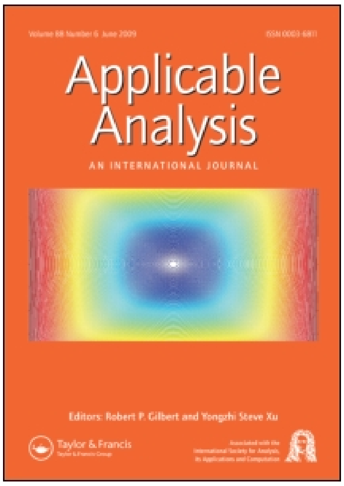

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Lyapunov functional and global asymptotic stability for an infection-age model

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Lyapunov functional and global asymptotic stability for an infection-age model

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We study an infection-age model of disease transmission, where both the infectiousness and the removal rate may depend on the infection age. In order to study persistence, the system is described using integrated semigroups. If the basic reproduction number $R_0 < 1$, then the disease-free equilibrium is globally asymptotically stable. For $R_0 > 1$, a Lyapunov functional is used to show that the unique endemic equilibrium is globally stable amongst solutions for which disease transmission occurs.

Keywords: Lyapunov functional; structured population; global stability; age of infection; integrated semigroup

AMS Subject Classifications: 34K20; 92D30

1. Introduction

In this article we first consider an infection-age model with a mass action law incidence function:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t)\displaystyle\int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I(a) i(t,a), \\ i(t,0) = \eta S(t)\displaystyle\int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a, \\ S(0) = S_0 \geq 0,\ i(0,.) = i_0 \in L^1_+(0,+\infty). \end{cases} \tag{1}$$

In the model (1), the population is decomposed into the class (S) of susceptible individuals and the class (I) of infected individuals. More precisely, the number of individuals in the class (S) at time t is $S(t)$. The age of infection $a \geq 0$ is the time since

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the infection began, and $i(t, a)$ is the density of infected individuals with respect to the age of infection. That is to say that for two given age values $a_1, a_2 : 0 \leq a_1 < a_2 \leq +\infty$ the number of infected individuals with age of infection a in between a_1 and a_2 is

$$\int_{a_1}^{a_2} i(t, a)\mathrm{d}a.$$

The infection age allows different interpretations for values of a. For example, an individual may be exposed (infected, but not yet infectious to susceptibles) from age $a = 0$ to $a = a_1$ and infectious to susceptibles from age a_1 to age a_2. In the model, the parameter $\gamma > 0$ is the entering flux into the susceptible class (S), and $\nu_S > 0$ is the exit (or (and) mortality) rate of susceptible individuals. The function $\beta(a)$ can be interpreted as the probability to be infectious (capable of transmitting the disease) with age of infection $a \geq 0$. The quantity

$$\int_0^{+\infty} \beta(a) i(t, a)\mathrm{d}a$$

is the number of infectious individuals within the subpopulation (I). The function $\beta(a)$ allows variable probability of infectiousness as the disease progresses within an infected individual.

Another interpretation of the density $i(t, a)$ of infected individuals is that

$$\int_{a_1}^{a_2} i(t, a)\mathrm{d}a, \quad 0 \leq a_1 < a_2$$

is the number of infected individuals in a particular class which is defined by the infection age interval $[a_1, a_2]$. For example, the infection age interval $[a_1, a_2]$ could correspond to an exposed (pre-infectious) phase, an infectious phase, an asymptomatic phase, or a symptomatic phase. Each of these phases of the disease course can be defined in terms an infection age interval common to all infected individuals. It is this interpretation of infection age that is used for the numerical work of Section 4.

Further, $\eta > 0$ is the rate at which an infectious individual infects the susceptible individuals. Finally, $\nu_I(a)$ is the exit (or (and) mortality or (and) recovery) rate of infected individuals with an age of infection $a \geq 0$. As a consequence the quantity

$$l_{\nu_I}(a) := \exp\left(-\int_0^a \nu_I(l)\mathrm{d}l\right)$$

is the probability for an individual to stay in the class (I) after a period of time $a \geq 0$.

In the sequel, we will make the following assumption.

ASSUMPTION 1.1 *We assume that the function* $a \to \beta(a)$ *is bounded and uniformly continuous from* $[0, +\infty)$ *to* $[0, +\infty)$, *and we assume that the function* $a \to \nu_I(a)$ *belongs to* $L^\infty_+(0, +\infty)$ *and satisfies*

$$\nu_I(a) \geq \nu_S \quad \textit{for almost every } a \geq 0.$$

In this article, we will be especially interested in analysing the dynamics of model (1) in the two situations described in Figure 1. In both cases, (A) and (B), we consider an incubation period of 10 time units – hours or days depending on the time scale. For case (A), after the incubation period the infectiousness function $\beta(a)$ increases

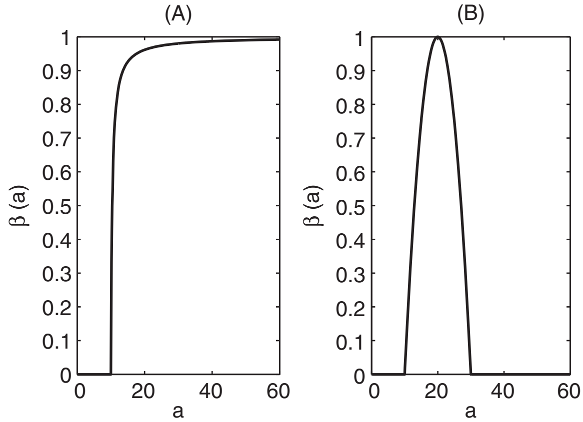

Figure 1. Probability to be infectious as a function of infection age a. The graph (A) is typical of diseases such as ebola and the graph (B) is typical of diseases such as influenza.

with the age of infection. This situation corresponds to a disease which becomes more and more transmissible with the age of infection. For case (B), after the incubation period, the infectiousness of infected individuals increases, passes through a maximum at $a = 20$, and then decreases and is eventually equal to 0 for large values of $a \geq 0$. Case (A) could be applied, for example, to *Ebola*, while case (B) could be applied to *Influenza* and various other diseases.

For model (1), the number R_0 of secondary infections produced by a single infected patient [1–4] is defined by

$$R_0 := \eta \frac{\gamma}{\nu_S} \int_0^{+\infty} \beta(a) l_{\nu_I}(a) \mathrm{d}a.$$

System (1) has at most two equilibria. The disease-free equilibrium

$$\left(\bar{S}_F, 0\right)$$

(with $\bar{S}_F = \frac{\gamma}{\nu_S}$) is always an equilibrium solution of system (1). Moreover when $R_0 > 1$, there exists a unique endemic equilibrium

$$\left(\bar{S}_E, \bar{\imath}_E\right)$$

(i.e. with $\bar{\imath}_E \in L^1_+(0, +\infty) \setminus \{0\}$) defined by

$$\bar{S}_E := 1 \Big/ \left(\eta \int_0^{+\infty} \beta(a) l_{\nu_I}(a) \mathrm{d}a \right) = \frac{\bar{S}_F}{R_0}$$

and

$$\bar{\imath}_E(a) := l_{\nu_I}(a) \bar{\imath}_E(0),$$

with

$$\bar{\imath}_E(0) := \gamma - \nu_S \bar{S}_E.$$

System (1) has been investigated by Thieme and Castillo-Chavez [5,6]. More precisely, in [5,6] they study the uniform persistence of the system and the local exponential asymptotic stability of the endemic equilibrium. The main question addressed in this article concerns the global asymptotic stability of the endemic equilibrium (when it exists).

In the context of SIR and SEIR models described by a system of ordinary differential equations, Lyapunov functions have been employed successfully to study the stability of endemic and the disease-free equilibrium [7–26]. We also refer to [27–31] for another geometrical approach which was also successfully applied in such a context. We refer to [32–34] for more results going in this direction.

One may observe that the problem is much more difficult here, since the system (1) yields an infinite-dimensional dynamical system. If one assumes, for example, that

$$\nu_I(a) \equiv \nu_I > 0, \quad \forall a \geq 0,$$

and that

$$\beta(a) = 1_{[\tau,+\infty)}(a)$$

for some $\tau \geq 0$. Then by setting

$$I(t) = \int_0^{+\infty} i(t,a)\mathrm{d}a$$

one derives the following delay differential equation:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t)e^{-\nu_I \tau} I(t-\tau), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \eta S(t)e^{-\nu_I \tau} I(t-\tau) - \nu_I I(t). \end{cases}$$

Also the system (1) can be viewed as a kind of distributed delay differential equation. Recently some work has been done on related epidemic models with delay, and we refer to [35–40] for more results on the subject.

The global asymptotic stability of the endemic equilibrium of (1) has been studied in [41] whenever the function $a \to e^{\nu_S a}\beta(a)l_{\nu_I}(a)$ is non-decreasing.

One may observe that $R_0 = 1$ corresponds to bifurcation points of the disease-free equilibrium. Also when $R_0 > 1$ and the parameters of the system are close to some bifurcation point (i.e. some parameter set for which $R_0 = 1$), it has been proved in [42] that the endemic equilibrium is also globally stable. Nevertheless, no global asymptotic stability results are known for the general case.

When $R_0 \leq 1$, we first obtain the following result extending the results proved in [41, Proposition 3.10] and in [6, Theorem 2].

THEOREM 1.2 *Assume that* $R_0 \leq 1$. *Then the disease-free equilibrium* $(\bar{S}_F, 0)$ *is globally asymptotically stable for the semiflow generated by system* (1).

When $R_0 > 1$ the behaviour is more delicate to study. We consider the extended real

$$\bar{a} = \sup\{a \geq 0 : \beta(a) > 0\}$$

and we define

$$\widehat{M}_0 = \left\{ i \in L^1_+(0, +\infty) : \int_0^{\bar{a}} i(a)\mathrm{d}a > 0 \right\}.$$

That is, $\widehat{M}_0$ consists of the distributions i that will generate new infectives either now or in future. Let

$$M_0 := [0, +\infty) \times \widehat{M}_0$$

and set

$$\partial M_0 := [0, +\infty) \times L^1_+(0, +\infty) \setminus M_0.$$

Then the state space is the set $M_0 \cup \partial M_0$.

The main result of this article is the following theorem.

