Electronic Journal of Differential Equations, Vol. 2001(2001), No. 65, pp. 1–35.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

COMPACT ATTRACTORS FOR TIME-PERIODIC AGE-STRUCTURED POPULATION MODELS

PIERRE MAGAL

Abstract. In this paper we investigate the existence of compact attractors for time-periodic age-structured models. So doing we investigate the eventual compactness of a class of abstract non-autonomous semiflow (non necessarily periodic). We apply this result to non-autonomous age-structured models. In the time periodic case, we obtain the existence of a periodic family of compact subsets that is invariant by the semiflow, and attract the solutions of the system.

1. Introduction

In this paper, we are interested in non-autonomous age-structured models. Usually this model takes the form

$$\begin{gathered} u(t)(0) = \int_0^{+\infty} \beta(t, u(t))(a)u(t)(a)da \\ \frac{\partial u}{\partial t}(t)(a) + \frac{\partial u}{\partial a}(t)(a) = -\mu u(t)(a) + M(t, u(t))u(t)(a) \\ u(0) = \varphi \end{gathered}$$

with $u \in C^1([0,T], L^1(0,+\infty;\mathbb{R})^N)$. We refer the reader to the books by Webb [18], Metz and Dieckmann [9], and Iannelli [6], for a nice survey on nonlinear age-structured population dynamic models. Here

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix},$$

where $u_i(t)$ represents i^{th} class of the population. For example the population can be divided into several species, and several patches (when there is a spatial

2000 *Mathematics Subject Classification.* 33L25, 47D62, 47H20, 92D25.
Key words and phrases. Integrated semigroups, nonlinear evolutionary system, non-autonomous semiflow, compact attractors, periodic semiflow, eventual compactness.

Submitted August 21, 2001. Published October 8, 2001.

structure). Moreover,

$$\mu u(t) = \begin{pmatrix} \mu_1 u_1(t) \\ \mu_2 u_2(t) \\ \vdots \\ \mu_N u_N(t) \end{pmatrix}, \quad \beta(t,u(t))u(t) = \begin{pmatrix} \sum_{j=1}^N \beta_{1j}(t,u(t))u_j(t) \\ \sum_{j=1}^N \beta_{2j}(t,u(t))u_j(t) \\ \vdots \\ \sum_{j=1}^N \beta_{Nj}(t,u(t))u_j(t) \end{pmatrix},$$

where $\mu_i(a)$ represents the natural mortality of class i, $\beta_{ij}(t,u(t))(a)$ represents the fertility of class j into class i, and

$$M(t,u(t))u(t) = \begin{pmatrix} \sum_{j=1}^N m_{1j}(t,u(t))u_j(t) \\ \sum_{j=1}^N m_{2j}(t,u(t))u_j(t) \\ \vdots \\ \sum_{j=1}^N m_{Nj}(t,u(t))u_j(t) \end{pmatrix}$$

represents for the application to fisheries problems, intra and inter-specific competition, fisheries, and migrations. One can note that it is very natural to introduce periodic births and periodic mortalities in fisheries problems. We refer to Pelletier and Magal [12] for the example of a fishery problem where the time periodicity is necessary for continuous time model.

In this paper we will consider an abstract formulation of that type of evolution problem. The results that we present here are in the line of Thieme's work [13, 16, 17, 15]. The main point here is to study (in abstract manner) the eventual compactness of the non-autonomous semiflow associated to this system. This problem is studied in the book by Webb [18] in the autonomous case, and with bounded mortality rates. In this paper, we obtain similar results to those in the book by Webb [18], but by using integrated solutions of the problem (see section 2 for a precise definition). Also, the first part of the paper (i.e. sections 2, 3 and 4)) is strongly related with the paper by Thieme [13]. But the goal of this article is not to show the existence, the uniqueness, and the positivity of the solutions. Our aim is to show the existence of compact attractors for the time periodic age structured population models.

We now present the plan of the paper. In section 2, we recall some results originating from the work of Da Prato and Sinestrari [4], concerning existence of integrated solutions. We also recall some results due to Arendt [1][2], Kellermann and Hieber [7], Neubrander [10], Thieme [14], concerning integrated semigroup. In section 3, we present some results based on the usual semi-linear approach. We adapt results of books by Cazenave and Haraux [3], and Webb [18] to this situation. In section 4, we study the time differentiability of the solutions. This part is strongly related with Proposition 3.6 and Theorem 3.7 in the paper by Thieme [13]. This part is based on the usual differentiability result that can be found in the book by Pazy [11] (see Theorem 1.5 p.187). This result is used in section 6 to prove the existence of an absorbing subset for the system. In section 5, we prove an eventual compactness result for a class of non-autonomous semiflow which is applied in section 6. Finally in section 6, we give conditions for existence, uniqueness, global existence, and eventual compactness of the nonlinear and non-autonomous semiflow generated by the age-structured models. These conditions are close to the conditions given by Webb in [18] for autonomous age-structured models. Finally, we prove the existence of a "global attractor" for the system when $t \to \beta(t,.)$ and $t \to M(t,.)$ are periodic maps.

2. Preliminaries

We consider the non homogeneous Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t), \quad t > t_0; \\ u(t_0) &= x_0. \end{aligned} \tag{2.1}$$

Assumption 2.1.

a) $A : D(A) \subset X \to X$ is a linear operator, and assume that there exist real constants $M \geq 1$, and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and

$$\|(\lambda Id - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \text{ for } n \in \mathbb{N} \setminus \{0\} \text{ and } lambda > \omega\,.$$

b) $x_0 \in X_0 = \overline{D(A)}$.
c) $f : [0, +\infty) \to X$ is continuous.

In the sequel, a linear operator $A : D(A) \subset X \to X$ satisfying Assumption 2.1 a) will be called *a Hille-Yosida operator*.

Definition 2.1. A continuous function $u : [t_0, +\infty[\to X$ is called an *integral solution* to (2.1) if

$$u(t) = x_0 + A \int_{t_0}^{t} u(s)ds + \int_{t_0}^{t} f(s)ds, \text{ for all } t \geq t_0. \tag{2.2}$$

Note that (2.2) implies that $\int_{t_0}^{t} u(s)ds \in D(A)$. The main result of this section is as follows.

Theorem 2.2 ([4, Thm 8.1])**.** *Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 2.1 a), and $x \in D(A)$. Let $F(t) = F(0) + \int_0^t f(s)ds$ (for $0 \leq t \leq T$) for some Bochner-integrable function $f : (0, T) \to X$, and assume that*

$$Ax + F(0) \in \overline{D(A)}.$$

Then there exists a unique function $U \in C^1([0, T], X) \cap C([0, T], D(A))$, such that

$$\begin{aligned} U'(t) &= AU(t) + F(t), \text{ for all } t \in [0, T] \\ U(0) &= x\,. \end{aligned}$$

We now recall some result concerning integrated semigroups. We refer the reader to Arendt [1, 2], Kellermann and Hieber [7], Neubrander [10], Thieme [14] for more details.

Definition 2.3. A family of bounded linear operators $S(t)$, $t \geq 0$, on a Banach space X is called an *integrated semigroup* if and only if

i) $S(0) = 0$
ii) $S(t)$ is strongly continuous in $t \geq 0$.
iii) $S(r)S(t) = \int_0^r (S(\tau + t) - S(\tau))d\tau = S(t)S(r)$ for all $t, r \geq 0$.

An integrated semigroup is non-degenerate if $S(t)x = 0$ for all $t > 0$ occurs only for $x = 0$. The *generator* A of a non-degenerate integrated semigroup is given by the requiring that, for $x, y \in X$,

$$x \in D(A), y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)yds, \quad \forall t \geq 0.$$

The following theorem is obtained by combining Theorem 4.1 in Arendt [2], Proposition 2.2, Theorem 2.4, and their proofs in Kellermann and Hieber [7]. This theorem is taken from Thieme [16, thm. 6].

Theorem 2.4. *The following three statements are equivalent for a linear closed operator A in a Banach space X:*

i) *A is the generator of an integrated semigroup S that is locally Lipschitz continuous in the sense that, for any $b > 0$, there exists a constant $\Lambda > 0$ such that*
$$\|S(t) - S(r)\| \leq \Lambda|t - r|, \text{ for all } 0 \leq r, t \leq b.$$
ii) *A is the generator of an integrated semigroup S and there exist constants $M \geq 1$, $\omega \in \mathbb{R}$, such that*
$$\|S(t) - S(r)\| \leq M \int_r^t e^{\omega s} ds, \text{ for all } 0 \leq r \leq t < +\infty.$$
iii) *There exist constants $M \geq 1, \omega \in \mathbb{R}$, such that $(\omega, +\infty)$ is contained in the resolvent set of A and*
$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \text{ for } n \in \mathbb{N} \setminus \{0\}, \text{ and } \lambda > \omega.$$

Moreover, if one (and then all) of i), ii), iii) holds, $\overline{D(A)}$ coincides with those $x \in X$ for which $S(t)x$ is continuously differentiable. The derivatives $S'(t)x, t \geq 0, x \in \overline{D(A)}$, provide bounded linear operators $S'(t)$ from $X_0 = \overline{D(A)}$ into itself forming a C_0-semigroup on X_0 which is generated by A_0 the part of A in X_0. That is the linear operator defined by

$$D(A_0) = \{x \in D(A) : Ax \in X_0\} \text{ and } A_0 x = Ax \text{ for all } x \in D(A_0).$$

Finally $S(t)$ maps X into X_0 and

$$S'(r)S(t) = S(t + r) - S(r), \text{ for all } r, t \geq 0.$$

In Kellermann and Hieber [7], a very short proof of Theorem 2.2 is given by using integrated semigroups. One has

$$\begin{aligned} u(t) =& T_0(t - t_0)x_0 + \frac{d}{dt}\int_{t_0}^t S(t - s)f(s)ds \\ =& T_0(t - t_0)x_0 + \int_{t_0}^t dS(t - s)f(s), \end{aligned} \tag{2.3}$$

where $S(t)$ is the integrated semigroup generated by A, and the last integral is a Stieltjes integral. Now by setting

$$u(t) = U'(t), \ x = 0, \text{ and } F(t) = x_0 + \int_0^t f(s)ds, \tag{2.4}$$

one immediately deduces the existence of a solution of equation (2.2).

Theorem 2.5 ([4]). *Under Assumption 2.1, there exists a unique solution to (2.2) with value in $X_0 = \overline{D(A)}$. Moreover, u satisfies the estimate*

$$\|u(t)\| \leq Me^{\omega(t-t_0)}\|x_0\| + \int_{t_0}^{t} Me^{\omega(t-s)}\|f(s)\|ds, \text{ for all } t \geq t_0. \tag{2.5}$$

Assume now that $f(t) \equiv 0$, then the family of operators $T_0(t) : X_0 \to X_0$, $t \geq 0$, defined by

$$T_0(t)x_0 = u(t), \text{ for all } t \geq 0,$$

is the C_0-semigroup generated by A_0 the part of A in X_0. For the rest of this article, we denote by $T_0(t)$ the semigroup generated by A_0.

In the paper by Thieme [13] the following approximation formula is obtained. Assume that u is a solution of (2.2), then one has

$$\frac{d}{dt}(\lambda Id - A)^{-1}u(t) = A_0(\lambda Id - A)^{-1}u(t) + (\lambda Id - A)^{-1}f(t), \tag{2.6}$$

so,

$$\lambda(\lambda Id - A)^{-1}u(t) = T_0(t)\lambda(\lambda Id - A)^{-1}x_0 + \int_0^t T_0(t-s)\lambda(\lambda Id - A)^{-1}f(s)ds, \tag{2.7}$$

thus $\lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda(\lambda Id - A)^{-1}f(s)ds$ exists because the other terms in equation (2.7) converge (since x_0 and $u(t)$ belong to X_0). So, we have

$$u(t) = T_0(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)\lambda(\lambda Id - A)^{-1}f(s)ds. \tag{2.8}$$

To conclude this section, we remark that Lemma 5.1 p.17 in Pazy [11] holds, even when the domain of the generator is non-dense. More precisely, let $|.|$ be the norm defined by

$$|x| = \lim_{\mu\to+\infty}\|x\|_\mu, \tag{2.9}$$

where

$$\|x\|_\mu = \sup_{n\geq 0}\|\mu^n(\mu Id - (A - \omega Id))^{-n}x\|, \text{ for all } \mu > 0.$$

Then one has the following two properties:

$$\|x\| \leq |x| \leq M\|x\|, \quad \forall x \in X,$$
$$|\lambda(\lambda Id - (A - \omega Id))^{-1}x| < |x|, \quad \forall x \in X, \forall \lambda > 0.$$

So, if $u \in C([0,T], X_0)$ is a solution of

$$u(t) = x_0 + (A - \omega Id)\int_0^t u(s)ds + \int_0^t f(s)ds, \text{ for all } t \in [0,T], \tag{2.10}$$

then one has (by using (2.5) with $M = 1$, and $\omega = 0$)

$$|u(t)| \leq |x_0| + \int_0^t |f(s)|ds, \text{ for all } t \geq 0. \tag{2.11}$$

3. Semi-linear problem

In this section, we first follow the approach of Cazenave and Haraux [3]. Here we consider the case where the nonlinearity is Lipschitz on bounded sets. We consider the following problem: $u \in C([0,T], X_0)$ satisfies

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \text{ for } t \in [0,T]. \tag{3.1}$$

Assumption 3.1.

a) $A : D(A) \subset X \to X$ is a linear operator, and there exist real constant $M \geq 1$, and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda-\omega)^n}, \text{ for } n \in \mathbb{N}\setminus\{0\}, \text{ and } \lambda > \omega.$$

b) $F : \mathbb{R}_+ \times X_0 \to X$ is continuous, and for all $C > 0$, there exists $K(C) > 0$, such that

$$\|F(t,x) - F(t,y)\| \leq K_F(C)\|x-y\|, \forall x,y \in \overline{B}(0,C) \cap X_0, \forall t \geq 0,$$

where $\overline{B}(0,C) = \{x \in X : \|x\| \leq C\}$.

Problem (3.1) is equivalent to

$$u(t) = x_0 + (A - \omega Id)\int_0^t u(s)ds + \int_0^t F(s,u(s)) + \omega u(s)ds, \quad \forall t \in [0,T].$$

Then by using the equivalent norm $|.|$ defined in (2.9), we can assume that $M = 1$, $\omega = 0$. Moreover, the map

$$G(t,x) = F(t,x) + \omega x, \forall x \in X_0, \forall t \geq 0,$$

satisfies for all $C > 0$,

$$|G(t,x) - G(t,y)| \leq (MK_F(C) + \omega)|x-y|, \forall x,y \in \overline{B}_{|.|}(0,C) \cap X_0, \forall t \geq 0,$$

where $\overline{B}_{|.|}(0,C) = \{x \in X : |x| \leq C\}$. So without loss of generality, we can assume that $M = 1$, and $w = 0$.

