

$$w(A) =^{def} \inf \left\{ w \in \mathbb{R} : \|T(t)\| \leq M_w e^{wt}, \forall t \geq 0 \text{ and suitable } M_w \right\}.$$

We also recall that a strongly continuous semigroup $T(t)$ on a Banach space X is called *quasi-compact* if

$$\lim_{t\to+\infty} dist(T(t), K(X)) = 0,$$

where $K(X)$ denotes the set of compact bounded linear operator on X, and for $T \in L(X)$

$$dist(T, K(X)) =^{def} \inf \left\{ \|T - L\| : L \in K(X) \right\}.$$

We will say that $\phi \in X_+$ is *quasi-interior* if $\phi^*(\phi) > 0$ $\forall \phi^* \in X_+^* \setminus \{0\}$, and $\phi^* \in X_+^*$ is *strictly positive* if $\phi^*(\phi) > 0$ $\forall \phi \in X_+ \setminus \{0\}$. We now consider the following bounded linear operator

$$A_{\gamma,\beta}\varphi = \gamma(L - Id)\phi + \beta\phi, \forall \phi \in X.$$

we denotes

$$B\phi = \beta\phi(y), \forall \phi \in X, \forall t \geq 0, a.e. \text{ on } (0,1).$$

Theorem 5.4. *The bounded linear operator* $A_{\gamma,\beta}$ *generates a uniformly continuous semigroup* $T_{\gamma,\beta}(t) = e^{A_{\gamma,\beta}t}$ *which satisfies the following properties:*
i) $T_{\gamma,\beta}(t) = e^{(-\gamma Id + B)t} + C_{\gamma,\beta}(t)$, *where* $C_{\gamma,\beta}(t) \in L(X)$ *is compact* $\forall t \geq 0$;
ii) $T_{\gamma,\beta}(t)$ *is positive and irreducible.*
iii) There exist $\widetilde{\phi}_0 \in X_+ \setminus \{0\}$ *quasi-interior, and* $\widetilde{\phi}_0^* \in X_+^* \setminus \{0\}$ *strictly positive, with* $\int_0^1 \widetilde{\phi}_0^*(y)\widetilde{\phi}_0(y)dy = 1$ *such that*

$$T_{\gamma,\beta}(t)\widetilde{\phi}_0 = e^{s(A_{\gamma,\beta})t}\widetilde{\phi}_0, \text{ and } T_{\gamma,\beta}^*(t)\widetilde{\phi}_0^* = e^{s(A_{\gamma,\beta})t}\widetilde{\phi}_0^*, \forall t \geq 0.$$

iv)

$$\underline{\beta} \leq s(A_{\gamma,\beta}) \leq \overline{\beta}$$

v) There exists $C > 0$ *such that*

$$\widetilde{\phi}_0^*(\phi) \geq C \|\phi\|, \forall \phi \in X_+.$$

vi) $s(A_{\gamma,\beta}) = w(A_{\gamma,\beta})$, *and*

$$\|T_{\gamma,\beta}(t)\| \leq \frac{1}{C} \left\| \widetilde{\phi}_0^* \right\|_{X^*} e^{s(A_{\gamma,\beta})t}, \forall t \geq 0.$$

Assume in addition that

$$\text{either a) } s(A_{\gamma,\beta}) \geq 0 \text{ and } \gamma > \overline{\beta}, \text{ or b) } \gamma > \overline{\beta} - \underline{\beta},$$

then $e^{-s(A_{\gamma,\beta})t}T_{\gamma,\beta}(t)$ *is a quasi-compact strongly continuous semigroup, and*
vii) $T_{\gamma,\beta}(t)$ *has the property of asynchronous exponential growth. Namely*

$$\lim_{t\to+\infty} e^{-s(A_{\gamma,\beta})t}T_{\gamma,\beta}(t)\phi = \widetilde{P}_0(\phi), \forall \phi \in X,$$
and
$$\left\| e^{-s(A_{\gamma,\beta})t}T_{\gamma,\beta}(t)\widetilde{P}_1\phi \right\| \leq Me^{-\delta t} \left\| \widetilde{P}_1\phi \right\|, \forall \phi \in X,$$

for some $M \geq 1$, *and some* $\delta > 0$, *with* $\widetilde{P}_0(\phi) = \int_0^1 \widetilde{\phi}_0^*(y)\phi(y)dy\widetilde{\phi}_0$, $\forall \phi \in X$, *and* $\widetilde{P}_1 = Id - \widetilde{P}_0$.

