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Existence of Periodic Solutions for a State Dependent Delay Differential Equation

P. Magal

Faculté des Sciences et Techniques, 25, rue Philippe Lebon, B.P. 540, 76058 Le Havre, France

and

O. Arino

Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, URA 1204, 64000 Pau, France

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1. INTRODUCTION

In this paper, we consider the problem of finding nontrivial periodic solutions for a state dependent delay differential equation. The equation under consideration was introduced in Arino *et al.* [2]. We also refer the reader to Nussbaum [12, 13], Alt [1], Kuang and Smith [6, 7], and Mallet-Paret and Nussbaum [10, 11], who consider different classes of state dependent delay equations. The equation reads

$$\begin{cases} \dot{x}(t) = -f(x(t-\tau(t))), \\ \dot{\tau}(t) = h(x(t), \tau(t)), \end{cases} \tag{1}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \times [\tau_1, \tau_2] \to \mathbb{R}$ (with $0 < \tau_1 < \tau_2$) are C^1 maps. The existence of nontrivial slowly oscillating periodic solutions was shown in [2] by application of the ejective fixed point theorem. In [2], a slowly oscillating solution means a solution whose distance between two consecutive zeros is not less than the maximum delay τ_2. Here, we will consider two types of slow oscillations: we will use the notation $\tau-$slow oscillating solution (with $\tau = \tau_1$ or τ_2) to denote a solution whose distance between two consecutive zeros is not less than τ (for a more precise definition, see Definition 2.3). In [2], a Poincaré map is defined on a set of initial values

$$X = \left\{ \begin{array}{r} (\varphi, \tau_0) \in \mathrm{Lip}([-\tau_2, 0]) \times [\tau_1, \tau_2] : \varphi \text{ is non-decreasing on } [-\tau_2, 0] \\ \text{and } \varphi(\theta) = 0, \text{ for some } \theta \in [-\tau_2, 0] \end{array} \right\},$$

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where τ_2 is the maximum delay. Conditions on f and h imply that the subset $\{0\} \times [\tau_1, \tau_2]$ is an invariant stable manifold for the Poincaré map. Therefore the trivial solution of the equation is not ejective. Thus, the authors introduce a cone of the type

$$K = \{(\varphi, \tau_0) \in X : |\tau - \tau^*| \leqslant K \|\varphi\|_\infty\},$$

and by restricting the Poincaré operator to this cone, they were able to prove the ejectivity of the trivial fixed point.

Here, we use another approach to apply this method. We first take the initial values of the problem in the set

$$\begin{aligned} E_0 = \{(\varphi, \tau_0) \in C^1([-\tau_2, 0]) \times [\tau_1, \tau_2] : \varphi'(s) \geqslant 0 \\ \forall s \in [-\tau_0, 0], \varphi(-\tau_0) = 0, \varphi'(0) = 0\}. \end{aligned}$$

As $t - \tau(t)$ is increasing, the solution starting from such initial values will only depend on the value of φ on $[-\tau_0, 0]$. From this remark, we deduce that the fixed point problem that we need to solve only concerns the part of φ on $[-\tau_0, 0]$. By identifying φ restricted to $[-\tau_0, 0]$ to a certain function ψ on $[-1, 0]$, we then obtain a Poincaré operator defined on

$$\begin{aligned} E_1 = \{(\psi, \tau_0) \in C^1([-1, 0]) \times [\tau_1, \tau_2] : \psi'(s) \geqslant 0 \\ \forall s \in [-1, 0], \psi(-1) = 0, \psi'(0) = 0\}. \end{aligned}$$

As in Arino *et al.* [2], the problem is the non ejectivity of the trivial fixed point of the Poincaré operator. To overcome this difficulty, we apply a semi-ejective fixed point theorem (see Theorem 1.1 in Magal and Arino [9]) to the Poincaré operator, and we obtain the existence of a nontrivial periodic solution of Eq. (1).

We now present the main result of this paper, for which we need to make the following assumptions.

- (H1) $f(x)\, x > 0$, for all $x \neq 0$.
- (H2) $h(x, \tau) \leqslant \frac{L}{L+1}$, for a certain $L > 0$, for all $(x, \tau) \in \mathbb{R} \times [\tau_1, \tau_2]$.
- (H3) $h(x, \tau_1) > 0$, $h(x, \tau_2) < 0$, for all $x \in \mathbb{R}$.
- (H4) $\{\exists m > 0, \exists G \geqslant 0$, such that $\forall (x, \tau) \in \mathbb{R} \times [\tau_1, \tau_2]$, $\frac{\partial h}{\partial \tau}(0, \tau) \leqslant -m$ and $|\frac{\partial h}{\partial x}(x, \tau)| \leqslant G$.
- (H5) $\exists r > 0$, $\exists \delta > \frac{1}{\tau_1}$, such that $|f(x)| \geqslant \delta\, |x|$, if $|x| < r$
- (H6) $|f(x)| \leqslant M$, $|f'(x)| \leqslant M'$, for all $x \in \mathbb{R}$.
- (H7) $(\tau_2 - \tau_1)\, |f(x)| \leqslant |x|$, for all $x \in \mathbb{R}$.

Looking at (H3) and (H4) we can see that the equation $h(0, \tau) = 0$ has a unique root, which we will denote τ^*. Then, it is clear that $(0, \tau^*)$ is an equilibrium solution of Eq. (1). The problem that we are interested in is finding another periodic solution.

The main result of this paper will depend first on the previous assumptions and then on some relation between τ_1, τ_2, τ^*, and m. For example in Arino *et al.* [2] one crucial assumption was $\tau_2 < 2\tau_1$, which will be weakened here to $\tau_2 < 2\tau^*$. One can note that these parameters are associated to the map h. Two different sets of relationships will be considered each of them corresponding, to a specific class of functions h.

Let us denote by $\tilde{h}_{\tau_1} = \tilde{h}_{\tau_1}(\xi_0, l_0, l_1, \xi_1, \xi_2)$ the class of C^1 maps $h: \mathbb{R} \times [\tau_1, \tau_2] \to \mathbb{R}$, satisfying assumptions (H2), (H3), (H4), where the parameters $\tau_1 = \tau_1(h)$, $\tau_2 = \tau_2(h)$, $\tau^* = \tau^*(h)$, and $m = m(h)$ may depend on h and, moreover, satisfy the inequalities

$$\tau_2 + \frac{\xi_0}{\tau_1^{l_0}} \leqslant 2\tau^* \leqslant l_1 \tau_1. \tag{2}$$

Here, $\xi_0 > 0$, $l_0 \geqslant 0$, $l_1 > 0$ are fixed constants, and

$$m \geqslant \xi_1 \tau_1^{\xi_2} \tag{3}$$

where $\xi_1 > 0$, $0 \geqslant \xi_2 > -1$ are fixed constants.

The other class denoted by $\tilde{h}_m = \tilde{h}_m(\xi_0, l_0, l_1, \xi_1, \xi_2, \xi_3, \xi_4)$ is obtained by changing condition (3) to

$$\xi_1 m^{\xi_2} \leqslant \tau_1 \leqslant \xi_3 m^{\xi_4}, \tag{4}$$

where $\xi_1 > 0$, $0 \geqslant \xi_2 > -1$, $\xi_3 > 0$, $\xi_4 \geqslant 0$ are fixed constants.

The following theorem is the main result of this paper.

THEOREM 1.1. *Under assumptions* (H1)–(H4) *and* (H6), *let* $\tilde{h}_{\tau_1}$ (*respectively* $\tilde{h}_m$) *be a class of maps h defined as above. Then there exists* $\tau_1^* > 0$ (*respectively,* $m^* > 0$) *such that for all* $h \in \tilde{h}_{\tau_1}$ (*respectively,* $h \in \tilde{h}_m$) *if*

$$\tau_1(h) > \tau_1^* \qquad (\textit{respectively, } m > m^*),$$

and

$$\tau^*(h)\, f'(0) > \frac{\pi}{2}, \qquad f'(0)\, \tau_1(h) > 1,$$

then Eq. (1) *has a* τ_1*-slowly oscillating periodic solution* (x, τ) *with* $x \neq 0$, $|x(t)| \leqslant \tau_2(h)\, M$, $\tau(t) \in [\tau_1(h), \tau_2(h)]$, *and the period is exactly the total length of two consecutive maximal intervals where the solution is positive and then negative. If, in addition,* (H7) *holds, then this periodic solution is* τ_2*-slowly oscillating.*

Remarks. One can note that if $f'(0)\,\tau_1(h)>1$, then the assumption (H5) holds. Moreover, from assertion (2) we have $\tau_2<2\tau^*$.

We now present some examples of applications in both cases, i.e., when $h\in\tilde{h}_{\tau_1}$, and $h\in\tilde{h}_m$. We start with the following system of equations where τ^* is understood as a parameter taken large enough.

$$\begin{cases}\dot{x}(t)=-f(x(t-\tau(t))),\\ \dot{\tau}(t)=h_0(x(t),\tau(t)-\tau^*).\end{cases}\tag{5}$$

Here the parameter is τ^*, and as a direct application of Theorem 1.1, with $h\in\tilde{h}_{\tau_1}$, one has the following corollary, which extends Theorem 1.1 in Arino *et al.* [2].

COROLLARY 1.2. *Consider Eq.* (5), *in which* $f\colon\mathbb{R}\to\mathbb{R}$, $h_0\colon\mathbb{R}\times[a,b]\to\mathbb{R}$, $a<0<b$, *and* f *and* h_0 *are* C^1*-functions and satisfy the following assumptions: for some* M, M', $G\geqslant 0$, L, $m>0$,

(i) $f(x)\,x>0, \qquad \forall x\neq 0.$

(ii) $|f(x)|\leqslant M, \qquad |f'(x)|\leqslant M',\ f'(0)>0.$

(iii) $h_0(x,\tau)\leqslant \frac{L}{L+1}, \qquad (x,\tau)\in\mathbb{R}\times[a,b].$

(iv) $\frac{\partial h_0}{\partial\tau}(0,\tau)\leqslant -m, \qquad \left|\frac{\partial h_0}{\partial x}(x,\tau)\right|\leqslant G.$

(v) $h_0(x,a)>0,\ h_0(x,b)<0,\ h_0(0,0)=0.$

Then, there exists $\tilde{\tau}^*>0$, *such that for each* $\tau^*\geqslant\tilde{\tau}^*$ *and each pair* (a,b) *verifying the above conditions, Eq.* (5) *has a* τ_1*-slowly oscillating nontrivial periodic solution* (x,τ), *with* $x\neq 0$ *and the delay* $\tau(t)\in[\tau^*+a,\tau^*+b]$. *If in addition* f *satisfies*

(vi) $(b-a)\,|f(x)|<|x|, \qquad$ *for* $\quad x\neq 0,$

then this periodic solution is τ_2*-slowly oscillating.*

Proof. Corollary 1.2 is a consequence of Theorem 1.1, with $\tau_1(h)=\tau^*+a$, $\tau^*(h)=\tau^*$, $\tau_2(h)=\tau^*+b$, and $m(h)=m$, and the class $\tilde{h}_{\tau_1}$, with $l_0=0$, $l_1=3$, $\xi_0>0$, $\xi_1>0$, $\xi_2=0$. ∎

Let us now consider another parametric example which corresponds to the case when $h\in\tilde{h}_m$.

$$\begin{cases}\dot{x}(t)=-f(x(t-\tau(t))),\\ \dot{\tau}(t)=h_0(x(t),\alpha(\tau(t)-\tau^*)).\end{cases}\tag{6}$$

Here the parameter is α. As a direct application of Theorem 1.1, with $h\in\tilde{h}_m$, we obtain the following corollary. The latter situation is in fact a small perturbation of the constant delay case as will be explained together with several comparison remarks, in the conclusion of the paper.