Lemma 3.1. *Under Assumption 3.1, for each $x_0 \in X_0$, (3.1) admits at most one solution $u \in C([0,T], X_0)$.*

Proof. Assume that (3.1) admits two solutions $u, v \in C([0,T], X_0)$. We denote

$$C = \sup_{t\in[0,T]} \max(|u(t)|, |v(t)|).$$

Then one has

$$u(t) - v(t) = A\int_0^t u(s) - v(s)ds + \int_0^t F(s,u(s)) - F(s,v(s))ds, \forall t \in [0,T],$$

thus

$$\begin{aligned} u(t) - v(t) =& (A - \omega Id)\int_0^t u(s) - v(s)ds \\ &+ \int_0^t F(s,u(s)) - F(s,v(s)) + \omega(u(s) - v(s))ds, \ \forall t \in [0,T]. \end{aligned}$$

So from (2.11), we have (by using the equivalent norm $|.|$ defined in (2.9))

$$|u(t)-v(t)| \leq \int_0^t |F(s,u(s))-F(s,v(s))+\omega(u(s)-v(s))|ds, \forall t \in [0,T],$$

thus

$$|u(t)-v(t)| \leq (MK_F(C)+\omega)\int_0^t |u(s)-v(s)|ds, \forall t \in [0,T],$$

and by Gronwall's lemma one deduces the result. □

Let $C_F = \max_{t\in[0,1/2]} |F(t,0)|$, $L_C = 2C + C_F$ for $C \geq 0$, and

$$T_C = [2K_G(2C+C_F)+2]^{-1} \in]0,1/2], \text{ for } C \geq 0,$$

where $K_G(C) = MK_F(C)+\omega$ for $C \geq 0$. The following proposition is adapted from Proposition 4.3.3 p.56 in the book by Cazenave and Haraux [3].

Proposition 3.2. *Let $C > 0$, and let $x_0 \in X_0$ with $|x_0| \leq C$. Under Assumption 3.1, there exists a unique solution of problem (3.1), $u \in C([0,T_C],X_0)$.*

Proof. Lemma 3.1 shows the uniqueness. Let $x_0 \in X_0$ with $|x_0| \leq C$, and let

$$E = \{u \in C([0,T_C],X_0) : |u(t)| \leq L_C, \forall t \in [0,T_C]\}$$

be equipped with the metric

$$d(u,v) = \max_{t\in[0,T_C]} |u(t)-v(t)|, \forall u,v \in E.$$

For $u \in E$, we define $\Phi_u \in C([0,T_C],X_0)$, as the solution of the following equation, $\forall t \in [0,T_C]$,

$$\Phi_u(t) = (A-\omega Id)\int_0^t \Phi_u(s)ds + x_0 + \int_0^t F(s,u(s)) + \omega u(s)ds. \tag{3.2}$$

We note that for all $s \in [0,T_C]$, one has $F(s,u(s)) = F(s,0)+(F(s,u(s))-F(s,0))$, thus

$$|F(s,u(s))+\omega u(s)| \leq C_F + L_C K_G(L_C) \leq (C+C_F)/T_C.$$

We deduce that

$$\begin{aligned} |\Phi_u(t)| \leq& |x_0| + \int_0^t |F(s,u(s))+\omega u(s)|ds \\ \leq& C + (C+C_F)t/T_C = L_C, \quad \forall t \in [0,T_C]. \end{aligned}$$

So, $\Phi : E \to E$. Moreover, for all $u,v \in E$, one has

$$|\Phi_u(t)-\Phi_v(t)| \leq K_G(L_C)\int_0^t |u(s)-v(s)|ds \leq 1/2d(u,v), \forall t \in [0,T_C].$$

So, Φ is a strict contraction and the theorem is proved. □

Theorem 3.3. *Under Assumption 3.1. Let*

$$T(x) = \sup\{T > 0 :\ \exists u \in C([0,T],X_0) \text{ solution of (3.1)}\}. \tag{3.3}$$

Then

$$2K_G(2|u(t)| + \sup_{t\in[0,T(x)[} |F(t,0)|) \geq \frac{1}{T(x)-t} - 2, \forall t \in [0,T(x)).$$

In particular, either $T(x) = +\infty$, or $T(x) < +\infty$ and $\lim_{t\uparrow T(x)} |u(t)| = +\infty$.

We refer the reader to Theorem 4.3.4 in the book by Cazenave and Haraux [3] for the proof of the above theorem.

Proposition 3.4. *Under Assumption 3.1, the following holds:*

i) $T : X \to (0, +\infty]$ *is lower semi-continuous.*
ii) *If* $x_n \to x$ *and if* $T < T(x)$*, then* $u_n \to u$ *in* $C([0, T], X_0)$*, where* u_n *and* u *are the solution of (3.1) corresponding respectively to the initial value* x_n *and* x*.*

We refer the reader to Proposition 4.3.7 p. 58 in the book by Cazenave and Haraux [3] for the proof of this proposition. We summarize Propositions 3.2 and 3.4 in the following theorem.

Theorem 3.5. *Under Assumption 3.1, the set* $D = \{(t, x) : x \in X_0,\ 0 \le t < T(x)\}$ *is open in* $[0, +\infty) \times X_0$*, and the map* $(t, x) \to u_x(t)$ *from* D *to* X_0 *is continuous.*

We are now interested in the positivity of the solutions, for which end we use the conditions used by Webb in [19]. Let $X_+ \subset X$ be a cone of X. That is to say that X_+ is a closed convex subset of X, satisfying

i) $\lambda x \in X_+, \forall x \in X_+$ for all $\lambda \ge 0$
ii) $x \in X_+$ and $-x \in X_+ \Rightarrow x = 0$.

It is clear that $X_{0+} = X_0 \cap X_+$ is also a cone of X_0. We recall that such a cone defines a partial order on the Banach space X which is defined by

$$x \ge y \quad \text{if and only if} \quad x - y \in X_+.$$

Assumption 3.2.

c) $(\lambda Id - A)^{-1} X_+ \subset X_+$ for $\lambda > \omega$.
d) For all $C > 0$ and all $T > 0$, there exists $\gamma(C, T) > 0$ such that

$$F(t, x) + \omega x + \gamma(C, T)x \in X_+, \quad \forall x \in \overline{B}(0, C) \cap X_{0+}, \forall t \in [0, T].$$

Proposition 3.6. *Under Assumptions 3.1 and 3.2, for each* $x_0 \in X_{0+}$*, the corresponding solution of equation (3.1)* u *satisfies*

$$u(t) \in X_{0+}, \quad \forall t \in [0, T(x)).$$

Proof. For $T \in [0, T_C]$, let

$$E_+^T = \{u \in C([0, T], X_0) : u(t) \in X_{0+}, |u(t)| \le L_C, \forall t \in [0, T]\}.$$

For $t \in [0, T]$, we define $\Phi_u^T(t)$ as the solution of the equation

$$\begin{aligned} \Phi_u^T(t) =& x_0 + (A - (\gamma(L_C, T_C) + \omega)Id) \int_0^t \Phi_u^T(s) ds \\ &+ \int_0^t F(s, u(s)) + (\gamma(L_C, T_C) + \omega)u(s) ds. \end{aligned} \tag{3.4}$$

Then we have to prove the positivity of $\Phi_u(t)$, for all $t \in [0,T]$. But, we have

$$\begin{aligned}\Phi_u^T(t) &= \lim_{\lambda\to+\infty} \lambda(\lambda Id - A)^{-1}\Phi_u^T(t)\\ &= \lim_{\lambda\to+\infty} \lambda(\lambda Id - A)^{-1}e^{-(\gamma(L_C,T_C)+\omega)t}T_0(t)x_0\\ &\quad + \int_0^t e^{-(\gamma(L_C,T_C)+\omega)(t-s)}T_0(t-s)\lambda\\ &\quad \times (\lambda Id - A)^{-1}[F(s,u(s)) + (\gamma(L_C,T_C)+\omega)u(s)]ds,\end{aligned}$$

so under Assumption 3.2 c) and d), we see that $\forall u \in E_+^T$, $\forall \lambda > \omega$, $\forall t \in [0,T]$,

$$\begin{aligned}\lambda(\lambda Id - A)^{-1}e^{-(\gamma(L_C,T_C)+\omega)t}T_0(t)x_0 + \int_0^t e^{-(\gamma(L_C,T_C)+\omega)(t-s)}T_0(t-s)\lambda\\ \times(\lambda Id - A)^{-1}[F(s,u(s)) + (\gamma(L_C,T_C)+\omega)u(s)]ds \in X_{0+}.\end{aligned}$$

So, by taking the limit as $\lambda \to +\infty$, and using the fact that X_{0+} is closed, we deduce that

$$\Phi_u^T(t) \in X_{0+}, \quad \forall t \in [0,T].$$

So $\Phi^T : E_+^T \to C([0,T], X_{0+})$. Finally, for all $T > 0$ small enough, Φ^T maps E_+^T into itself, and Φ^T is a strict contraction. The result follows. $\square$

We recall, that a cone X_+ of a Banach space $(X, \|.\|)$ is *normal,* if there exists a norm $\|.\|_1$ equivalent to $\|.\|$, which is monotone, that is to say

$$\forall x, y \in X,\ 0 \le x \le y \quad \text{implies} \quad \|x\|_1 \le \|y\|_1.$$

Assumption 3.3.

e) There exist $G_1 : \mathbb{R}_+ \times X_0 \to X$ and $G_2 : \mathbb{R}_+ \times X_0 \to X$ continuous maps, such that

$$F(t,x) = G_1(t,x) + G_2(t,x), \quad \forall x \in X_0, \forall t \ge 0,$$

where $G_1(t,x) \in -X_+$ for all $x \in X_{0+}$, all $t \ge 0$, and

$$\|G_2(t,x)\| \le k_{G_2}\|x\|, \quad \forall x \in X_{0+}, \forall t \ge 0.$$

f) X_+ is a normal cone of $(X, \|.\|)$.

Proposition 3.7. *Under Assumptions 3.1, 3.2, and 3.3, for each $x_0 \in X_{0+}$, there exists a unique $u \in C([0,+\infty), X_{0+})$ solution of (3.1). Moreover, there exist $C_0 > 0$ and $C_1 > 0$, such that for all $x_0 \in X_{0+}$,*

$$\|u(t)\| \le \|x_0\|C_0 e^{(C_1 k_{G_2}+\omega)t}, \quad \forall t \ge 0.$$

Proof. We start by letting $|x|_1 = \lim_{\mu\to+\infty} \|x\|_{1\mu}$, where

$$\|x\|_{1\mu} = \sup_{n\ge0} \|\mu^n(\mu Id - (A - \omega Id))^{-n}x\|_1, \mu > 0.$$

Then since $\|.\|_1$ is monotone, and $(\lambda Id - A)^{-1}$ is a positive operator for $\lambda > \omega$, we deduce that $|.|_1$ is monotone, and satisfies

$$|\mu(\mu Id - (A - \omega Id))^{-1}x|_1 \le |x|_1, \text{ for } \mu > 0, \text{ and } x \in X.$$

Consider now

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \text{ for } t \in [0, T(x_0)).$$

Let $T \in [0, T(x_0))$. Then by definition of $T(x_0)$, we have

$$C_u = \sup_{t\in[0,T]} \|u(t)\| < +\infty.$$

By Assumption 3.2 d), there exists $\gamma(C_u, T) > 0$ such that

$$F(t,x) + \omega x + \gamma(C_u, T)x \in X_+, \forall x \in \overline{B}(0, C_u) \cap X_{0+}, \quad \forall t \in [0,T].$$

We fix $\alpha > 0$ such that $\alpha + w > \gamma(C_u, T)$. Then for all $t \in [0,T]$,

$$u(t) = x_0 + (A - (\omega+\alpha)Id)\int_0^t u(s)ds + \int_0^t F(s,u(s)) + (\omega+\alpha)u(s)ds\,.$$

Therefore, for all $t \in [0,T]$,

$$|u(t)|_1 \le e^{-\alpha t}|x_0|_1 + \int_0^t e^{-\alpha(t-s)}|F(s,u(s)) + (\omega+\alpha)u(s)|_1 ds\,.$$

Using the monotonicity of $|.|_1$ one has

$$|u(t)|_1 \le e^{-\alpha t}|x_0|_1 + \int_0^t e^{-\alpha(t-s)}|G_2(s,u(s)) + (\omega+\alpha)u(s)|_1 ds\,.$$

Since the norm $\|.\|$ and $|.|_1$ are equivalent, we have for some constant $C_1 > 0$,

$$|u(t)|_1 \le e^{-\alpha t}|x_0|_1 + \int_0^t e^{-\alpha(t-s)}[C_1 k_{G_2} + \omega + \alpha]|u(s)|_1 ds, \forall t \in [0,T].$$

By using Gronwall's lemma we obtain

$$|u(t)|_1 \le |x_0|_1 e^{(C_1 k_{G_2} + \omega)t}, \quad \forall t \in [0,T].$$

Existence of a global solution follows from Theorem 3.3. □

4. Time differentiability of the solutions

In this section, we study only the time differentiability of the solutions. We refer to Thieme [13] Theorem 3.4 and Corollary 3.5 for the differentiability with respect to the space variable. Consider $u \in C([0,T], \overline{D(A)})$ a solution of

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \text{ for } t \in [0,T], \tag{4.1}$$

and assume that $x_0 \in D(A)$ and that $F : [0,T] \times X_0 \to X$ is a C^1 map. Then when the domain is dense it is well known (see Pazy [11] Theorem 6.1.5 p. 187) that $t \to u(t)$ is continuously differentiable, $u(t) \in D(A)$ for all $t \in [0,T]$, and satisfies

$$\begin{gathered} u'(t) = Au(t) + f(t), \forall t \in [0,T], \\ u(0) = x_0. \end{gathered}$$

We are now interested in the same type of result when the domain is non-dense. We will use the following theorem.

