A semi-ejective fixed point theorem.

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1 Introduction

In this paper we are interested in the existence of non trivial fixed point for completely continuous maps. In a large number of applications, one has a trivial fixed point, and the main difficulty is to assure the existence of another fixed point. To solve this problem Krasnoselskii [15] introduced first conditions on the nonlinear eigenvalues. Another approach to solve this type of problem is the one due to Browder [3] which use the notion of ejective fixed points. Following those results, a large number of results appear on the subject, and we refer to Gustafson and Schmitt [13], Turner [19], Amann [1], Gatica and Smith [12], Fournier [8][9], Fournier and Peitgen [10][11], Brown[4], Dajun [6], Canada and Zertiti [5], Magal and Pelletier [16].

In Arino, Hadeler, and Hbid [2] the authors introduce a Poincaré map of the following type. Let K be a cone of a Banach space $(X, \|.\|)$, i.e. K is closed convex subset of X, satisfying

$$tK \subset K, \text{ for } t > 0, \text{ and } K \cap -K = \{0\}.$$

Let $a < 0 < b$ be two real numbers. The map considered in Arino et al. [2], denoted here by $f : K \times [a,b] \to K \times [a,b]$, has the following properties:
i) $f(0) = 0$
ii) $f(\{0\} \times [a,b]) \subset \{0\} \times [a,b]$
iii) For each $M > 0$, there exists $C > 0$, and $0 \leq \gamma < 1$, such that for all $x = (x_1, x_2) \in K \times [a,b]$, with $\|x\| \leq M$, and $\|x_1\| \leq C|x_2|$,

$$|f_2(x_1, x_2)| \leq \gamma |x_2|,$$

where $f_2(x_1, x_2)$ denote the second component of f.

One can see that under *ii)* and *iii)*, the trivial fixed 0 can not be ejective, because for all $x \in \{0\} \times [a,b]$

$$\lim_{m \to +\infty} f^m(x) = 0.$$

So the Browder's ejective fixed point theorem does not apply. In Arino et al. [2] to encompass this difficulty, the authors first prove the existence of a subcone $K_\alpha = \{(x_1, x_2) \in K \times [a,b] : \|x_1\| \geq \alpha |x_2|\}$ for a certain $\alpha > 0$, such that $f(K_\alpha) \subset K_\alpha$. Then, by showing that 0 is an ejective fixed point $f|_{K_\alpha}$, the authors were able to prove the existence of a non trivial fixed point of F.

The purpose of this paper is to use the notion of semi-ejectivity instead of the previous construction. More precisely, we want only local conditions around 0, to assure the existence of non trivial fixed point. Let C be a of a Banach space $(X, \|.\|)$, $f : C \to C$ be a map, $A \subset C$, and let $x_0 \in \partial_C A$ be a fixed point of f. We will say that x_0 is a semi-ejective fixed point of f on $C \setminus A$, if there exists a neighborhood V of x_0 in C, satisfying for all $y \in V \setminus A$, there exists an integer $m \in I\!N$, such that $f^m(y) \in C \setminus V$. One may see that the notion of semi-ejectivity coincide with the notion of ejectivity, when $A = \{x_0\}$. Such a neighborhood V of x_0 will be call a semi-ejective neighborhood of x_0 for f on $C \setminus A$.

The following theorem is used in Magal and Arino [17] to prove the existence of a non trivial periodic solution for a state dependent delay equation.

Theorem 1.1 *: Let $(X_1, \|.\|_1)$ and $(X_2, \|.\|_2)$ be two Banach spaces, and let K_1 be a cone of $(X_1, \|.\|_1)$, and let C_2 be a bounded closed convex subset of X_2 containing 0_{X_2}. Let $C = \overline{B}_{K_1}(0, r_1) \times C_2$, and let $f : C \to C$ be a compact map satisfying $f(0) = 0$.*

Assume in addition that:
i) 0 *is a semi-ejective fixed point of* f *on* $C \setminus A$, *with* $A = \{0_{X_1}\} \times C_2$,
ii) There exists $r_0 > 0$, $\eta > 0$, *and* $0 < \gamma < 1$, *such that*
for all $(x_1, x_2) \in C$, *with* $\|\tilde{x}_1\|_1 + \|\tilde{x}_2\|_2 \leq r_0$, *and* $\|\tilde{x}_1\|_1 \leq \eta \|\tilde{x}_2\|_2$ *imply*

$$\|f_2((x_1, x_2))\|_2 \leq \gamma \|x_2\|_2,$$

where $f_2((x_1, x_2))$ *is the second component of* $f((x_1, x_2))$.
iii) $f(C \setminus A) \subset C \setminus A$.
Then f *has a fixed point* $\overline{x} \in C \setminus \{0\}$.

