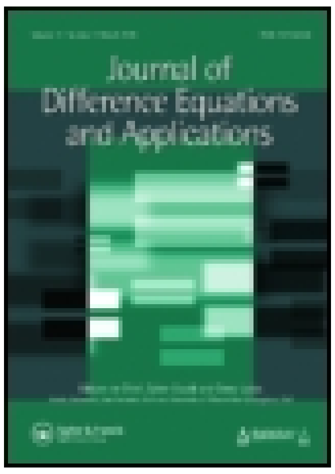

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A Fixed Point Theorem with Application to a Model of Population Dynamics

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In this paper, we investigate the existence of a nontrivial fixed point for a continuous map $F: R_+^n \to R_+^n$ which has 0 as trivial fixed point. We apply our result to a discrete time model of population dynamics of exploited fish. Finally, we give for this model, a necessary and sufficient condition for existence of a nontrivial fixed point.

Keywords: Fixed point; Map; Cone; Population dynamics; Differences equation

Classification Categories: 47H09, 47H10, 92D25.

1 INTRODUCTION

In this paper, the problem of interest to us is to show the existence of a non trivial steady state for an exploited population of fish, described by a discrete time model. More precisely the dynamic of the population is supposed described by a difference equation, of the following form

$\forall t = 0, 1, 2, \ldots$

$$\begin{cases} X(t+1) = H(X(t)) \\ X(0) = X_0 \in R_+^n \end{cases} \tag{1}$$

where X(t) is the state variable of the population, and $H: R_+^n \to R_+^n$ is a continuous map, which satisfies

$$H(0) = 0 \tag{2}$$

Here we denote by R^n the set of the n component real vectors, and by R^n_+ the subset of R^n of all the vectors which have all non negative components.

Moreover, a steady state for the population corresponds to a fixed point of equation (1), and such a fixed point will be called non trivial if it is not zero. This problem has already been studied in another context by Krasnoselskii [*KRA*] and Browder [*BROW*], and they use two different methods to prove an existence result of a non trivial fixed point. For a comparison of these two results one can see Hale's book [*HA*]. Here we prove the existence of a non trivial fixed point when both Krasnoselskii's and Browder's theorem do not apply.

Before giving the results, we present the model which motivated this study. To do it, we need first to recall some other classical discrete time models, which are called density-dependent models of age structured populations. If we consider a population of exploited fish which is structured in n age classes ($n \geq 2$), with some assumptions on the population, we can represent the dynamics of the population by the following system of difference equations:

$\forall t = 0, 1, 2, \ldots$

$$\begin{cases} x_1(t+1) = f_1(\sum_{i=1}^{n} b_i . x_i(t)) \\ x_2(t+1) = x_1(t).\exp(-(M_1 + q_1 E)) \\ x_3(t+1) = x_2(t).\exp(-(M_2 + q_2 E)) \\ \vdots \\ x_n(t+1) = x_{n-1}(t).\exp(-(M_{n-1} + q_{n-1} E)) \end{cases} \tag{3}$$

with the initial condition:

$$x_i(0) = x_i \geq 0, \forall i = 1, 2, \ldots, n$$

where f_1: $R_+ \to R_+$ is a continuous map satisfying

$$f_1(0) = 0,$$

and

$$b_i \geq 0, \forall i = 1, 2, \ldots, n$$

$$M_i \geq 0, q_i \geq 0, \forall i = 1, 2, \ldots, n-1,$$

$$E \geq 0,$$

b_i: is the number of individuals produced by individuals of the ith age class,
M_i: is the natural mortality of the individuals of the ith age class,
q_i: is the catchability of the individuals of the ith age class, and
E: is the fishing effort.

Several examples of models of this type exist. For fish population, the most quoted ones are the following:

The Beverton and Holt model [*BH*] in which:

$$f_1(x) = \frac{x}{1 + \beta.x}, \forall x \geq 0, (\beta > 0);$$

The Ricker model [*RI*]:

$$f_1(x) = x.\exp(-\beta.x), \forall x \geq 0, (\beta > 0),$$

and the Shepherd model [*SH*]:

$$f_1(x) = \frac{x}{1 + \beta.x^c}, \forall x \geq 0, (\beta > 0, c > 0).$$

Another example of a density-dependent population dynamics model is the Liu and Cohen model [*LC*], which takes the following form:
$\forall t = 0, 1, 2 \ldots$