Theorem 4.1 ([Thm. 6.3][4])**.** *Let $A : D(A) \to X$ be a Hille-Yosida operator. Let $f \in C([0,T], X)$ and $x_0 \in X_0$. If u is a solution of*

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \forall t \in [0,T],$$

belonging to $C^1([0,T],X)$ *or to* $C([0,T],D(A))$*, then*

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad \forall t \in [0,T], \\ u(0) &= x_0. \end{aligned}$$

with $u \in C^1([0,T],X) \cap C([0,T],D(A))$*.*

Assumption 4.1.

a) $A : D(A) \subset X \to X$ is a linear operator, and there exist two real constants $M \geq 1$, and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \text{ for } n \in \mathbb{N} \setminus \{0\}, \text{ and } \lambda > \omega.$$

b) $F : [0,T] \times X_0 \to X$ is continuously differentiable from $[0,T] \times X_0$ into X.
c) There exists $u \in C([0,T], \overline{D(A)})$ solution of

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \text{ for all } t \in [0,T]. \tag{4.2}$$

Theorem 4.2. *Under Assumption 4.1, if in addition* $x_0 \in D(A_0)$ *(i.e.* $x_0 \in D(A)$ *and* $Ax_0 \in \overline{D(A)}$*) and* $F(0,x_0) \in \overline{D(A)}$*; then there exists* $u \in C^1([0,T],X) \cap C([0,T],D(A))$ *satisfying*

$$\begin{aligned} u'(t) &= Au(t) + F(t,u(t)), \ \forall t \in [0,T], \\ u(0) &= x_0. \end{aligned}$$

Proof. We use the idea in the proof of Theorem 6.1.5 in Pazy [11]. Let $w \in C([0,T], \overline{D(A)})$ be a solution of the equation

$$\begin{aligned} w(t) =& Ax_0 + F(0,x_0) + A\int_0^t w(s)ds \\ &+ \int_0^t \frac{\partial}{\partial t}F(s,u(s)) + D_xF(s,u(s))w(s)ds, \forall t \in [0,T]. \end{aligned} \tag{4.3}$$

It is clear that the solution $w(t)$ exists and is uniquely determined, since $u(t)$ exists on $[0,T]$. Let $t \geq 0$. For $h > 0$, we have

$$\begin{aligned} &\frac{u(t+h) - u(t)}{h} \\ =& \frac{1}{h}A[\int_0^{t+h} u(s)ds - \int_0^t u(s)ds] + \frac{1}{h}[\int_0^{t+h} F(s,u(s))ds - \int_0^t F(s,u(s))ds] \\ =& A[\int_0^t \frac{u(s+h) - u(s)}{h}ds] + \frac{1}{h}A\int_0^h u(s)ds \\ &+ \int_0^t \frac{F(s+h,u(s+h)) - F(s,u(s))}{h}ds + \frac{1}{h}\int_0^h F(s,u(s))ds\,. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{u(t+h)-u(t)}{h}-w(t)\\
=&A\int_0^t \frac{u(s+h)-u(s)}{h}-w(s)ds\\
&+\frac{1}{h}A\int_0^h u(s)ds+\frac{1}{h}\int_0^h F(s,u(s))ds-Ax_0-F(0,x_0)\\
&+\int_0^t \frac{F(s+h,u(s+h))-F(s+h,u(s))}{h}-D_xF(s,u(s))w(s)ds\\
&+\int_0^t \frac{F(s+h,u(s))-F(s,u(s))}{h}-\frac{\partial}{\partial t}F(s,u(s))ds.
\end{aligned}
$$

So by using (2.5), and Gronwall's lemma, the right differentiability of $u(t)$ will follow if we prove that

$$
\lim_{h\searrow 0}\frac{1}{h}A\int_0^h u(s)ds+\frac{1}{h}\int_0^h F(s,u(s))ds+Ax_0+F(0,x_0)=0.
$$

If $t\geq h>0$,

$$
\begin{aligned}
&\frac{u(t-h)-u(t)}{-h}\\
=&\frac{1}{-h}A[\int_0^{t-h}u(s)ds-\int_h^t u(s)ds]-\frac{1}{-h}A\int_0^h u(s)ds\\
&+\frac{1}{-h}[\int_0^{t-h}F(s,u(s))ds-\int_h^t F(s,u(s))ds]-\frac{1}{-h}\int_0^h F(s,u(s))ds\\
=&A\int_h^t \frac{u(s-h)-u(s)}{-h}ds+\int_h^t \frac{F(s-h,u(s-h))-F(s,u(s))}{-h}ds\\
&+\frac{1}{h}A\int_0^h u(s)ds+\frac{1}{h}\int_0^h F(s,u(s))ds\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{u(t-h)-u(t)}{-h}-w(t)=&A\int_h^t \frac{u(s-h)-u(s)}{-h}ds-A\int_h^t w(s)ds-A\int_0^h w(s)ds\\
&-Ax_0-F(0,x_0)+\frac{1}{h}A\int_0^h u(s)ds+\frac{1}{h}\int_0^h F(s,u(s))ds\\
&+\int_h^t \frac{F(s-h,u(s-h))-F(s,u(s))}{-h}ds\\
&-\int_0^t \frac{\partial}{\partial t}F(s,u(s))+D_xF(s,u(s))w(s)ds\,.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\frac{u(t-h)-u(t)}{-h} - w(t) \\
&= A\int_h^t \frac{u(s-h)-u(s)}{-h} - w(s)ds - A\int_0^h w(s)ds \\
&\quad - Ax_0 - F(0,x_0) + \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds \\
&\quad + \int_h^t \frac{F(s-h,u(s-h)) - F(s,u(s))}{-h}ds \\
&\quad - \int_h^t \frac{\partial}{\partial t}F(s-h,u(s-h)) + D_xF(s-h,u(s-h))w(s-h)ds \\
&\quad - \int_0^h \frac{\partial}{\partial t}F(s,u(s)) + D_xF(s,u(s))w(s)ds.
\end{aligned}
$$

Since by construction, we have

$$\lim_{h\searrow 0} A\int_0^h w(s)ds = 0,$$

to prove the left differentiability of u it is sufficient to prove that

$$\lim_{h\searrow 0} \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds = Ax_0 + F(0,x_0).$$

Taking into account Theorem 4.1, Theorem 4.2 is a consequence of the following lemma. □

Lemma 4.3. *Under the assumptions of Theorem 4.2, one has*

$$\lim_{h\searrow 0} \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds = Ax_0 + F(0,x_0).$$

Proof. This lemma will be proved if we show that

$$\lim_{h\searrow 0} \frac{u(h)-x_0}{h} = Ax_0 + F(0,x_0).$$

We remark that $u(t) = T_0(t)x_0 + v(t)$, where $T_0(t)$ is the semigroup generated by A_0 the part of A in $\overline{D(A)}$, and $v \in C([0,T],\overline{D(A)})$ is the solution of

$$v(t) = A\int_0^t v(s)ds + \int_0^t F(s,v(s)+T_0(s)x_0)ds\,.$$

Since x_0 belongs to the domain of A_0, it remains to prove that

$$\lim_{h\searrow 0} \frac{v(h)}{h} = F(0,x_0).$$

Clearly one has for $\lambda > \omega$

$$\begin{aligned}
&v(t) - t\lambda(\lambda Id - A)^{-1}F(0,x_0) \\
&= A[\int_0^t v(s) - s\lambda(\lambda Id - A)^{-1}F(0,x_0)ds] \\
&+ \frac{t^2}{2}\lambda A(\lambda Id - A)^{-1}F(0,x_0) - t\lambda(\lambda Id - A)^{-1}F(0,x_0) + tF(0,x_0) \\
&+ \int_0^t F(s, v(s) + T_0(s)x_0) - F(0,x_0)ds
\end{aligned}$$

Now using the fact that $F(0,x_0) \in \overline{D(A)}$ one has

$$\lim_{\lambda\to+\infty} \lambda(\lambda Id - A)^{-1}F(0,x_0) = F(0,x_0),$$

and using (2.5),

$$\begin{aligned}
\|v(t) - tF(0,x_0)\| \leq &\|v(t) - t\lambda(\lambda Id - A)^{-1}F(0,x_0)\| \\
&+ t\|\lambda(\lambda Id - A)^{-1}F(0,x_0) - F(0,x_0)\| \\
\leq &Me^{\omega t}\|\frac{t^2}{2}\lambda A(\lambda Id - A)^{-1}F(0,x_0)\| \\
&+ \int_0^t Me^{\omega(t-s)}\|F(s, v(s) + T_0(s)x_0) - F(0,x_0)\|ds \\
&+ 2t\|\lambda(\lambda Id - A)^{-1}F(0,x_0) - F(0,x_0)\|.
\end{aligned}$$

$\square$

To extend the differentiability result to the case where $F(0,x_0) \notin \overline{D(A)}$, we remark that, since $u(t) \in \overline{D(A)}$ for all $t \in [0,T]$, a necessary condition for the differentiability is

$$Ax_0 + F(0,x_0) \in \overline{D(A)}.$$

In fact, this condition is also sufficient. Indeed, taking any bounded linear operator $B \in \mathcal{L}(X)$, if u satisfies

$$u(t) = x_0 + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds,\ \forall t \in [0,T],$$

we have

$$u(t) = x_0 + (A+B)\int_0^t u(s)ds + \int_0^t F(s,u(s)) - Bu(s)ds,\ \text{for } t \in [0,T].$$

So to prove the differentiability of $u(t)$ it is sufficient to find B such that $(A+B)x_0 \in \overline{D(A)}$. By taking $B(\varphi) = -x^*(\varphi)Ax_0$, where $x^* \in X^*$ is a continuous linear form, with $x^*(x_0) = 1$ if $x_0 \neq 0$, which is possible by the Hahn-Banach theorem. So we have

$$x_0 \in D(A) = D(A+B),\ \text{and } (A+B)x_0 \in \overline{D(A)} = \overline{D(A+B)}.$$

Moreover, assuming that $Ax_0 + F(0,x_0) \in \overline{D(A)}$, we obtain $F(0,x_0) - Bx_0 \in \overline{D(A)}$.

So, by using classical perturbation technics (see Pazy [11] Chapter 3), we deduce that $A+B$ is a Hille-Yosida operator, and we have the following theorem.

Theorem 4.4. *Under Assumption 4.1, if $x_0 \in D(A)$, and $Ax_0 + F(0,x_0) \in \overline{D(A)}$, then there exists $u \in C^1([0,T],X) \cap C([0,T],D(A))$ satisfying*

$$\begin{gathered} u'(t) = Au(t) + F(t,u(t)), \forall t \in [0,T], \\ u(0) = x_0 \,. \end{gathered}$$

We now consider the nonlinear generator,

$$A_N\varphi = A\varphi + F(0,\varphi), \text{ for } \varphi \in D(A_N) = D(A),$$

As in the linear case, one may define $A_{N,0}$ the part A_N in $\overline{D(A)}$ as follows

$$A_{N,0} = A_N \text{ on } D(A_{N,0}) = \left\{ y \in D(A) : A_N y \in \overline{D(A)} \right\}.$$

Of course, one may ask about the density of the domain $D(A_{N,0})$ in $\overline{D(A)}$. This property will be useful in section 6 to obtain a priori estimates (more precisely to obtain the existence of an absorbing set).

Assumption 4.2.

d) $F(0,.) : X_0 \to X$ is Lipschitz on bounded sets i.e. $\forall C > 0$, $\exists K(C) > 0$, such that

$$\|F(0,x) - F(0,y)\| \le K(C)\|x-y\|, \quad \forall x,y \in \overline{B}(0,C) \cap X_0.$$

Lemma 4.5. *Under Assumptions 4.1 a) and 4.2, $D(A_{N,0})$ is dense in $X_0 = \overline{D(A)}$.*

Proof. Let $y \in \overline{D(A)}$ be fixed. Consider the following fixed point problem: $x_\lambda \in D(A)$ satisfying

$$(Id - \lambda A - \lambda F)x_\lambda = y \Leftrightarrow x_\lambda = (Id-\lambda A)^{-1}y + \lambda(Id-\lambda A)^{-1}F(0,x_\lambda).$$

We denote

$$\Phi_\lambda(x) = (Id-\lambda A)^{-1}y + \lambda(Id-\lambda A)^{-1}F(0,x), \forall x \in X_0.$$

Then $r > 0$ being fixed, one can prove that there exists $\eta = \eta(r) > 0$ (with $[\eta^{-1}, +\infty[\subset \rho(A))$ such that

$$\Phi_\lambda(B(y,r)) \subset B(y,r), \forall \lambda \in (0,\eta],$$

where $B(y,r)$ denotes the ball of center y with radius r in X_0. Moreover, one can assume that Φ_λ is a strict contraction on $B(y,r)$. So, $\forall \lambda \in]0,\eta]$, there exists $x_\lambda \in B(y,r)$, such that $\Phi_\lambda(x_\lambda) = x_\lambda$. Finally, by using the fact that $y \in \overline{D(A)}$, we deduce

$$\lim_{\lambda \searrow 0} (Id-\lambda A)^{-1}y = \lim_{\lambda \searrow 0} \lambda^{-1}(\lambda^{-1}Id - A)^{-1}y = y,$$

so $\lim_{\lambda \to +\infty} x_\lambda = y$. □

5. Eventual compactness

In this section we are interested in the eventual compactness of the nonlinear non-autonomous semiflow generated by

$$u_{x_0}(t) = x_0 + A\int_0^t u_{x_0}(s)ds + \int_0^t F(s,u_{x_0}(s))ds, \text{ for } t \in [0,T]. \tag{5.1}$$

We recall that a family of operators $U(t,s)$ (with $t \ge s \ge 0$) is called a non-autonomous semiflow (see Thieme [13]) if

$$U(t,r)U(r,s) = U(t,s) \text{ if } t \ge r \ge s, \text{ and } U(t,t) = Id.$$

Here we are interested by the non-autonomous semiflow defined by

$$U(t,s)x_0 = x_0 + A\int_s^t U(l,s)x_0dl + \int_s^t F(l,U(l,s)x_0)dl, \text{ for } t \geq s,$$

and we want to investigate the eventual compactness of the family of nonlinear operators $\{U(s+t,s)\}_{t\geq 0}$(i.e. the complete continuity of $U(s+t,s)$ for $t \geq 0$ large enough). In the sequel, we only consider the case where $s=0$, the case $s>0$ being similar.

This problem is studied in the linear autonomous case by Thieme in [16], and we refer to the paper by Webb [18] for the semi-linear case with dense domain. This problem is also investigated in the book by Webb [19] for nonlinear age structured model with bounded mortality rate, in the autonomous case. Here, we follow Webb's approach in [19] , and we adapt his approach to the abstract problem.