One may rewrite the previous result into a more compact form. To do this we need to introduce some additional definition. We now, consider a compact map $f : C \to C$, where C is a closed convex subset of a Banach space $(X, \|.\|)$. Let $x_0 \in C$ be a fixed point of f. In theorem 1.1 one can first note that, since K_1 is a cone, the subset $A = \{0_{X_1}\} \times C_2$ is extremal in C, that is if $x, y \in C$, $0 < t < 1$, and $tx + (1-t)y \in A$, then $x \in A$, and $y \in A$.

On the other hand, the property *ii)* of theorem 1.1 can be formulated by introducing the notion of conditional stability. In the sequel, given a subset M of X, and $z \in X$, we will denote by $B_M(z, \rho) = \{y \in M : \|y - z\| < \rho\}$, and $\overline{B}_M(z, \rho) = \{y \in M : \|y - z\| \leq \rho\}$. Let U be a subset of $C \subset X$ containing x_0, let $f : U \to C$ be a map such that $f(x_0) = x_0$. Then we will say that x_0 is conditional stable for f, if for all $\rho_1 > 0$, there exists $0 < \rho_2 < \rho_1$, such that for all $x \in \overline{B}_U(x_0, \rho_2)$, if $f^j(x) \in U$, for all $j = 1, ..., m$, then $f^j(x) \in \overline{B}_U(x_0, \rho_1)$, for all $j = 1, ..., m$. We will say that x_0 is conditional globally asymptotically stable for f, if x_0 is conditional stable for f, and for all $\rho > 0$, and all $y \in U$, there exists an integer $m_0 \in \mathbb{N}$, such that

$$f^{m_0}(y) \in \overline{B}_U(x_0, \rho) \cup (X \setminus U).$$

One may note that if we assume that $f(U) \subset U$, then the conditional stability gives the classical stability for discrete time systems (see Hale [14]). Moreover, if x_0 is conditional globally asymptotically stable for f, and $x \in U$ is such that $f^m(x) \in U$ for all $m \in \mathbb{N}$, then

$$\lim_{m \to +\infty} f^m(x) = x_0.$$

Indeed, let $\rho_1 > 0$, then since x_0 is conditional stable for f, there exists $0 < \rho_2 < \rho_1$, such that for all $x \in \overline{B}_U(x_0, \rho_2)$, if $f^j(x) \in U$, for all $j = 1, ..., m$, then $f^j(x) \in \overline{B}_U(x_0, \rho_1)$, for all $j = 1, ..., m$.

But, since x_0 is conditional globally asymptotically stable for f, and $f^m(x) \in U$ for all $m \in I\!N$, there exists $m_0 \in I\!N$, such that

$$f^{m_0}(x) \in \overline{B}_U(x_0, \rho_2), \text{ and } f^{m_0+l}(x) \in U, \text{ for all } l \in I\!N,$$

so

$$f^{m_0+l}(x) \in \overline{B}_U(x_0, \rho_1), \text{ for all } l \in I\!N.$$

Finally, as $\rho_1 > 0$, is shoosen arbitrary, we deduce that $f^m(x)$ converge to x_0, as m goes to infinity.