THEOREM 1.3 *Assume that $R_0 > 1$. Then every solution of system* (1) *with initial value in ∂M_0* (*respectively in M_0*) *stays in ∂M_0* (*respectively stays in M_0*). *Moreover each solution with initial value in ∂M_0 converges to $(\bar{S}_E, 0)$. Furthermore, every solution with an initial value in M_0 converges to the endemic equilibrium $(\bar{S}_E, \bar{\imath}_E)$. Furthermore, this equilibrium $(\bar{S}_E, \bar{\imath}_E)$ is locally asymptotically stable.*

One important consequence of the above theorem, concerns the uniform persistence in the context of nosocomial infection. As presented in [41,43,44], one may derive from the above results some uniform persistence result of individuals infected by resistant strain. These consequences will be presented elsewhere, but this was our original motivation to study such a problem.

The method employed here to prove Theorem 1.3 is the following. In Section 2, we will first use integrated semigroup theory in order to obtain a comprehensive spectral theory for the linear C_0-semigroups obtained by linearizing the system around equilibriums. We refer to Webb [45,46] and Engel and Nagel [47] for more results of this topic. We will also use a uniform persistence result due to Hale and Waltman [48] combined with the results in Magal and Zhao [49] to assure the existence of global attractor A_0 of the system in M_0. We also refer to Magal [42] for a continuous time version of these results. Then in Section 3, we will first show that it is sufficient to consider the special case

$$\nu_I(a) = \nu_S, \quad \forall a \geq 0 \text{ and } \gamma = \nu_S$$

since a change of variables converts the general form of Equation (1) to this special case. We will then define V the Lyapunov functional

$$V(S(t), i(t, .)) = g\left(\frac{S(t)}{\bar{S}_E}\right) + \int_0^\infty \alpha(a)\, g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\mathrm{d}a,$$

where

$$g(x) := x - 1 - \ln x,$$

and

$$\alpha(a) := \int_a^\infty \eta\beta(l)\bar{\imath}_E(l)\mathrm{d}l.$$

We will observe that V is well defined on the attractor A_0, while V is not defined on M_0 because of the function g under the integral. (For instance, if $i(t,.)$ is zero on an interval, then $V(S(t),\ i(t,.)$ is not defined.) Then we prove that this functional is decreasing over the complete orbits on A_0. We conclude this article by proving that this implies that A_0 is reduced to the endemic equilibrium.

The plan of this article is the following. In Section 2, we present some results about the semiflow generated by (1) and we will present some results about uniform persistence and about the existence of global attractors. In Section 3, we study the Lyapunov functional for complete orbits passing through a point of the global attractor. In Section 4, we will apply the model (1) to the severe acute respiratory syndrome (SARS) epidemic in 2003.

2. Preliminary

To describe the semiflow generated by (1) we can use both Volterra's integral formulation [45,50,51] and integrated semigroup formulation [52–55].

Without loss of generality, we can add the class of recovered individuals to the system (1) and obtain the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \gamma - \nu_S S(t) - \eta S(t) \displaystyle\int_0^{+\infty} \beta(a) i(t,a) da, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I(a) i(t,a), \\ i(t,0) = \eta S(t) \displaystyle\int_0^{+\infty} \beta(a) i(t,a) da, \\ S(0) = S_0 \geq 0, \\ i(0,.) = i_0 \in L^1_+(0,+\infty), \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{dR(t)}{dt} = \displaystyle\int_0^{+\infty} (\nu_I(a) - \nu_S) i(t,a) da - \nu_S R(t), \quad \forall t \geq 0. \\ R(0) \geq 0. \end{cases}$$

By assumption $\nu_I\ (a) \geq \nu_S$ for almost every $a \geq 0$, we deduce that

$$R(0) \geq 0 \Rightarrow R(t) \geq 0, \quad \forall t \geq 0.$$

Now, by setting

$$N(t) = S(t) + \int_0^{+\infty} i(t,a) da + R(t)$$

we deduce that $N(t)$ satisfies the following ordinary differential equation:

$$\frac{dN(t)}{dt} = \gamma - \nu_S N(t) \tag{3}$$

and so $N(t)$ converges to $\frac{\gamma}{\nu_S}$. Moreover, since $R(t) \geq 0$, $\forall t \geq 0$, we obtain the following estimate:

$$S(t) + \int_0^{+\infty} i(t,a) da \leq N(t), \quad \forall t \geq 0. \tag{4}$$

2.1. *Volterra's formulation*

The Volterra integral formulation of age-structured models has been used successfully in various contexts and provides explicit (or implicit) formulas for the solutions of age-structured models. In this context, system (2) can be formulated as follows:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t)\int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a$$

and

$$i(t,a) = \begin{cases} \exp\left(\int_{a-t}^{a} \nu_I(l)\mathrm{d}l\right) i_0(a-t) & \text{if } a-t \geq 0 \\ \exp\left(\int_0^{a} \nu_I(l)\mathrm{d}l\right) b(t-a) & \text{if } a-t \leq 0, \end{cases}$$

where $t \to b(t)$ is the unique continuous function satisfying

$$b(t) = \eta S(t)\left[\begin{array}{l} \int_0^t \beta(a)\exp\left(\int_0^a \nu_I(l)\mathrm{d}l\right) b(t-a)\mathrm{d}a \\ \quad + \int_t^{+\infty} \beta(a)\exp\left(\int_{a-t}^a \nu_I(l)\mathrm{d}l\right) i_0(a-t)\mathrm{d}a \end{array}\right]. \tag{5}$$

By using this approach one may derive the following results by using the results given in [44] (see also [6]). Instead, we use the following approach.

2.2. *Integrated semigroup formulation*

We use the approach introduced by Thieme [55]. In order to take into account the boundary condition, we extend the state space and we consider

$$\widehat{X} = \mathbb{R} \times L^1(0, +\infty)$$

and $\widehat{A} : D(\widehat{A}) \subset \widehat{X} \to \widehat{X}$ the linear operator on $\widehat{X}$ defined by

$$\widehat{A}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \nu_I \varphi \end{pmatrix}$$

with

$$D\left(\widehat{A}\right) = \{0\} \times W^{1,1}(0, +\infty).$$

If $\lambda \in \mathbb{C}$, with $\mathrm{Re}(\lambda) > -\nu_S$, then $\lambda \in \rho(\widehat{A})$ (the resolvent set of $\widehat{A}$), and we have the following explicit formula for the resolvent of $\widehat{A}$

$$\left(\lambda I - \widehat{A}\right)^{-1}\begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \Leftrightarrow \varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda \mathrm{d}l}\alpha + \int_0^a e^{-\int_s^a \nu_I(l)+\lambda \mathrm{d}l}\psi(s)\mathrm{d}s.$$

Then by noting that

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t) \displaystyle\int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 0 \\ i(t,.) \end{pmatrix} = \widehat{A}\begin{pmatrix} 0 \\ i(t,.) \end{pmatrix} + \begin{pmatrix} \eta S(t) \displaystyle\int_0^{+\infty} \beta(a) i(t,a)\mathrm{d}a \\ 0 \end{pmatrix}, \\ S(0) = S_0 \geq 0, \\ i(0,.) = i_0 \in L^1_+(0,+\infty). \end{cases} \tag{6}$$

Moreover by defining $\widehat{i}(t) = \begin{pmatrix} 0 \\ i(t,.) \end{pmatrix}$ the partial differential equation (PDE) Equation (6) can be rewritten as an ordinary differential equation coupled with a non-densely defined Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = -\nu_S S(t) + F_1(S(t), \widehat{i}(t)) \\ \dfrac{\mathrm{d}\widehat{i}(t)}{\mathrm{d}t} = \widehat{A}\,\widehat{i}(t) + F_2(S(t), \widehat{i}(t)), \end{cases}$$

where

$$F_1\left(S, \begin{pmatrix} 0 \\ i \end{pmatrix}\right) = \gamma - \eta S \int_0^{+\infty} \beta(a) i(a)\mathrm{d}a$$

and

$$F_2\left(S, \begin{pmatrix} 0 \\ i \end{pmatrix}\right) = \begin{pmatrix} \eta S \displaystyle\int_0^{+\infty} \beta(a) i(a)\mathrm{d}a \\ 0 \end{pmatrix}.$$

Set

$$X = \mathbb{R} \times \mathbb{R} \times L^1(0,+\infty) \quad \text{and} \quad X_+ = \mathbb{R}_+ \times \mathbb{R}_+ \times L^1(0,+\infty)$$

and let $A : D(A) \subset X \to X$ be the linear operator defined by

$$A\begin{pmatrix} S \\ \begin{pmatrix} 0 \\ i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\nu_S S \\ \widehat{A}\begin{pmatrix} 0 \\ i \end{pmatrix} \end{pmatrix} = \begin{bmatrix} -\nu_S & 0 \\ 0 & \widehat{A} \end{bmatrix}\begin{pmatrix} S \\ \begin{pmatrix} 0 \\ i \end{pmatrix} \end{pmatrix}$$

with

$$D(A) = \mathbb{R} \times D\left(\widehat{A}\right).$$

Then $\overline{D(A)} = \mathbb{R} \times \{0\} \times L^1(0,+\infty)$ is not dense in X. We consider $F : \overline{D(A)} \to X$ the non-linear map defined by

$$F\begin{pmatrix} S \\ \begin{pmatrix} 0 \\ i \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F_1\left(S, \begin{pmatrix} 0 \\ i \end{pmatrix}\right) \\ F_2\left(S, \begin{pmatrix} 0 \\ i \end{pmatrix}\right) \end{pmatrix}.$$

Set

$$X_0 := \overline{D(A)} = \mathbb{R} \times \{0\} \times L^1(0, +\infty)$$

and

$$X_{0+} := \overline{D(A)} \cap X_+ = \mathbb{R}_+ \times \{0\} \times L^1_+(0, +\infty).$$

We can rewrite the system (2) as the following abstract Cauchy problem:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + F(u(t)) \quad \text{for } t \geq 0, \text{ with } u(0) = x \in \overline{D(A)}. \tag{7}$$

By using the fact that the non-linearities are Lipschitz continuous on bounded sets, by using (4), and by applying the results given in [52], we obtain the following proposition.