COROLLARY 1.3. *Consider Eq.* (6), *in which* $f\colon \mathbb{R} \to \mathbb{R}$, $h_0\colon \mathbb{R} \times [a, b] \to \mathbb{R}$, $a < 0 < b$, *and* f *and* h_0 *are* C^1*-functions and satisfy the assumptions: for some* M, M', $G \geqslant 0$, L, $m > 0$,

(i) $f(x)\, x > 0, \qquad \forall x \neq 0.$

(ii) $|f(x)| \leqslant M, \qquad |f'(x)| \leqslant M',\ f'(0) > 0.$

(iii) $h_0(x, \tau) \leqslant \frac{L}{L+1}, \qquad (x, \tau) \in \mathbb{R} \times [a, b].$

(iv) $\frac{\partial h_0}{\partial \tau}(0, \tau) \leqslant -m, \qquad |\frac{\partial h_0}{\partial x}(x, \tau)| \leqslant G.$

(v) $h_0(x, a) > 0, h_0(x, b) < 0, \qquad h_0(0, 0) = 0.$

Then, there exists $\tilde{\alpha} > 0$, *such that for each* $\alpha \geqslant \tilde{\alpha}$ *and each pair* (a, b) *verifying the above conditions, and* $f'(0)\, \tau^* > \frac{\pi}{2}$, *Eq.* (6) *has a nontrivial* τ_1*-slowly oscillating periodic solution,* (x, τ), *with* $x \neq 0$ *and the delay* $\tau(t) \in [\tau^* + \frac{a}{\alpha}, \tau^* + \frac{b}{\alpha}]$. *If, in addition,* f *satisfies*

(vi) $(b - a)\, |f(x)| < |x|, \qquad$ *for* $\quad x \neq 0,$

then this periodic solution is τ_2*-slowly oscillating.*

Proof. Corollary 1.3 is a consequence of Theorem 1.1, when $\tau_1(h) = \tau^* + \frac{a}{\alpha}$, $\tau^*(h) = \tau^*$, $\tau_2(h) = \tau^* + \frac{b}{\alpha}$, and $m(h) = \alpha m$, and of the class $\widetilde{h}_m$, with $l_0 = 0$, $l_1 = 2$, $\xi_0 > 0$, $\xi_1 < \tau^*$, $\xi_2 = 0$, $\xi_3 > 0$, and $\xi_4 > 0$. ∎

2. EXISTENCE OF OSCILLATING SOLUTIONS

In the following, we consider solutions starting from initial values in the set

$$E = \{(\varphi, \tau_0) \in \mathrm{Lip}([-\tau_2, 0]) \times [\tau_1, \tau_2] : \varphi(-\tau_0) = 0 \text{ and } \varphi \text{ is non-decreasing on } [-\tau_0, 0]\}.$$

This special class of initial values was already introduced in the paper by Kuang and Smith [7].

The following result is proved in Arino *et al.* [2, Proposition 2.3].

PROPOSITION 2.1. *Under assumptions* (H2) *and* (H3), *for each* $(\varphi, \tau_0) \in E$ *there exists a unique solution* $(x(t), \tau(t))$ *of Eq.* (1) *such that* $x(s) = \varphi(s)$ *on* $[-\tau_2, 0]$ *and* $\tau(0) = \tau_0$. *Moreover,* $\tau(t) \in [\tau_1, \tau_2]$, *for all* $t \geqslant 0$, *and* $t - \tau(t)$ *is increasing on* $\mathbb{R}_+$.

Let $(\varphi, \tau_0) \in E$. We then denote $t_0 = -\tau_0$, $t_0^* = 0$, and

$$t_1 = t_1(\varphi, \tau_0) = \inf\{\tau > 0 : x(\varphi, \tau_0)(t) = 0\}.$$

The following lemma is an adaptation of proposition 5.18 in Arino *et al.* [2].

LEMMA 2.2. *Assume* (H1), (H2), (H3), *and* (H5).

Let (φ, τ_0) *be given in* E. *Then, if* $\varphi(0) \leqslant R$ (*with* $R \geqslant r$) *one has*

$$t_1(\varphi, \tau_0) \leqslant T(R),$$

where $T(R) = 3\tau_2 + (R - r)/C_{r,R}$ *and* $C_{r,R} = \inf\{f(s) : s \in [r, R]\} > 0$.

DEFINITION 2.3. Let x be a function defined on some interval $[t_0, +\infty[$. We will say that x is τ-slowly ($\tau = \tau_1$ or τ_2) oscillating if the set of zeros of x is a disjoint union of closed intervals, the distance between the left end of two successive intervals being not less than τ, and x is alternatively >0 and <0 in between such intervals.

By adapting the proof of theorem 3.9 in Arino *et al.* [2], we obtain the following theorem.

THEOREM 8. *Assume* (H1), (H2), (H3) *and* (H5). *Let* $\varepsilon \in \{-1, 1\}$ *and let* $(x(t), \tau(t))$ *be the solution of Eq.* (1), *with* $(\varepsilon\varphi, \tau_0) \in E$ *as its initial value. Then we can define two sequences* $\{t_i^*\}_{i \geqslant 0}$ *and* $\{t_i\}_{i \geqslant 0}$, *such that for all* $i \geqslant 0$,

$$t_0 = -\tau_0, \qquad t_0^* = 0,\ t_i^* \leqslant t_{i+1}, \qquad t_i = t_i^* - \tau(t_i^*),$$

and $\varepsilon(-1)^{i+1} x(t)$ *is non increasing on* $[t_i^*, t_{i+1}^*]$, *with* $x(t_i) = 0$ *and* $x(t_i^*) \neq 0$ *if* $\varphi(0) \neq 0$.

So $(\varepsilon(-1)^{i+1} x_{t_i^*}, \tau(t_i^*)) \in E$, *for* $i \geqslant 0$, *and* $x(t)$ *is* τ_1*-slowly oscillating. Moreover, if in addition we assume* (H7) *and* $\varphi(0) \neq 0$, *then* x *is* τ_2*-slowly oscillating.*

3. COMPLETE CONTINUITY OF THE POINCARÉ OPERATOR

From now on, we will use a different approach compared with Arino *et al.* [2]. To construct the Poincaré operator we will first prove the following lemma, which says that $(x(\varphi, \tau_0)(t), \tau(\varphi, \tau_0)(t))$ depends only on the value of φ on $[-\tau_0, 0]$.

LEMMA 3.1. *Under assumptions* (H2) *and* (H3), *let* $(\varphi, \tau_0) \in E$, *then for all* $t \geqslant 0$,

$$(x(\varphi, \tau_0)(t), \tau(\varphi, \tau_0)(t)) = (x(\varphi^+, \tau_0)(t), \tau(\varphi^+, \tau_0)(t))$$

where $\varphi^+(s) = \varphi(s)$, $\forall s \in [-\tau_0, 0]$, *and* $\varphi^+(s) = 0$, $\forall s \in [-\tau_2, -\tau_0]$.

Proof. As $t - \tau(t)$ is increasing, one has

$$-\tau_0 \leqslant t - \tau(t) \leqslant 0, \qquad \forall t \in [0, \tau_1],$$

So, for all $t \in [0, \tau_1]$,

$$x(\varphi, \tau_0)(t) = \varphi^+(0) + \int_0^t -f(\varphi^+(s - \tau(\varphi, \tau_0)(s)))\, ds,$$

and

$$\tau(\varphi, \tau_0)(t) = \tau_0 + \int_0^t h(x(s), \tau(\varphi, \tau_0)(s)).$$

Since, by construction, $(x(\varphi^+, \tau_0)(t), \tau(\varphi^+, \tau_0)(t))$ is the unique solution satisfying the previous integral equation, one has

$$(x(\varphi, \tau_0)(t), \tau(\varphi, \tau_0)(t)) = (x(\varphi^+, \tau_0)(t), \tau(\varphi^+, \tau_0)(t)), \qquad \forall t \in [0, \tau_1].$$

The proof for all $t \geqslant 0$ follows by induction on $k \geqslant 1$ by considering the intervals of the form $[0, k\tau_1]$. ∎

From Lemma 3.1, in order to have existence of a nontrivial periodic solution of Eq. (1), it is sufficient that there exist $p_0 \geqslant 1$, and $(\varphi, \tau_0) \in E$, $\varphi(0) > 0$, such that

$$\tau_0 = \tau(\varphi, \tau_0)(t^*_{2p_0})$$

and (7)

$$\varphi(s) = x_{t^*_{2p_0}}(\varphi, \tau_0)(s), \qquad \text{for all} \quad s \in [-\tau_0, 0].$$

The fixed point problem (7) can be rewritten in the following manner.

Consider the spaces

$$X_0 = C^1([-\tau_2, 0]) \times [\tau_1, \tau_2], \qquad \text{and} \qquad X_1 = C^1([-1, 0]) \times [\tau_1, \tau_2].$$

In the following, X_0 and X_1 will be supposed to be respectively endowed with the metrics induced by the norms

$$\|(\varphi, \tau_0)\|_0 = \|\varphi\|_{\infty, [-\tau_2, 0]} + \|\dot{\varphi}\|_{\infty, [-\tau_2, 0]} + |\tau_0|, \qquad \text{for all} \quad (\varphi, \tau_0) \in X_0$$

and

$$\|(\psi, \tau_0)\|_1 = \|\psi\|_{\infty, [-1, 0]} + \|\dot{\psi}\|_{\infty, [-1, 0]} + |\tau_0|, \qquad \text{for all} \quad (\psi, \tau_0) \in X_1.$$

We then denote

$$E_0 = \{(\varphi, \tau_0) \in X_0 : \varphi'(s) \geqslant 0 \ \forall s \in [-\tau_0, 0], \varphi(-\tau_0) = 0, \text{ and } \varphi'(0) = 0\},$$

$$E_0^- = \{(\varphi, \tau_0) \in X_0 : (-\varphi, \tau_0) \in E_0\},$$

and

$$E_1 = \{(\psi, \tau_0) \in X_1 : \psi'(s) \geqslant 0 \ \forall s \in [-1, 0], \psi(-1) = 0, \text{ and } \psi'(0) = 0\},$$

$$E_1^- = \{(\psi, \tau_0) \in X_1 : (-\psi, \tau_0) \in E_1\}$$

For each $j \geqslant 1$, denote by P_j, the Poincaré operator defined on E_0 by

$$P_j(\varphi, \tau_0) = (x_{t_j^*}(\varphi, \tau_0), \tau(\varphi, \tau_0)(t_j^*)),$$

and

$$P_j^+(\varphi, \tau_0) = ((-1)^j x_{t_j^*}(\varphi, \tau_0), \tau(\varphi, \tau_0)(t_j^*)).$$

We remark that, by construction if $(\varphi, \tau_0) \in E_0$ then $x(\varphi, \tau_0)(t)$ is continuously differentiable on $[-\tau_2, +\infty[$, since $\varphi'_-(0) = 0$, and $0 = f(\varphi(-\tau_0)) = x'_+(\varphi, \tau_0)(0)$. From this remark and by using Theorem 2.4, we deduce that

$$P_{p_0}^+ : E_0 \to E_0, \qquad \text{for} \quad p_0 \geqslant 1.$$

So, in particular, $P_{2p_0}: E_0 \to E_0$, for $p_0 \geqslant 1$.

Lemma 3.1 shows that we can restrict our attention to pairs (φ, τ_0) with φ defined on $[-\tau_0, 0]$. It will be convenient to represent the function φ in terms of functions defined on a fixed interval. We will use functions defined on $[-1, 0]$. On the other hand, E_0 is not convex (because of the condition $\varphi(-\tau_0) = 0$), so in order to apply fixed point techniques, we really need to identify E_0 with E_1. To do this, we introduce $Q: X_1 \to X_0$ the operator defined by

$$Q(\psi, \tau_0) = (\varphi, \tau_0),$$

where

$$\varphi(s) = \psi\left(\frac{s}{\tau_0}\right), \qquad \text{for all} \quad s \in [-\tau_0, 0],$$

and

$$\varphi(s) = \varphi'_+(-\tau_0)(s-\tau_0) = \frac{\psi'_+(-1)}{\tau_0}(s-\tau_0), \qquad \text{for all} \quad s \in [-\tau_2, -\tau_0],$$

and we introduce L: $X_0 \to X_1$, the operator defined by

$$L(\varphi, \tau_0) = (\psi, \tau_0),$$

where

$$\psi(s) = \varphi(\tau_0 s) \qquad \text{for all} \quad s \in [-1, 0].$$

We then have

$$Q(E_1) \subset E_0,\ Q(E_1^-) \subset E_0^-,\ L(E_0) \subset E_1, \qquad \text{and} \qquad L(E_0^-) \subset E_1^-.$$

With the previous notations the fixed point problem (7) can be rewritten as follows: Find $(\psi, \tau_0) \in E_1$, with $\psi(0) > 0$ satisfying

$$(\psi, \tau_0) = F_{2p_0}(\psi, \tau_0) \tag{8}$$

for a certain $p_0 \geqslant 1$, where F_{2p_0}: $E_1 \to E_1$, $p_0 \geqslant 1$, and F_{2p_0+1}: $E_1 \to E_1^-$, $p_0 \geqslant 1$, are defined by

$$F_{p_0} = L \circ P_{p_0} \circ Q.$$

LEMMA 3.2. *Under assumptions* (H2) *and* (H3), *one has*

$$F_{p_0} = F_1^{p_0},$$

where F_1^m *is defined by* $F_1^{m+1} = F_1 \circ F_1^m$, *for* $m \geqslant 1$, *and* $F_1^1 = F_1$.

Proof. This result follows directly from Lemma 3.1 and from the definitions of F_{p_0}, L, and Q. ∎

From the previous lemma, we deduce that it is sufficient to study the compactness of F_2 to deduce the compactness of F_{2p_0}.

PROPOSITION 3.3. *Under assumptions* (H1) *through* (H6), $F_2(E_1)$ *is relatively compact in* E_1.