In theorem 1.1, if we denote by $U = \{(x_1, x_2) \in C : \|\tilde{x}_1\|_1 < \eta \|\tilde{x}_2\|_2\}$, then the closer of U in C, is $\overline{U} = \{(x_1, x_2) \in C : \|\tilde{x}_1\|_1 \leq \eta \|\tilde{x}_2\|_2\}$, and under assumption $ii)$ of theorem 1.1, the trivial fixed point 0 is conditional stable for $f \mid_{\overline{U} \cap \overline{B}_U(x_0, r_0)}$. Indeed, let $\rho_1 > 0$, and $0 < \rho_2 < \min\left(\rho_1, \frac{r_0}{1+\eta}\right)$. Then by choosing the norm $\|(x_1, x_2)\| = \max(\|\tilde{x}_1\|_1, \eta \|\tilde{x}_2\|_2)$, if we assume that there exists $(x_1, x_2) \in \overline{B}_{\overline{U}}(0, \rho_2)$, and $f^j(x) \in \overline{U}$, for all $j = 1, ..., m$, one has

$$\|f(x)\| = \max(\|f_1(x)\|, \eta \|f_2(x)\|_2),$$

where $f_i(x)$ denote the i^{th} component of $f(x)$, and since $f(x) \in \overline{U}$, by using assumption $ii)$ of theorem 1.1 one has

$$\|f(x)\| = \eta \|f_2(x)\|_2 \leq \gamma\eta \|x_2\|_2 \leq \gamma \|x\|.$$

By induction one has $f^j(x) \in \overline{U}$, for all $j = 1, ..., m$,

$$\left\|f^j(x)\right\| \leq \gamma^j \|x\|.$$

So, assumption $ii)$ of theorem 1.1 implies the conditional stability of 0, for $f \mid_{\overline{U} \cap \overline{B}_U(x_0, r_0)}$. Moreover, it is clear that given a certain $\rho > 0$, if $y \in \overline{U}$, then there exists $m_0 \in I\!N$, such that $f^{m_0}(y) \in \overline{B}_{\overline{U}}(0, \rho) \cup \left(C \setminus \overline{U}\right)$. Indeed, assume that this not the case, then $f^m(y) \in \overline{U} \setminus \overline{B}_{\overline{U}}(0, \rho)$, for all $m \in I\!N$, then as previously one has

$$\left\|f^j(x)\right\| \leq \gamma^m \|x\|, \text{ for all } m \in I\!N,$$

and we obtain a contradiction. This prove that 0 is conditional globally asymptotically stable for $f \mid_{\overline{U}}$.

To prove theorem 1.1 it remains so to prove the following theorem.

Theorem 1.2 : *Let C be a closed convex subset of a Banach space $(X, \|.\|)$, and $r_0 > 0$. Let $f : C \to C$ be a compact map, $x_0 \in C$ be a fixed point of f, and let $A \subset C$ be closed extremal subset of C, such that $x_0 \in \partial_C A$.*
Assume in addition that:
i) x_0 is a semi-ejective fixed point of f on $C \setminus A$.
ii) There exists $U \subset C$ a neighborhood of $A \backslash \{0\}$, open in C, such that $x_0 \in \overline{U}$, and x_0 is conditionally globally asymptotically stable for $f\,|_{\overline{B}_C(x_0,r_0)\cap\overline{U}}$.
iii) $f\left(\overline{B}_C(x_0, r_0) \cap \overline{U}\right) \subset \overline{B}_C(x_0, r_0)$, and $f\left(\overline{B}_C(x_0, r_0) \setminus A\right) \subset C \setminus A$.
Then f has a fixed point $x_1 \in C \setminus \{x_0\}$.

The rest of the paper is devoted to the proof of theorem 1.2. To prove this theorem we will first verify in section 2, that the Browder's ejective fixed point theorem can be extended to the case where the fixed point is replaced by an extremal set, which is closed and ejective for the map. In section 2, we will follow the method used by Nussbaum [18] chapter 3. We will then deduce from this preliminary result a consequence on the value of the fixed point index locally around the semi-ejective fixed point, with the additional assumptions of theorem 1.2.

2 Preliminary result

In the sequel, we use fixed point index, and we refer to Amann [1] for more precisions on this subject.