Assumption 5.1.

a) $A : D(A) \subset X \to X$ is a Hille-Yosida operator.

b) $F : [0,T] \times X_0 \to X$ is a continuous map, which satisfies

$$F(t,x) = F_1(t,x) + F_2(t,x),$$

where $F_1 : [0,T] \times X_0 \to X$, and $F_2 : [0,T] \times X_0 \to X_0$ satisfy: $\forall C > 0$, $\exists K(C) > 0$, such that

$$\|F_i(t,x) - F_i(t,y)\| \leq K(C)\|x-y\|, \forall x,y \in \overline{B}(0,C) \cap X_0, \forall t \in [0,T], \quad i = 1,2.$$

c) There exists a bounded set $B \subset X_0$ such that, for each $x_0 \in B$, there exists a continuous solution $u_{x_0} : [0,T] \to X_0$ of (5.1), and

$$\sup_{x_0\in B}\sup_{t\in[0,T]}\|u_{x_0}(t)\| \leq \alpha_0.$$

d) For each $t \in [0,T]$, $\varphi \to F_1(t,\varphi)$ is continuous and maps bounded sets into relatively compact sets, and $\forall C > 0$, $\exists k(C) > 0$, such that

$$\|F_1(t,x) - F_1(l,x)\| \leq k(C)|t-l|, \ \forall x \in \overline{B}(0,C) \cap X_0, \ \forall t,l \in [0,T].$$

e) There exists $k = k(B) \geq 0$, such that

$$\|F_1(t,u_{x_0}(t)) - F_1(l,u_{x_0}(l))\| \leq k|t-l|, \ \forall x_0 \in B, \ \forall t,l \in [0,T].$$

We now consider the system of equations

$$u_{1x_0}(t) = A\int_0^t u_{1x_0}(s)ds + \int_0^t F_1(s,u_{x_0}(s))ds, \text{ for } t \in [0,T],$$

$$u_{2x_0}(t) = x_0 + A\int_0^t u_{2x_0}(s)ds + \int_0^t F_2(s,u_{x_0}(s))ds, \text{ for } t \in [0,T].$$

Then the solution of the previous system clearly exists, and by uniqueness of the solution of the problem

$$v(t) = x_0 + A\int_0^t v(s)ds + \int_0^t F(s,u_{x_0}(s))ds, \text{ for all } t \in [0,T],$$

we have

$$u_{x_0}(t) = u_{1_{x_0}}(t) + u_{2_{x_0}}(t), \text{ for all } t \in [0,T].$$

Theorem 5.1. *Under Assumption 5.1, the set* $\{u_{1x_0}(t) : t \in [0,T], x_0 \in B\}$ *has compact closure.*

Proof. For each $x_0 \in B$, we denote $v_{n,x_0} \in C([0,T], X_0)$ the solution of the problem

$$v_{n,x_0}(t) = A \int_0^t v_{n,x_0}(s)ds + \int_0^t \rho_n * \widetilde{F}_1(., u_{x_0}(.))(s)ds, \forall t \in [0,T],$$

where $\rho_n : \mathbb{R} \to \mathbb{R}_+$ is a mollifier, with support in $[-\frac{1}{n}, \frac{1}{n}]$,

$$\widetilde{F}_1(t, u_{x_0}(t)) = \begin{cases} F_1(0, u_{x_0}(0)), \text{ if } t \le 0, \\ F_1(t, u_{x_0}(t)), \text{ if } 0 \le t \le T, \\ F_1(T, u_{x_0}(T)), \text{ if } t \ge T, \end{cases}$$

$$\rho_n * \widetilde{F}_1(., u_{x_0}(.))(t) = \int_{-\infty}^{+\infty} \rho_n(\theta)\widetilde{F}_1(t-\theta, u_{x_0}(t-\theta))d\theta.$$

On the other hand, we know (see Thieme [13]) that

$$v_{n,x_0}(t) = \int_0^t dS(s)\rho_n * \widetilde{F}_1(., u_{x_0}(.))(t-s),$$

where $S(t)$ denotes the integrated semigroup generated by A. We then have,

$$\begin{aligned} v_{n,x_0}(t) =& S(t)[\rho_n * \widetilde{F}_1(., u_{x_0}(.))(0)] - S(0)[\rho_n * \widetilde{F}_1(., u_{x_0}(.))(t)] \\ &+ \int_0^t S(s)\rho_n' * \widetilde{F}_1(., u_{x_0}(.))(t-s)ds\,. \end{aligned}$$

Since $S(0)x = 0$, for all $x \in X$,

$$v_{n,x_0}(t) = S(t)[\rho_n * \widetilde{F}_1(., u_{x_0}(.))(0)] + \int_0^t S(s)\rho_n' * \widetilde{F}_1(., u_{x_0}(.))(t-s)ds\,.$$

By using Assumption 5.1 d), one deduces that $\overline{F_1([0,T] \times (B(0,\alpha_0) \cap X_0))}$ is compact, and Mazur's theorem $\overline{\text{conv}}(F_1([0,T] \times (B(0,\alpha_0) \cap X_0)))$ is compact. Indeed, let $k > 0$ be fixed such that

$$\|F_1(t,x) - F_1(r,x)\| \le k|t-r|, \forall x \in B(0,\alpha_0) \cap X_0, \forall t, r \in [0,T].$$

For each $n \in \mathbb{N} \setminus \{0\}$, let $t_i^n = \frac{i}{n}T$ for $i = 0, \dots, n$. Then for $i = 0, 1, \dots, n-1$,

$$\|F_1(t_i^n, x) - F_1(t,x)\| \le kT/n, \quad \forall t \in [t_i^n, t_{i+1}^n].$$

Let $\varepsilon > 0$ and $n \in \mathbb{N} \setminus \{0\}$ be such that $kT/n \le \varepsilon/2$. As $F_1(t_i^n, B(0,\alpha_0) \cap X_0)$ is relatively compact, so there exists $\{x_1^i, x_2^i, \dots, x_{k(i)}^i\} \subset B$, such that for all $x \in B$, there exists $j \in \{1, 2, \dots, k(i)\}$, satisfying

$$\|F_1(t_i^n, x_j^i) - F_1(t_i^n, x)\| \le \frac{\varepsilon}{2}.$$

We have

$$F_1([0,T] \times (B(0,\alpha_0) \cap X_0)) \subset \cup_{i=1,\dots,n\ j=1,\dots,k(i)} \overline{B(x_j^i, \varepsilon)},$$

and the compactness of $\overline{F_1([0,T] \times (B(0,\alpha_0) \cap X_0))}$ follows. From Theorem 2.4, we know that $S(t)$ is locally Lipschitz, so by using the same argument as above, we deduce that

$$\cup_{t \in [0,T]} \cup_{x_0 \in B} S(t)[\rho_n * \widetilde{F}_1(., u_{x_0}(.))(0)]$$

is relatively compact, and

$$\cup_{t \in [0,T]\ s \in [0,T]} \cup_{x_0 \in B} S(t)\rho_n' * \widetilde{F}_1(., u_{x_0}(.))(s)$$

is also relatively compact. Thus for each $n \in \mathbb{N} \setminus \{0\}$, there exists a compact subset $C_n \subset X_0$, such that

$$v_{n,x_0}(t) \in C_n, \forall x_0 \in B, \forall t \in [0,T].$$

To complete the proof, it remains to prove the uniform convergence of $v_{n,x_0}(t)$ to $u_{1,x_0}(t)$. That is to say $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \setminus \{0\}$, such that

$$\|u_{1,x_0}(t) - v_{n,x_0}(t)\| \leq \varepsilon, \forall x_0 \in B, \forall t \in [0,T], \forall n \geq n_0.$$

We have $\forall t \in [0,T]$,

$$\begin{aligned} &u_{1,x_0}(t) - v_{n,x_0}(t) \\ &= A\int_0^t u_{1,x_0}(s) - v_{n,x_0}(s)ds + \int_0^t F_1(s, u_{x_0}(s)) - \rho_n * \widetilde{F}_1(., u_{x_0}(.))(s)ds, \end{aligned}$$

so that

$$\|u_{1,x_0}(t) - v_{n,x_0}(t)\| \leq M\int_0^t e^{\omega(t-s)}\|F_1(s, u_{x_0}(s)) - \rho_n * \widetilde{F}_1(., u_{x_0}(.))(s)\|ds.$$

Then $\forall s \in [0,T]$,

$$\begin{aligned} &F_1(s, u_{x_0}(s)) - \rho_n * \widetilde{F}_1(., u_{x_0}(.))(s) \\ &= F_1(s, u_{x_0}(s)) - \int_{-\infty}^{+\infty} \rho_n(\theta)\widetilde{F}_1(s-\theta, u_{x_0}(s-\theta))d\theta \\ &= \int_{-\infty}^{+\infty} \rho_n(\theta)\widetilde{F}_1(s, u_{x_0}(s)) - \widetilde{F}_1(s-\theta, u_{x_0}(s-\theta))d\theta\,. \end{aligned}$$

So by using Assumption 5.1 e), one has

$$\|F_1(s, u_{x_0}(s)) - \rho_n * \widetilde{F}_1(., u_{x_0}(.))(s)\| \leq k\int_{-\infty}^{+\infty} \rho_n(\theta)|\theta|d\theta \leq \frac{k}{n}\,,$$

and we have

$$\|u_{1,x_0}(t) - v_{n,x_0}(t)\| \leq Me^{\omega T}T\frac{k}{n}, \forall x_0 \in B, \forall t \in [0,T], \forall n \geq 1.$$

□

Assumption 5.2.

f) Let $(Z, \|.\|_Z)$ be a Banach space, let $H : Z \times X_0 \to X_0$ be a continuous bilinear map, and let be a Lipschitz continuous map $G : \mathbb{R}_+ \times X_0 \to Z$ which maps bounded sets into relatively compact sets. We assume that

$$F_2(t,x) = H(G(t,x), x), \forall x \in X_0, \forall t \geq 0.$$

g) Let $w_{x_0} \in C([0,T], X_0)$ be the solution of

$$w_{x_0}(t) = T_0(t)x_0 + \int_0^t T_0(t-s)H(G(s, u_{x_0}(s)), w_{x_0}(s))ds, \forall x_0 \in B, \forall t \in [0,T].$$

We assume that there exists T', $0 < T' \leq T$, such that

$$w_{x_0}(t) = 0, \forall x_0 \in B, \quad \forall t \geq T'.$$

The following theorem gives the eventual compactness of the non-autonomous semi-flow.

Theorem 5.2. *Under Assumptions 5.1-5.2, the set* $\{u_{x_0}(t) : t \in [T', T], x_0 \in B\}$ *has compact closure.*

Proof. By taking into account Theorem 5.1, it remains to investigate the eventual compactness of the second component $u_{2x_0}(t)$. We have

$$u_{2x_0}(t) = T_0(t)x_0 + L_{x_0}(u_{2x_0}(.))(t) + L_{x_0}(u_{1x_0}(.))(t), \forall t \in [0, T],$$

where

$$L_{x_0}(\psi(.))(t) = \int_0^t T_0(t-s)H(G(s, u_{x_0}(s)), \psi(s))ds, \forall t \in [0, T].$$

As an immediate consequence one has

$$u_{2x_0}(t) = \sum_{k=0}^{\infty} L_{x_0}^k(T_0(.)x_0)(t) + \sum_{k=1}^{\infty} L_{x_0}^k(u_{1x_0}(.))(t), , \forall t \in [0, T],$$

thus

$$u_{2x_0}(t) = w_{x_0}(t) + \sum_{k=1}^{\infty} L_{x_0}^k(u_{1x_0}(.))(t), , \forall t \in [0, T],$$

where $w_{x_0}(t)$ is the continuous solution of

$$w_{x_0}(t) = T_0(t)x_0 + \int_0^t T_0(t-s)H(G(s, u_{x_0}(s)), w_{x_0}(s))ds, \forall t \in [0, T].$$

By Assumption 5.2 g), one deduces that

$$u_{2x_0}(t) = \sum_{k=1}^{\infty} L_{x_0}^k(u_{1x_0}(.))(t), \forall t \in [T', T] .$$

So for each integer $m \geq 1$,

$$u_{2x_0}(t) = \sum_{k=1}^{m} L_{x_0}^k(u_{1x_0}(.))(t) + \sum_{k=m+1}^{\infty} L_{x_0}^k(u_{1x_0}(.))(t), \forall t \in [T', T].$$

We recall that

$$L_{x_0}(u_{1x_0}(.))(t) = \int_0^t T_0(t-s)H(G(s, u_{x_0}(s)), u_{1x_0}(s))ds, \forall t \in [0, T].$$

Moreover, by using Assumption 5.2 f), and Theorem 5.1, one deduces that

$$M_0 \stackrel{\text{def}}{=} \{H(G(t, u_{x_0}(t)), u_{1x_0}(t)) : x_0 \in B, t \in [0, T]\}$$

is relatively compact. By compactness of $[0, T] \times \overline{M_0}$, and by continuity of $(t, x) \to T_0(t)x$, one deduces that

$$M_1 \stackrel{\text{def}}{=} \{T_0(t)x : x \in M_0, t \in [0, T]\}$$

is also relatively compact. Therefore,

$$L_{x_0}(u_{1x_0}(.))(t) \in \overline{\text{conv}}(M_1) \stackrel{\text{def}}{=} E_0, \quad \forall t \in [0, T],$$

and by Mazur's theorem E_0 is compact. By using induction arguments we deduce that for each $m \geq 1$, there exists a compact subset $E_m \subset X_0$, such that

$$\sum_{k=1}^{m} L_{x_0}^k(u_{1x_0}(.))(t) \in E_m, \quad \forall t \in [0, T].$$

Moreover, by using Assumption 5.1 c), we know that there exists a constant $C > 0$, such that

$$\|L_{x_0}^k(u_{1x_0}(.))(t)\| \leq \alpha_0 e^{\omega T} C^k \frac{T^k}{k!},$$

where $C = M\|H\|_{\mathcal{L}(Z\times X_0,X_0)}[\|G(0,0)\| + \|G\|_{\mathrm{Lip}}[\alpha_0 + T]]$, $\alpha_0 > 0$ is the constant introduced in Assumption 5.1 c), and where $\mathbb{R}\times X_0$ is endowed with the norm $\|(t,x)\| = |t| + \|x\|$. So, we deduce that $\forall t \in [0,T]$,

$$\begin{aligned}\|\sum_{k=m+1}^{\infty} L_{x_0}^k(u_{1x_0}(.))(t)\| &\leq \alpha_0 e^{\omega T}\sum_{k=m+1}^{\infty}\frac{(CT)^k}{k!}\\ &\leq \alpha_0 e^{\omega T}(e^{CT} - \sum_{k=0}^{m}\frac{(CT)^k}{k!}) \stackrel{\text{def}}{=} \gamma_m \to 0\end{aligned}$$

as $m \to +\infty$. Let $E = \cup_{t\in[T',T]\ x_0\in B}\{u_{2x_0}(t)\}$. Then for all $x \in E$ there exists $y \in E_m$ such that $\|x-y\| \leq \gamma_m$. Let $\varepsilon > 0$, and let be $m > 0$ such that $\gamma_m \leq \frac{\varepsilon}{2}$. Since E_m is compact we can find a finite sequence $\{y_j\}_{j=1,\dots,p}$ such that

$$E_m \subset \cup_{j=1,\dots,p} B(y_j, \frac{\varepsilon}{2}),$$

and since $\gamma_m \leq \frac{\varepsilon}{2}$, we also have $E \subset \cup_{j=1,\dots,p} B(y_j,\varepsilon)$. So E is relatively compact. □

We are now in position to investigate compact global attractors for periodic non-autonomous semiflow generated by the Cauchy problem

$$\begin{gathered}\frac{dU(t,s)x_0}{dt} = AU(t,s)x_0 + F(t,U(t,s)x_0),\ \text{for } t \geq s,\\ U(s,s)x_0 = x_0,\end{gathered}$$

where F is time periodic.