Proof. Let $(\tilde{\psi}, \tilde{\tau}_0) \in E_1$. Denote by $(x(t), \tau(t))$ the solution of Eq. (1) with initial value $(\varphi, \tilde{\tau}_0) = L(\tilde{\psi}, \tilde{\tau}_0)$. Consider now $(\psi, \tau_0) = F_2(\tilde{\psi}, \tilde{\tau}_0)$. One has

$$\tau_0 = \tau(\varphi, \tilde{\tau}_0)(t_2^*), \qquad \text{and} \qquad \psi(s) = x(\varphi, \tilde{\tau}_0)(\tau_0 s + t_2^*) \text{ on } [-1, 0].$$

So

$$\dot{\psi}(s) = \tau_0 \dot{x}(\tau_0 s + t_2^*) = -\tau_0 f(x(\tau_0 s + t_2^* - \tau(\tau_0 s + t_2^*))),$$

and since $\tau_0 \leqslant \tau_2$,

$$\|\dot{\psi}\|_{\infty, [-1, 0]} \leqslant \tau_2 M.$$

Since $\psi(-1) = 0$, one has

$$\psi(0) = -\tau_0 \int_{-1}^{0} f(x(\tau_0 s + t_2^* - \tau(\tau_0 s + t_2^*)))\, ds,$$

so

$$\|\psi\|_{\infty, [-1, 0]} \leqslant \tau_2 M.$$

On the other hand,

$$\dot{\psi}(s) = -\tau_0 f(x(\tau_0 s + t_2^* - \tau(\tau_0 s + t_2^*)))$$

and $\tau_0 s + t_2^* \in [t_2, t_2^*]$, $\forall s \in [-1, 0]$.

Moreover, we have by definition

$$t_2^* - \tau(t_2^*) = t_2, \qquad \text{and} \qquad t_1^* - \tau(t_1^*) = t_1,$$

and as $t_2 > t_1^*$, we deduce from the monotonicity of $t - \tau(t)$ that

$$t_2 \geqslant t - \tau(t) \geqslant t_1, \qquad \forall t \in [t_2, t_2^*].$$

So, we deduce that

$$\dot{x}(t - \tau(t)) = -f(x(t - \tau(t) - \tau(t - \tau(t)))), \qquad \forall t \in [t_2, t_2^*].$$

So, we deduce that $\dot{\psi}$ is differentiable, and

$$\ddot{\psi}(s) = \tau_0^2 f'(x(s_1)) f(s_1 - \tau(s_1))(1 - h(x(\tau_0 s + t_2^*), \tau(\tau_0 s + t_2^*)))$$

where $s_1 = \tau_0 s + t_2^* - \tau(\tau_0 s + t_2^*)$.

Finally, we have

$$\|\ddot{\psi}\|_{\infty, [-1, 0]} \leqslant (\tau_2)^2 MM' \sup_{|y| \leqslant \tau_2 M,\ \tau \in [\tau_1, \tau_2]} |1 - h(y, \tau)|,$$

and the conclusion on compactness follows by standard Ascoli–Arzela arguments. ∎

The remainder of this section is devoted to proving the continuity of F_{2p_0}.

LEMMA 3.4. *The operators* $Q: X_1 \to X_0$, $L: X_0 \to X_1$ *are continuous.*

Proof. We will not detail this proof. ∎

LEMMA 3.5. *Assume* (H1) *through* (H6) *hold. Let* $(\tilde{\psi}, \tilde{\tau}_0) \in E_1$ *and denote* $(\tilde{\varphi}, \tilde{\tau}_0) = Q(\tilde{\psi}, \tilde{\tau}_0)$. *Assume that the function* $(\varphi, \tau_0) \to t_1^*(\tilde{\varphi}, \tilde{\tau}_0)$ *is continuous at* $(\tilde{\varphi}, \tilde{\tau}_0)$ *in* E_0. *Then* F_1 *is continuous at* $(\tilde{\psi}, \tilde{\tau}_0)$.

Proof. This result is a direct consequence of the continuous dependence of the system on its initial values. ∎

The following proposition corresponds to proposition 4.12 in Arino *et al.* [2].

PROPOSITION 3.6. *Assume* (H1) *through* (H6) *hold. Then, the operator* F_2 *is continuous at each* $(\tilde{\psi}, \tilde{\tau}_0) \in E_1$ *such that* $\tilde{\psi}(0) > 0$.

Proof. By using Lemma 3.5 and the continuous dependence of the solutions with respect to its initial values, one can adapt the proof of Proposition 4.12 in Arino *et al.* [2] and the result follows. ∎

The only problem for the continuity of F_{2p_0} is for the second component of F_{2p_0} when $(\psi, \tau_0) \to (0, \tilde{\tau}_0)$ with $\tilde{\tau}_0 \neq \tau^*$. This problem comes from the fact that we do not know if $\lim_{(\psi, \tau_0) \to (0, \tilde{\tau}_0)} t_{2p_0}^*(\varphi, \tau_0)$ exists when $\tilde{\tau}_0 \neq \tau^*$. To encompass the difficulty, we will transform the map F_{2p_0}.

Denote for $p_0 \geqslant 1$, $F_{2p_0}^1: E_1 \to \Gamma_1$, and $F_{2p_0}^2: E_1 \to [\tau_1, \tau_2]$, the operators defined for $(\psi, \tau_0) \in E_1$ by

$$F_{2p_0}^1(\psi, \tau_0)(s) = x_{t_{2p_0}^*(\varphi, \tau_0)}(\varphi, \tau_0)(\tau(\varphi, \tau_0)(t_{2p_0}^*(\varphi, \tau_0))\, s), \qquad \text{on } [-1, 0]$$

and

$$F_{2p_0}^2(\psi, \tau_0) = \tau(\varphi, \tau_0)(t_{2p_0}^*(\varphi, \tau_0))$$

where $(\varphi, \tau_0) = Q(\psi, \tau_0)$.

Then by definition of F_{2p_0}, one has

$$F_{2p_0}(\psi, \tau_0) = (F_{2p_0}^1(\psi, \tau_0), F_{2p_0}^2(\psi, \tau_0)), \qquad \forall (\psi, \tau_0) \in E_1.$$

LEMMA 3.7. *Assume* (H1) *through* (H6) *hold. Then, one has*

$$\lim_{(\psi, \tau_0) \to (0, \tilde{\tau}_0)_{E_1}} \|F_{2p_0}^1(\psi, \tau_0)\|_{1, [-1, 0]} = 0$$

and

$$\lim_{(\psi,\tau_0)\to(0,\tilde{\tau}_0)_{E_1}} F^2_{2p_0}(\psi,\tau_0)=\tau^*.$$

Proof. Let $\{(\psi^n,\tau_0^n)\}_{n\geqslant 0}$ be a sequence in E_1 which converges to $(0,\tilde{\tau}_0)\in E_1$ as $n\to+\infty$. Then as Q is continuous, if we denote $(\varphi^n,\tau_0^n)=Q(\psi^n,\tau_0^n)$, we have

$$F^1_{2p_0}(\psi^n,\tau_0^n)=x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)(\tau(\varphi^n,\tau_0^n)(t^*_{2p_0}(\varphi^n,\tau_0^n))\,s)|_{[-1,0]},$$

and as Q is continuous we also have $\|\varphi^n\|_{1,[-\tau_2,0]}\to 0$.

From Lemma 2.2, one has for a certain $R>r$,

$$t^*_{2p_0}(\varphi,\tau_0)\leqslant 2p_0(\tau_2+T(R))=t^*,$$

and

$$\|F^1_{2p_0}(\psi^n,\tau_0^n)\|_{\infty,[-1,0]}\leqslant \|x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)\|_{\infty,[0,t^*]}.$$

So from the continuous dependence of the solutions with respect to its initial values, one has

$$\lim_{(\psi,\tau_0)\to(0,\tilde{\tau}_0)} \|x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)\|_{\infty,[0,t^*]}=0.$$

Moreover,

$$\begin{aligned}&\|x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)(\tau(\varphi^n,\tau_0^n)(t^*_{2p_0}(\varphi^n,\tau_0^n))\cdot)'\|_{\infty,[-1,0]}\\&\quad\leqslant \tau_2 M'\,\|x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)\|_{\infty,[-\tau_2,t^*]},\end{aligned}$$

so one has

$$\lim_{n\to+\infty}\|x_{t_2^*(\varphi,\tau_0)}(\varphi^n,\tau_0^n)\|_{\infty,[-\tau_2,t^*]}=0,$$

from which we deduce that

$$\lim_{n\to+\infty}\|F^1_{2p_0}(\psi^n,\tau_0^n)\|_{1,[-1,0]}=0.$$

For the second limit, we note first that $\tau(0,\tau^*)(t)=\tau^*$, for all t$\geqslant 0$, so

$$\begin{aligned}|\tau^*-\tau(\varphi^n,\tau_0^n)(t^*_{2p_0}(\varphi^n,\tau_0^n))|&\leqslant|\tau(0,\tau^*)(t^*_{2p_0}(\varphi^n,\tau_0^n))-\tau(\varphi^n,\tau_0^n)(t^*_{2p_0}(\varphi^n,\tau_0^n))|\\&\leqslant\|\tau(0,\tau^*)-\tau(\varphi^n,\tau_0^n)\|_{\infty,[0,t^*]},\end{aligned}$$

and, once again using continuous dependence with respect to the initial values on bounded time interval, one deduces that

$$\lim_{n \to +\infty} \tau(\varphi^n, \tau_0^n)(t^*_{2p_0}(\varphi^n, \tau_0^n)) = \tau^*. \quad \blacksquare$$

Denote for $\varepsilon > 0$, $\widetilde{F}_{2p_0, \varepsilon}: E_1 \to E_1$ the map defined by

$$\widetilde{F}_{2p_0, \varepsilon}(\psi, \tau_0) = (F^1_{2p_0}(\psi, \tau_0), \widetilde{F}^2_{2p_0, \varepsilon}(\psi, \tau_0)), \qquad \forall (\psi, \tau_0) \in E_1,$$

where

$$\widetilde{F}^2_{2p_0, \varepsilon}(\psi, \tau_0) = \begin{cases} \phi\left(\frac{\|\psi\|_1}{\varepsilon\,|\tau_0 - \tau^*|}\right) F^2_{2p_0}(\psi, \tau_0) + \left(1 - \phi\left(\frac{\|\psi\|_1}{\varepsilon\,|\tau_0 - \tau^*|}\right)\right)\tau^*, & \text{if } \tau_0 \neq \tau^*, \\ \tau^*, & \text{if } \tau_0 = \tau^*, \end{cases}$$

and $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous map satisfying

$$\phi(s) = 1, \forall s \geqslant 1, \phi(s) \in [1, 0], \forall s \in [1, 0], \phi(0) = 0.$$

The following theorem summarizes the previous results.

THEOREM 3.8. *Assume* (H1) *through* (H6) *hold. For each* $\varepsilon > 0$, *the operator* $\widetilde{F}_{2p_0, \varepsilon}: E_1 \to E_1$ *is completely continuous and* $\widetilde{F}_{2p_0, \varepsilon}(E_1)$ *is relatively compact.*

4. EXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS

In this section we use a new technique to prove the existence of nontrivial periodic solutions. Compared to [2], the authors consider the map denoted here by $F_2: E_1 \to E_1$. Their idea was to apply the Browder ejective fixed point theorem [3]. Let us recall that

$$E_1 = K_1 \times [\tau_1, \tau_2],$$

with

$$K_1 = \{\psi \in C^1([-1, 0]) : \psi'(s) \geqslant 0,\ \forall s \in [-1, 0],\ \psi(-1) = 0, \text{ and } \psi'(0) = 0\}.$$

The difficulty that one has to encompass to apply the Browder ejective fixed point theorem is the lack of ejectivity of 0 in the set $\{0\} \times [\tau_1, \tau_2]$.

In [2], in order to encompass this difficulty, the authors first proved the existence of a subcone $K_\alpha = \{(\psi, \tau_0) \in K_1 \times [\tau_1, \tau_2] : \|\psi\|_1 \geqslant \alpha\, |\tau_0 - \tau^*|\}$, such that $F_2(K_\alpha) \subset K_\alpha$ for a certain $\alpha > 0$. Then, by showing that 0 is an ejective fixed point of $F_2|_{K_\alpha}$, the authors were able to prove the existence of a nontrivial fixed point of F_2.

Here, we use a different approach. We first remark that $\tilde{F}_{2p_0, \varepsilon}$ has the following properties:

(i) $\tilde{F}_{2p_0, \varepsilon}(0) = 0$

(ii) $\tilde{F}_{2p_0, \varepsilon}(\{0\} \times [\tau_1, \tau_2]) \subset \{0\} \times [\tau_1, \tau_2]$.