Let (M, d) be a metric space, $f : M \to M$ be a map, $A \subset M$, then we will say that A is ejective for f, if there exists a neighborhood U of A such that for all $x \in U \setminus A$ there exists an integer $m \in \mathbb{N}$, such that $f^m(x) \notin U$. Such a neighborhood U of A will call an ejective neighborhood of A for f. Moreover, if we replace U by $\overline{U}$ in the previous definition, then $\overline{U}$ will be call a closed ejective neighborhood of A *for f.*
The following lemma is a direct extension of a lemma due to Browder [3].

Lemma 2.1 : *Let (M, d) be a compact metric space, $f : M \to M$, $A \subset M$ be a closed subset of M, such that $f(M \setminus A) \subset M \setminus A$, with $M \setminus A \neq \emptyset$.*
Then if A is ejective for f, there exists a neighborhood W of A, such that for all open neighborhood V of A, there exists an integer $m(V) \in \mathbb{N}$ such that

$$f^m(M \setminus V) \subset M \setminus W, \text{ for all } m \geq m(V).$$

Proof: The proof can obtained using the same kind of arguments as in the proof of lemma 3.6 p:81 in Nussbaum [18], and we will not detail further this proof.□

Let us now recall, the corollary 3.1 p:79 in Nussbaum [18]. Here we only state a particular version of this result, because that is all need in the sequel.

Lemma 2.2 *: Let C be a closed convex subset of a Banach space $(X, \|.\|)$, $f : C \to C$ be a continuous map, and let G be relatively open subset of C, such that $f\mid_G$ is compact.*
If there exists a compact subset $B \subset C$, which is homotope to a point in G, and $f^m(G) \subset B \subset G$ for all $m \geq m_0$, then

$$i_C(f, G) = 1.$$

Theorem 2.3 *: Let C be a closed convex subset of a Banach space $(X, \|.\|)$, $f : C \to C$ be a compact map, and let A be an extremal closed subset of C, which is an ejective subset for f, and $f(C \setminus A) \subset C \setminus A$.*
If U is a relatively open neighborhood of A in C, such that all the fixed points of f in U are include in A, and $C \neq A$, then

$$i_C(f, U) = 0.$$

Proof: Since the map f is compact, the subset $C_1 = \overline{co}(f(C))$ is a compact convex subset, and from permanence property of the fixed point index, we deduce that

$$i_C(f, U) = i_{C_1}(f, U \cap C_1).$$

So by stating $U_1 = U \cap C_1$, $A_1 = A \cap C$, and $f_1 = f\mid_{C_1}$, it remain to prove that

$$i_{C_1}(f_1, U_1) = 0.$$

where C_1 is a compact convex subset of $(X, \|.\|)$, $f_1 : C_1 \to C_1$ is a continuous map, $A_1 \subset C_1$ is a closed extremal subset of C_1, which is ejectif for f_1. Moreover, $f(C_1 \setminus A_1) \subset C_1 \setminus A_1$, and U_1 is a relatively open neighborhood of A_1 in C_1. To conclude the proof it remains to apply the following lemma.□

Lemma 2.4 *: Let C be a compact convex subset of a Banach space $(X, \|.\|)$, $f : C \to C$ be a continuous map, $f : C \to C$ be a compact map, and let A*

be an extremal closed subset of C, which is an ejective subset for f, and $f(C \setminus A) \subset C \setminus A$.
If U is a relatively open neighborhood of A in C, such that all the fixed points of f in U are include in A, and $C \neq A$, then

$$i_C(f, U) = 0.$$

Proof: Let U be such a neighborhood, then if we denote by $V_\rho = \{x \in C : d(x, A) < \rho\}$ (where $d(x, A) = \inf_{y \in A} \|x - y\|$), one has by compactness of A, the existence of a certain $\rho_0 > 0$ such that

$$V_\rho(A) \subset U, \text{ for all } \rho \in]0, \rho_0].$$

Moreover, the additivity property of the fixed point index implies that

$$i_C(f, U) = i_C(f, V_\rho(A)), \text{ for all } \rho \in]0, \rho_0].$$

Since A is compact, we may also choose $\rho_1 \in]0, \rho_0]$, such that $\overline{V}_{\rho_1}(A)$ is a closed ejective neighborhood of A for f. We are now in the conditions of lemma 2.1, and there exists an open neighborhood W of A in C such that