Proof: *i*) follows directly from the fact that L is compact, and the variation of constant formula

$$T_{\gamma,\beta}(t)\phi = e^{(-\gamma Id+B)t}\phi + \int_0^t e^{(-\gamma Id+B)(t-s)}\gamma L T_{\gamma,\beta}(s)(\phi)ds, \forall t \geq 0.$$

So if one denotes $C_{\gamma,\beta}(t)(\phi) = \int_0^t e^{(-\gamma Id+B)(t-s)}\gamma L T_{\gamma,\beta}(s)(\phi)ds$, it is clear that $C_{\gamma,\beta}(t)$ is compact.
Assertion *ii*) follows from the fact that $\forall \mu \in \mathbb{R}, \forall \phi \in X, \forall t \geq 0$,

$$T_{\gamma,\beta}(t)\phi = e^{-\mu t}e^{\gamma(L-Id)t}\phi + \int_0^t e^{-\mu(t-s)}e^{\gamma(L-Id)(t-s)}(\mu Id + B)T_{\gamma,\beta}(s)(\phi)ds.$$

So by taking

$$\mu \geq -\underline{\beta},$$

we obtain

$$T_{\gamma,\beta}(t)\phi \geq e^{-\mu t}e^{\gamma(L-Id)t}\phi, \forall \phi \in X_+, \forall t \geq 0, \tag{54}$$

and since by Theorem 5.3 the semigroup $e^{\gamma(L-Id)t}$ is irreducible, assertion *ii*) follows.
To prove *iii*) it is sufficient to use the facts that $X = L^1(0,1)$ is a Banach lattice, and that $T_{\gamma,\beta}(t)$ is irreducible, then by applying Proposition 3.5 p: 310 in Nagel [6] *iii*) follows.
We now prove *iv*). From *iii*) there exists $\widetilde{\phi}_0 \in X_+ \setminus \{0\}$ such that

$$s(A_{\gamma,\beta})\widetilde{\phi}_0 = \gamma(L - Id)\left(\widetilde{\phi}_0\right) + \beta\widetilde{\phi}_0,$$

thus

$$(s(A_{\gamma,\beta}) + \gamma)\int_0^1 \widetilde{\phi}_0(y)dy = \gamma\int_0^1 L(\widetilde{\phi}_0)(y)dy + \int_0^1 \beta(y)\widetilde{\phi}_0(y)dy.$$

By using *ii*) in Theorem 5.2 we obtain

$$(s(A_{\gamma,\beta}) + \gamma)\int_0^1 \widetilde{\phi}_0(y)dy = \gamma\int_0^1 \widetilde{\phi}_0(y)dy + \int_0^1 \beta(y)\widetilde{\phi}_0(y)dy,$$

thus

$$s(A_{\gamma,\beta})\int_0^1 \widetilde{\phi}_0(y)dy = \int_0^1 \beta(y)\widetilde{\phi}_0(y)dy,$$

and *iv*) follows.
We now prove *v*). From equation (54) we have

$$\begin{aligned}
&e^{s(A_{\gamma,\beta})t}\int_0^1 \widetilde{\phi}_0^*(y)\phi dy = \int_0^1 \widetilde{\phi}_0^*(y)T_{\gamma,\beta}(t)(\phi)(y)dy \geq e^{-\mu t}\int_0^1 \widetilde{\phi}_0^*(y)e^{\gamma(L-Id)t}(\phi)(y)dy \\
&\geq e^{-\alpha t}\left[\int_0^1 \widetilde{\phi}_0^*(y)e^{\gamma(L-Id)t}P_0(\phi)(y)dy + \int_0^1 \widetilde{\phi}_0^*(y)e^{\gamma(L-Id)t}P_1(\phi)(y)dy\right] \\
&\geq e^{-\alpha t}\left[\int_0^1 \phi(y)dy\int_0^1 \widetilde{\phi}_0^*(y)\phi_0(y)dy - \left\|\widetilde{\phi}_0^*\right\|_{L^\infty(0,1)} Me^{-\gamma(1-\alpha)t}\left\|P_1\phi\right\|\right] \\
&\geq e^{-\alpha t}\int_0^1 \phi(y)dy\left[\int_0^1 \widetilde{\phi}_0^*(y)\phi_0(y)dy - \left\|\widetilde{\phi}_0^*\right\|_{L^\infty(0,1)} Me^{-\gamma(1-\alpha)t}\left\|P_1\right\|_{L(X)}\right]
\end{aligned}$$

and for $t > 0$ large enough we have

$$C_t = \int_0^1 \widetilde{\phi}_0^*(y)\phi_0(y)dy - \left\|\widetilde{\phi}_0^*\right\|_{L^\infty(0,1)} Me^{-\gamma(1-\alpha)t}\left\|P_1\right\|_{L(X)} > 0,$$

since $\widetilde{\phi}_0^*$ is strictly positive, and ϕ_0 is quasi-interior. Thus by choosing $t>0$ large enough we have $C_t>0$, and

$$\widetilde{\phi}_0^*(\phi) \geq C_t e^{-(s(A_{\gamma,\beta})+\alpha)t} \|\phi\| .$$