Moreover, we will prove that

(iii) for each $M > 0$, there exists $C > 0$ and $0 \leqslant \gamma < 1$, such that for all $(\psi, \tau_0) \in K_1 \times [\tau_1, \tau_2]$, with $\|\psi\|_1 + |\tau_0| \leqslant M$ and $\|\psi\|_1 \leqslant C\, |\tau_0 - \tau^*|$,

$$|\tilde{F}^2_{2p_0, \varepsilon}(\psi, \tau_0) - \tau^*| \leqslant \gamma\, |\tau_0 - \tau^*|,$$

where $\tilde{F}^2_{2p_0, \varepsilon}(\psi, \tau_0)$ denotes the second component of $\tilde{F}_{2p_0, \varepsilon}(\psi, \tau_0)$.

One can see that under (ii) and (iii), the trivial fixed point 0 can not be ejective, because for all $(\psi, \tau_0) \in \{0\} \times [\tau_1, \tau_2]$,

$$\lim_{m \to +\infty} \tilde{F}^m_{2p_0, \varepsilon}(\psi, \tau_0) = \tau^*.$$

In order to circumvent this problem, we will use a notion weaker than ejectivity, namely semi-ejectivity as defined in [8]. Let C be a subset of a Banach space $(X, \|.\|)$, f: $C \to C$ be a map, $A \subset C$, and let $x_0 \in \partial_C A$ be a fixed point of f. We will say that x_0 is a semi-ejective fixed point of f on $C \backslash A$ if there exists a neighborhood V of x_0 in C such that for all $y \in V \backslash A$, there exists an integer $m \in \mathbb{N}$, with $f^m(y) \in C \backslash V$. One can see that the notion of semi-ejectivity coincides with the notion of ejectivity when $A = \{x_0\}$.

For the following theorem we refer to Magal and Arino [9, Theorem 1.1].

THEOREM 4.1. *Let $(X_1, \|.\|_1)$ and $(X_2, \|.\|_2)$ be two Banach spaces, K_1 a cone of $(X_1, \|.\|_1)$, and C_2 a bounded closed convex subset of X_2 containing 0_{X_2}. Let $C = \bar{B}_{K_1}(0, r_1) \times C_2$, and let f: $C \to C$ be a compact map satisfying $f(0) = 0$.*

Assume in addition that:

(i) 0 *is a semi-ejective fixed point of* f *on* $C\backslash A$, *with* $A=\{0_{X_1}\}\times C_2$.

(ii) *There exists* $r_0>0, \eta>0$, *and* $0<\gamma<1$, *such that for all* $(x_1, x_2)\in C$, *with* $\|\tilde{x}_1\|_1+\|\tilde{x}_2\|_2\leqslant r_0$, *and* $\|\tilde{x}_1\|_1\leqslant\eta\,\|\tilde{x}_2\|_2$ *it holds that*

$$\|f_2((x_1, x_2))\|_2\leqslant\gamma\,\|x_2\|_2,$$

where $f_2((x_1, x_2))$ *is the second component of* $f((x_1, x_2))$.

(iii) $f(C\backslash A)\subset C\backslash A$.

Then f *has a fixed point* $\bar{x}\in C\backslash\{0\}$.

We will verify that assumptions (i)–(iii) of Theorem 4.1 are satisfied for $\tilde{F}_{2p_0, \varepsilon_0}$ for a certain $\varepsilon_0>0$ small enough with $U_{\varepsilon_0}=\{(\psi, \tau_0)\in E_1$: $\|\psi\|_1<\varepsilon_0|\tau_0-\tau^*|\}$. So, in view of Theorem 9, $\tilde{F}_{2p_0, \varepsilon_0}$ will have a non trivial fixed point. To conclude, we will have to verify that this nontrivial fixed point $(\bar{\psi}, \bar{\tau}_0)$ is also a fixed point for F_{2p_0}. But under assumption (ii) of Theorem 4.1, and since F_{2p_0} coincides with $\tilde{F}_{2p_0, \varepsilon_0}$ on $E_1\backslash U_{\varepsilon_0}$ (with $U_{\varepsilon_0}=\{(\psi, \tau_0)\in E_1:\|\psi\|_1<\varepsilon_0\,|\tau_0-\tau^*|\}$), this latter fact will be automatically verified.

We now prove that Theorem 4.1 applies to $\tilde{F}_{2p_0, \varepsilon_0}$ for $p_0=1$, with $X_1=C^1([-1, 0])$, $X_2=\mathbb{R}$, and $C_2=[\tau_1, \tau_2]$.

LEMMA 4.2. *Assume* (H1) *through* (H6) *hold. Then, for each integer* $p_0\geqslant 1$ *and each* $\varepsilon>0$,

$$\tilde{F}_{2p_0, \varepsilon}(E_1)\subset\bar{B}_{\Gamma_1}(0, r_1)\times[\tau_1, \tau_2],$$

with $r_1=\tau_2[M+M']$.

Proof. By construction of F_2 one has

$$F_2(E_1)\subset\bar{B}_{\Gamma_1}(0, r_1)\times[\tau_1, \tau_2],$$

with $r_1=\tau_2[M+M']$ (see the proof of Proposition 3.3).

Moreover, from Lemma 3.2 we have $F_{2p_0}=F_2^{p_0}$, so

$$F_{2p_0}(E_1)\subset\bar{B}_{\Gamma_1}(0, r_1)\times[\tau_1, \tau_2]. \quad \blacksquare$$

In the following we denote

$$C=\bar{B}_{\Gamma_1}(0, \tau_2[M+M'])\times[\tau_1, \tau_2], \qquad \text{and} \qquad A=\{0\}\times[\tau_1, \tau_2].$$

The following lemma shows that Assumption (iii) of Theorem 4.1 is satisfied.

LEMMA 4.3. *Assume* (H1) *through* (H6) *hold. Then for each integer* $p_0 \geqslant 1$ *and each* $\varepsilon > 0$,

$$\widetilde{F}_{2p_0, \varepsilon}(C \backslash A) \subset C \backslash A.$$

Proof. Let $(\psi, \tau_0) \in C$ with $\psi(0) > 0$, and denote $(\varphi, \tau_0) = Q(\psi, \tau_0)$. Then $\varphi(0) = \psi(0) > 0$, and from Theorem 2.4 we have $x(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0)) > 0$. ∎

In the following, we will assume that

$$f'(0)\, \tau_2 > 1,$$

so this will imply that

$$M' \tau_2 > 1.$$

Moreover, we will always take $(\varphi, \tau_0) \in E_0$ such that $0 < \varphi(0) < r_1$. We remark that the constant r in Assumption (H5) can be chosen in order to satisfy

$$r < r_1 = \tau_2[M + M'],$$

and from Lemma 2.2, we will have for all integers $i \geqslant 0$

$$t_{i+1}(\varphi, \tau_0) - t^*_i(\varphi, \tau_0) \leqslant T(r_1).$$

LEMMA 4.4. *Assume* (H1) *through* (H6) *hold and* $f'(0)\, \tau_2 > 1$. *Then for each* $(\varphi, \tau_0) \in E_0$, *for all* $t \in [0, t^*_{2p_0}(\varphi, \tau_0)]$,

$$|x(\varphi, \tau_0)(t)| \leqslant (\tau_2 M')^{2p_0}\, |\varphi(0)|$$

and

$$|\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant e^{-mt}\, |\tau_0 - \tau^*| + \frac{1}{mG}\, (\tau_2 M')^{2p_0}\, |\varphi(0)|.$$

Proof. Let $(\varphi, \tau_0) \in E_0$ with $0 < \varphi(0) < r_1$. We have by construction

$$x(\varphi, \tau_0)(t^*_1(\varphi, \tau_0)) = \int_{t_1}^{t^*_1} -f(x(s - \tau(s)))\, ds,$$

and since $t - \tau(t)$ is increasing we have

$$|x(\varphi, \tau_0)(t^*_1(\varphi, \tau_0))| \leqslant (t^*_1 - t_1)\, M'\, |x(0)| \leqslant \tau_2 M'\, |\varphi(0)|.$$

Using the same argument, we also have

$$|x(\varphi, \tau_0)(t^*_{i+1}(\varphi, \tau_0))| \leqslant \tau_2 M' \, |x(\varphi, \tau_0)(t^*_i(\varphi, \tau_0))|,$$

and so

$$|x(\varphi, \tau_0)(t^*_i)| \leqslant (\tau_2 M')^{2p_0} \, |\varphi(0)|, \qquad \forall i = 1, ..., 2p_0,$$

and the first inequality is proved.

Under assumption (H4), we have for almost every $t \in [0, t^*_{2p_0}(\varphi, \tau_0)]$,

$$\frac{d}{dt} \, |\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant -m \, |\tau(\varphi, \tau_0)(t) - \tau^*| + G \, |x(\varphi, \tau_0)(t)|,$$

so

$$\frac{d}{dt} \, |\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant -m \, |\tau(\varphi, \tau_0)(t) - \tau^*| + G(\tau_2 M')^{2p_0} \, |\varphi(0)|$$

and

$$|\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant e^{-mt} \, |\tau_0 - \tau^*| + \int_0^t e^{-m(t-s)} G(\tau_2 M')^{2p_0} \, |\varphi(0)| \, ds$$

$$|\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant e^{-mt} \, |\tau_0 - \tau^*| + \frac{1}{mG} \, (\tau_2 M')^{2p_0} \, |\varphi(0)|. \quad \blacksquare$$

The following proposition shows that Assumption (ii) of Theorem 4.1 is satisfied and that all the nontrivial fixed points of $\widetilde{F}_{2p_0, \varepsilon_0}$ are also nontrivial fixed points of F_{2p_0} when $\varepsilon_0 > 0$ is suitably chosen.

PROPOSITION 4.5. *Assume* (H1) *through* (H6) *hold. Then for each integer* $p_0 \geqslant 1$, *there exist* $\varepsilon_0 = \varepsilon_0(p_0) > 0$ *and* $0 \leqslant \gamma = \gamma(p_0) < 1$ *such that for all* $(\psi, \tau_0) \in C$,

$$\|\psi\|_{1, [-1, 0]} \leqslant \varepsilon_0 \, |\tau_0 - \tau^*| \Rightarrow |\widetilde{F}^2_{2p_0, \varepsilon_0}(\psi, \tau_0) - \tau^*| \leqslant \gamma \, |\tau_0 - \tau^*|,$$

where $\widetilde{F}^2_{2p_0, \varepsilon_0}(\psi, \tau_0)$ *is the second component of* $\widetilde{F}_{2p_0, \varepsilon_0}(\psi, \tau_0)$.

Proof. Since

$$\widetilde{F}^2_{2p_0, \varepsilon_0}(\psi, \tau_0) = \alpha F^2_{2p_0}(\psi, \tau_0) + (1 - \alpha) \, \tau^*$$

for a certain $\alpha \in [0, 1]$, it is sufficient to prove the result by replacing $\widetilde{F}^2_{2p_0, \varepsilon_0}$ by $F^2_{2p_0}$.

But

$$F^2_{2p_0}(\psi, \tau_0) = \tau(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0))$$

where $(\varphi, \tau_0) = Q(\psi, \tau_0)$.

Now, from Lemma 4.4, we have

$$|\tau(\varphi, \tau_0)(t^*_{2p_0}) - \tau^*| \leqslant e^{-mt^*_{2p_0}} |\tau_0 - \tau^*| + \frac{1}{mG} (\tau_2 M')^{2p_0} |\varphi(0)|,$$

and as $t^*_{2p_0} \geqslant 2p_0\tau_1 > 0$, we have

$$|\tau(\varphi, \tau_0)(t^*_{2p_0}) - \tau^*| \leqslant e^{-m2p_0\tau_1} |\tau_0 - \tau^*| + \frac{1}{mG} (\tau_2 M')^{2p_0} |\varphi(0)|.$$

Now taking $\varepsilon_0 > 0$ small enough to have

$$\gamma = e^{-m2p_0\tau_1} + \frac{1}{mG} (\tau_2 M')^{2p_0} \varepsilon_0 < 1$$

and assuming that $|\varphi(0)| \leqslant \varepsilon_0 |\tau_0 - \tau^*|$, we obtain

$$|\tau(\varphi, \tau_0)(t^*_{2p_0}) - \tau^*| \leqslant \gamma |\tau_0 - \tau^*|. \quad \blacksquare$$

From now on, we are interested in proving the semi-ejectivity of the trivial fixed point $(0, \tau^*)$. To prove this, we first need to obtain some estimations locally around $(0, \tau^*)$.

In the following lemma we use classes of maps $\tilde{h}_{\tau_1}$ and $\tilde{h}_m$, as defined in Section 1.