$$W \subset \overline{V}_{\rho_1}(A),$$

and for all open neighborhood V of A in C, there exists an integer $m(V) \in I\!N$ such that

$$f^m(C \setminus V) \subset C \setminus W, \text{ for all } m \geq m(V).$$

Let $x_1 \in C \setminus A$, we denote by

$$B = \{(1-t)y + tx_1 : 0 \leq t \leq 1, \ y \in C \setminus W\}.$$

The subset $B \subset C$ is compact, $A \cap B = \emptyset$ (because A is extremal in C), and $x_1 \notin A$.
Choosing a open neighborhood V of A in C, such that $\overline{V} \cap B = \emptyset$. We now may apply lemma 2.2 with $G = C \setminus \overline{V}$, since by construction

$$f^m(G) \subset B \subset C \setminus W \subset G, \text{ for all } m \geq m(V),$$

and B is homotope to x_1, we deduce from lemma 2.2 that

$$i_C\left(f, C \setminus \overline{V}\right) = 1,$$

and since C is convex, we deduce that

$$i_C(f, C) = 1.$$

Finally, the additivity property of the fixed point index implies that

$$1 = i_C(f, C) = i_C(f, V) + i_C\left(f, C \setminus \overline{V}\right),$$

thus

$$i_C(f, V) = 0.$$

To conclude it remains to see that $\overline{V}_{\rho_1}(A)$ is an ejective neighborhood of A for f, and $\overline{V} \subset W \subset \overline{V}_{\rho_1}(A)$, we obtain $i_C(f, V_{\rho_1}(A)) = i_C(f, V) = 0.\square$

3 Consequences of semi-ejectivity.

Theorem 3.1 *: With the notations, and under the assumptions of theorem 1.2. If $V \subset C$ is a neighborhood of x_0, such that x_0 is the only fixed point f in V, then*

$$i_C(f, V) = 0.$$

Proof: From the additivity property of the fixed point index, $i_C(f, V)$ is independent of V, for V as in theorem 3.1. So it is sufficiently to prove the theorem for a neighborhood V of x_0 sufficient small. Let us choose $\rho_1 \in]0, r_0]$, such that for all $\rho \in]0, \rho_1]$,

$$f\left(\overline{B}_C(x_0, \rho)\right) \subset \overline{B}_C(x_0, r_0),$$

and $\overline{B}_C(x_0, \rho_1)$ is a semi-ejective neighborhood of x_0 for f on $C \setminus A$. Since x_0 is conditionally globally asymptotically stable for $f\ |_{\overline{U} \cap B_C(x_0, r_0)}$, we deduce that x_0 is the only fixed point of f in $\overline{U} \cap B_C(x_0, r_0)$. From the additivity property of the fixed point index, one has

$$i_C(f, V) = i_C(f, B_C(x_0, \rho)),\ \text{for all}\ \rho \in]0, \rho_1].$$

Using again the additivity property of the fixed point index, one also has

$$i_C(f, V) = i_C(f, B_C(x_0, \rho)) = i_C(f, B_C(x_0, \rho) \cup (U \cap B_C(x_0, r_0))),$$

and as $f\left(\overline{B_C(x_0,\rho)\cup(U\cap B_C(x_0,r_0))}\right)\subset\overline{B}_C(x_0,r_0)$, one deduce from the permanence property of the fixed point index that

$$i_C(f,B_C(x_0,\rho))=i_{\overline{B}_C(x_0,r_0)}(f,B_C(x_0,\rho)\cup(U\cap B_C(x_0,r_0))).$$

Consider now the map $R:C\to\overline{B}_C(x_0,r_0)$, the map defined by

$$R(x)=\begin{cases} x, \text{ If } \|x-x_0\|\leq r_0, \\ tx+(1-t)x_0, \text{ with } t=\frac{r_0}{\|x-x_0\|}, \text{ If } \|x-x_0\|>r_0. \end{cases}$$