Proposition 2.1 *There exists a uniquely determined semiflow* $\{U(t)\}_{t \geq 0}$ *on* X_{0+}, *such that for each* $x = \begin{pmatrix} S_0 \\ \begin{pmatrix} 0 \\ i_0 \end{pmatrix} \end{pmatrix} \in X_{0+}$, *there exists a unique continuous map* $U \in C([0, +\infty), X_{0+})$ *which is an integrated solution of the Cauchy problem* (7), *that is to say that*

$$\int_0^t U(s)x\mathrm{d}s \in D(A), \quad \forall t \geq 0$$

and

$$U(t)x = x + A\int_0^t U(s)x\mathrm{d}s + \int_0^t F(U(s)x)\mathrm{d}s, \quad \forall t \geq 0.$$

Moreover

$$\limsup_{t \to +\infty} S(t) \leq \frac{\gamma}{\nu_S}.$$

By using the results in [56] (see also [57]), and by using the fact that $a \to \beta(a)$ is uniformly continuous, we deduce that the semiflow $\{U(t)\}_{t \geq 0}$ is asymptotically smooth (see [58] for a precise definition). Moreover by using again (4), we deduce that U is bounded dissipative, and by using the results of [58], we obtain the following proposition.

Proposition 2.2 *There exists a compact set* $\mathcal{A} \subset X_{0+}$, *such that*

(i) $\mathcal{A}$ *is invariant under the semiflow* $U(t)$ *that is to say that*

$$U(t)\mathcal{A} = \mathcal{A}, \quad \forall t \geq 0;$$

(ii) $\mathcal{A}$ *attracts the bounded sets of* X_{0+} *under* U, *that is to say that for each bounded set* $\mathcal{B} \subset X_{0+}$,

$$\lim_{t \to +\infty} \delta(U(t)\mathcal{B}, \mathcal{A}) = 0,$$

where the semi-distance $\delta(.,.)$ *is defined as*

$$\delta(\mathcal{B}, \mathcal{A}) = \sup_{x \in \mathcal{B}} \inf_{y \in \mathcal{A}} \|x - y\|.$$

Moreover, the subset $\mathcal{A}$ *is locally asymptotically stable.*

2.3. *Linearized equation at the disease-free equilibrium*

We now turn to the linearized equation at the disease-free equilibrium. Our goal is to compute the projector on the eigenspace associated with the dominant eigenvalue, in order to study the uniform persistence property. The linearized equation at the disease-free equilibrium $(\bar{S}_F, 0)$ is

$$\begin{cases} \frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\nu_S S(t) - \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \\ \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\nu_I(a) i(t,a), \\ i(t,0) = \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \\ S(0) = S_0 \geq 0, \\ i(0,.) = i_0 \in L^1_+(0, +\infty). \end{cases}$$

For this linearized system, the dynamics of i do not depend on S and so, in order to study the uniform persistence of disease we need to focus on the linear system

$$\begin{cases} \frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} = -\nu_I(a) i(t,a), \\ i(t,0) = \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \\ i(0,.) = i_0 \in L^1_+(0, +\infty), \end{cases}$$

where $\bar{S}_F = \frac{\gamma}{\nu_S}$. We define

$$\hat{B}_\kappa \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \kappa \int_0^{+\infty} \beta(a) \phi(a) \mathrm{d}a \\ 0 \end{pmatrix}$$

with

$$\kappa := \eta \frac{\gamma}{\nu_S}.$$

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$, we defined the characteristic function $\Delta(\lambda)$ as

$$\Delta(\lambda) := 1 - \kappa \int_0^{+\infty} \beta(a) e^{-\int_0^a \nu_I(l) + \lambda \mathrm{d}l} \mathrm{d}a.$$

Moreover since $\lambda I - \widehat{A}$ is invertible, we deduce that $\lambda I - (\widehat{A} + \hat{B}_\kappa)$ is invertible if and only if $I - \hat{B}_\kappa (\lambda I - \widehat{A})^{-1}$ is invertible or for short

$$\lambda \in \rho\left(\widehat{A} + \hat{B}_\kappa\right) \Leftrightarrow 1 \in \rho\left(\hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1}\right)$$

and we have

$$\left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1} = \left(\lambda I - \widehat{A}\right)^{-1} \left[I - \hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1}\right]^{-1}.$$

But we have

$$\begin{pmatrix}\alpha\\ \psi\end{pmatrix} - \hat{B}_\kappa\left(\lambda I - \widehat{A}\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}\gamma\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow \begin{cases}\alpha - \left[\kappa\int_0^{+\infty}\beta(a)\left[e^{-\int_0^a \nu_I(l)+\lambda dl}\alpha + \int_0^a e^{-\int_s^a \nu_I(l)+\lambda \mathrm{d}l}\psi(s)\mathrm{d}s\right]\mathrm{d}a\right] = \gamma\\ \psi = \varphi\end{cases}.$$

We can isolate α only if $\Delta(\lambda) \neq 0$. So, we deduce that for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$, the linear operator $I - \hat{B}_\kappa(\lambda I - \widehat{A})^{-1}$ is invertible if and only if $\Delta(\lambda) \neq 0$, and we have

$$\left[I - \hat{B}_\kappa\left(\lambda I - \widehat{A}\right)^{-1}\right]^{-1}\begin{pmatrix}\gamma\\ \varphi\end{pmatrix} = \begin{pmatrix}\Delta(\lambda)^{-1}\left[\kappa\int_0^{+\infty}\beta(a)\int_0^a e^{-\int_s^a \nu_I(l)+\lambda \mathrm{d}l}\varphi(s)\mathrm{d}s\,\mathrm{d}a + \gamma\right]\\ \varphi\end{pmatrix}.$$

It follows that for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$ and $\Delta(\lambda) \neq 0$, we have

$$\left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow$$
$$\varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda \mathrm{d}l}\left\{\Delta(\lambda)^{-1}\left[\kappa\int_0^{+\infty}\beta(a)\int_0^a e^{-\int_s^a \nu_I(l)+\lambda \mathrm{d}l}\psi(s)\mathrm{d}s\,\mathrm{d}a + \alpha\right]\right\}$$
$$+ \int_0^a e^{-\int_s^a \nu_I(l)+\lambda \mathrm{d}l}\psi(s)\mathrm{d}s.$$

Assume that $R_0 = [\kappa\int_0^{+\infty}\beta(a)e^{-\int_0^a \nu_I(l)\mathrm{d}l}\mathrm{d}a]_{\lambda=0} > 1$. Then we can find $\lambda_0 \in \mathbb{R}$, such that

$$\kappa\int_0^{+\infty}\beta(a)e^{-\int_0^a \nu_I(l)+\lambda_0 \mathrm{d}l}\mathrm{d}a = 1$$

and $\lambda_0 > 0$ is a dominant eigenvalue of $\widehat{A} + \hat{B}_\kappa$ [46]. Moreover, we have

$$\frac{\mathrm{d}\Delta(\lambda_0)}{\mathrm{d}\lambda} = \kappa\int_0^{+\infty} a\beta(a)e^{-\int_0^a \nu_I(l)+\lambda_0 \mathrm{d}l}\mathrm{d}a > 0$$

and the expression

$$\widehat{\Pi}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)\left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix}$$

exists and satisfies

$$\widehat{\Pi}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow \varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda_0 \mathrm{d}l}\left\{\left(\frac{\mathrm{d}\Delta(\lambda_0)}{\mathrm{d}\lambda}\right)^{-1}\left[\kappa\int_0^{+\infty}\beta(a)\int_0^a e^{-\int_s^a \nu_I(l)+\lambda_0 \mathrm{d}l}\psi(s)\mathrm{d}s\,\mathrm{d}a + \alpha\right]\right\}.$$

The linear operator $\widehat{\Pi} : \widehat{X} \to \widehat{X}$ is the projector onto the generalized eigenspace of $\widehat{A} + \hat{B}_\kappa$, associated with the eigenvalue λ_0. We define $\Pi : X \to X$

$$\Pi\begin{pmatrix}S\\ \alpha\\ i\end{pmatrix} = \begin{pmatrix}0\\ \widehat{\Pi}\begin{pmatrix}\alpha\\ \psi\end{pmatrix}\end{pmatrix}.$$

We observe that the subset $\widehat{M}_0$ defined in Section 1 can be identified with

$$M_0 = \left\{x \in X_{0+} : \Pi x \neq 0\right\}$$

and

$$\partial M_0 = X_{0+} \setminus M_0.$$

LEMMA 2.3 *The subsets M_0 and ∂M_0 are both positively invariant under the semiflow $\{U(t)\}_{t\geq 0}$, that is to say that*

$$U(t)M_0 \subset M_0 \quad \text{and} \quad U(t)\partial M_0 \subset \partial M_0.$$

Moreover for each $x \in \partial M_0$,

$$U(t)x \to \overline{x}_F, \quad \text{as } t \to +\infty,$$

where $\overline{x}_F = \left(\begin{smallmatrix} \bar{S}_F \\ \left(\begin{smallmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{smallmatrix}\right) \end{smallmatrix}\right)$ is the disease-free equilibrium of $\{U(t)\}_{t\geq 0}$.

Proof of Theorem 1.2 Assume that $R_0 \leq 1$. Then we first observe that

$$R_0 = \eta \frac{\gamma}{\nu_S} \int_0^{+\infty} \beta(a) l_{\nu_I}(a) \mathrm{d}a \leq 1 \Leftrightarrow \frac{\gamma}{\nu_S} \leq \bar{S}_E. \tag{8}$$

We set

$$\Gamma_I(a) = \eta \bar{S}_E \int_a^{+\infty} e^{-\int_a^s \nu_I(l)\mathrm{d}l} \beta(s) \mathrm{d}s, \quad \forall a \geq 0.$$

Then since

$$\eta \bar{S}_E = \left(\int_0^{+\infty} e^{-\int_0^s \nu_I(l)\mathrm{d}l} \beta(s) \mathrm{d}s\right)^{-1},$$

we have

$$\begin{cases} \Gamma_I'(a) = \nu_I(a)\Gamma_I(a) - \eta \bar{S}_E \beta(a) & \text{for almost every } a \geq 0, \\ \Gamma_I(0) = 1. \end{cases}$$

We define

$$D((A+F)_0) = \left\{x \in D(A) : Ax + F(x) \in \overline{D(A)}\right\}.$$

Let $x \in D((A+F)_0) \cap X_{0+}$, then we know [52,55] that

$$i(t,.) \in W^{1,1}(0, +\infty), \quad \forall t \geq 0,$$

and we have for each $\forall t \geq 0$,

$$i(t,0) = S(t)\eta \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a, \quad \forall t \geq 0,$$

the map $t \to i(t,.)$ belongs to $C^1([0,+\infty), L^1\,(0,+\infty))$ and $\forall t \geq 0$,

$$\frac{\mathrm{d}i(t,.)}{\mathrm{d}t} = -\frac{\partial i(t,.)}{\partial a} - \nu_I(a)i(t,a) \quad \text{for almost every } a \in (0,+\infty).$$