LEMMA 4.6. *Assume* (H1) *through* (H6) *hold*, $f'(0)\tau_2 > 1$, *and* $2\tau^* > \tau_2$. *Then for each* $p_0 \geqslant 1$ *there exists* $\varepsilon_1 = \varepsilon_1(p_0) > 0$, *and* $C_1 > 0$ *such that*;

$$\forall(\varphi, \tau_0) \in E_0, \qquad |\tau_0 - \tau^*| + |\varphi(0)| \leqslant \varepsilon_1$$
$$\Rightarrow |\varphi(0)| \leqslant C_1^{2p_0} |x(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0))|.$$

Moreover, there exists $\tau_1^* = \tau_1^*(\xi_0, l_0, l_1, \xi_1, \xi_2) > 0$ *such that for all* $h \in \tilde{h}_{\tau_1}$

$$\tau_1 > \tau_1^* \Rightarrow e^{-m\tau_1} C_1 < 1,$$

and there exists $m^* = m^*(\xi_0, l_0, l_1, \xi_1, \xi_2, \xi_3, \xi_4) > 0$ *such that for all* $h \in \tilde{h}_m$

$$m > m^* \Rightarrow e^{-m\tau_1} C_1 < 1.$$

Remark 1. In the previous lemma the delay is characterized by the class $\tilde{h}_{\tau_1}$, or $\tilde{h}_m$, of map h, and the map h can be any function in this class.

By contrast, the map f remains the same and so all the parameters corresponding to this map remain constant.

Remark 2. In comparison with the paper by Arino *et al.* [2] (see Proposition 5.19), semi-ejectivity allows us to replace the condition $\tau_2 < 2\tau_1$ by the weaker condition $\tau_2 \leqslant 2\tau^*$. This is possible, because we will only need the previous estimations for small values of $|\tau_0 - \tau^*| + |\varphi(0)|$.

Proof of Lemma 4.6. Let us start by noting that since $f'(0) > 0$, and under assumption (H1), there exists $\delta' > 0$ such that

$$|f(x)| \geqslant \delta' \, |x|, \qquad \forall x \in [-r_1, r_1], \tag{9}$$

where $r_1 = \tau_2[M + M']$. Using Lemma 4.4, one can see that to prove Lemma 4.6 it is sufficient to show that there exists $\varepsilon_1 \in]0, r_1]$ and $C_1 > 0$ such that for all $(\varphi, \tau_0) \in E_0$,

$$|\tau_0 - \tau^*| + |\varphi(0)| \leqslant \varepsilon_1 \Rightarrow |\varphi(0)| \leqslant C_1 \, |x(\varphi, \tau_0)(t_1^*(\varphi, \tau_0))|.$$

In fact, it is clear that from lemma 9, given an integer $p_0 \geqslant 1$, we may choose $\varepsilon_2 > 0$ small enough to have

$$|\tau_0 - \tau^*| + |\varphi(0)| \leqslant \varepsilon_2$$
$$\Rightarrow |x(\varphi, \tau_0)(t)| + |\tau(\varphi, \tau_0)(t) - \tau^*| \leqslant \varepsilon_1, \qquad \forall t \in [0, t_{2p_0}^*(\varphi, \tau_0)],$$

and Lemma 4.6 follows by induction. Let $\varepsilon_1 \in]0, r_1]$ be fixed such that

$$\left[1 + \frac{1}{mG} \, (\tau_2 M')^4\right] \varepsilon_1 \leqslant \eta, \qquad \text{with} \quad 0 < \eta < \frac{[2\tau^* - \tau_2]}{4}.$$

Let $(\varphi, \tau_0) \in E_0$, with $|\tau_0 - \tau^*| + |\varphi(0)| \leqslant \varepsilon_1$. Denote by $(x(t), \tau(t))$ the solution of Eq. (1) with initial condition (φ, τ_0). Then from Lemma 4.4, we have for all $t \in [0, t_4^*]$,

$$|x(t)| \leqslant (\tau_2 M')^4 \, |\varphi(0)| \leqslant \eta,$$

and (10)

$$|\tau(t) - \tau^*| \leqslant e^{-mt} \, |\tau_0 - \tau^*| + \frac{1}{mG} \, (\tau_2 M')^4 \, |\varphi(0)| \leqslant \eta.$$

Moreover since $0 \leqslant \varphi(0) \leqslant r_1$, we deduce that $|x(t)| \leqslant r_1$, $\forall t \geqslant 0$, because $|x(t)| \leqslant \sup_{i \in \mathbb{N}} |x(t_i^*)|$, $\forall t \geqslant 0$, and $\tau_2 M \leqslant r_1$.

We either have $x(t) \geqslant x(0)/2$, for $-\tau_1/2 \leqslant t \leqslant 0$, or $\varphi(-\tau_1/2) \leqslant x(0)/2$. In the latter case, $0 \leqslant x(t) \leqslant x(0)/2$, for $-\tau_0 \leqslant t \leqslant -\tau_1/2$, so for all $t \in [0, \tau_1/2]$ and for all $M'' \geqslant M'$

$$\dot{x}(t) = -f(x(t-\tau(t))) \geqslant -M'' \frac{x(0)}{2},$$

and by integration we obtain

$$x(t) \geqslant x(0) - \frac{M''}{2} x(0)\, t, \qquad \text{for} \quad t \in \left[0, \frac{\tau_1}{2}\right] \text{ and all } M'' \geqslant M',$$

and

$$x(t) \geqslant \frac{x(0)}{2}, \qquad \text{for} \quad t \in \left[0, \frac{1}{M''}\right].$$

Putting the two cases together, we have $x(t) \geqslant \frac{x(0)}{2}$ in some interval of length

$$l = \min\left(\frac{1}{M''}, \frac{\tau_1}{2}\right),$$

contained in $[-\tau_1/2, \tau_1/2]$. We denote by $[t_1^{(0)}, t_2^{(0)}]$ such an interval. We denote by a_i, $i = 1, 2$, the solutions of the equation $t_i^{(0)} = a_i - \tau(a_i)$. We have

$$a_2 - a_1 \geqslant \frac{t_2^{(0)} - t_1^{(0)}}{L+1} = \frac{l}{L+1}; \qquad a_1 \geqslant \frac{\tau_1}{2},$$

where L is the constant introduced in assumption (H2).

For $t \in [a_1, a_2]$, we have $t - \tau(t) \in [t_1^{(0)}, t_2^{(0)}]$, so from (9) we have

$$|\dot{x}(t)| = |f(x(t-\tau(t)))| \geqslant \delta' \, |x(t-\tau(t))| \geqslant \delta' \frac{x(0)}{2}, \qquad \text{for} \quad t \in [a_1, a_2].$$

By integration of the above inequality, we deduce that there exists an interval of length $l/4(L+1)$ on which

$$|x(t)| \geqslant \frac{l}{4(L+1)} \delta' \frac{x(0)}{2}.$$

Set

$$\chi = \frac{1}{4(L+1)}.$$

With the above notations, we have $|x(t)| \geqslant \chi l \delta' \frac{x(0)}{2}$ for each t in an interval of length χl to the right of $\tau_1/2$.

Let $[t_1^{(1)}, t_2^{(1)}]$ be such an interval. Again, we denote by a_i, $i=1, 2$, the solutions of the equation $t_i^{(1)} = a_i - \tau(a_i)$, and we have $a_1 \leqslant a_2$ and $a_1 \geqslant t_1^{(1)} + \tau_1 \geqslant 3\tau_1/2$. Moreover, $a_2 - a_1 \geqslant \chi l/(L+1)$, and for each $t \in [a_1, a_2]$ we have $t - \tau(t) \in [t_1^{(1)}, t_2^{(1)}]$, so from (9) we have

$$|\dot{x}(t)| = |f(x(t-\tau(t)))| \geqslant \delta' \, |x(t-\tau(t))|$$

$$\geqslant (\delta')^2 \chi l \, \frac{x(0)}{2}, \qquad \text{for} \quad t \in [a_1, a_2].$$

By the same arguments, we deduce that there exists an interval $[t_1^{(2)}, t_2^{(2)}] \subset [a_1, a_2]$, $t_1^{(2)} \geqslant 3\tau_1/2$, such that $|x(t)| \geqslant \chi^3 l^{2\delta'2}(x(0)/2)$, for $t \in [t_1^{(2)}, t_2^{(2)}]$, with $t_2^{(2)} - t_1^{(2)} = \chi^2 l$.

By induction, we construct two sequences $\{t_i^{(j)}\}_{j \in \mathbb{N}}$, $i=1, 2$, such that

$$|x(t)| \geqslant \mu_j \, \frac{x(0)}{2}, \qquad \text{for all} \quad t \in [t_1^{(j)}, t_2^{(j)}],$$

with

$$\mu_{j+1} = \delta' l \chi^{j+1} \mu_j, \qquad \mu_0 = 1 \tag{11}$$

$$t_2^{(j)} - t_1^{(j)} = \chi^j l, \; t_1^{(j)} \geqslant \tau_1\left(\tfrac{1}{2} + (j-1)\right) \tag{12}$$

and

$$t_1^{(j-1)} \leqslant t_1^{(j)} - \tau(t_1^{(j)}) \leqslant t_2^{(j)} - \tau(t_2^{(j)}) \leqslant t_2^{(j-1)}. \tag{13}$$

From (12) there exists $j_0 \in \mathbb{N}$ such that

$$t_1^{(j_0-1)} \leqslant t_1^* \leqslant t_1^{(j_0)}.$$

So, either $(t_1^{(j_0-1)} > t_1$ or $t_1^{(j_0)} < t_2)$ or $(t_1^{(j_0-1)} \leqslant t_1$ and $t_1^{(j_0)} \geqslant t_2)$. Let us first examine the second situation, that is, $t_1^{(j_0-1)} \leqslant t_1$ and $t_1^{(j_0)} \geqslant t_2$. We will show that this situation cannot occur. Assume that this situation occurs, we then have from (13)

$$t_1^{(j_0-1)} \leqslant t_1 \leqslant t_1 + \tau_1 \leqslant t_1^* \leqslant t_2 \leqslant t_1^{(j_0)} \leqslant t_2^{(j_0-1)} + \tau(t_1^{(j_0)}) \leqslant t_1^{(j_0-1)} + l + \tau_2. \tag{14}$$

Assuming that $t_2^* \leqslant t_1^{(j_0)}$, we would have from (10)

$$t_1^{(j_0)} - t_1^{(j_0-1)} \geqslant t_2^* - t_2 + t_1^* - t_1 \geqslant \tau(t_2^*) + \tau(t_1^*) \geqslant 2\tau^* - 2\eta.$$

On the other hand, from (14) we know that

$$t_1^{(j_0)} - t_1^{(j_0-1)} \leqslant \tau_2 + l = \tau_2 + \min\left(\frac{1}{M''}, \frac{\tau_1}{2}\right),$$

with $M'' \geqslant M'$. We obtain

$$2\tau^* - \tau_2 \leqslant 2\eta + \min\left(\frac{1}{M''}, \frac{\tau_1}{2}\right),$$

and since $\eta < [2\tau^* - \tau_2]/4$, we obtain a contradiction by taking

$$\frac{1}{M''} = \frac{[2\tau^* - \tau_2]}{4q_0},$$

with $q_0 \geqslant 1$ large enough.

So, we have $t_2^* \geqslant t_1^{(j_0)} \geqslant t_2$. In this case $\dot{x}(t_1^{(j_0)}) \geqslant 0$, so $f(x(t_1^{(j_0)} - \tau(t_1^{(j_0)}))) \leqslant 0$, and we deduce that

$$x(t_1^{(j_0)} - \tau(t_1^{(j_0)})) \leqslant 0.$$

From this inequality, we deduce that $t_1^{(j_0)} - \tau(t_1^{(j_0)}) \leqslant t_2$, and as $t_1^{(j_0)} \geqslant t_1^*$ we have $t_1^{(j_0)} - \tau(t_1^{(j_0)}) \geqslant t_1$. So

$$t_1^{(j_0-1)} \leqslant t_1 \leqslant t_2^{(j_0-1)},$$

which yields a contradiction since $x(t_1) = 0$ while, by construction $|x(t)|$ is > 0 at each point of $[t_1^{(j_0-1)}, t_2^{(j_0-1)}]$. We conclude that the second situation can not occur.