One may note that $R(C\setminus A)\subset C\setminus A$, since A is extremal in C, and $x_0\in A$. We denote by $g:\overline{B}_C(x_0,r_0)\to\overline{B}_C(x_0,r_0)$, the map defined by

$$g(x)=R(f(x)), \text{ for all } x\in\overline{B}_C(x_0,r_0).$$

By stating $C_1=\overline{B}_C(x_0,r_0)$, $A_1=A\cap C_1$, and $U_1=U\cap C_1$, and from the permanence property of the fixed point index, one has

$$i_C(f,B_C(x_0,\rho))=i_{C_1}(g,B_{C_1}(x_0,\rho)\cup U_1),$$

since by construction f and g coincide on $B_C(x_0,\rho)\cup U_1$, and from the property of conditional global asymptotic stability, there is no fixed point of f in $U_1\setminus\{x_0\}$.
It remains to show that for $\rho>0$ small enough one has

$$i_{C_1}(g,B_{C_1}(x_0,\rho)\cup U_1)=0.$$

To prove this fact, we now have to show that we are in the conditions of theorem 2.3 (with $A=A_1$, $C=C_1$, $U=B_{C_1}(x_0,\rho)\cup U_1$, $f=g$). It is clear that C_1 is a closed convex subset of $(X,\|.\|)$, A_1 is close and extremal of C_1, and by construction of the map $R:C\to\overline{B}_C(x_0,r_0)$, one also have $g(C_1\setminus A_1)\subset C_1\setminus A_1$. So, it only remains to prove that for $\rho>0$ small enough,

$$U_\rho=B_{C_1}(x_0,\rho)\cup U_1$$

is an ejective neighborhood of A_1 for g. Since f and g coincide on U_{ρ_1}, we deduce that x_0 is an semi-ejective fixed point of g on $C_1\setminus A_1$, and x_0 is conditionally globally asymptotically stable for $g\mid_{\overline{U}_1}$. We now prove that U_ρ is an ejective neighborhood of A_1 for g, for all $\rho>0$ small enough.

Let $\mu_0 \in]0, \rho_1]$ such that $\overline{B}_{C_1}(x_0, \mu_0)$ is a semi-ejective neighborhood of x_0 for g on $K_1 \setminus A_1$. Let $\mu_1 \in]0, \mu_0]$ such that

$$g\left(\overline{B}_{C_1}(x_0, \mu_1)\right) \subset \overline{B}_{C_1}(x_0, \mu_0).$$

Now, as x_0 is conditionally stable for $g|_{\overline{U}_1}$, one may choose $\mu_2 \in]0, \mu_1]$ such that for all $y \in \overline{B}_{C_1}(x_0, \mu_2)$, if $f^j(y) \in \overline{U}_1$, for all $j = 1, ..., m_0$, then

$$f^j(y) \in \overline{B}_{C_1}(x_0, \mu_1), \text{ for all } j = 1, ..., m_0.$$

Let us prove by contradiction that $U_{\mu_2} = \overline{B}_{C_1}(x_0, \mu_2) \cup \overline{U}_1$ is an ejective neighborhood of A_1 for g. If U_{μ_2} is not an ejective neighborhood of A_1 for g, then there exists $y \in U_{\mu_2} \setminus A_1$ such that $f^m(y) \in U_{\mu_2}$ for all $m \in I\!N$.
But since x_0 is conditionally globally asymptotically stable for $g|_{\overline{U}_1}$, there exists an integer $m_0 \in I\!N$ such that

$$g^{m_0}(y) \in \overline{B}_{C_1}(x_0, \mu_2),$$

and by construction, one obtain

$$g^{m_0+l}(y) \in \overline{B}_{C_1}(x_0, \mu_1), \text{ for all } l \in I\!N.$$

This give a contradiction, because by construction $\overline{B}_{C_1}(x_0, \mu_1)$ is a semi-ejective neighborhood of x_0 for g on $C_1 \setminus A_1$.□

Proof: (of theorem 1.2)
Since C is convex, one has $i_C(f, C) = 1$, and from theorem 3.1, we know that $i_C(f, B_C(x_0, r_0)) = 0$. So, by using the additivity property of the fixed point index, one has

$$1 = i_C(f, C) = i_C\left(f, C \setminus \overline{B}_C(x_0, r_0)\right),$$

and f has a fixed point $x_1 \in C \setminus \{x_0\}$.□

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