So $\forall t \geq 0$,

$$\frac{\mathrm{d}\int_0^{+\infty} \Gamma_I(a)i(t,a)\mathrm{d}a}{\mathrm{d}t} = -\int_0^{+\infty} \Gamma_I(a)\frac{\partial i_R(t,a)}{\partial a}\mathrm{d}a - \int_0^{+\infty} \Gamma_I(a)\nu_I(a)i_I(t,a)\mathrm{d}a.$$

By using the fact that $i(t,.) \in W^{1,1}\,(0,+\infty)$, we deduce that

$$i(t,a) \to 0 \quad \text{as } a \to +\infty,$$

so by integrating by part we obtain

$$\begin{aligned}\frac{\mathrm{d}\int_0^{+\infty} \Gamma_I(a)i(t,a)\mathrm{d}a}{\mathrm{d}t} &= -[\Gamma_I(a)i(t,a)]_0^{+\infty} + \int_0^{+\infty} \Gamma_I'(a)i(t,a)\mathrm{d}a \\ &\quad - \int_0^{+\infty} \Gamma_I(a)\nu_I(a)i_R(t,a)\mathrm{d}a \\ &= i(t,0) - \eta\bar{S}_E\int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a \\ &= \eta\big(S(t) - \bar{S}_E\big)\int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a\end{aligned}$$

so

$$\frac{\mathrm{d}\int_0^{+\infty} \Gamma_I(a)i(t,a)\mathrm{d}a}{\mathrm{d}t} = \eta\big(S(t) - \bar{S}_E\big)\int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a, \quad \forall t \geq 0 \tag{9}$$

and by density of $D((A+F)_0) \cap X_{0+}$ into X_{0+}, the above equality hold for any initial value $x \in X_{0+}$.

Let $x \in \mathcal{A}$. Since there exists a complete orbit $\left\{u(t) = \left(\begin{smallmatrix} S(t) \\ \left(\begin{smallmatrix} 0 \\ i(t,.)\end{smallmatrix}\right)\end{smallmatrix}\right)\right\}_{t \in \mathbb{R}} \subset \mathcal{A}$ and since

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t)\int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a \tag{10}$$

it follows that for each $t < 0$,

$$S(0) = e^{-\int_t^0 \nu_S + \int_0^{+\infty} \beta(a)i(l,a)\mathrm{d}a\,\mathrm{d}l}S(t) + \int_t^0 e^{-\int_t^s \nu_S + \int_0^{+\infty} \beta(a)i(l,a)\mathrm{d}a\,\mathrm{d}l}\gamma\mathrm{d}s,$$

thus

$$S(0) \leq e^{-\int_t^0 \nu_S + \int_0^{+\infty} \beta(a)i(l,a)\mathrm{d}a\,\mathrm{d}l}S(t) + \int_t^0 e^{-\int_t^s \nu_S \mathrm{d}l}\gamma\mathrm{d}s$$

and by taking the limit when $t \to -\infty$, we obtain

$$S(0) \leq \frac{\gamma}{\nu_S}.$$

Now since the above arguments hold for any $x \in \mathcal{A}$, we deduce that

$$S(t) \leq \frac{\gamma}{\nu_S}, \quad \forall t \in \mathbb{R}. \tag{11}$$

Now by combining (8), (9) and (11), we deduce that $t \to \int_0^{+\infty} \Gamma_I(a) i(t,a) \mathrm{d}a$ is non-decreasing along the complete orbit.

Now assume that $\mathcal{A} \not\subseteq M_0$. Let $x \in M_0 \backslash \mathcal{A}$. By using the definition of Γ_I and the definition of M_0, it follows that

$$\int_0^{+\infty} \Gamma_I(a) i(0,a) \mathrm{d}a > 0$$

and since $t \to \int_0^{+\infty} \Gamma_I(a) i(t,a) \mathrm{d}a$ is non-decreasing it follows that

$$\int_0^{+\infty} \Gamma_I(a) i(t,a) \mathrm{d}a \geq \int_0^{+\infty} \Gamma_I(a) i(0,a) \mathrm{d}a > 0, \quad \forall t \in \mathbb{R}.$$

Thus, the alpha-limit set of the complete orbit passing through x satisfies

$$\alpha(x) := \cap_{t \leq 0} \overline{\cup_{s \leq t} \{u(t)\}} \subset \mathcal{A} \cap M_0.$$

Moreover, there exists a constant $C > 0$, such that for each $\hat{x} = \left(\begin{smallmatrix} \widehat{S} \\ \left(\begin{smallmatrix} 0 \\ \hat{i} \end{smallmatrix} \right) \end{smallmatrix} \right) \in \alpha(x)$, we have

$$\int_0^{+\infty} \Gamma_I(a) \hat{i}(a) \mathrm{d}a = C > 0 \tag{12}$$

and

$$\widehat{S} \leq \frac{\gamma}{\nu_S}. \tag{13}$$

Let $\left\{ \widehat{u}(t) = \left(\begin{smallmatrix} \hat{S}(t) \\ \left(\begin{smallmatrix} 0 \\ \hat{i}(t,.) \end{smallmatrix} \right) \end{smallmatrix} \right) \right\}_{t \geq 0}$ be the solution of the Cauchy problem (7) with initial value $\hat{x} \in \alpha(x)$. Then (12) implies that $\hat{x} \in M_0$, and by using (5) we deduce that there exists $t_1 > 0$, such that

$$\int_0^{+\infty} \beta(a) \hat{i}(t,a) \mathrm{d}a > 0, \quad \forall t \geq t_1.$$

Now by using the invariance of the alpha-limit set $\alpha(x)$ by the semiflow generated by (7), and by using (10) and (13), we deduce that for each $t_2 > t_1$, we have

$$\hat{S}(t) < \frac{\gamma}{\nu_S}, \quad \forall t \geq t_2.$$

Finally since by (8), we have $\frac{\gamma}{\nu_S} \leq \bar{S}_E$ and by using (9) we obtain

$$\frac{\mathrm{d} \int_0^{+\infty} \Gamma_I(a) \hat{i}(t,a) \mathrm{d}a}{\mathrm{d}t} < 0, \quad \forall t \geq t_2$$

so the map $t \to \int_0^{+\infty} \Gamma_I(a)\hat{\imath}(t,a)\mathrm{d}a$ is not constant. This contradiction assures that

$$\mathcal{A} \subset \partial M_0$$

and it follows that

$$\mathcal{A} = \{\overline{x}_F\},$$

the result follows. ∎

By applying the results in [49] (or [42]), we obtain the following proposition.

PROPOSITION 2.4 *Assume that*

$$R_0 > 1.$$

The semiflow $\{U(t)\}_{t\geq 0}$ is uniformly persistent with respect to the pair $(\partial M_0, M_0)$, that is to say that there exists $\varepsilon > 0$, such that

$$\liminf_{t\to+\infty} \left\| \Pi U(t)x \right\| \geq \varepsilon, \quad \forall x \in M_0.$$

Moreover, there exists $\mathcal{A}_0$ a compact subset of M_0 which is a global attractor for $\{U(t)\}_{t\geq 0}$ in M_0, that is to say that

(i) *$\mathcal{A}_0$ is invariant under U, that is to say that*

$$U(t)\mathcal{A}_0 = \mathcal{A}_0, \quad \forall t \geq 0;$$

(ii) *For each compact subset $\mathcal{C} \subset M_0$,*

$$\lim_{t\to+\infty} \delta(U(t)\mathcal{C}, \mathcal{A}_0) = 0.$$

Moreover, the subset $\mathcal{A}_0$ is locally asymptotically stable.

Proof Since $\overline{x}_F = \begin{pmatrix} \bar{S}_F \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}$ the disease-free equilibrium is globally asymptotically stable in ∂M_0, to apply Theorem 4.1 in [48], we only need to study the behaviour of the solutions starting in M_0 in some neighbourhood of $\overline{x}_F$. It is sufficient to prove that there exists $\varepsilon > 0$, such that for each $x = \begin{pmatrix} S_0 \\ \begin{pmatrix} 0 \\ i_0 \end{pmatrix} \end{pmatrix} \in \{ y \in M_0 : \|\overline{x}_F - y\| \leq \varepsilon\}$, there exists $t_0 \geq 0$, such that

$$\left\| \overline{x}_F - U(t_0)x \right\| > \varepsilon.$$

This will show that $\{ y \in X_{0+} : \|\overline{x}_F - y\| \leq \varepsilon\}$ is an isolating neigbourhood of $\{\overline{x}_F\}$ (i.e. there exists a neigbourhood of $\{\overline{x}_F\}$ in which $\{\overline{x}_F\}$ is the largest invariant set for U) and

$$W^s(\{\overline{x}_F\}) \cap M_0 = \varnothing,$$

where

$$W^s(\{\overline{x}_F\}) = \left\{ x \in X_{0+} : \lim_{t\to+\infty} U(t)x = \overline{x}_F \right\}.$$

Assume by contradiction that for each $n \geq 0$, we can find $x_n = \begin{pmatrix} S_0^n \\ \begin{pmatrix} 0 \\ i_0^n \end{pmatrix} \end{pmatrix} \in \{ y \in M_0 : \|\overline{x}_F - y\| \leq \frac{1}{n+1}\}$, such that

$$\left\|\overline{x}_F - U(t)x_n\right\| \leq \frac{1}{n+1}, \quad \forall t \geq 0. \tag{14}$$

Set

$$\begin{pmatrix} S^n(t) \\ \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} \end{pmatrix} := U(t)x_n$$

and we have

$$\left|S^n(t) - \bar{S}_F\right| \leq \frac{1}{n+1}, \quad \forall t \geq 0.$$

Moreover, the map $t \to \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix}$ is an integrated solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} = \widehat{A}\begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} + F_2\left(S^n(t), \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix}\right) & \text{for } t \geq 0, \\ \text{with } \begin{pmatrix} 0 \\ i^n(0,.) \end{pmatrix} = \begin{pmatrix} 0 \\ i_0^n \end{pmatrix}. \end{cases}$$