Assumption 5.3.

a) $A : D(A) \subset X \to X$ is a Hille-Yosida operator.

b) $F : [0,+\infty)\times X_0 \to X$ is a continuous map, which satisfies
$$F(t,x) = F_1(t,x) + F_2(t,x),$$
where $F_1 : [0,+\infty)\times X_0 \to X$, and $F_2 : [0,+\infty)\times X_0 \to X_0$ satisfying: $\forall C > 0$, $\exists K(C) > 0$, such that
$$\|F_i(t,x) - F_i(t,y)\| \leq K(C)\|x-y\|, \forall x,y \in \overline{B}(0,C)\cap X_0, \forall t \geq 0, \quad i = 1,2.$$

c) There exists a closed convex subset $E_0 \subset X_0$ such that, for each $s \geq 0$, and each $x_0 \in E_0$ there exists a continuous solution $U(.,s)x_0 : [s,+\infty) \to X_0$ of
$$\begin{gathered}U(t,s)x_0 = x_0 + A\int_s^t U(l,s)x_0 dl + \int_s^t F(l,U(l,s)x_0)dl,\ \forall t \geq s,\\ U(t,s)E_0 \subset E_0, \forall t \geq s \geq 0,\end{gathered}$$
and for each $s \geq 0$, each $T \geq 0$, and each bounded subset $B \subset E_0$, the set
$$\{U(t+s,s)x_0 : 0 \leq t \leq T, x_0 \in B\} \text{ is bounded.}$$

d) For each $t \geq 0$, $\varphi \to F_1(t,\varphi)$ is continuous and maps bounded sets into relatively compact sets, and for each $C > 0$, for each $T \geq 0$, there exists $k = k(C,T) > 0$, such that
$$\|F_1(t,x) - F_1(l,x)\| \leq k|t-l|, \forall x \in \overline{B}(0,C)\cap X_0, \forall t,l \in [0,T].$$

e) For each bounded subset $B \subset E_0$, for each $s \geq 0$, and for each $T \geq s$, there exists $k = k(B, s, T) \geq 0$, such that

$$\|F_1(t, U(t,s)x_0) - F_1(l, U(l,s)x_0)\| \leq k|t-l|, \forall x_0 \in B, \forall t, l \in [s,T].$$

f) Let $(Z, \|.\|_Z)$ be a Banach space, let $H : Z \times X_0 \to X_0$ be a continuous bilinear map, and let be a Lipschitz continuous map $G : \mathbb{R}_+ \times X_0 \to Z$ which maps bounded sets into relatively compact sets. We assume that

$$F_2(t,x) = H(G(t,x), x), \forall x \in X_0, \forall t \geq 0.$$

g) For each $s \geq 0$, and for each $x_0 \in E_0$, let $w_{x_0}(., s) \in C([s, +\infty), X_0)$ be the solution of

$$w_{x_0}(t,s) = T_0(t-s)x_0 + \int_s^t T_0(t-l)H(G(l, U(l,s)x_0), w_{x_0}(l,s))dl, \forall t \geq s.$$

We assume that there exists $T' > 0$, such that

$$w_{x_0}(t,s) = 0, \forall x_0 \in E_0, \forall t \geq T' + s.$$

h) There exists $\omega > 0$ such that

$$F(t+\omega, x) = F(t,x), \forall t \geq 0, \forall x \in X_0.$$

i) There exists a closed bounded subset $E_1 \subset E_0$ such that for each $s \geq 0$, for each bounded subset $B \subset E_0$, there exists $t_0 = t_0(B,s) \geq s$ such that

$$U(t,s)B \subset E_1, \forall t \geq t_0.$$

In section 6, we will verify Assumption 5.3 for the age-structured model with $E_0 = X_{0+}$, and $E_1 = \overline{B}(0,M) \cap X_{0+}$ for some $M > 0$. But it is possible to consider different situations.

The following theorem describes the global attractor for a periodic non-autonomous semiflow. The compactness of A and its attractor properties have already been proved by Zhao [20] under more general assumptions. Zhao's proof also contains *vi*), but not *iii*).

Theorem 5.3. *Under Assumption 5.3, the non-autonomous semiflow $U(t,s)$ restricted to E_0 is ω-periodic, that is to say that*

$$U(t+\omega, s+\omega)x_0 = U(t,s)x_0, \text{ for all } x_0 \in E_0, \text{ for all } t \geq s \geq 0.$$

Moreover, there exists a family $\{A_t\}_{t\geq 0}$ of subsets of E_0, satisfying:

i) $A_t = A_{t+\omega}$ *for all* $t \geq 0$.
ii) *For all* $t \geq 0$, A_t *is compact and connected.*
iii) *For all* $t \geq s \geq 0$, $U(t,s)A_s = A_t$.
iv) $A = \cup_{0\leq t\leq\omega} A_t$ *is compact.*
v) *The map* $t \to A_t$ *is continuous with respect to the Hausdorff metric, that is to say that* $h(A_t, A_{t_0}) \to 0$, *as* $t \to t_0$, *where*

$$h(A,B) = \max(\mathrm{dist}(A,B), \mathrm{dist}(B,A)),$$

with $\mathrm{dist}(A,B) = \sup_{x\in A} \mathrm{dist}(x,B)$, *and* $\mathrm{dist}(x,B) = \inf\{\|x-y\| : y \in B\}$.

vi) *For each bounded set* $B \subset E_0$, *and for each* $s \geq 0$,

$$\lim_{t\to+\infty} \mathrm{dist}(U(t,s)B, A_t) = 0.$$

Proof. One can first note that under Assumption 5.3 a)-g), Assumptions 5.1 and 5.2 are satisfied for any bounded set $B \subset E_0$ and for any $T \geq T'$. So Theorem 5.2 implies that for each $s \geq 0$, and for each $T \geq T'$

$$\{U(t+s,s)x_0 : t \in [T',T], x_0 \in B\} \text{ has compact closure.}$$

The periodicity of $U(t,s)$ is immediate. Let us denote for each $t \geq 0$, the map $T_t : E_0 \to E_0$, defined by

$$T_t(x) = U(t+\omega,t)x, \forall x \in E_0.$$

From Assumption 5.3 i), it is not difficult to see that E_1 is an absorbing set for T_t, that is to say that for each bounded set $B \subset E_0$, there exists an integer $k_0 \in \mathbb{N}$, such that

$$T_t^k(B) \subset E_1, \forall k \geq k_0.$$

Moreover, from Theorem 3.5 and Theorem 5.2 we know that for all $m \in \mathbb{N}$, such that $m\omega \geq T'$, T_t^m is continuous and maps bounded sets into relatively compact sets. Thus from Theorem 2.4.2 p.17 in the book by Hale [5], we deduce that for each $t \geq 0$, there exists $A_t \subset E_0$ a global attractor for T_t. Namely *i)* A_t is compact; *ii)* $T_t A_t = A_t$; and *iii)* for every bounded subset $B \subset E_0$,

$$\lim_{m\to+\infty} \operatorname{dist}(T_t^m(B), A_t) = 0. \tag{5.2}$$

Furthermore, since E_0 is closed and convex, we deduce that $\overline{\operatorname{conv}}(A_t) \subset E_0$. Moreover by Mazur's theorem $\overline{\operatorname{conv}}(A_t)$ is compact, so A_t attracts $\overline{\operatorname{conv}}(A_t)$ with respect to the map T_t. By applying the method of the proof of Lemma 2.4.1 p.17 in the book by Hale [5], we deduce that A_t is connected. We now prove that

$$U(t,s)A_s = A_t, \forall t \geq s \geq 0.$$

Let $t \geq s \geq 0$ be fixed, and let us denote

$$B_t = U(t,s)A_s.$$

Then

$$\begin{aligned} T_t B_t =& U(t+\omega,t)U(t,s)A_s = U(t+\omega,s+\omega)U(s+\omega,s)A_s \\ =& U(t,s)T_s A_s = U(t,s)A_s = B_t. \end{aligned}$$

So B_t is compact and invariant by T_t. We deduce from (5.2) that $B_t \subset A_t$. Moreover if $k \in \mathbb{N}$ is such that $s + k\omega > t$, and $m \geq k$

$$\begin{aligned} A_t &= T_t^m(A_t) = U(t+m\omega,t)A_t \\ &= U(t+m\omega,s+k\omega)U(s+k\omega,t)A_t \\ &= U(t+m\omega,s+m\omega)U(s+m\omega,s+k\omega)U(s+k\omega,t)A_t \\ &= U(t+m\omega,s+m\omega)T_s^{m-k}U(s+k\omega,t)A_t \\ &= U(t,s)T_s^{m-k}U(s+k\omega,t)A_t. \end{aligned}$$

So by using again (6.2), and by taking the limit when m goes to infinity, one deduces that

$$A_t \subset U(t,s)A_s = B_t.$$

So,

$$A_t = U(t,s)A_s, \forall t \geq s \geq 0. \tag{5.3}$$

We now prove *iv)*. Let $\{x_n\}_{n\geq 0}$ be a sequence in A. Then there exists numbers $t_n \in [0,\omega]$ such that $x_n \in A_{t_n}$. Since $A_t = U(t,0)A_0$, there exists elements y_n in

A_0 such that $x_n = U(t_n, 0)y_n$. Since A_0 is compact, $y_n \to y \in A_0$ after choosing a subsequence. Further $t_n \to t \in [0, \omega]$ after choosing a further subsequence. Since the map $(t, x) \to U(t, 0)x$ is continuous by Theorem 3.5, $x_n \to U(t, 0)y \in A_t$ after choosing a subsequence.

We now prove v). Claim 1: $\operatorname{dist}(A_s, A_t) \to 0$ as $s \to t$. Suppose that is not the case, then there exists some $\varepsilon > 0$ and a sequence $t_n \to t$ such that $\operatorname{dist}(A_{t_n}, A_t) \geq \varepsilon$ for all $n \in \mathbb{N}$. By definition of $\operatorname{dist}(.,.)$, there exist elements $x_n \in A_{t_n}$ such that $\operatorname{dist}(x_n, A_t) \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $A_{t_n} = U(t_n, 0)A_0$, there exist elements $y_n \in A_0$ such that $x_n = U(t_n, 0)y_n$ for all $n \in \mathbb{N}$. Since A_0 is compact, $y_n \to y \in A_0$ after choosing a subsequence. So $x_n \to U(t, 0)y \in A_t$ after choosing a subsequence, and $\operatorname{dist}(x_n, A_t) \to 0$, which gives a contradiction.
Claim 2: $\operatorname{dist}(A_t, A_s) \to 0$ as $s \to t$. Suppose that is is not the case, then there exists some $\varepsilon > 0$ and a sequence $t_n \to t$ such that $\operatorname{dist}(A_t, A_{t_n}) \geq \varepsilon$ for all $n \in \mathbb{N}$. By definition of $\operatorname{dist}(.,.)$, there exist elements $x_n \in A_t$ such that $\operatorname{dist}(x_n, A_{t_n}) \geq \varepsilon$ for all $n \in \mathbb{N}$ Then there exist elements $y_n \in A_0$ such that $x_n = U(t, 0)y_n$. After choosing a subsequence, $x_n \to U(t, 0)y$ for some $y \in A_0$. By definition of $\operatorname{dist}(.,.)$,

$$\|U(t,0)y_n - U(t_n,0)y\| = \|x_n - U(t_n,0)y\| \geq \operatorname{dist}(x_n, A_{t_n}) \geq \varepsilon > 0.$$

But

$$\|U(t,0)y_n - U(t_n,0)y\| \to \|U(t,0)y - U(t,0)y\| = 0.$$

To complete the proof it remains to prove vi). Assume vi) does not hold. Then there exists a sequence $t_n \to +\infty$ and some $\varepsilon > 0$ such that

$$\operatorname{dist}(U(t_n, s)B, A_{t_n}) \geq \varepsilon > 0, \quad \forall n \in \mathbb{N}.$$

Let $\theta_n \in [0, \omega]$, and $m_n \in \mathbb{N}$, be such that $t = m_n\omega + \theta_n + s$, then one has

$$\begin{aligned} U(t_n, s) &= U(m_n\omega + \theta_n + s, s) \\ &= U(m_n\omega + \theta_n + s, m_n\omega + s)U(m_n\omega + s, s) \\ &= U(\theta_n + s, s)U(m_n\omega + s, s) = U(\theta_n + s, s)T_s^{m_n}. \end{aligned} \tag{5.4}$$

By (5.4) there exist elements $x_n \in B$ such that

$$\operatorname{dist}(U(\theta_n + s, s)T_s^{m_n} x_n, A_{\theta_n + s}) \geq \varepsilon > 0, \forall n \in \mathbb{N},$$

and $m_n \to +\infty$ as $n \to +\infty$. Since A_s attracts B under T_s and A_s is compact, $y_n = T_s^{m_n} x_n \to y \in A_s$ after choosing a subsequence. Since $U(\theta_n + s, s)y \in A_{\theta_n + s}$,

$$0 \leq \varepsilon \leq \operatorname{dist}(U(\theta_n + s, s)T_s^{m_n} x_n, A_{\theta_n + s}) \leq \|U(\theta_n + s, s)y_n - U(\theta_n + s, s)y\|.$$

After choosing another subsequence $\theta_n \to \theta$,

$$0 < \varepsilon \leq \|U(\theta + s, s)y - U(\theta + s, s)y\|,$$

which gives a contradiction. □

6. Applications

In this section, we consider the following system, for $i = 1, \dots, N$,

$$
\begin{gathered}
u_i(t)(0) = \int_0^{+\infty} \sum_{j=1}^{N} \beta_{ij}(t, u(t))(a) u_j(t)(a) da, \\
\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial a} = -\mu_i(a) u_i(t)(a) + \sum_{j=1}^{N} m_{ij}(t, u(t))(a) u_j(t)(a), \ a.e. \ a \in (0, a_\dagger^i), \\
u_i(t)(a) = 0, \ a.e. \ a \in (a_\dagger^i, +\infty), \\
u_i(0)(a) = \psi_i(a), \ a.e. \ a \in (0, +\infty),
\end{gathered}
$$

where $a_\dagger^i > 0$ is the maximum age of the class i, $\beta_{ij}(t, u(t))(a)$ the birth rate of individuals of the class j in the class i, $\mu_i(a)$ is the mortality of individuals of class i, and $m_{ij}(t, u(t))(a)$ represents: 1) if $i \neq j$ a migration from class j to class i; 2) if $i = j$ intra-specific or inter-specific competition, and loss due to migration from class i to another class.