In the first situation, we have $t_1^{(j_0)} \leqslant t_2$ or $t_1^{(j_0-1)} \geqslant t_1$. Together with $t_1^{(j_0-1)} \leqslant t_1^* \leqslant t_1^{(j_0)}$, we have either

$$|x(t_1^*)| \geqslant |x(t_1^{(j_0-1)})| \geqslant \mu_{j_0-1} \frac{x(0)}{2},$$

or

$$|x(t_1^*)| \geqslant |x(t_1^{(j_0)})| \geqslant \mu_{j_0-1} \frac{x(0)}{2}.$$

To conclude, it remains to remark that from Lemma 2.2 we have $t_1^* \leqslant \tau_2 + T(r_1)$, and from (12), there exists an integer $j_{\max} \in \mathbb{N}$ such that $0 \leqslant j_0 \leqslant j_{\max}$. Finally, by setting

$$C_1^{-1} = \tfrac{1}{2} \min_{0 \leqslant j \leqslant j_{\max} : t_1^{(j-1)} \leqslant t_1^*} \mu_j \geqslant \min_{0 \leqslant j \leqslant j_{\max}} \mu_j > 0$$

we have

$$x(0) \leqslant C_1 |x(t_1^*)|.$$

It remains to prove the second part of the lemma. Let $h \in \widetilde{h}_{\tau_1}$. By construction we have

$$C_1^{-1} = \tfrac{1}{2} \min_{0 \leqslant j \leqslant j_{\max}\,:\, t_1^{(j-1)} \leqslant t_1^*} \mu_j,$$

and for $0 \leqslant j \leqslant j_{\max}$ such that $t_1^{(j-1)} \leqslant t_1^*$ we have

$$\mu_j = \tfrac{1}{2}(\delta')^j \, l^j \chi^{(j+1)+j+(j-1)+\cdots+1} = \tfrac{1}{2}(\delta')^j \, l^j \chi^{(j+2)/2}$$

with

$$\chi = \frac{1}{4(L+1)},$$

$$l = \min\left(\frac{1}{M''}, \frac{\tau_1}{2}\right).$$

We can take $M'' > 0$ large enough to have $l = \frac{1}{M''}$ and $l \leqslant 1$; we also can assume that $\delta' \leqslant 1$. From Equation (12), we have

$$\tau_1\left(\tfrac{1}{2} + (j-2)\right) \leqslant t_1^{(j-1)} \leqslant t_1^* \leqslant \tau_2 + T(r_1),$$

and since we have supposed that $l_1 \tau_1 > \tau_2$, we have

$$j \leqslant \frac{\tau_2 + T(r_1)}{\tau_1} + 3 \leqslant l_1 + \frac{T(r_1)}{\tau_1} + 3 \leqslant 4 l_1 + \frac{1}{\tau_1} \frac{r_1 - r}{C_{r, r_1}} = \kappa_1(\tau_1),$$

where $C_{r,R} = \inf\{f(s) : s \in [r, R]\} > 0$. So, we have

$$e^{m\tau_1} C_1^{-1} \geqslant e^{m\tau_1} \frac{1}{2} (\delta')^j \left(\min\left(\frac{1}{M''}, \frac{\tau_1}{2}\right)\right)^j \left(\frac{1}{4(L+1)}\right)^{(j+2)/2}.$$

So for $\tau_1 > 0$ large enough, and since $1/M'' = [2\tau^* - \tau_2]/4q_0 \geqslant \xi_0/4q_0\tau_1^{l_0}$, and as we have supposed that $m \geqslant \xi_1 \tau_1^{\xi_2}$, with $\xi_1 > 0$, and $\xi_2 > -1$, we have

$$e^{m\tau_1} C_1^{-1} \geqslant e^{\xi_1 \tau_1^{1+\xi_2}} \frac{1}{2} (\delta')^{\kappa_1(\tau_1)} \left(\frac{\xi_0}{4 q_0 \tau_1^{l_0}}\right)^{\kappa_1(\tau_1)} \left(\frac{1}{4(L+1)}\right)^{(\kappa_1(\tau_1)+2)/2}. \quad (15)$$

Now it is not difficult to see that the right side of the previous inequality goes to infinity when τ_1 goes to infinity. So, there exists a certain $\tau_1^* > 0$ such that

$$\tau_1 > \tau_1^* \Rightarrow e^{m\tau_1} C_1^{-1} > 1.$$

Finally, using Eq. (15), it is not difficult to see that there exists $m^* > 0$, such that for all $h \in \widetilde{h}_m$,

$$m > m^* \Rightarrow e^{-m\tau_1} C_1 < 1. \quad \blacksquare$$

LEMMA 4.7. *Assume* (H1) *through* (H6) *hold*, $f'(0)\,\tau_2 > 1$, $h \in \widetilde{h}_{\tau_1}$ *(respectively,* $h \in \widetilde{h}_m$*), and assume that* $\tau_1 > \tau_1^*$ *(respectively,* $m > m^*$*).*

Then, for each $p_0 \geqslant 1$, *there exist* $\varepsilon_1 > 0$ *and* $C_2 > 0$ *such that for each* $\varepsilon_0 > 0$ *and each* $(\psi^{(0)}, \tau_0^{(0)}) \in E_1$ *satisfying*

$$\psi^{(0)}(0) > 0, \qquad \text{and} \qquad |\tau_0^{(n)} - \tau^*| + |\psi^{(n)}(0)| \leqslant \varepsilon_1,\ \forall n \in \mathbb{N}$$

(where $(\psi^{(n)}, \tau_0^{(n)}) = \widetilde{F}^n_{2p_0, \varepsilon_0}(\psi^{(0)}, \tau_0^{(0)})$*), there exists* $n_1 \geqslant 0$ *such that*

$$\begin{aligned}|\tau_0^{(n+1)} - \tau^*| &\leqslant |\tau(\varphi^{(n)}, \tau_0^{(n)})(t^*_{2p_0}(\varphi^{(n)}, \tau_0^{(n)})) - \tau^*| \\ &\leqslant C_2 |\varphi^{(n+1)}(0)|,\ \forall n \geqslant n_1,\end{aligned}$$

with $(\varphi^{(n)}, \tau_0^{(n)}) = Q(\psi^{(n)}, \tau_0^{(n)})$.

Proof. From Lemma 4.6, there exists $\varepsilon_1 > 0$ and $C_1 > 0$ such that for all $(\varphi, \tau_0) \in E$,

$$|\tau_0 - \tau^*| + |\varphi(0)| \leqslant \varepsilon_1 \Rightarrow |\varphi(0)| \leqslant C_1^{2p_0}\, |x(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0))|. \tag{16}$$

Let $(\psi^{(0)}, \tau_0^{(0)}) \in E_1$, satisfying for each $n \in \mathbb{N}$

$$\psi^{(0)}(0) > 0, \qquad \text{and} \qquad |\tau_0^{(n)} - \tau^*| + |\psi^{(n)}(0)| \leqslant \varepsilon_1,$$
$$\text{where } (\psi^{(n)}, \tau_0^{(n)}) = \widetilde{F}^n_{2p_0, \varepsilon_0}(\psi^{(0)}, \tau_0^{(0)}),$$

and denote for each $n \in \mathbb{N}$

$$(\varphi^{(n)}, \tau_0^{(n)}) = Q(\psi^{(n)}, \tau_0^{(n)}).$$

From Lemma 4.4 we have

$$|\tau(\varphi^{(0)}, \tau_0^{(0)})(t^*_{2p_0}) - \tau^*| \leqslant e^{-mt^*_{2p_0}}\, |\tau_0^{(0)} - \tau^*| + \frac{1}{mG}\, (\tau_2 M')^{2p_0}\, |\varphi^{(0)}(0)|,$$

and by construction we have $t^*_{2p_0} \geqslant 2p_0 \tau_1$ and

$$|\tau(\varphi^{(0)}, \tau_0^{(0)})(t^*_{2p_0}) - \tau^*| \leqslant e^{-m2p_0\tau_1} |\tau_0^{(0)} - \tau^*| + \frac{1}{mG} (\tau_2 M')^{2p_0} |\varphi^{(0)}(0)|. \quad (17)$$

Let $\bar{u} > 0$ such that

$$\left[C_1^{2p_0} e^{-m2p_0\tau_1} \frac{1}{\bar{u}} + C_1^{2p_0} \frac{1}{m} G(\tau_2 M')^{2p_0} \right] = \frac{1}{\bar{u}}.$$

Solving for $\bar{u}$ is possible, since we have assumed that $\tau_1 > \tau_1^*$ (respectively $m > m^*$), and so $C_1^{2p_0} e^{-m2p_0\tau_1} < 1$.

Then as $\varphi^{(0)}(0) = \psi^{(0)}(0) > 0$, there exists $0 < \gamma_0 \leqslant \bar{u}$ such that $\gamma_0 |\tau_0 - \tau^*| \leqslant |\varphi(0)|$, and using (16–18), we have

$$|\tau(\varphi^{(0)}, \tau_0^{(0)})(t^*_{2p_0}) - \tau^*|$$
$$\leqslant \left[e^{-m2p_0\tau_1} \frac{1}{\gamma_0} + \frac{1}{m} G(\tau_2 M')^{2p_0} \right] C_1^{2p_0} |x(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0))|.$$

Moreover, by construction of $\tilde{F}_{2p_0, \varepsilon_0}$ we have

$$\tau_0^{(1)} = \mu\tau(\varphi^{(0)}, \tau_0^{(0)})(t^*_{2p_0}) + (1-\mu)\, \tau^*$$

for a certain $\mu \in [0, 1]$ and

$$x(\varphi, \tau_0)(t^*_{2p_0}(\varphi, \tau_0)) = \psi^{(1)}(0).$$

We obtain

$$|\tau_0^{(1)} - \tau^*| \leqslant \left[e^{-m2p_0\tau_1} \frac{1}{\gamma_0} + \frac{1}{m} G(\tau_2 M')^{2p_0} \right] C_1^{2p_0} |\psi^{(1)}(0)|.$$

By induction, we obtain

$$|\tau_0^{(n)} - \tau^*| \leqslant \frac{1}{u_n} |\psi^{(n)}(0)|.$$

where the sequence $\{u_n\}_{n \geqslant 0}$ is defined by

$$\frac{1}{u_{n+1}} = \left[e^{-m2p_0\tau_1} \frac{1}{u_n} + \frac{1}{m} G(\tau_2 M')^{2p_0} \right] C_1^{2p_0}, \quad \forall n \geqslant 0, \text{ with } u_0 = \gamma_0.$$

The previous difference equation can be rewritten in the following manner:

$$u_{n+1} = \frac{u_n}{[\theta_1 + \theta_2 u_n]}, \qquad \text{for all} \quad n \geqslant 0, \text{ with } u_0 = \gamma_0, \tag{18}$$

where $\theta_1 = C_1^{2p_0} e^{-m2p_0\tau_1}$, $0 < \theta_1 < 1$, $\theta_2 = C_1^{2p_0} \frac{1}{m} G(\tau_2 M')^{2p_0} > 0$, and $0 < \gamma_0 \leqslant \bar{u}$.
It is not difficult to see that since $0 < \theta_1 < 1$, we have

$$\lim_{m \to +\infty} u_n = \bar{u}.$$

So it is sufficient to take $C_2 = \frac{\bar{u}}{2}$, and the result follows. ▮

The following lemma can be found in Hale [4].

Lemma 4.8. *For each $\tau^* > 0$, such that $\tau^* f'(0) > \frac{\pi}{2}$, the characteristic equation associated to the linear equation*

$$\frac{dx}{dt} = -f'(0)\, x(t - \tau^*) \tag{19}$$

has roots with positive real part.

In the following, we will always assume that $\tau^* f'(0) > \frac{\pi}{2}$, and from the previous lemma the characteristic equation of (19) has two dominant roots,

$$\lambda_{\pm} = \alpha \pm \beta,$$

with $\alpha > 0$, $\beta > 0$. We denote by $U = \text{vect}\{e^{\alpha\theta}\cos(\beta\theta), e^{\alpha\theta}\sin(\beta\theta)\}$ the corresponding eigenspace. Let us decompose $C([-\tau^*, 0]) = U \oplus V$ in the usual manner, and let Π_U be the usual projection on U. Let us denote

$$\Gamma_2 = \{\varphi \in C([-\tau^*, 0]) : \varphi(s) \geqslant 0 \text{ on } [-\tau^*, 0] \text{ and } \varphi \text{ is non-decreasing}\}.$$

The following lemma can be found in Hale [4].

Lemma 4.9. *Assume that $\tau^* f'(0) > \frac{\pi}{2}$. Then*

$$\inf_{\varphi \in \Gamma_2,\ \varphi(0) = 1} |\Pi_U(\varphi)| > 0.$$

We denote

$$\gamma_1 = \inf_{\varphi \in \Gamma_2, \varphi(0) = 1} |\Pi_U(\varphi)| \qquad \text{and} \qquad \gamma_2 = \sup_{\varphi \in C([-\tau^*, 0]),\ \|\varphi\|_\infty = 1} |\Pi_U(\varphi)|.$$

The following result shows that $(0, \tau^*)$ is a semi-ejective fixed point of $\tilde{F}_{2p_0, \varepsilon_0}$ on $C \backslash \{0\} \times [\tau_1, \tau_2]$, and this completes the proof of Theorem 1.1.

PROPOSITION 4.10. *Assume* (H1) *through* (H6), $f'(0)\,\tau_2>1$, *and let* $h\in\tilde{h}_{\tau_1}$ (*respectively* $h\in\tilde{h}_m$), *and assume that* $\tau_1>\tau_1^*$ (*respectively* $m>m^*$). *Then for each* ?$_0\in\,]0,\frac{1}{C_2}[$, $(0,\tau^*)$ *is a semi-ejective fixed of* $\tilde{F}_{2,\varepsilon_0}$ *on* $C\backslash\{0\}\times[\tau_1,\tau_2]$.