Now since $\widehat{A}$ is resolvent positive and F_2 monotone non-decreasing, we deduce that

$$i^n(t,.) \geq \tilde{\imath}^n(t,.), \tag{15}$$

where $t \to \widehat{\imath^n}(t,.)$ is a solution of the linear Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix} = \widehat{A}\begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix} + F_2\left(\bar{S}_F + \dfrac{1}{n+1}, \begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix}\right) & \text{for } t \geq 0, \\ \text{with } \begin{pmatrix} 0 \\ \tilde{\imath}^n(0,.) \end{pmatrix} = \begin{pmatrix} 0 \\ i_0^n \end{pmatrix}, \end{cases}$$

or $\tilde{\imath}^n(t,a)$ is a solution of the PDE problem

$$\begin{cases} \dfrac{\partial \tilde{\imath}^n(t,a)}{\partial t} + \dfrac{\partial \tilde{\imath}^n(t,a)}{\partial a} = -\nu_I(a)\tilde{\imath}^n(t,a), \\ \tilde{\imath}^n(t,0) = \eta\left(\bar{S}_F - \dfrac{1}{n+1}\right)\displaystyle\int_0^{+\infty} \beta(a)\tilde{\imath}^n(t,a)\mathrm{d}a, \\ \tilde{\imath}^n(0,.) = i_0^n \in L^1_+(0, +\infty). \end{cases}$$

We observe that

$$F_2\left(\bar{S}_F - \frac{1}{n+1}, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \hat{B}_{\kappa_n}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

with

$$\kappa_n = \eta\left(\bar{S}_F - \frac{1}{n+1}\right).$$

Now since $R_0 > 1$, we deduce that for all $n \geq 0$ large enough, the dominanted eigenvalue of the linear operator $\widehat{A} + \hat{B}_{\kappa_n} : D(A) \subset X \to X$ satisfies the characteristic equation

$$\eta\left(\bar{S}_F - \frac{1}{n+1}\right)\int_0^{+\infty} \beta(a) e^{-\int_0^a \nu_I(l)+\lambda_{0n}\mathrm{d}l}\mathrm{d}a = 1.$$

It follows that $\lambda_{0n} > 0$ for all $n \geq 0$ large enough. Now $x_n \in M_0$, we have

$$\widehat{\Pi}_n\begin{pmatrix} 0 \\ i_0^n \end{pmatrix} \neq 0,$$

where $\widehat{\Pi}_n$ is the projector on the eigenspace associated to the dominant eigenvalue λ_{0n}. It follows that

$$\lim_{t\to+\infty}\left\|\tilde{\imath}^n(t,.)\right\| = +\infty$$

and by using (15) we obtain

$$\lim_{t\to+\infty}\left\|i^n(t,.)\right\| = +\infty.$$

So we obtain a contradiction with (14) and the result follows. ■

The following proposition was proved by Thieme and Castillo-Chavez [6]. For completeness we will prove this result.

Proposition 2.5 *Assume that*

$$R_0 > 1.$$

Then the endemic equilibrium $\overline{x}_E = \begin{pmatrix} \bar{S}_E \\ \begin{pmatrix} 0 \\ \bar{\imath}_E \end{pmatrix} \end{pmatrix}$ *is locally asymptotically stable for* $\{U(t)\}_{t\geq 0}$.

Proof The linearized equation of (7) around the endemic equilibrium $\overline{x}_E$ is

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = Av(t) + DF(\overline{x}_E)(v(t)) \quad \text{for } t \geq 0, \text{ with } v(0) = x \in \overline{D(A)},$$

which corresponds to the following PDE:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -\nu_S x(t) - \eta\bar{S}_E\displaystyle\int_0^{+\infty}\beta(a)\, y(t,a)\mathrm{d}a - x(t)\eta\int_0^{+\infty}\beta(a)\bar{\imath}_E(a)\mathrm{d}a, \\ \dfrac{\partial y(t,a)}{\partial t} + \dfrac{\partial y(t,a)}{\partial a} = -\nu_I(a)\, y(t,a), \\ y(t,0) = \eta\bar{S}_E\displaystyle\int_0^{+\infty}\beta(a)\, y(t,a)\mathrm{d}a + x(t)\eta\int_0^{+\infty}\beta(a)\bar{\imath}_E(a)\mathrm{d}a, \\ x(0) = x_0 \in \mathbb{R}, \\ y(0,.) = y_0 \in L^1(0,+\infty). \end{cases}$$

Since $\{T_{A_0}(t)\}_{t\geq 0}$ the semigroup generated by A_0 the part of A in $\overline{D(A)}$ satisfies

$$\left\|T_{A_0}(t)\right\| \leq \hat{M}e^{-\nu_S t}, \quad \forall t \geq 0,$$

for some constant $\hat{M} > 0$. It follows that $\omega_{\text{ess}}(A_0)$ the essential growth of rate of $\{T_{A_0}(t)\}_{t\geq 0}$ is $\leq -\nu_S$. Let $\{T_{(A+DF(\overline{x}_E))_0}(t)\}_{t\geq 0}$ be the linear C_0-semigroup generated by $(A + DF(\overline{x}_E))_0$ the part of $A + DF(\overline{x}_E) : D(A) \subset X \to X$ in $\overline{D(A)}$. Since $DF(\overline{x}_E)$ is a compact bounded linear operator, it follows that [59,60] that

$$\omega_{\text{ess}}((A + DF(\overline{x}_E))_0) \leq -\nu_S.$$

So it remains to study the ponctual spectrum of $(A + DF(\overline{x}_E))_0$. So we consider the exponential solutions (i.e. solutions of the form $u(t) = e^{\lambda t}x$ with $x \neq 0$) to derive the characteristic equation and we obtain the following system:

$$\begin{cases} \lambda x = -\nu_S x - \eta \bar{S}_E \int_0^{+\infty} \beta(a)\, y(a) \mathrm{d}a - x\eta \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) \mathrm{d}a, \\ \lambda y(a) + \dfrac{\mathrm{d}y(a)}{\mathrm{d}a} = -\nu_I(a)\, y(a), \\ y(0) = \eta \bar{S}_E \int_0^{+\infty} \beta(a)\, y(a) \mathrm{d}a + x\eta \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) \mathrm{d}a, \end{cases}$$

where $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$ and $(x, y) \in \mathbb{R} \times W^{1,1}\ (0, +\infty) \setminus \{0\}$. By integrating $y(a)$ we obtain the system of two equations for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$,

$$\left(\lambda + \nu_S + \eta \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) \mathrm{d}a\right) x = -\eta \bar{S}_E y(0) \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a$$

and

$$\left(1 - \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a\right) y(0) = +x\eta \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) \mathrm{d}a,$$

where

$$\bar{S}_E := \left(\eta \int_0^{+\infty} \beta(a) l_{\nu_I}(a) \mathrm{d}a\right)^{-1} \quad \text{and} \quad \eta \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) \mathrm{d}a = \gamma \bar{S}_E^{-1} - \nu_S$$

and

$$(x, y(0)) \in \mathbb{R}^2 \setminus \{0\}.$$

We obtain

$$\begin{aligned} 1 &= \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a \\ &\quad - \frac{\left(\gamma \bar{S}_E^{-1} - \nu_S\right)}{\left(\lambda + \nu_S + \gamma \bar{S}_E^{-1} - \nu_S\right)} \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a \\ &= \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a \left[1 - \frac{\left(\gamma \bar{S}_E^{-1} - \nu_S\right)}{\left(\lambda + \gamma \bar{S}_E^{-1}\right)}\right], \end{aligned}$$

thus it remains to study the characteristic equation $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$,

$$1 = \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} \mathrm{d}a \left[\frac{(\lambda + \nu_S)}{\left(\lambda + \gamma \bar{S}_E^{-1}\right)}\right]. \tag{16}$$

By considering the real and the imaginary part of λ, we obtain

$$\frac{\left[\left(\mathrm{Re}(\lambda)+\gamma\bar{S}_E^{-1}\right)+i\,\mathrm{Im}(\lambda)\right]\left[(\mathrm{Re}(\lambda)+\nu_S)-i\,\mathrm{Im}(\lambda)\right]}{\left[(\mathrm{Re}(\lambda)+\nu_S)^2+\mathrm{Im}(\lambda)^2\right]}$$
$$=\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}[\cos(a\,\mathrm{Im}(\lambda))+i\,\sin(a\,\mathrm{Im}(\lambda))]\mathrm{d}a\right].$$

So by identifying the real and the imaginary parts, we obtain for the real part

$$\left(\mathrm{Re}(\lambda)+\gamma\bar{S}_E^{-1}\right)(\mathrm{Re}(\lambda)+\nu_S)+\mathrm{Im}(\lambda)^2$$
$$=\left[(\mathrm{Re}(\lambda)+\nu_S)^2+\mathrm{Im}(\lambda)^2\right]\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\cos(a\,\mathrm{Im}(\lambda))\mathrm{d}a\right],$$

thus

$$\left(\gamma\bar{S}_E^{-1}-\nu_S\right)(\mathrm{Re}(\lambda)+\nu_S)$$
$$=\left[(\mathrm{Re}(\lambda)+\nu_S)^2+\mathrm{Im}(\lambda)^2\right]\left[\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\cos(a\,\mathrm{Im}(\lambda))\mathrm{d}a\right]-1\right].$$

Assume that there exists $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$ satisfying (16). Then since $\bar{S}_E=(\eta\int_0^{+\infty}\beta(a)l_{\nu_I}(a)\mathrm{d}a)^{-1}$ we deduce that

$$\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-aRe(\lambda)}\cos(a\,\mathrm{Im}(\lambda))\mathrm{d}a\right]\leq 1$$

and since $R_0=\eta\frac{\gamma}{\nu_S}\int_0^{+\infty}\beta(a)l_{\nu_I}(a)\mathrm{d}a=\frac{\gamma}{\nu_S}\frac{1}{\bar{S}_E}>1$, we obtain $\gamma\bar{S}_E^{-1}-\nu_S>0$, thus

$$\left(\gamma\bar{S}_E^{-1}-\nu_S\right)(\mathrm{Re}(\lambda)+\nu_S)>0.$$

It follows that the characteristic Equation (16) has no root with non-negative real part. The proof is complete. ■

3. Lyapunov functional and global asymptotic stability

In this section, we assume that

$$R_0>1.$$

By using Proposition 2.4 (since $\mathcal{A}_0$ is invariant under U), we can find $\{u(t)\}_{t\in\mathbb{R}}\subset\mathcal{A}_0$ a complete orbit of $\{U(t)\}_{t\geq 0}$, that is to say that

$$u(t)=U(t-s)u(s),\quad \forall t,s\in R,\ \text{with } t\geq s.$$

So we have

$$u(t)=\begin{pmatrix} S(t)\\ \begin{pmatrix} 0\\ i(t,.)\end{pmatrix}\end{pmatrix}\in\mathcal{A}_0,\quad \forall t\in\mathbb{R}$$

and $\{(S(t),i(t,.))\}_{t\in\mathbb{R}}$ is complete orbit of system (1).