Let us consider

$$
Y = Y_1 \times Y_2 \times \dots \times Y_N,
$$

with

$$
Y_i = \left\{ \varphi \in L^1(0, +\infty) : \varphi(a) = 0, \ a.e. \ a \in (a_\dagger^i, +\infty) \right\}, \quad i = 1, \dots, N.
$$

Here the Banach space X is

$$
X = \mathbb{R}^N \times Y
$$

which is endowed with a usual product norm of $\mathbb{R}^N \times L^1(0, +\infty)^N$, and

$$
X_+ = \mathbb{R}_+^N \times Y_+
$$

where $Y_+ = Y_{1+} \times Y_{2+} \times \dots \times Y_{N+}$, with $Y_{i+} = Y_i \cap L_+^1(0, +\infty)$, for $i = 1, \dots, N$. Following Thieme's approach [13, p. 1037], we define $A : D(A) \subset X \to X$

$$
A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = (-\varphi(0), B\varphi), \quad \text{for } \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A) = \{0_{\mathbb{R}^N}\} \times D(B),
$$

where $B : D(B) \subset Y \to Y$ is defined by

$$
(B\varphi)_i(a) = \begin{cases} -\varphi_i'(a) - \mu_i(a)\varphi_i(a), & \text{a.e. } a \in (0, a_\dagger^i) \\ 0, & \text{a.e. } a \in (a_\dagger^i, +\infty), \end{cases}
$$

and

$$
D(B) = \left\{ \varphi \in W^{1,1}(0, +\infty)^N : \mu_i \varphi_i \in L^1(0, +\infty), \varphi_i(a) = 0, \text{ a.e. } a \in (a_\dagger^i, +\infty) \right\}.
$$

So here $X_0 = \overline{D(A)} = \{0_{\mathbb{R}^N}\} \times Y$, and $X_{0+} = \{0_{\mathbb{R}^N}\} \times Y_+$. We also introduce the nonlinear map $F : \mathbb{R}_+ \times X_0 \to X$ is,

$$
F(t, (0_{\mathbb{R}^N}, \varphi)) = F_1(t, \varphi) + F_2(t, \varphi), \text{ for } t \geq 0, \text{ and } \varphi \in Y,
$$

where $F_1 : \mathbb{R}_+ \times Y \to X$ is

$$
F_1(t, \varphi) = \begin{pmatrix} \widetilde{F}_1(t, \varphi) \\ 0_Y \end{pmatrix}, \text{ with } \widetilde{F}_1(t, \varphi)_i = \int_0^{+\infty} \sum_{j=1}^{N} \beta_{ij}(t, \varphi)(a) \varphi_j(a) da
$$

and $F_2 : \mathbb{R}_+ \times Y \to Y$ is

$$F_2(t,\varphi) = \begin{pmatrix} 0_{\mathbb{R}^N} \\ \widetilde{F}_2(t,\varphi) \end{pmatrix}, \text{ with } \widetilde{F}_2(t,\varphi)_i = \sum_{j=1}^{N} m_{ij}(t,\varphi)\varphi_j$$

Assumption 6.1.

a) (Concerning the unbounded linear operator A) For $i = 1,\dots,N$, $\mu_i \in L^1_{\mathrm{loc}}([0,a^i_\dagger),\mathbb{R})$ and satisfies

$$\int_0^a \mu_i(s)ds < +\infty, \forall a \in [0,a^i_\dagger[, \quad \lim_{a \nearrow a^i_\dagger} \int_0^a \mu_i(s)ds = +\infty,$$

$$\mu_i(a) \geq 0, \ a.e. \ a \in (0,a^i_\dagger), \ \mu_i(a) = 0, \ a.e. \ a \in (a^i_\dagger, +\infty)$$

b) (Concerning the existence and uniqueness of the solutions) For all $t \geq 0$, $\forall \varphi \in Y$, $\forall i,j = 1,\dots,N$, the functions $\beta_{ij} : \mathbb{R}_+ \times Y \to L^\infty(0,+\infty)$ and $m_{ij} : \mathbb{R}_+ \times Y \to L^\infty(0,+\infty)$ are continuous maps, and

$$\text{if } i \neq j, \quad m_{ij}(t,\varphi)(a) = 0, \text{ a.e. } a \geq a^i_\dagger.$$

Moreover, for $i,j = 1,\dots,N$, $\forall C > 0$, there exist $k_1^{\beta_{ij}}(C) > 0$, $k_2^{\beta_{ij}}(C) > 0$, $k_1^{m_{ij}}(C) > 0$, and $k_2^{m_{ij}}(C) > 0$, such that $\forall \varphi_1,\varphi_2 \in Y \cap B(0,C)$, $\forall t \geq 0$,

$$\begin{gathered}
\|\beta_{ij}(t,\varphi_1) - \beta_{ij}(t,\varphi_2)\|_{L^\infty(0,+\infty)} \leq k_1^{\beta_{ij}}(C)\|\varphi_1 - \varphi_2\|_{L^1(0,+\infty)^N}, \\
\|m_{ij}(t,\varphi_1) - m_{ij}(t,\varphi_2)\|_{L^\infty(0,+\infty)} \leq k_1^{m_{ij}}(C)\|\varphi_1 - \varphi_2\|_{L^1(0,+\infty)^N}, \\
\|\beta_{ij}(t,\varphi_1)\|_{L^\infty(0,+\infty)} \leq k_2^{\beta_{ij}}(C), \\
\|m_{ij}(t,\varphi_1)\|_{L^\infty(0,+\infty)} \leq k_2^{m_{ij}}(C).
\end{gathered}$$

c) (Positivity of the solutions) For all $i,j = 1,\dots,N$, $\forall \varphi \in Y_+, \forall t \geq 0$, we have

$$\beta_{ij}(t,\varphi) \geq 0, \text{ and if } i \neq j, \ m_{ij}(t,\varphi) \geq 0.$$

d) (Global existence of the nonnegative solutions) For all $i,j = 1,\dots,N$, $\exists k_3^{\beta_{ij}} > 0$, $\exists k_3^{m_{ij}} > 0$, $\forall \varphi \in Y_+, \forall t \geq 0$,

$$\begin{gathered}
\sup_{t \geq 0, \varphi \in Y_+} \|\beta_{ij}(t,\varphi)\|_{L^\infty(0,+\infty)} \leq k_3^{\beta_{ij}}, \\
\sup_{t \geq 0, \varphi \in Y_+} \|m_{ij}(t,\varphi)^+\|_{L^\infty(0,+\infty)} \leq k_3^{m_{ij}},
\end{gathered}$$

where $m_{ij}(t,\varphi)^+(a) = \max(0, m_{ij}(t,\varphi)(a))$, a.e. $a \geq 0$.

Theorem 6.1. *Under Assumption 6.1 a), the operator $A : D(A) \subset X \to X$ satisfies $(0,+\infty) \subset \rho(A)$, and for all $\lambda > 0$,*

$$\|(\lambda Id - A)^{-1}\| \leq \frac{1}{\lambda}.$$

Moreover, for all $\lambda > 0$, $(\lambda Id - A)^{-1}X_+ \subset X_+$. Also $T_0(t) = (0, \widetilde{T}_0(t))$ the C_0-semigroup generated by A_0, the part of A in $\overline{D(A)}$, is given by

$$\widetilde{T}_0(t)\varphi_i(a) = \begin{cases} 0, & a.e. \ a \in (0, \min(t,a^i_\dagger)), \\ \exp(-\int_{a-t}^a \mu_i(\sigma)d\sigma)\varphi_i(a-t), & a.e. \ a \in (\min(t,a^i_\dagger), a^i_\dagger), \\ 0, & a.e. \ a \in (a^i_\dagger, +\infty). \end{cases}$$

Proof. We consider the case $N = 1$, the case $N > 1$ being similar. Let us start by computing the resolvent of A. Let $\binom{\alpha}{\psi} \in \mathbb{R} \times Y$ (since $N = 1$ we have $Y = Y_1$). We look for $\varphi \in Y_1$ solution for $\lambda > 0$ of

$$\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = (\lambda Id - A)\begin{pmatrix}0\\ \varphi\end{pmatrix} \Leftrightarrow \begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}\varphi(0)\\ \lambda\varphi + \varphi' + \mu\varphi\end{pmatrix}$$

Thus

$$\varphi(a) = e^{-\int_0^a \lambda+\mu(l)dl}\alpha + \int_0^a e^{-\int_s^a \lambda+\mu(l)dl}\psi(s)ds, \text{ a.e. } a \in (0, a_\dagger^1).$$

We conclude that the resolvent operator is positive i.e. $(\lambda Id - A)^{-1}(\mathbb{R}_+ \times Y_+) \subset \{0\} \times Y_+$. Moreover, for $\lambda > 0$,

$$\begin{aligned}
\|(\lambda Id - A)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix}\| &= \int_0^{a_\dagger^1} |e^{-\int_0^a \lambda+\mu(l)dl}\alpha + \int_0^a e^{-\int_s^a \lambda+\mu(l)dl}\psi(s)ds|da \\
&\leq \int_0^{a_\dagger^1} e^{-\int_0^a \lambda+\mu(l)dl}da|\alpha| + \int_0^{a_\dagger^1}\int_0^a e^{-\int_s^a \lambda+\mu(l)dl}|\psi(s)|dsda \\
&\leq \int_0^{a_\dagger^1} e^{-\lambda a}da|\alpha| + \int_0^{a_\dagger^1}\int_a^{a_\dagger^1} e^{-\lambda a}dae^{\lambda s}|\psi(s)|ds \\
&\leq \frac{(1-e^{-\lambda a_\dagger^1})}{\lambda}|\alpha| + \int_0^{a_\dagger^1}\frac{(e^{-\lambda s} - e^{-\lambda a_\dagger^1})}{\lambda}e^{\lambda s}|\psi(s)|ds \\
&\leq \frac{1}{\lambda}[|\alpha| + \int_0^{a_\dagger^1}|\psi(s)|ds] = \frac{1}{\lambda}\|\begin{pmatrix}\alpha\\ \psi\end{pmatrix}\|,
\end{aligned}$$

so A is a Hille-Yosida operator. To complete the proof of this theorem, it remains to give the explicit formula for the linear semigroup $T_0(t)$. Let $\begin{pmatrix}0\\ \varphi\end{pmatrix} \in \overline{D(A)}$ (i.e. $\varphi \in Y_1$). We denote

$$T_1(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ \widetilde{T}_1(t)\varphi\end{pmatrix},$$

where

$$\widetilde{T}_1(t)(\varphi)(a) = \begin{cases} 0, & \text{a.e. } a \in (0, \min(t, a_\dagger^1)), \\ \exp(-\int_{a-t}^a \mu_i(\sigma)d\sigma)\varphi_i(a-t), & \text{a.e. } a \in (\min(t, a_\dagger^1), a_\dagger^1), \\ 0, & \text{a.e. } a \in (a_\dagger^1, +\infty). \end{cases}$$

From section 2, to prove that $T_0(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = T_1(t)\begin{pmatrix}0\\ \varphi\end{pmatrix}, \forall t \geq 0$, it is sufficient to verify that

$$T_1(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix} + A\int_0^t T_1(s)\begin{pmatrix}0\\ \varphi\end{pmatrix}ds, \forall t \geq 0. \tag{6.1}$$

To show this, we need to compute $\int_0^t \widetilde{T}_1(\varphi)(s)ds$. We define $\psi_1 \in C(\mathbb{R}, Y)$, for all $t \geq 0$, by

$$\psi_1(t)(a) = \begin{cases} \int_0^a \exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)ds, & \text{a.e. } a \in (0, \min(t, a_\dagger^1)), \\ \int_0^t \exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)ds, & \text{a.e. } a \in (\min(t, a_\dagger^1), a_\dagger^1), \\ 0, & \text{a.e. } a \in (a_\dagger^1, +\infty). \end{cases}$$

We now want to prove that $\psi_1(t) = \int_0^t \widetilde{T}_1(\varphi)(s)ds$, for all $t \geq 0$. By the Hahn-Banach theorem, it is sufficient to show that

$$\int_0^{+\infty} f(a)(\int_0^t \widetilde{T}_1(s)(\varphi)ds)(a)da = \int_0^{+\infty} f(a)\psi_1(t)(a)da, \forall f \in L^\infty(0,+\infty).$$

Moreover, it is not difficult to see that $T_1(t)$ has the semigroup property, and it is sufficient to prove (6.1) for $t \leq a_\dagger^1$. We deduce that it is sufficient to prove the above equality for $t \leq a_\dagger^1$. Let $t \in [0, a_\dagger^1]$. and $f \in L^\infty(0,+\infty)$,

$$\begin{aligned}
&\int_0^{+\infty} f(a)(\int_0^t \widetilde{T}_1(s)(\varphi)ds)(a)da \\
&= \int_0^t \int_0^{+\infty} f(a)\widetilde{T}_1(s)(\varphi)(a)dads \\
&= \int_0^t \int_s^{a_\dagger^1} f(a)\widetilde{T}_1(s)(\varphi)(a)dads \\
&= \int_0^t \int_s^t f(a)\exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)dads \\
&\quad + \int_0^t \int_t^{a_\dagger^1} f(a)\exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)dads \\
&= \int_0^t f(a)\int_0^a \exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)dsda \\
&\quad + \int_t^{a_\dagger^1} f(a)\int_0^t \exp(-\int_{a-s}^a \mu_i(\sigma)d\sigma)\varphi(a-s)dsda \\
&= \int_0^t f(a)\psi_1(t)(a)da + \int_t^{a_\dagger^1} f(a)\psi_1(t)(a)da \\
&= \int_0^{+\infty} f(a)\psi_1(t)(a)da.
\end{aligned}$$

Now it remains to replace $T_1(t)$ and $\psi_1(t)$ in equation (6.1). □

Theorem 6.2. *Under Assumption 6.1, for each $s \geq 0$, and each $x_0 \in X_0$, there exists a unique maximal solution $U(.,s)x_0 \in C([s, T_s(x_0)), X_0)$ of*

$$U(t,s)x_0 = x_0 + A\int_s^t U(l,s)x_0 dl + \int_s^t F(l, U(l,s)x_0)dl, \forall t \in [s, T_s(x_0)), \tag{6.2}$$

where the map $T_s : X_0 \to (s,+\infty]$ is

$$T_s(x_0) = \sup\{T > s : \exists u \in C([s,T], X_0) \text{ solution of (6.2)}\},$$

and $U(t,s)$ is a non-autonomous semiflow, that is to say that

$$U(t,r)U(r,s)x_0 = U(t,s)x_0, \forall x_0 \in X_0, \forall 0 \leq s \leq r \leq t < T_s(x_0).$$

Moreover for each $s \geq 0$, the set $D_s = \{(t,x) : x \in X_0, s \leq t < T_s(x_0)\}$ is an open set in $[s,+\infty) \times X_0$, and the map $(t,x) \to U(t,s)x$ from D to X_0 is continuous. Furthermore, for each $s \geq 0$, and each $x_0 \in X_{0+}$, there exists a unique solution $U(.,s)x_0 \in C([s,+\infty), X_0)$ of (6.2) (i.e. $T_s(x_0) = +\infty$),

$$U(t,s)X_{0+} \subset X_{0+}, \forall t \geq s \geq 0,$$

and there exists $C_0 > 0$, *and* $C_1 > 0$ *such that*

$$\|U(t,s)x_0\|_{L^1(0,+\infty)^N} \leq \|x_0\|_{L^1(0,+\infty)^N} C_0 e^{C_1(t-s)}, \forall t \geq s \geq 0, \forall x_0 \in X_{0+}. \quad (6.3)$$

Proof. From Theorem 6.1, we know that Assumption 3.1 a) is satisfied with $M = 1$, and $\omega = 0$. By using Assumption 6.1 b) it is not difficult to see that Assumption 3.1 b) is satisfied. By using Theorem 6.1, and Assumption 6.1 c), one can easily see that Assumption 3.2 is satisfied. Finally, we define $\forall t \geq 0, \forall \varphi \in Y$

$$G_1(t,\varphi) = \begin{pmatrix} 0 \\ \widetilde{G}_1(t,\varphi) \end{pmatrix}, \text{ and } G_2(t,\varphi) = \begin{pmatrix} \widetilde{F}_1(t,\varphi) \\ \widetilde{F}_2(t,\varphi) - \widetilde{G}_1(t,\varphi) \end{pmatrix},$$

where for $i = 1, \dots, N$,

$$\widetilde{G}_1(t,\varphi)_i = -m_{ii}(t,\varphi)^- \varphi_i$$

where $m_{ii}(t,\varphi)^-(a) = \max(-m_{ii}(t,\varphi)(a), 0)$, a.e. $a \geq 0$. Then clearly Assumption 3.3 is satisfied. □

Assumption 6.2.

e) (Differentiability of the solutions) For $i, j = 1, \dots, N$, $\beta_{ij} : \mathbb{R}_+ \times Y \to L^\infty(0,+\infty)$ and $m_{ij} : \mathbb{R}_+ \times Y \to L^\infty(0,+\infty)$ are continuously differentiable.