Proof. Let $h\in\tilde{h}_{\tau_1}$ (respectively, $h\in\tilde{h}_m$) and assume that $\tau_1>\tau_1^*$ (respectively $m>m^*$). Let $\varepsilon_1>0$, such that the conclusions of Lemmas 4.6 and 4.7 hold when $p_0=1$, and when $p_0=\tilde{p}_0$ with $\tilde{p}_0\geqslant 1$ such that

$$\frac{\gamma_1}{\gamma_2}\,e^{2\alpha\tau_1\tilde{p}_0}>1.$$

Assume that $(0,\tau^*)$ is not a semi-ejective fixed point of $\tilde{F}_{2,\varepsilon_0}$ on $C\backslash\{0\}\times[\tau_1,\tau_2]$. Then for each $\varepsilon\in\,]0,\varepsilon_1]$, there exists $(\psi^{(0)},\tau_0^{(0)})\in\tilde{C}$, with $\|\psi^{(0)}\|_1+|\tau_0^{(0)}-\tau^*|\leqslant\varepsilon$, such that

$$d_1(\tilde{F}^n_{2,\varepsilon_0}(\psi^{(0)},\tau_0^{(0)}),(0,\tau^*))\leqslant\varepsilon,\qquad \forall n\in\mathbb{N}. \tag{20}$$

We set for each $n\in\mathbb{N}$

$$(\psi^{(n)},\tau_0^{(n)})=\tilde{F}^n_{2,\varepsilon_0}(\psi^{(0)},\tau_0^{(0)})\qquad\text{and}\qquad(\varphi^{(n)},\tau_0^{(n)})=Q(\psi^{(n)},\tau_0^{(n)}).$$

Then from Lemma 4.6, there exists $C_1>0$ such that

$$|\varphi^{(n)}(0)|\leqslant C_1^2\,|\varphi^{(n+1)}(0)|,\qquad\forall n\in\mathbb{N},$$

and from Lemma 4.7, there exists $C_2>0$ and $n_1\in\mathbb{N}$ such that

$$|\tau_0^{(n)}-\tau^*|\leqslant C_2\,|\varphi^{(n)}(0)|,\qquad\forall n\geqslant n_1.$$

In the following, we will assume that $n_1=0$, and the problem is unchanged because we can replace $(\psi^{(0)},\tau_0^{(0)})$ by $(\psi^{(n_1)},\tau_0^{(n_1)})$. Moreover, from Eq. (20), we also have

$$|\tau_0^{(n)}-\tau^*|+|\varphi^{(n)}(0)|\leqslant\varepsilon,\qquad\forall n\in\mathbb{N}. \tag{21}$$

Moreover, since C_1, and C_2 are fixed independently of $\varepsilon_0>0$, we can choose ε_0 in $]0,\frac{1}{C_2}[$. In this case by definition of $\tilde{F}_{2,\varepsilon_0}$ we have

$$|\tau_0^{(n)}-\tau^*|\leqslant C_2\,|\varphi^{(n)}(0)|\Rightarrow\tilde{F}_{2,\varepsilon_0}(\psi^{(n)},\tau_0^{(n)})=F_2(\psi^{(n)},\tau_0^{(n)}).$$

One can apply Lemma 3.2, and we deduce that for all $p\geqslant 1$ and all $n\geqslant 0$,

$$\tilde{F}^p_{2,\varepsilon_0}(\psi^{(n)},\tau_0^{(n)})=F_2^p(\psi^{(n)},\tau_0^{(n)})=F_{2p}(\psi^{(n)},\tau_0^{(n)}). \tag{22}$$

Denote for each $n \in \mathbb{N}, m \geqslant 1$

$$x^{(n)}(t) = x(\varphi^{(n)}, \tau_0^{(n)})(t),\ x_+^{(n)}(t) = x(\varphi_+^{(n)}, \tau_0^{(n)})(t), \qquad \forall t \geqslant -\tau_2,$$
$$\tau^{(n)}(t) = \tau(\varphi^{(n)}, \tau_0^{(n)})(t),\ \tau_+^{(n)}(t) = \tau(\varphi_+^{(n)}, \tau_0^{(n)})(t), \qquad \forall t \geqslant 0,$$

and

$$t^*_{2m,\,n} = t^*_{2m}(\varphi^{(n)}, \tau_0^{(n)}).$$

From assertion (22), one has, for all $n \geqslant 0, m \geqslant 0$,

$$x_+^{(n)}(t^*_{2m,\,n}) = \varphi^{(n+m)}(0), \tag{23}$$

where $\varphi_+^{(n)}$ is defined by

$$\varphi_+^{(n)}(s) = 0, \qquad \text{on} \quad [-\tau_2, -\tau_0^{(n)}],$$

and

$$\varphi_+^{(n)}(s) = \varphi^{(n)}(s), \qquad \text{on} \quad [-\tau_0^{(n)}, 0].$$

Then, from Lemma 3.1, we have

$$x^{(n)}(t) = x_+^{(n)}(t),\ \forall t \geqslant -\tau_0^{(n)}, \qquad \text{and} \qquad \tau^{(n)}(t) = \tau_+^{(n)}(t),\ \forall t \geqslant 0,$$

so

$$t^*_{2,\,n} = t^*_2(\varphi^{(n)}, \tau_0^{(n)}) = t^*_2(\varphi_+^{(n)}, \tau_0^{(n)}), \qquad \forall n \in \mathbb{N},$$

and

$$\varphi^{(n+1)}(0) = \psi^{(n+1)}(0) = x_+^{(n)}(t^*_{2,\,n}), \qquad \forall n \in \mathbb{N}.$$

Moreover, $\forall t \geqslant 0$,

$$\frac{dx_+^{(n)}(t)}{dt} = -f(x_+^{(n)}(t - \tau_+^{(n)}(t)))$$

$$\begin{aligned}\frac{dx_+^{(n)}(t)}{dt} = {} & -f'(0)\, x_+^{(n)}(t-\tau^*) \\ & + [f'(0)\, x_+^{(n)}(t-\tau^*) - f(x_+^{(n)}(t-\tau^*))] \\ & + [f(x_+^{(n)}(t-\tau^*)) - f(x_+^{(n)}(t - \tau_+^{(n)}(t)))].\end{aligned}$$

Then, by using Lemmas 4.4 and 4.6, one can prove that for all $n \geqslant 1$ and all $0 \leqslant t \leqslant t^*_{2\tilde{p}_0, n}$,

$$\frac{dx^{(n)}_{+}}{dt}(t) = -f'(0)\, x^{(n)}_{+}(t-\tau^*) + o(\varphi^{(n)}(0)). \tag{24}$$

From now on, we denote $x_t = x|_{[t-\tau^*, t]}$. Then, by projecting $x^{(n)}_{+t}$ onto U, and denoting $y^{(n)}(t) = \Pi_U\, x^{(n)}_{+t}$, the above equation leads to an ordinary differential equation with a forcing term (see Hale [4]), namely,

$$\frac{dy^{(n)}}{dt}(t) = A_U y^{(n)} + o(\varphi^{(n)}(0))\, \Pi_U(X_0),$$

where X_0 is the integral of the Dirac distribution δ_0 (see Hale [4]). Select a basis of U. Then the vectors of U are represented by their components on the basis and A_U by a (2×2)-matrix. We can choose the basis in such a way that

$$A_U = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Using the canonical scalar product on $\mathbb{R}^2$ and taking the scalar product of the above equation with $y^{(n)}(t)$, we then arrive at

$$\begin{aligned} \frac{d}{dt}\, |y^{(n)}(t)|^2 &= 2(y^{(n)}(t), \dot{y}^{(n)}(t)) \\ &= 2\alpha\, |y^{(n)}(t)|^2 + o(\varphi^{(n)}(0))\, 2(y^{(n)}(t), \Pi_U(X_0)). \end{aligned}$$

But we have $\forall t \in [0, t^*_{2\tilde{p}_0, n}]$,

$$\begin{aligned} (y^{(n)}(t), \Pi_U(X_0)) &\leqslant C_5\, |y^{(n)}(t)| \\ &\leqslant C_5 \gamma_2\, \|x^n_{+t}\|_{\infty, [-\tau_2, t^*_{2\tilde{p}_0, n}]} \leqslant C_5 \gamma_2 (\tau_2 M')^{2\tilde{p}_0}\, |\varphi^{(n)}(0)|, \end{aligned}$$

for a certain $C_5 > 0$. So, $\forall t \in [0, t^*_{2\tilde{p}_0, n}]$,

$$\frac{d}{dt}\, |y^{(n)}(t)|^2 = 2\alpha\, |y^{(n)}(t)|^2 + o(\varphi^{(n)}(0)^2).$$

Thus, by integrating, we obtain $\forall t \in [0, t^*_{2\tilde{p}_0, n}]$,

$$\begin{aligned} |y^{(n)}(t)|^2 &\geqslant e^{2\alpha t}\, |y^{(n)}(0)|^2 - |o(\varphi^{(n)}(0)^2)| \int_0^t e^{2\alpha(t-s)}\, ds, \\ &\geqslant e^{2\alpha t} \left[|y^{(n)}(0)|^2 - \frac{1}{2\alpha}\, |o(\varphi^{(n)}(0)^2)| \right]. \end{aligned}$$

Now, as $\varphi_{+}^{(n)} \in \Gamma_2$, we may apply Lemma 4.9 and we have

$$|y^{(n)}(0)|^2 \geqslant \gamma_1^2 \, |\varphi^{(n)}(0)|^2,$$

so

$$|y^{(n)}(t^*_{2\tilde{p}_0, n})|^2 \geqslant e^{2\alpha 2\tilde{p}_0 \tau_1} \left[\gamma_1^2 - \frac{1}{2\alpha} \, |o(1)| \right] \, |\varphi^{(n)}(0)|^2.$$

Let us remark that

$$y^{(n)}(t^*_{2\tilde{p}_0, n}) = \Pi_U(x^{(n)}_{+t^*_{2\tilde{p}_0, n}}) = \Pi_U(z_1^{(n)}) + \Pi_U(z_2^{(n)}),$$

with

$$\begin{aligned} z_1^{(n)}(s) &= x^{(n)}_{+t^*_{2\tilde{p}_0, n}}(s), \qquad && \text{on} \quad [-\tau^*, 0] \cap [-\tau_+^{(n)}(t^*_{2\tilde{p}_0, n}), 0], \\ z_1^{(n)}(s) &= 0, && \text{elsewhere}, \end{aligned}$$

and

$$z_2^{(n)}(s) = x^{(n)}_{+t^*_{2\tilde{p}_0, n}}(s) - z_1^{(n)}(s), \qquad \text{on } [-\tau^*, 0].$$

We have

$$z_1^{(n)} \in \Gamma_2, \qquad \text{and} \qquad z_1^{(n)}(0) = x_+^{(n)}(t^*_{2\tilde{p}_0, n}),$$

so

$$|\Pi_U(z_1^{(n)})| \leqslant \gamma_2 \, |x_+^{(n)}(t^*_{2\tilde{p}_0, n})| \leqslant \gamma_2 \, |\varphi^{(n+\tilde{p}_0)}(0)|.$$

For convenience, we recall the formula of the formal dual product: for $\psi \in C([0, \tau^*])$, $\varphi \in C([-\tau^*, 0])$, we have

$$\langle \psi, \varphi \rangle = \psi(0) \, \varphi(0) - f'(0) \int_{-\tau^*}^{0} \psi(\xi + \tau^*) \, \varphi(\xi) \, d\xi.$$

By construction, we have $z_2^{(n)}(0) = 0$, and the support of $z_2^{(n)}(0)$ is contained in an interval of length less than or equal to $|\tau^* - \tau_+^{(n)}(t^*_{2\tilde{p}_0, n})|$, so from the form of the formal dual product, we deduce that

$$|\Pi_U(z_2^{(n)})| \leqslant \gamma_2 \, |\tau^* - \tau_+^{(n)}(t^*_{2\tilde{p}_0, n})| \, \|x^{(n)}_{+t^*_{2\tilde{p}_0, n}}\|_{\infty, [-\tau_2, \tau_+^{(n)}(t^*_{2\tilde{p}_0, n})]},$$

and as above, by using Lemmas 4.4, 4.6, and 4.7, we have

$$|\Pi_U(z_2^{(n)})| = |o(\varphi^{(n)}(0))|.$$

So, finally we obtain

$$[\gamma_2^2 |\varphi^{(n+\tilde{p}_0)}(0)|^2 + |o(\varphi^{(n)}(0))|]^2 \geqslant e^{2\alpha 2\tilde{p}_0 \tau_1} \left[\gamma_1 - \frac{1}{2\alpha} |o(1)| \right] |\varphi^{(n)}(0)|^2.$$