Moreover, by using the same arguments as in Lemma 3.6 and Proposition 4.3 in [41], we have the following lemma.

Lemma 3.1 *There exist constants $M > \varepsilon > 0$, such that for each complete orbit $\left\{\left(\begin{pmatrix} S(t) \\ 0 \\ i(t,.) \end{pmatrix}\right)\right\}_{t\in\mathbb{R}}$ of U in $\mathcal{A}_0$, we have*

$$\varepsilon \leq S(t) \leq M, \quad \forall t \in \mathbb{R}$$

and

$$\varepsilon \leq \int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a \leq M, \quad \forall t \in \mathbb{R}.$$

Moreover

$$O = \overline{\cup_{t\in\mathbb{R}}\{(S(t), i(t,.))\}}$$

is compact in $\mathbb{R} \times L^1(0, +\infty)$.

3.1. *Change of variable*

By using Volterra's formulation of the solution, we have

$$i(t,a) = \exp\left(\int_0^a -\nu_I(r)\mathrm{d}r\right)b(t-a),$$

where

$$b(t) = \eta S(t)\int_0^{+\infty} \beta(a)i(t,a)\mathrm{d}a.$$

Set

$$u(t,a) := \exp\left(\int_0^a (\nu_I(r) - \nu_S)\mathrm{d}r\right)i(t,a) = e^{-\nu_S a}b(t-a),$$

$$\widehat{l}(a) := \exp\left(-\int_0^a (\nu_I(r) - \nu_S)\mathrm{d}r\right)$$

and

$$\widehat{\beta}(a) := \beta(a)\widehat{l}(a).$$

Then we have

$$i(t,a) = \widehat{l}(a)u(t,a)$$

and $(S(t), u(t,a))_{t\in\mathbb{R}}$ is a complete orbit of the following system:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \gamma - \nu_S S(t) - \eta S(t)\displaystyle\int_0^{+\infty} \widehat{\beta}(a)u(t,a)\mathrm{d}a, \\ \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\nu_S u(t,a), \\ u(t,0) = \eta S(t)\displaystyle\int_0^{+\infty} \widehat{\beta}(a)u(t,a)\mathrm{d}a, \\ S(0) = S_0 \geq 0, \\ u(0,.) = u_0 \in L^1_+(0, +\infty). \end{cases} \tag{17}$$

Moreover, by using (17) we deduce that

$$\frac{\mathrm{d}\left[S(t)+\int_0^{+\infty}u(t,a)\mathrm{d}a\right]}{\mathrm{d}t}=\gamma-\nu_S\left[S(t)+\int_0^{+\infty}u(t,a)\mathrm{d}a\right] \tag{18}$$

and since $t\to[S(t)+\int_0^{+\infty}u(t,a)\mathrm{d}a]$ is a bounded complete orbit of the above ordinary differential equation, we deduce that

$$\gamma=\nu_S\left[S(t)+\int_0^{+\infty}u(t,a)\mathrm{d}a\right],\quad \forall t\in\mathbb{R}.$$

Moreover, by multiplying $S(t)$ and $u(t,a)$ by $\frac{\nu_S}{\gamma}$, we can assume that

$$\frac{\gamma}{\nu_S}=1.$$

So without loss of generality, we can assume that system (1) satisfies the following assumption. (For clarity, we emphasize that through the change of variables given above, the general form of system (1) is equivalent to the special case obtained by using Assumption 3.2.)

Assumption 3.2 *We assume that*

$$\nu_I(a)=\nu_S,\quad \forall a\geq 0 \text{ and } \gamma=\nu_S.$$

Then system (1) becomes

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\nu_S-\nu_S S(t)-\eta S(t)\displaystyle\int_0^{+\infty}\beta(a)i(t,a)\mathrm{d}a,\\ \dfrac{\partial i(t,a)}{\partial t}+\dfrac{\partial i(t,a)}{\partial a}=-\nu_S i(t,a),\\ i(t,0)=\eta S(t)\displaystyle\int_0^{+\infty}\beta(a)i(t,a)\mathrm{d}a,\\ S(0)=S_0\geq 0,\ i(0,.)=i_0\in L^1_+(0,+\infty),\end{cases} \tag{19}$$

and from here on, we consider this system. In this special case, the endemic equilibrium satisfies the following system of equations:

$$\begin{aligned}0&=\nu_S-\nu_S\bar{S}_E-\eta\bar{S}_E\int_0^{+\infty}\beta(a)\bar{\imath}_E(a)\mathrm{d}a\\ \bar{\imath}_E(a)&=e^{-\nu_S a}\bar{\imath}_E(0)\end{aligned} \tag{20}$$

with

$$1=\eta\bar{S}_E\int_0^{+\infty}\beta(a)e^{-\nu_S a}\mathrm{d}a.$$

Moreover, by Lemma 3.1, we can consider $\{(S(t),i(t,.))\}_{t\in\mathbb{R}}$ a complete orbit of system (19) satisfying

$$\varepsilon\leq S(t)\leq M,\quad \forall t\in\mathbb{R}$$

and

$$\varepsilon \leq \int_0^{+\infty} \beta(a) i(t,a) \mathrm{d}a \leq M, \quad \forall t \in \mathbb{R}.$$

Moreover,

$$O = \overline{\cup_{t\in\mathbb{R}}\{(S(t), i(t,.))\}}$$

is compact in $\mathbb{R} \times L^1(0, +\infty)$.

Furthermore, we have

$$\frac{i(t,a)}{\bar{\imath}_E(a)} = \frac{b(t-a)}{\bar{\imath}_E(0)} = \frac{\eta S(t-a) \int_0^{+\infty} \beta(l) i(t-a,l) \mathrm{d}l}{\bar{\imath}_E(0)}$$

and thus

$$\frac{\eta}{\bar{\imath}_E(0)} \varepsilon^2 \leq \frac{i(t,a)}{\bar{\imath}_E(a)} \leq \frac{\eta}{\bar{\imath}_E(0)} M^2.$$

3.2. *Lyapunov functional*

Let

$$g(x) = x - 1 - \ln x.$$

Note that $g'(x) = 1 - (1/x)$. Thus, g is decreasing on $(0, 1]$ and increasing on $[1, \infty)$. The function g has only one extremum which is a global minimum at 1, satisfying $g(1) = 0$. We first define expressions $V_S(t)$ and $V_i(t)$ and calculate their derivatives. Then, we will analyse the Lyapunov functional $V = V_S + V_i$. Let

$$V_S(t) = g\left(\frac{S(t)}{\bar{S}_E}\right).$$

Then

$$\begin{aligned}
\frac{\mathrm{d}V_S}{\mathrm{d}t} &= g'\left(\frac{S(t)}{\bar{S}_E}\right) \frac{1}{\bar{S}_E} \frac{\mathrm{d}S}{\mathrm{d}t} \\
&= \left(1 - \frac{\bar{S}_E}{S(t)}\right) \frac{1}{\bar{S}_E} \left[\nu_S - \nu_S S(t) - \int_0^{\infty} \eta \beta(l) i(t,l) S(t) \mathrm{d}l\right] \\
&= \left(1 - \frac{\bar{S}_E}{S(t)}\right) \frac{1}{\bar{S}_E} \left[\nu_S \big(\bar{S}_E - S(t)\big) + \int_0^{\infty} \eta \beta(l) \big(\bar{\imath}_E(l) \bar{S}_E - i(t,l) S(t)\big) \mathrm{d}l\right] \\
&= -\nu_S \frac{\big(S(t) - \bar{S}_E\big)^2}{S(t)\bar{S}_E} + \int_0^{\infty} \eta \beta(l) \bar{\imath}_E(l) \left(1 - \frac{i(t,l)}{\bar{\imath}_E(l)} \frac{S(t)}{\bar{S}_E} - \frac{\bar{S}_E}{S(t)} + \frac{i(t,l)}{\bar{\imath}_E(l)}\right) \mathrm{d}l. \qquad (21)
\end{aligned}$$

Let

$$V_i(t) = \int_0^{\infty} \alpha(a)\, g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right) \mathrm{d}a,$$

where

$$\alpha(a) := \int_a^\infty \eta\beta(l)\bar{\imath}_E(l)\mathrm{d}l. \tag{22}$$

Then

$$\begin{aligned}\frac{\mathrm{d}V_i}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \alpha(a)\, g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\mathrm{d}a\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \alpha(a)\, g\left(\frac{b(t-a)}{\bar{\imath}_E(0)}\right)\mathrm{d}a\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t \alpha(t-s)\, g\left(\frac{b(s)}{\bar{\imath}_E(0)}\right)\mathrm{d}s\\ &= \alpha(0)\, g\left(\frac{b(t)}{\bar{\imath}_E(0)}\right) + \int_{-\infty}^t \alpha'(t-s)\, g\left(\frac{b(s)}{\bar{\imath}_E(0)}\right)\mathrm{d}a\end{aligned}$$

and thus

$$\frac{\mathrm{d}V_i}{\mathrm{d}t} = \alpha(0)\, g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) + \int_0^\infty \alpha'(a)\, g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\mathrm{d}a. \tag{23}$$

Moreover, by the definition of α we have

$$\alpha(0)\, g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) = \int_0^\infty \eta\beta(l)\bar{\imath}_E(l)\, g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right)\mathrm{d}l. \tag{24}$$

Noting additionally, that $\alpha'(a) = -\eta\beta(a)\bar{\imath}_E(a)$, we may combine Equations (23) and (24) to get

$$\frac{\mathrm{d}V_i}{\mathrm{d}t} = \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) - g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\right]\mathrm{d}a.$$

Filling in for the function g, we obtain

$$\frac{\mathrm{d}V_i}{\mathrm{d}t} = \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{i(t,a)}{\bar{\imath}_E(a)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\right]\mathrm{d}a. \tag{25}$$

Let

$$V(t) = V_S(t) + V_i(t).$$

Then, by combining (21) and (25), we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = &-\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E}\\ &+\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\begin{bmatrix} 1 - \dfrac{i(t,a)}{\bar{\imath}_E(a)}\dfrac{S(t)}{\bar{S}_E} - \dfrac{\bar{S}_E}{S(t)} + \dfrac{i(t,0)}{\bar{\imath}_E(0)}\\ -\ln\dfrac{i(t,0)}{\bar{\imath}_E(0)} + \ln\dfrac{i(t,a)}{\bar{\imath}_E(a)}\end{bmatrix}\mathrm{d}a.\end{aligned} \tag{26}$$