Theorem 6.3. *Under Assumptions 6.1 and 6.2, let* $s \geq 0$, *and let* $x_0 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_0$ *be such that*

$$x_0 \in \left\{ y \in D(A) : Ay + F(s,y) \in \overline{D(A)} \right\},$$

namely

$$\varphi \in Y \cap W^{1,1}(0,+\infty)^N, \ \mu\varphi = \begin{pmatrix} \mu_1\varphi_1 \\ \mu_2\varphi_2 \\ \vdots \\ \mu_N\varphi_N \end{pmatrix} \in L^1(0,+\infty)^N, \ \textit{and } \varphi(0) = \widetilde{F}_1(s,\varphi).$$

Then the map $t \to U(t,s)x_0$ *is continuously differentiable, and*

$$\frac{dU(t,s)x_0}{dt} = AU(t,s)x_0 + F(t, U(t,s)x_0), \forall t[s, T_s(x_0)).$$

Moreover, for the domain $D(A_{N,0,s}) = \left\{ x_0 \in D(A) : Ax_0 + F(s,x_0) \in \overline{D(A)} \right\}$ *we have that* $D(A_{N,0,s}) \cap X_{0+}$ *is dense in* X_{0+}.

Proof. The proof of the first part of Theorem 6.3 is a direct consequence of the results of section 4. It remains to show that we can approximate an element of X_{0+} by an element of $D(A_{N,0,s}) \cap X_{0+}$. We denote $F_s(x) = F(s,x)$, $\forall s \geq 0$, $\forall x \in X_0$. Let $y \in X_{0+}$. Then as in section 4, we consider the following fixed point problem,

$$\begin{gathered} (Id + \lambda\mu Id - \lambda A - \lambda(F_s + \mu Id))x_{\lambda\mu} = y \\ \Leftrightarrow x_{\lambda\mu} = (Id - \lambda(A - \mu Id))^{-1}y + \lambda(Id - \lambda(A - \mu Id))^{-1}(F_s + \mu Id)(x_{\lambda\mu}) \end{gathered}$$

By fixing $\mu > 0$, large enough, such that

$$(F_s + \mu Id)(z) \in X_+, \forall z \in B(0, 2\|y\|) \cap X_{0+},$$

then for all $\lambda > 0$ small enough, the map

$$\Phi_\lambda(x) = (Id - \lambda(A - \mu Id))^{-1}y + \lambda(Id - \lambda(A - \mu Id))^{-1}(F_s + \mu Id)(x),$$

maps $B(0,2\|y\|)\cap X_{0+}$ into itself, and for all $\lambda>0$ small enough Φ_λ restricted to $B(0,2\|y\|)\cap X_{0+}$ is a contraction. The result follows. $\square$

Assumption 6.3.

f) (Eventual Compactness) For all $C>0$, $\forall T>0$, for $i,j=1,\dots,N$, there exists $k_4^{\beta_{ij}}(C,T)>0$, such that $\forall t,l\in[0,T]$,

$$\|\beta_{ij}(t,\varphi)-\beta_{ij}(l,\varphi)\|_{L^\infty(0,+\infty)}\le k_4^{\beta_{ij}}(C,T)|t-l|.$$

g) For $i,j=1,\dots,N$, $\forall t>0$, $\forall\varphi\in Y$, one has $\beta_{ij}(t,\varphi)\mu_j\in L^\infty(0,+\infty)$, and $\forall C>0$, $\forall T>0$,

$$\sup_{\substack{\varphi\in Y\cap B(0,C)\\ t\in[0,T]}}\|\beta_{ij}(t,\varphi)\mu_j\|_{L^\infty(0,+\infty)}<+\infty;$$

h) For $i,j=1,\dots,N$, $\forall t>0$, $\forall\varphi\in Y$, one has $\beta_{ij}(t,\varphi)\in W^{1,\infty}(0,+\infty)$, and $\forall C>0,\forall T>0$,

$$\sup_{\varphi\in Y\cap B(0,C)\ t\in[0,T]}\|\frac{d}{da}\beta_{ij}(t,\varphi)\|_{L^\infty(0,+\infty)}<+\infty;$$

i) There exists $M\in\mathbb{N}\setminus\{0\}$, for $i,j=1,\dots,N$, for all $C>0$, for all $T>0$, there exists $k_5^{\beta_{ij}}(C,T)>0$, such that for all $t\in[0,T]$, for all $\varphi_1,\varphi_2\in B(0,C)\cap Y$,

$$\begin{aligned}&\|\beta_{ij}(t,\varphi_1)-\beta_{ij}(t,\varphi_2)\|_\infty\\ &\le k_5^{\beta_{ij}}(C,T)\sum_{l=1}^{M}\sum_{p=1}^{N}|\int_0^{+\infty}f_{lp}^{ij}(t,\varphi_1,\varphi_2)(a)(\varphi_1(a)_p-\varphi_2(a)_p)da|,\end{aligned}$$

where for $i,j,p=1,\dots,N$, $l=1,\dots,M$,

$$\begin{gathered}f_{lp}^{ij}(t,\varphi_1,\varphi_2)(.)\in W^{1,\infty}(0,+\infty),\\ \mu_p(.)f_{lp}^{ij}(t,\varphi_1,\varphi_2)(.)\in L^\infty(0,+\infty),\\ \sup_{\substack{\|\varphi_1\|\le C,\|\varphi_2\|\le C\\ 0\le t\le T}}\|\frac{d}{da}f_{lp}^{ij}(t,\varphi_1,\varphi_2)\|_{L^\infty(0,+\infty)}<+\infty,\\ \sup_{\substack{\|\varphi_1\|\le C,\|\varphi_2\|\le C\\ 0\le t\le T}}\|\mu_pf_{lp}^{ij}(t,\varphi_1,\varphi_2)\|_{L^\infty(0,+\infty)}<+\infty.\end{gathered}$$

Lemma 6.4. *Under Assumptions 6.1 and 6.3, Assumptions 5.3 a)-e) are satisfied with $E_0=X_{0+}$.*

Proof. Assumptions 5.3 a)-d) are clearly satisfied, and we only have to prove Assumption 5.3 e). We must prove that given a bounded set $B\subset X_{0+}$, and given $s\ge0$, and $T>s$, there exists a constant $k=k(B,s,T)\ge0$, such that

$$\|F_1(t,U(t,s)x_0)-F_1(l,U(l,s)x_0)\|\le k|t-l|,\forall t,l\in[s,T],\forall x_0\in B.$$

We assume that $s=0$, the case $s\ge0$ being similar, and we denote

$$u_{x_0}(t)=U(t,0)x_0,\quad\forall t\ge0,\forall x_0\in X_{0+}.$$

Let $x_0\in B$. It is sufficient to consider, for each $i,j=1,\dots,N$,

$$I=\Big|\int_0^{+\infty}\beta_{i,j}(t,u_{x_0}(t))(a)u_{x_0j}(t)(a)-\beta_{i,j}(l,u_{x_0}(l))(a)u_{x_0j}(l)(a)da\Big|,$$

where $u_{x_0 j}(t)$ denotes the j^{th} component of $u_{x_0 2}(t)$, with $u_{x_0}(t) = \begin{pmatrix} 0_{\mathbb{R}R^N} \\ u_{x_0 2}(t) \end{pmatrix}$. Then

$$\begin{aligned} I \leq & \Big| \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a)[u_{x_0 j}(t)(a) - u_{x_0 j}(l)(a)]da \Big| \\ & + \Big| \int_0^{+\infty} [\beta_{i,j}(t, u_{x_0}(t))(a) - \beta_{i,j}(l, u_{x_0}(l))(a)] u_{x_0 j}(l)(a) da \Big| . \end{aligned}$$

Note that

$$u_{x_0}(t) = x_0 + A \int_0^t u_{x_0}(s)ds + \int_0^t F(s, u_{x_0}(s))ds, \quad \forall t \geq 0,$$

and we know that there exist two constants $C_0 > 0$, and $C_1 > 0$, such that

$$\|u_{x_0}(t)\| \leq C_0 \|x_0\| e^{C_1 k_{G_2} t}, \quad \forall t \geq 0.$$

So

$$\begin{aligned} & u_{x_0}(t) - u_{x_0}(l) \\ & = A[\int_0^t u_{x_0}(s)ds - \int_0^l u_{x_0}(s)ds] + [\int_0^t F(s, u_{x_0}(s))ds - \int_0^l F(s, u_{x_0}(s))ds] . \end{aligned}$$

Therefore,

$$\begin{aligned} I \leq & | \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a) \frac{\partial}{\partial a} [(\int_0^t u_{x_0 j}(s)ds)(a) - (\int_0^l u_{x_0 j}(s)ds)(a)] da | \\ & + | \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a) \mu_j(a) [(\int_0^t u_{x_0 j}(s)ds)(a) - (\int_0^l u_{x_0 j}(s)ds)(a)] da | \\ & + | \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a) [(\int_0^t F(s, u_{x_0}(s))ds)_{2j}(a) - (\int_0^l F(s, u_{x_0}(s))ds)_{2j}(a)] da | \\ & + \|u_{x_0}(l)\|_{L^1(0,+\infty)^N} \|\beta_{i,j}(t, u_{x_0}(t)) - \beta_{i,j}(l, u_{x_0}(l))\|_{L^\infty(0,+\infty)}, \end{aligned}$$

and since $(\int_0^t u_{x_0 j}(s)ds)(a_\dagger^j) = 0$, $\forall t \geq 0$, and $\beta_{i,j}(t, u_{x_0}(t))(a_\dagger^i) = 0$ (because of Assumptions 6.1 a) 6.3 g) and 6.3 h)), by integrating by parts we get

$$\begin{aligned} I \leq & |\beta_{i,j}(t, u_{x_0}(t))(0) [(\int_0^t u_{x_0 j}(s)ds)(0) - (\int_0^l u_{x_0 j}(s)ds)(0)]| \\ & + | \int_0^{+\infty} \frac{\partial}{\partial a} \beta_{i,j}(t, u_{x_0}(t))(a) [(\int_0^t u_{x_0 j}(s)ds)(a) - (\int_0^l u_{x_0 j}(s)ds)(a)] da | \\ & + | \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a) \mu_j(a) [(\int_0^t u_{x_0 j}(s)ds)(a) - (\int_0^l u_{x_0 j}(s)ds)(a)] da | \\ & + | \int_0^{+\infty} \beta_{i,j}(t, u_{x_0}(t))(a) [(\int_0^t F(s, u_{x_0}(s))ds)_{2j}(a) - (\int_0^l F(s, u_{x_0}(s))ds)_{2j}(a)] da | \\ & + \|u_{x_0}(l)\|_{L^1(0,+\infty)^N} \|\beta_{i,j}(t, u_{x_0}(t)) - \beta_{i,j}(l, u_{x_0}(l))\|_{L^\infty(0,+\infty)}. \end{aligned}$$

So, one has

$$
\begin{aligned}
I \leq & \sup_{0\leq t\leq T} \|\beta_{i,j}(t, u_{x_0}(t))\|_{L^\infty(0,+\infty)} \Big| \int_l^t \|\widetilde{F}_1(s, u_{x_0 j}(s))\| ds\Big| \\
& + \sup_{0\leq t\leq T} \|\frac{\partial}{\partial a}\beta_{i,j}(t, u_{x_0}(t))\|_{L^\infty(0,+\infty)} \Big| \int_l^t \|u_{x_0 j}(s)\|_{L^1(0,+\infty)} ds\Big| \\
& + \sup_{0\leq t\leq T} \|\mu_j\beta_{i,j}(t, u_{x_0}(t))\|_{L^\infty(0,+\infty)} \Big| \int_l^t \|u_{x_0 j}(s)\|_{L^1(0,+\infty)} ds\Big| \\
& + \sup_{0\leq t\leq T} \|\beta_{i,j}(t, u_{x_0}(t))\|_{L^\infty(0,+\infty)} \Big| \int_l^t \|F(s, u_{x_0}(s))_{2j}\|_{L^1(0,+\infty)} ds\Big| \\
& + \|u_{x_0}(l)\|_{L^1(0,+\infty)^N} \|\beta_{i,j}(t, u_{x_0}(t)) - \beta_{i,j}(l, u_{x_0}(l))\|_{L^\infty(0,+\infty)}.
\end{aligned}
$$

It remains to consider

$$
\begin{aligned}
J = & \|\beta_{i,j}(t, u_{x_0}(t)) - \beta_{i,j}(l, u_{x_0}(l))\|_{L^\infty(0,+\infty)} \\
\leq & \|\beta_{i,j}(t, u_{x_0}(t)) - \beta_{i,j}(t, u_{x_0}(l))\|_{L^\infty(0,+\infty)} \\
& + \|\beta_{i,j}(t, u_{x_0}(l)) - \beta_{i,j}(l, u_{x_0}(l))\|_{L^\infty(0,+\infty)}.
\end{aligned}
$$

By using Assumption 6.3 f), it remains to consider

$$
K = \|\beta_{i,j}(t, u_{x_0}(l)) - \beta_{i,j}(t, u_{x_0}(l))\|_{L^\infty(0,+\infty)}.
$$

But by using Assumption 6.3 i), and arguments similar to the previous part of the proof, the result follows. □

Assumption 6.4.

(1) j) (Eventual Compactness) For $i, j = 1, \dots, N$, $m_{ij} : \mathbb{R}_+ \times Y \to L^\infty(0, +\infty)$ is completely continuous.

Theorem 6.5. *Under Assumptions 6.1, 6.3, and 6.4, Assumptions 5.3 a)-g) are satisfied with $E_0 = X_{0+}$, and $T' = \max_{i=1,\dots,n}(a^i_\dagger)$. Moreover for each $s \geq 0$, for each bounded set $B \subset X_{0+}$, and for each $T \geq \max_{i=1,\dots,n}(a^i_\dagger)$, the set*

$$
\Big\{U(t+s, s)x_0 : \max_{i=1,\dots,n}(a^i_\dagger) \leq t \leq T, x_0 \in B\Big\}
$$

has compact closure.