Let $n_0 \geqslant 1$ be fixed. Since the sequence $\{\varphi^{(n_0+q\tilde{p}_0)}(0)\}_{q\geqslant 0}$ is bounded, for each $1 < C \leqslant 2$ there exists an integer $q_0 \in \mathbb{N}$ such that

$$C\varphi^{(n_0+q_0\tilde{p}_0)}(0) \geqslant \varphi^{(n_0+(q_0+1)\tilde{p}_0)}(0),$$

so we obtain for all $1 < C \leqslant 2$,

$$[\gamma_2^2 C + o(1)]^2 \geqslant e^{2\alpha 2\tilde{p}_0 \tau_1} \left[\gamma_1^2 - \frac{1}{2\alpha} |o(1)| \right],$$

and when $\varepsilon \to 0$, $C \to 1$, we obtain a contradiction with

$$\frac{\gamma_1}{\gamma_2} e^{2\alpha\tilde{p}_0\tau_1} > 1. \quad \blacksquare$$

5. CONCLUSION

In this paper, we have extended the result by Arino *et al.* concerning the existence of slowly oscillating periodic solutions, thus providing a significant improvement with respect to the assumptions made in [2]. The improvements are mainly due to the use of two different types of arguments. First, we consider solutions starting from initial values in the subset

$$K = \{(\varphi, \tau_0) \in C([-\tau_2, 0], \mathbb{R}) \times [\tau_1, \tau_2] : \varphi(-\tau_0) = 0,$$
$$\text{and } \varphi \text{ is increasing on } [-\tau_0, 0]\}.$$

This special class of initial values was already introduced in the paper by Kuang and Smith [7]. Here, this remark is useful because $t - \tau(t)$ is increasing, and this allows us to consider the Poincaré operator on a subset bigger than the one introduced in Arino *et al.* [2]. In this framework, we are able to relax two fundamental restrictions made in Arino *et al.* [2] as follows: (1) Assumption (H7) is only optional here and, is only necessary if we want to ensure that the periodic solutions (whose existence is ascertained by Theorem 1.1) are slowly oscillating. (2) The condition

$(\tau_2 - \tau_1)(H(\tau^*) + 1) < 1$ given in Arino *et al.* [2] to ensure that the projection onto the linear unstable manifold is nondegenerate is not needed here (see proposition 6.27 in Arino *et al.* [2]).

Another improvement in this paper concerns the technique used in proving the existence of the nontrivial fixed point of the Poincaré operator. Here the main problem is the lack of ejectivity of the trivial fixed point. In order to encompass this problem, we employ the notion of semi-ejectivity which extends the previous notion of ejectivity. Finally, by applying a semi-ejective fixed point theorem proved in Magal and Arino [9] we have obtained the existence of a nontrivial fixed point for the Poincaré operator.

The values $\tau_1^* > 0$ and $m^* > 0$ of Theorem 1.1 are given by the formula (15). So those constants are fully specified. On the other hand, if we introduce a multiplicative factor $\lambda > 1$ before f in Eq. (1), then the results are unchanged. This result has to be compared with the constant delay case,

$$x'(t) = -\lambda f(x(t - \tau^*)),$$

in which slowly oscillating periodic solutions are shown to exist for all $\lambda > (\pi/2)(1/\tau^* f'(0))$. Here we reach the same result. Indeed the constraints given in the conclusion of Theorem 1.1 are related to the state dependent delay. Also, when $b - a$ goes to zero the statement obtained in Corollary 1.3 approaches the corresponding constant delay result.

In this paper we concentrated on explaining a technique that can be used to prove the existence of periodic solutions, the semi-ejective fixed point theorem. We left untouched all other dynamical aspects associated with the system.

With regard the stability of the trivial equilibrium solution $(0, \tau^*)$, one may verify that when $\lambda = \tau^* f'(0)) < \frac{\pi}{2}$ the linearized equation is stable, and the same holds for the state dependent delay equation.

Let us now consider the super-critical case for the Hopf bifurcation of the fixed delay equation. In this situation, when $\lambda = \tau^* f'(0) > \frac{\pi}{2}$ (close enough to $\frac{\pi}{2}$) the fixed delay equation

$$x'(t) = -f(x(t - \tau^*))$$

and the state dependent delay equation both have slowly oscillating solutions. The point $\lambda = \tau^* f'(0) = \frac{\pi}{2}$ is a Hopf bifurcation point for both equations. Moreover, for $\lambda = \tau^* f'(0) > \frac{\pi}{2}$ (close enough to $\frac{\pi}{2}$) both equations have the same characteristics and the bifurcations are on the same side. So we can conclude that if the bifurcating solution of the constant delay equation is stable, it is the same for the state dependent delay equation (and the converse). But since we do not know that the bifurcation branch is the one defined by the periodic solutions of Theorem 1.1 we cannot reach a conclusion on the stability of these periodic solutions.

Numerically, we have dealt with an example which verifies the assumptions of Theorem 1.1,

$$x'(t) = -f(x(t-\tau(t)))$$

$$\tau'(t) = 10f(x(t)) - 100(\tau(t) - \tau^*),$$

where $f(x) = xe^{-|x|}$ and $1.5 \leqslant \tau^*$.

For numerical simulations, we have used the initial values

$$x(t) = -\cos(t) + \mathrm{Tr}, \qquad \forall t \in [-\tau_2, 0],$$

$$\tau(0) = \tau^*,$$

where Tr is a parameter representing a shift of the origin.

When $\lambda = \tau^* f'(0) = \tau^* > \frac{\pi}{2}$, we observe globally stable periodic oscillating solutions (see Figs. 1 and 2 where $\tau^* = 10$ and $\mathrm{Tr} = -3$, 0, or 3).

We finally observe (see Fig. 3) that when $\lambda = \tau^* f'(0) = \tau^* \to \frac{\pi}{2}$ the nontrivial periodic solution tends to zero. It is possible to see that the periodic solutions found are on the bifurcation branch and that the branch is super-critical. In both the constant delay case and the time dependent delay case one observes numerically a Hopf bifucation (see Fig. 3 in which $1.5 \leqslant \tau^* \leqslant 10$ and $\mathrm{Tr} = 0$).

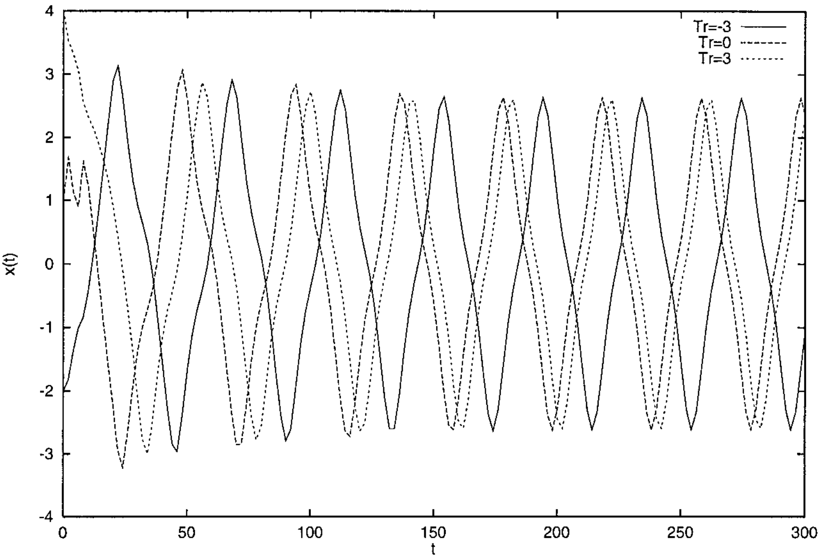

FIG. 1. First component of the solution.

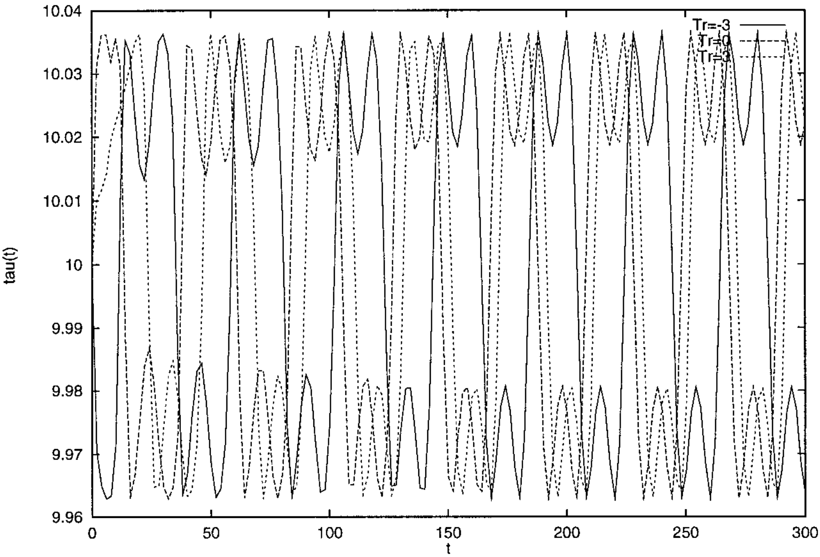

FIG. 2. Time-dependent delay.

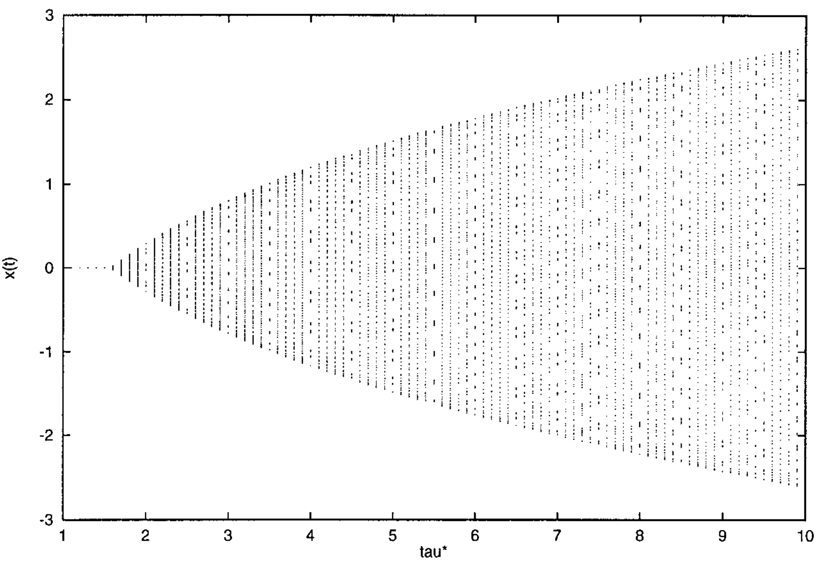

FIG. 3. Hopf bifurcation graph, each vertical line representing the periodic orbit reached asymptotically from an arbitrary initial value.

REFERENCES

1. W. Alt, "Periodic Solutions of Some Autonomous Differential Equations with Variable Time Delay," Lecture Notes in Mathematics, Vol. 730, pp. 16–31, Springer-Verlag, New York/Berlin, 1979.
2. O. Arino, K. P. Hadeler, and M. L. Hbid, Existence of periodic solutions for delay differential equations with state dependent delay, *J. Differential Equations* **144**, No. 2 (1998), 263–301.
3. F. Browder, A further generalization of the Schauder fixed point theorem, *Duke Math. J.* **32** (1965), 575–578.
4. J. K. Hale, "Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.
5. J. K. Hale and S. M. Verduyn Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, New York/Berlin, 1993.
6. Y. Kuan and H. L. Smith, Slowly oscillating periodic solutions of autonomous state-dependent delay equations, *Nonlinear Anal. Theory Methods Appl.* **19**, No. 9 (1992), 855–872.
7. Y. Kuan and H. L. Smith, Periodic solutions of differential delay equations with threshold-type delays, oscillations and dynamics in delay equations, *Contemp. Math.* **129** (1992).
8. P. Magal, "Contribution à l'étude des systèmes dynamiques dicrets préservant un cône et application à des modèles de dynamiques de population," Ph.D. thesis, Pau, France, 1996.
9. P. Magal and O. Arino, A semi-ejective fixed point theorem, working paper, 1998.
10. J. Mallet-Paret and R. Nussbaum, Boundary layer phenomena for differential delay with state-dependent time lags, *Arch. Rational Mech. Anal.* **120** (1992), 99–146.
11. J. Mallet-Paret, R. Nussbaum, and P. Paraskevopulos, Periodic solutions to functional differential equations with multiple state-dependent time lags, *in* "Topological Methods in Nonlinear Analysis," Vol. 3, pp. 101–162, J. Juliusz Schauder Center, 1994.
12. R. Nussbaum, Periodic solutions of some nonlinear autonomous functional differential equations, *Ann. Math. Pure Appl.* **10** (1974), 263–306.
13. R. Nussbaum, "Periodic Solutions of Nonlinear Autonomous Functional Differential Equations," Lecture Notes in Math., Vol. 730, pp. 283–325, Springer-Verlag, New York/Berlin, 1979.