The object now, is to show that $\frac{\mathrm{d}V}{\mathrm{d}t}$ is non-positive. To help with this, we demonstrate that two of the terms above cancel out

$$\begin{aligned}
&\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\right]\mathrm{d}a \\
&= \frac{1}{\bar{S}_E}\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\bar{S}_E\,\mathrm{d}a\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{1}{\bar{S}_E}\int_0^\infty \eta\beta(a)i(t,a)S(t)\,\mathrm{d}a \\
&= \frac{1}{\bar{S}_E}\bar{\imath}_E(0)\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{1}{\bar{S}_E}i(t,0) \\
&= 0.
\end{aligned} \tag{27}$$

Using this to simplify Equation (26) gives

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= -\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E} \\
&\quad + \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[1 - \frac{\bar{S}_E}{S(t)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\right]\mathrm{d}a.
\end{aligned} \tag{28}$$

Noting that $\bar{\imath}_E(0)/i(t,0)$ is independent of a, we may multiply both sides of (27) by this quantity to obtain

$$\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[1 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right]\mathrm{d}a = 0. \tag{29}$$

We now add (29) to (28) and also add and subtract $\ln(S(t)/\bar{S}_E)$ to get

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E} + \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)C(a)\,\mathrm{d}a,$$

where

$$\begin{aligned}
C(a) &= 2 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)} - \frac{\bar{S}_E}{S(t)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)} + \ln\frac{S(t)}{\bar{S}_E} - \ln\frac{S(t)}{\bar{S}_E} \\
&= \left(1 - \frac{\bar{S}_E}{S(t)} + \ln\frac{\bar{S}_E}{S(t)}\right) + \left(1 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right) \\
&= -\left[g\left(\frac{\bar{S}_E}{S(t)}\right) + g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right)\right] \\
&\leq 0.
\end{aligned}$$

Thus, $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$ with equality if and only if

$$\frac{\bar{S}_E}{S(t)} = 1 \quad \text{and} \quad \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{\bar{\imath}_E(0)}{i(t,0)} = 1. \tag{30}$$

Using (20), this second condition is equivalent to

$$i(t,a) = i(t,0)e^{-\nu_S a} \tag{31}$$

for all $a \geq 0$.

Look for the largest invariant set Q for which (30) holds. In Q, we must have $S(t) = \bar{S}_E$ for all t and so we have $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$. Combining this with (31), we obtain

$$\begin{aligned} 0 &= \nu_S - \nu_S \bar{S}_E - \int_0^\infty \eta\beta(a) i(t,a)\,\mathrm{d}a\,\bar{S}_E \\ &= \nu_S - \nu_S \bar{S}_E - \int_0^\infty \eta\beta(a) i(t,0) e^{-\nu_S a}\,\mathrm{d}a\,\bar{S}_E \\ &= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \int_0^\infty \eta\beta(a) \bar{\imath}_E(0) e^{-\nu_S a}\,\mathrm{d}a\,\bar{S}_E \\ &= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \int_0^\infty \eta\beta(a) \bar{\imath}_E(a)\,\mathrm{d}a\,\bar{S}_E \\ &= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \left(\nu_S - \nu_S \bar{S}_E\right) \\ &= \left(1 - \frac{i(t,0)}{\bar{\imath}_E(0)}\right)\left(\nu_S - \nu_S \bar{S}_E\right). \end{aligned}$$

Since $\bar{S}_E$ is not equal to 1, we must have $i(t,0) = \bar{\imath}_E(0)$ for all t. Thus, the set Q consists of only the endemic equilibrium.

Proof of Theorem 1.3 Assume that $\mathcal{A}_0$ is larger than $\{\overline{x}_E\}$. Then there exists $x \in \mathcal{A}_0 \setminus \{\overline{x}_E\}$, and we can find $\{u(t)\}_{t\in\mathbb{R}} \subset \mathcal{A}_0$, a complete orbit of U, passing through x at $t=0$, with alpha-limit set $\alpha(x)$. Since

$$u(0) = x \neq \overline{x}_E, \tag{32}$$

we deduce that $t \to V(u(t))$ is a non-increasing map. Thus, V is a constant functional on the alpha-limit set $\alpha(x)$. Since $\alpha(x)$ is invariant under U, it follows that

$$\alpha(x) = \{\overline{x}_E\}. \tag{33}$$

Recalling from Proposition 2.5 that the endemic equilibrium is locally asymptotically stable, Equation (33) implies $x = \overline{x}_E$ which contradicts (32). ■

4. Numerical examples

We present three examples to illustrate the infection-age model (1). In the examples, infection age is used to track the period of incubation, the period of infectiousness, the appearance of symptoms and the quarantine of infectives.

Example 1 In the first example we interpret infection age corresponding to an exposed period (infected, but not yet infectious) from $a=0$ to $a=a_1$ and an infectious period from $a=a_1$ to $a=a_2$. The total number of exposed infectives $E(t)$ and infectious infectives $I(t)$ at time t are

$$E(t) = \int_0^{a_1} \imath(t,a)\mathrm{d}a, \quad I(t) = \int_{a_1}^{a_2} \imath(t,a)\mathrm{d}a.$$

This interpretation of the model is typical of a disease such as influenza, in which there is an initial non-infectious period followed by a period of increasing than decreasing infectiousness. We investigate the role of quarantine in controlling an epidemic using infection age to track quarantined individuals. We consider a population of initially γ/ν_S susceptible individuals with an on-going influx at rate γ and efflux at rate ν_S. These rates influence the extinction or endemicity of the epidemic; specifically, the continuing arrival of new susceptibles enables a disease to persist, which might otherwise extinguish.

Set $\gamma = 365$, $\nu_S = 1/365$ (time units are days). For a human population of ~100,000 people these rates may be interpreted in terms of daily immigration and emigration into and out of the population. Set $a_1 = 5$ and $a_2 = 21$. We use the form of the transmission function $\beta(a)$ in Figure 2:

$$\beta(a) = \begin{cases} 0.0, & \text{if } 0.0 \le a \le 5.0; \\ 0.66667(a-5.0)^2 e^{-0.6(a-5.0)} & \text{if } a > 5.0. \end{cases}$$

We set the transmission rate $\eta = 1.5 \times 10^{-5}$. We set $\nu_I(a) = \nu_Q(a) + \nu_H(a) + \nu_S$, where

$$\nu_Q(a) = \begin{cases} -\log(0.95), & \text{if } 0.0 \le a \le 5.0; \\ 0.0 & \text{if } a > 5.0 \end{cases} \qquad \nu_H(a) = \begin{cases} 0.0, & \text{if } 0.0 \le a \le 5.0; \\ -\log(0.5) & \text{if } a > 5.0. \end{cases}$$

The function $\nu_Q(a)$ represents quarantine of exposed infectives at a rate of 5% per day and the function $\nu_H(a)$ represents hospitalized (or removed) infectious infectives at a rate of 50% per day. It is assumed that exposed infectives are asymptomatic (only asymptomatic individuals are quarantined) and only infectious infectives are

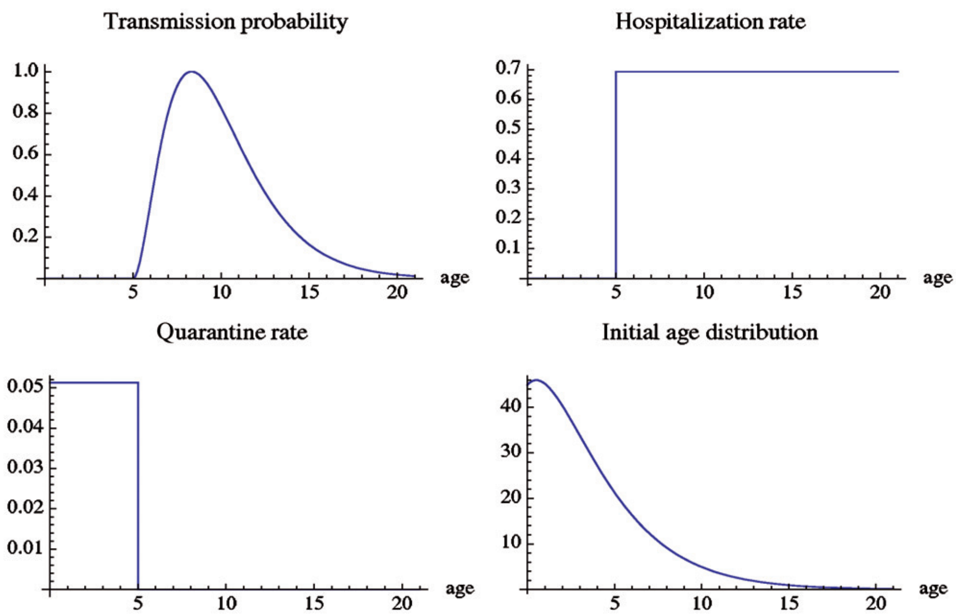

Figure 2. The period of infectiousness begins at day 5 and lasts 16 days. The transmission probability peaks at 8.33333 days. Symptoms appear at day 5, which coincides with the beginning of the infectious period. Infected individuals are hospitalized (or otherwise removed) at a rate of 50% per day after day 5. Pre-symptomatic infectives are quarantined at a rate of 5% per day during the pre-symptomatic period. From the initial infection age distribution we obtain $E(0) = 179.9$ and $I(0) = 72.5$.