Proof. By taking into account Lemma 6.4, it only remains to show Assumptions 5.3 f) and g). We denote

$$
Z = L^\infty(0, +\infty)^{N^2}.
$$

We define $H : Z \times X_0 \to X_0$, for all $\alpha = (\alpha_{ij}) \in L^\infty(0, +\infty)^{N^2}$, and all $\varphi \in Y$ by

$$
H(\alpha, \begin{pmatrix} 0_{\mathbb{R}^N} \\ \varphi \end{pmatrix}) = \begin{pmatrix} 0_{\mathbb{R}^N} \\ H_2(\alpha, \varphi) \end{pmatrix}
$$

with

$$
H_2(\alpha, \varphi)_i = \sum_{j=1}^N \alpha_{ij}(a)\varphi_j(a).
$$

Under Assumption 6.4, $G : \mathbb{R}_+ \times Y \to Z$ defined by

$$
G(t, \varphi)_{ij} = m_{ij}(t, \varphi),
$$

is completely continuous, and we have

$$F_2(t, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}) = H(G(t,\varphi), \begin{pmatrix} 0 \\ \varphi \end{pmatrix}), \forall \varphi \in Y, \forall t \geq 0.$$

So Assumption 5.3 f) is satisfied by F_2. It remains to prove Assumption 5.3 g). We assume that $s = 0$, the case $s \geq 0$ being similar, and we denote

$$u_{x_0}(t) = U(t,0)x_0, \forall t \geq 0, \forall x_0 \in X_{0+}.$$

Let $x_0 \in X_{0+}$. We must show that if $w_{x_0}(t)$ is solution of

$$w_{x_0}(t) = T_0(t)x_0 + \int_0^t T_0(t-s)H(G(s,u_{x_0}(s)), w_{x_0}(s))ds, \forall t \geq 0,$$

then $w_{x_0}(t) = 0$ for all $t \geq \max_{i=1,\dots,N}(a_\dagger^i)$. We have

$$w_{x_0}(t) = T_0(t)x_0 + L_{x_0}(w_{x_0}(.))(t), \forall t \geq 0,$$

where

$$L_{x_0}(\psi(.))(t) = \int_0^t T_0(t-s)H(G(s,u_{x_0}(s)), \psi(s))ds, \forall t \geq 0,$$

thus

$$w_{x_0}(t) = \sum_{k=0}^{\infty} L_{x_0}^k(T_0(.)x_0)(t), \forall t \geq 0,$$

where

$$L_{x_0}^0 = Id, \text{ and } L_{x_0}^{k+1} = L_{x_0} \circ L_{x_0}^k, \text{ for } k \geq 0.$$

So, it remains to prove that,

$$L_{x_0}^k(w_{x_0}(.))(t) = 0, \forall t \geq \max_{i=1,\dots,N}(a_\dagger^i), \forall k \geq 0.$$

For $k = 0$, the result follows from the explicit formulation of $T_0(t)$ given in Theorem 6.1. For $k = 1$, we have

$$\begin{aligned}
&\|L_{x_0}(T_0(.)x_0)(t)\|_{L^1(0,+\infty)^N} \\
&\leq \int_0^t \|T_0(t-s)H(G(s,u_{x_0}(s)), T_0(s)x_0)\|_{L^1(0,+\infty)^N} ds \\
&\leq \int_0^t \||T_0(t-s)H(G(s,u_{x_0}(s)), T_0(s)x_0)|\|_{L^1(0,+\infty)^N} ds \\
&\leq \int_0^t \|T_0(t-s)|H(G(s,u_{x_0}(s)), T_0(s)x_0)|\|_{L^1(0,+\infty)^N} ds \\
&\leq k_6(C,T) \int_0^t \|T_0(t-s)JT_0(s)|x_0|\|_{L^1(0,+\infty)^N} ds
\end{aligned}$$

where

$$J = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ J_{n1} & \cdots & \cdots & J_{nn} \end{bmatrix},$$

with $i,j = 1,\dots,N, \forall \varphi_j \in Y_j$

$$J_{ij}(\varphi_j)(a) = \begin{cases} \varphi_j(a), & \text{a.e. } a \in (0, a_\dagger^i), \\ 0, & \text{a.e. } a \in (a_\dagger^i, +\infty). \end{cases}$$

By using the explicit formulation of $T_0(t)$ given in Theorem 6.1 , we get for $i,j = 1,\dots,N$,

$$\text{if } t_1 \geq 0,\ t_2 \geq 0, \text{ and } t_1 + t_2 \geq \max_{i=1,\dots,N}(a_\dagger^i), \text{ then } T_{0i}(t_1)J_{ij}T_{0j}(t_2) = 0,$$

and we deduce that

$$L_{x_0}(T_0(.)x_0)(t) = 0, \text{ for } t \geq \max_{i=1,\dots,N}(a_\dagger^i).$$

For $k \geq 2$, the result follows from the fact that $\forall (i_1, i_2, \dots, i_k) \in \{1,\dots,N\}$, $\forall (t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k$,

$$\begin{gathered}\text{if } t_i \geq 0, \forall i = 1,\dots,k, \text{ and } t_1 + t_2 + \dots + t_k \geq \max_{i=1,\dots,N}(a_\dagger^i),\\ \text{then } T_{0i_1}(t_1)J_{i_1 i_2}T_{0i_2}(t_2)\dots .J_{i_{k-1}i_k}T_{0i_k}(t_k) = 0,\end{gathered}$$

and by using similar arguments we deduce that

$$L_{x_0}^k(T_0(.)x_0)(t) = 0, \text{ for } t \geq \max_{i=1,\dots,N}(a_\dagger^i).$$

□

Assumption 6.5.

k) (Existence of absorbing set) For $i = 1,\dots,N$, there exists $\delta_i > 0$, such that

$$-m_{ii}(t,\varphi)(a) \geq \delta_i \int_0^{+\infty} \varphi_i(a)da, \forall \varphi \in Y_+.$$

Theorem 6.6. *Under Assumptions 6.1, 6.2, 6.3, 6.4, and 6.5, Let us denote* $\delta = N^2 \frac{\sum_{i=1}^N \max_{j=1,\dots,N}(k_3^{\beta_{ij}} + k_3^{m_{ij}})}{\min_{i=1,\dots,N}(\delta_i)} > 0$. *Then for each* $\varepsilon > 0$, *for any bounded set* $B \subset X_{0+}$, *and for each* $s \geq 0$, *there exists* $t_0 = t_0(\varepsilon, B) \geq 0$, *such that*

$$\begin{gathered}U(t+s,s)B \subset \overline{B}(0,\delta+\varepsilon) \cap X_{0+}, \forall t \geq t_0,\\ U(t+s,s)\overline{B}(0,\delta+\varepsilon) \cap X_{0+} \subset \overline{B}(0,\delta+\varepsilon) \cap X_{0+}, \forall t \geq 0.\end{gathered}$$

Proof. To prove the theorem we consider the case $s = 0$, the case $s > 0$ being similar. Let $\varphi \in D(A_{N,0}) \cap X_{0+} = \left\{\psi \in D(A) : A\psi + F(0,\psi) \in \overline{D(A)}\right\} \cap X_{0+}$. Then from Theorem 6.3, $u(t) = U(t,0)\varphi$ satisfies

$$\frac{du_{2i}}{dt} = -\frac{\partial u_{2i}}{\partial a} - \mu_i u_{2i} + \sum_{i=1}^N m_{ij}(t,u(t))u_{2j}(t)$$

so

$$\begin{aligned}&\frac{d}{dt}\int_0^{+\infty} u_{2i}(t)(a)da\\ &= u_{2i}(t)(0) - \int_0^{+\infty} \mu_i(a)u_{2i}(t)(a)da + \sum_{i=1}^N \int_0^{+\infty} m_{ij}(t,u(t))(a)u_{2j}(t)(a)da\\ &\leq (\max_{j=1,\dots,N}(k_3^{\beta_{ij}} + k_3^{m_{ij}}))\sum_{j=1}^N \int_0^{+\infty} u_{2j}(t)(a)da - \delta_i(\int_0^{+\infty} u_{2j}(t)(a)da)^2 .\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N}\int_0^{+\infty} u_{2i}(t)(a)da &\leq (\sum_{i=1}^{N}\max_{j=1,\dots,N}(k_3^{\beta_{ij}}+k_3^{m_{ij}}))\sum_{j=1}^{N}\int_0^{+\infty} u_{2j}(t)(a)da \\
&- \min_{i=1,\dots,N}(\delta_i)\sum_{j=1}^{N}(\int_0^{+\infty} u_{2j}(t)(a)da)^2 \\
&\leq (\sum_{i=1}^{N}\max_{j=1,\dots,N}(k_3^{\beta_{ij}}+k_3^{m_{ij}}))\sum_{j=1}^{N}\int_0^{+\infty} u_{2j}(t)(a)da \\
&- \frac{1}{N^2}\min_{i=1,\dots,N}(\delta_i)(\sum_{j=1}^{N}\int_0^{+\infty} u_{2j}(t)(a)da)^2.
\end{aligned}$$

From this inequality, we deduce that

$$\|u(t)\| \leq \|\varphi\| + \int_0^t C_1(\delta - \|u(s)\|)\|u(s)\|ds, \forall t \geq 0, \tag{6.4}$$

with $C_1 = \sum_{i=1}^{N}\max_{j=1,\dots,N}(k_3^{\beta_{ij}}+k_3^{m_{ij}})$. By density of $D(A_{N,0}) \cap X_{0+}$ in X_{0+}, we deduce that inequality (6.4) holds for all $\varphi \in X_{0+}$, and the result follows. □

Assumption 6.6.

1) (Periodicity) There exists $\omega > 0$, $\forall i, j = 1, \dots, N, \forall t \geq 0$, $m_{ij}(t+\omega, .) = m_{ij}(t, .)$, and $\beta_{ij}(t+\omega, .) = \beta_{ij}(t, .)$.

The next result gives the existence of a family of compact attracting subsets.

Theorem 6.7. *Under Assumptions 6.1-6.5, the non-autonomous semiflow $U(t,s)$ restricted to X_{0+} is ω-periodic, that is to say that*

$$U(t+\omega, s+\omega)x_0 = U(t,s)x_0, \text{ for all } x_0 \in X_{0+}, \text{ for all } t \geq s \geq 0.$$

Moreover, there exists a family $\{A_t\}_{t\geq 0}$ of subsets of X_{0+}, satisfying:

i) $A_t = A_{t+\omega}, \forall t \geq 0$.
ii) *For all $t \geq 0$, A_t is compact and connected.*
iii) *For all $t \geq s \geq 0, U(t,s)A_s = A_t$. iv) $A = \cup_{0\leq t\leq \omega} A_t$ is compact.*
v) *The map $t \to A_t$ is continuous with respect to the Hausdorff metric, that is to say that*

$$h(A_t, A_{t_0}) \to 0, \text{ as } t \to t_0,$$

where $h(A,B) = \max(\mathrm{dist}(A,B), \mathrm{dist}(B,A))$.

vi) *For each bounded set $B \subset X_{0+}$, and for each $s \geq 0$,*

$$\lim_{t\to+\infty} \mathrm{dist}(U(t,s)B, A_t) = 0.$$

Proof. To prove Theorem 6.7 it is sufficient to apply Theorem 5.3 with $E_0 = X_{0+}$, $T' = \max_{i=1,\dots,n}(a_\dagger^i)$, and $E_1 = B(0, \delta+\varepsilon)$, for a certain $\varepsilon > 0$, where $\delta \geq 0$ is the constant introduced in Theorem 6.6. □

Acknowledgment. I would like to thank the referee for his helpful suggestions, and for his/her with the the proof of assertions $iv) - vi)$ in Theorem 5.3.

References

[1] Arendt, W: Resolvent Positive Operators and Integrated Semigroups, Proc. London Math. Soc. (3) 54, 321-349, (1987).

[2] Arendt, W: Vector Valued Laplace Transforms and Cauchy Problems. Israel J. Math. 59, 327-352, (1987).

[3] Cazenave, T and A. Haraux: Introduction aux problème d'évolution semi-linéaires. S.M.A.I. (1990).

[4] Da Prato, G and E. Sinestrari: Differential Operators with Non-Dense Domain. Ann. Sc. Norm. Pisa, 14, 285-344, (1987).

[5] Hale, J. K: Asymptotic Behavior of Dissipative Systems. Amer. Math. Soc. Providence (1988).

[6] Iannelli, M: Mathematical Theory of Age-Structured Population Dynamics. Giadini Editori e Stampatori in Pisa. (1994).

[7] Kellermann, H, and M. Hieber: Integrated Semigroups. J. Funct. Anal. 84, 160-180, (1989).

[8] Matsumoto, T, S. Oharu, H. R. Thieme: Nonlinear Perturbations of a Class of Integrated Semigroups. Hiroshima Mathematical Journal vol. 26, no. 3, 433-473 (1996).

[9] Metz, J. A. J, and Diekmann: The Dynamics of Physiologically Structured Populations. Springer-Verlag. Lecture Notes in Biomathematics, 68 (1986).

[10] Neubrander, F: Integrated Semigroups and their Application to the Abstract Cauchy Problem, Pac. J. Math. 135, 111-155, (1988).

[11] Pazy, A: Semigroups of Linear Operator and Applications to Partial Differential Equations. Applied Mathematical Sciences vol. 44. Springer, (1983).

[12] Pelletier, D, and P. Magal: Dynamics of a migratory population under different fishing effort allocation schemes in time and space. Can. J. Fish. Aquat. Sci. vol. 53, p.1186-1199, (1996).

[13] Thieme, H. R: Semiflows Generated by Lipschitz Perturbations of Non-Densely Defined Operators. Differential and Integral Equations. vol. 3, 6, 1035-1066, (1990).

[14] Thieme, H. R: "Integrated Semigroups" and Integrated Solutions to Abstract Cauchy Problems. Journ. Math. Anal. Appl. vol. 152, no 2, 416-447, (1990).

[15] Thieme, H. R: Analysis of Age-Structured Population Models with an Additional Structure. Proceedings of the second international conference (O.Arino, D.E.Axelrod, M.Kimmel edts), Marcel Dekker, 115-125, (1991).

[16] Thieme, H. R: Quasi-Compact Semigroups via Bounded Perturbation. Advances in Mathematical Population Dynamics-Molecules, Cells and Man. (O. Arino, D. Axelrod, and M. Kimmel: edts.). Worlds Scientific. 691-713, (1997).

[17] Thieme, H. R: Positive Perturbation of Operator Semigroups, Growth Bounds, Essential Compactness, and Asymchronous Exponential Growth. Discrete and Continuous Dynamical Systems. Vol. 4, no 4, 735-764, (1998).

[18] Webb, G. F: Compactness of Bounded Trajectories of Dynamical Systems in Infinite Dimensional Spaces. Proc. Roy. Soc. Edinburgh. 84A, 19-33, (1979).

[19] Webb, G. F: Theory of Nonlinear Age-Dependent Population Dynamics. Marcel Dekker, (1985).

[20] Zhao, X. Q: Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications. Canadian Appl. Math. Quart. 3, 473-495, (1995).

Pierre Magal

Unité de Biométrie, INRA, 78352 Jouy-en-Josas Cedex France

E-mail address: magal.pierre@wanadoo.fr