We prove $vi)$. We first have to note that

$$\left\| T_{\gamma,\beta}(t)\widetilde{\phi}_0 \right\| = e^{s(A_{\gamma,\beta})t} \left\| \widetilde{\phi}_0 \right\|,$$

so $w(A_{\gamma,\beta}) \geq s(A_{\gamma,\beta})$. From $v)$ we have $\forall \phi \in X$

$$\begin{aligned} C\|T_{\gamma,\beta}(t)\phi\| &\leq C\|T_{\gamma,\beta}(t)(|\phi|)\| \leq \textstyle\int_0^1 \widetilde{\phi}_0^*(y)T_{\gamma,\beta}(t)(|\phi|)(y)dy \\ &= e^{s(A_{\gamma,\beta})t}\textstyle\int_0^1 \widetilde{\phi}_0^*(y)\,|\phi|\,(y)dy \leq \left\|\widetilde{\phi}_0^*\right\|_{X^*} e^{s(A_{\gamma,\beta})t}\|\phi\|, \end{aligned}$$

thus $w(A_{\gamma,\beta}) \leq s(A_{\gamma,\beta})$.

It remains to prove $vii)$. From assertion $i)$ and since $\gamma > \overline{\beta}$ we have $\{T_{\gamma,\beta}(t)\}_{t\geq 0}$ quasi-compact. Assume in addition that $s(A_{\gamma,\beta}) \geq 0$, then it is clear that

$$\widetilde{T}(t) = e^{-s(A_{\gamma,\beta})t} T_{\gamma,\beta}(t),$$

is also quasi-compact. Moreover, the semigroup

$$e^{-s(A_{\gamma,\beta})t} e^{(-\gamma Id + B)t}$$

converges to zero in norm of operator if

$$-s(A_{\gamma,\beta}) - \gamma + \beta(y) \leq -\delta < 0, \text{ a.e. on } (0,1),$$

so by using $iv)$ it is sufficient to verify

$$\gamma > \overline{\beta} - \underline{\beta}.$$

So in both cases we obtain a quasi-compact semigroup. Applying $vi)$ or Theorem 2.10 p:216 in Nagel [6] one deduces that

$$w(A_{\gamma,\beta}) = s(A_{\gamma,\beta}),$$

so

$$w(A_{\gamma,\beta} - s(A_{\gamma,\beta})Id) = 0.$$

We are now in position to apply Theorem 2.1 p:343 , and remark (d) p:344 in Nagel [6], and $vii)$ follows.□

We now return back to equation (53), and we apply Theorem 4.1.

Theorem 5.5. *: Let $\gamma > 0$, and $\beta \in L^\infty(0,1)$ be fixed, and assume that*

either a) $s(A_{\gamma,\beta}) > 0$ and $\gamma > \overline{\beta}$; or b) $\gamma > \overline{\beta} - \underline{\beta}$.

Consider equation (53), namely

$$\begin{cases} \frac{du}{dt} = \gamma\left[L(u(t)) - u(t)\right] + \beta u(t) + \tau\left[R(u(t)) - u(t)\right] - F(u(t))u(t), \\ u(0) = u_0 \in L^1_+(0,1). \end{cases}$$

Then there exists $\tau^* > 0$, such that for all $\tau \in [0,\tau^*]$, there exists $u_\tau \in L^1_+(0,1) \setminus \{0\}$, with $\|u_\tau\| = 1$, and $\mu_\tau \in \mathbb{R}$, such that

$$(s(A_{\gamma,\beta}) + \mu_\tau)u_\tau = \gamma\left[L(u_\tau) - u_\tau\right] + \beta u_\tau + \tau\left[R(u_\tau) - u_\tau\right].$$

and $\forall \tau \in [0,\tau^*]$, $\forall u_0 \in X_+ \setminus \{0\}$,

$$\frac{u(t)}{\|u(t)\|} \to u_\tau \text{ as } t \to +\infty.$$

Moreover $\forall \tau \in [0, \tau^*]$, $\forall u_0 \in X_+ \setminus \{0\}$,
a) If $s(A_{\gamma,\beta}) + \mu_\tau \leq 0$, *then*

$$u(t) \to 0, \text{ as } t \to +\infty.$$

b) If $s(A_{\gamma,\beta}) + \mu_\tau > 0$,

$$u(t) \to \overline{u}_\tau = (s(A_{\gamma,\beta}) + \mu_\tau) \frac{u_\tau}{F(u_\tau)}, \text{ as } t \to +\infty.$$

Assume in addition that $s(A_{\gamma,\beta}) > 0$, *there exists* $\widehat{\tau} \in (0, \tau^*]$, $\forall \tau \in [0, \widehat{\tau}]$,

$$s(A_{\gamma,\beta}) + \mu_\tau > 0, \text{ and } \overline{u}_\tau \text{ is exponentially stable.}$$

Proof: The proof of this theorem is a direct consequence of Theorem 4.1, and Theorem 5.4. □

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