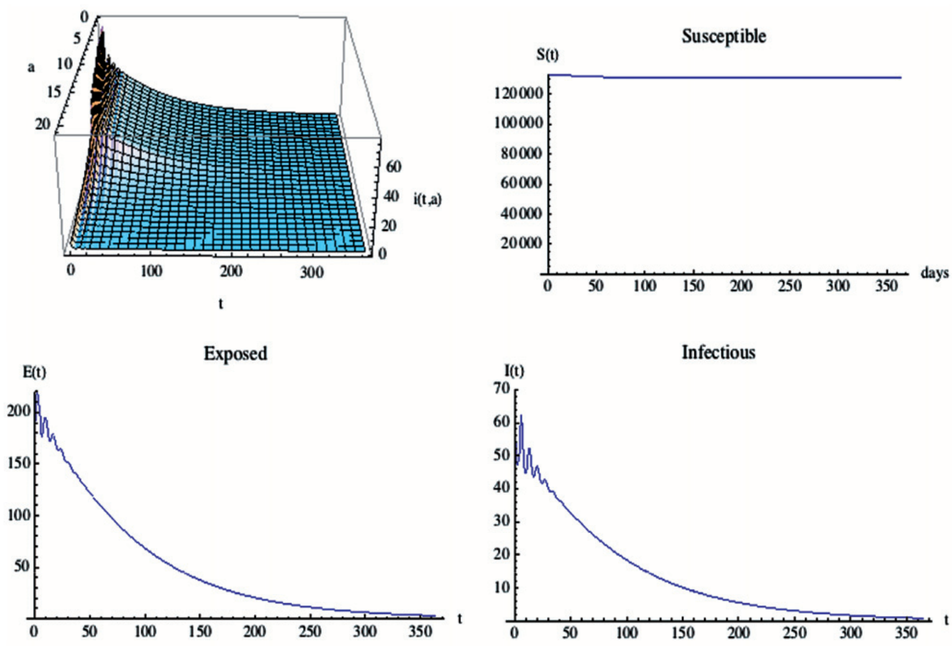

Figure 3. With quarantine of asymptomatic infectives at a rate of 5% per day, the disease is extinguished and the susceptible population converges to the disease-free steady state $\bar{S}_F = \gamma/\nu_S = 133,225$, $\bar{I}_F = 0.0$; $R_0 = 0.939$.

symptomatic (symptomatic infectives are hospitalized or otherwise isolated from susceptibles). These assumptions were valid for the SARS epidemic in 2003, but may not hold for other influenza epidemics. In fact, in the 1918 influenza pandemic the infectious period preceded the symptomatic period by several days, resulting in much higher transmission. We assume that the initial susceptible population is $S(0) = \gamma/\nu_S = 133,225$ and initial age distribution of infectives (see Figure 2) is $\iota(0,a) = 50.0(a+2.0)e^{-0.4(a+2.0)}$, $a \geq 0.0$. For these parameters $R_0 < 1.0$ as in Section 1 and the epidemic is extinguished in approximately 1 year (Figure 3).

Example 2 In our second example we simulate Example 1 without quarantine measures implemented (i.e., all parameters and initial conditions are as in Example 1 except that $\nu_Q(a) \equiv 0.0$). Without quarantine control the disease becomes endemic (Figure 4). In this case $R_0 > 1.0$ and the solutions converge to the endemic equilibrium, as in Section 1. From Figure 4 it is seen that the solutions oscillate as they converge to the disease equilibrium over a period of years. The on-going source γ of susceptibles allows the disease to persist albeit at a relatively low level. At equilibrium the population of susceptibles is significantly lower than the disease-free susceptible population.

Example 3 In our third example we assume that the infectious period and the symptomatic period are not coincident, as in the two examples above. In this case the severity of the epidemic may be much greater, since the transmission potential of some infectious individuals will not be known during some part of their period

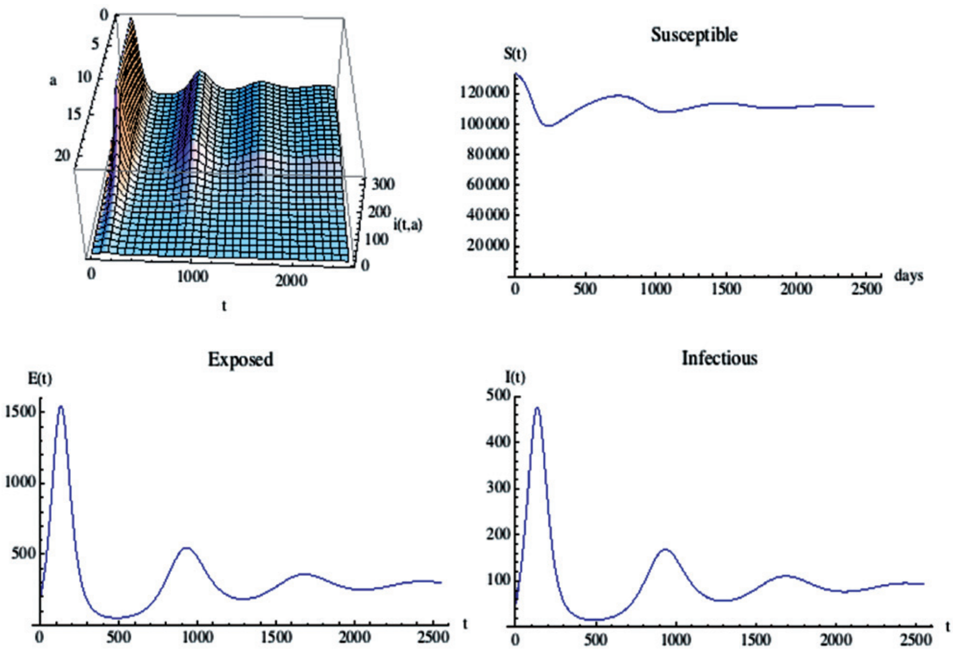

Figure 4. Without quarantine implemented, the disease becomes endemic and the populations converge with damped oscillations to the disease steady state $\bar{S}_E = \frac{\gamma}{R_0\,\nu_S} = 102,480$, $\bar{E} = 369$, $\bar{I} = 120$ and $\bar{\imath}_E(a) = l_{\nu_I}(a)\bar{\imath}_E(0)$, $l_{\nu_I}(a) = \exp\left(-\int_0^a \nu_I(l)\mathrm{d}l\right)$, $\bar{\imath}_E(0) = \gamma - \nu_S\bar{S}_E = 85.8$; $R_0 = 1.26$.

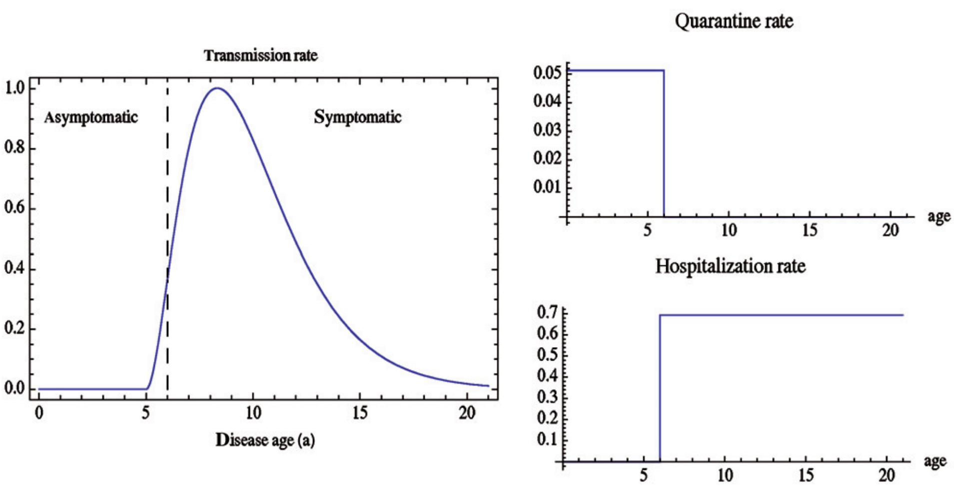

Figure 5. The period of infectiousness overlaps by 1 day the asymptomatic period. Quarantine of asymptomatic infectives ends on day 6. Hospitalization of symptomatic infectives begins on day 6.

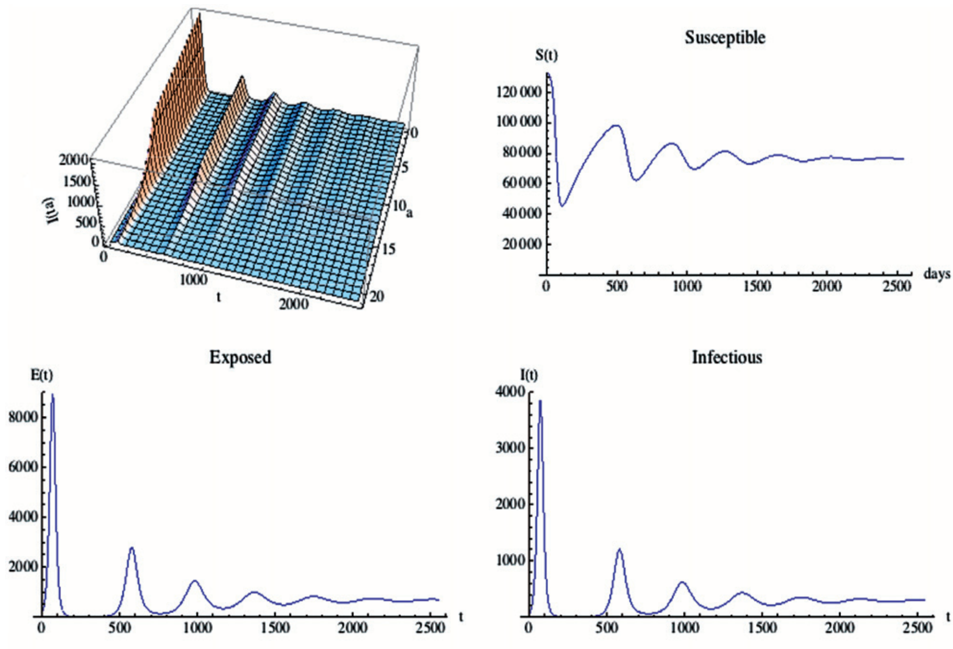

Figure 6. When the infectious period overlaps the asymptomatic period, even with quarantine implemented, the disease becomes endemic and the populations converge with damped oscillations to the disease steady state $\bar{S}_E = \frac{\gamma}{R_0\, \nu_S} = 74,834$, $\bar{E} = 690$, $\bar{I} = 303$ and $\bar{\imath}_E(a) = l_{\nu_I}(a)\bar{\imath}_E(0)$, $l_{\nu_I}(a) = \exp\left(-\int_0^a \nu_I(l)\mathrm{d}l\right)$, $\bar{\imath}_E(0) = \gamma - \nu_S \bar{S}_E = 160.0$; $R_0 = 1.78$.

of infectiousness. We illustrate this case in an example in which the infectious period overlaps the asymptomatic period by 1 day. All parameters and initial conditions are as in Example 1, except that symptoms first appearing on day 6, which means that hospitalization (or removal) of infectious individuals does not begin until 1 day after the period of infectiousness begins. It is also assumed that quarantine of infected individuals does not end until day 6 (Figure 5). In this scenario the epidemic, even with quarantine measures implemented as in Example 1, becomes endemic. The epidemic populations exhibit extreme oscillations, with the infected populations attaining very low values, as the population converges to steady state as in Section 1 (Figure 6).

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