$$\begin{cases} x_1(t+1) = (\sum_{i=1}^{n} b_i.x_i(t)) .\exp(-\sum_{j=1}^{n} \mu_{ij}.x_j(t)) \\ x_2(t+1) = x_1(t).\exp(-(M_1 + q_1 E + \sum_{j=1}^{n} \gamma_{1j}.x_j(t))) \\ x_3(t+1) = x_2(t).\exp(-(M_2 + q_2 E + + \sum_{j=1}^{n} \gamma_{2j}.x_j(t))) \\ \vdots \\ x_n(t+1) = x_{n-1}(t).\exp(-(M_{n-1} + q_{n-1} E + \sum_{j=1}^{n} \gamma_{n-1,j}.x_j(t))) \end{cases} \tag{4}$$

with the initial condition:

$$x_i(0) = x_i \geq 0, \forall i = 1, 2, \ldots, n,$$

Here,

$$\mu_{ij} \geq 0, \forall i, j = 1, 2, \ldots, n$$

$$\gamma_{ij} \geq 0, \forall i = 1, 2, \ldots, n-1, \forall j = 1, 2, \ldots, n.$$

In the case of system (4), existence of non trivial fixed points is less easy to prove directly. Nevertheless, a method for showing the result is to use the Browder ejective fixed point theorem [*BROW*], or the Krasnoselskii fixed point theorem [*KRA*]. In Liu and Cohen's paper, those methods to prove existence of non trivial fixed were not used, but they gave a direct proof of existence and uniqueness of non trivial fixed point.

On the other hand, in fishery problems, it seems natural to suppose that the fishing effort is a function of time. Also, by considering the adaptation of the fishing effort to the yield of catch per unit of fishing effort, one can write:

$$E(t+1) = f_2(Y(t)); \forall t = 0, 1, 2, \ldots$$

where

$Y(t)$: is the Yield of catch per unit of fishing effort, at time t.
$E(t)$: is the fishing effort at time t.

Classically, the yield per unit of fishing effort is:

$$Y(t) = \sum_{i=1}^{n-1} W_i . x_i(t) . \frac{q_i}{(q_i E(t) + M_i)} (1 - \exp(-(M_i + q_i . E(t))))$$

where $\forall i = 1, \ldots, n-1$, $W_i > 0$.

A work in this direction is in preparation, and will be published elsewhere. We give here a possible model which takes into account the adaptation of fishing effort.

The model is as follows
$\forall t = 0, 1, 2, \ldots$

$$
\begin{cases}
x_1(t+1) = f_1(\sum_{i=1}^{n} b_i.x_i(t)) \\
x_2(t+1) = x_1(t).\exp(-(M_1 + q_1 E(t))) \\
x_3(t+1) = x_2(t).\exp(-(M_2 + q_2 E(t))) \\
\vdots \\
x_n(t+1) = x_{n-1}(t).\exp(-(M_{n-1} + q_{n-1} E(t))) \\
E(t+1) = f_2\left(\sum_{i=1}^{n-1} W_i.x_i(t).\dfrac{q_i}{(q_i E(t) + M_i)}(1 - \exp(-(M_i + q_i.E(t))))\right)
\end{cases}
\tag{5}
$$

with the initial condition:

$$x_i(0) = x_i \geq 0; \forall i = 1, 2, \ldots, n$$

and

$$E(0) = E_0 \geq 0$$

and we make the following assumptions on f_1 and f_2.

Assumptions on f_1 and f_2:

(H1) f_1: $R_+ \to R_+$ is a continuous map defined by:

$$f_1(x) = x.h_1(x); \forall x \geq 0,$$

where h_1: $R_+ \to [0, 1]$ is a strictly decreasing continuous function, satisfying:

$$h_1(0) = 1$$

$$\lim_{x \to +\infty} h_1(x) = 0.$$

(H2) f_2: $R_+ \to R_+$ is a continuous map satisfying:

$$f_2(0) = 0$$

Now, if we denote by F: $R_+^{n+1} \to R_+^{n+1}$ the map associated to system (5), this means that system (5) can be rewritten

$\forall t = 0, 1, 2 \ldots$

$$X(t+1) = F(X(t)) \tag{6}$$

with $X(t) = (x_1(t), x_2(t), \ldots, x_n(t), E(t))^T$.

Then,

$$F(^T(0, \ldots, 0, E)) = 0, \forall E \geq 0. \tag{7}$$

So, here the trivial fixed point cannot be ejective, and so the Browder ejective fixed point theorem does not apply. For the same reason, the Krasnoselskii theorem is also not applicable. This has motivated the search for a result which would allow us to obviate this problem.

In the sequel, we will denote M_n (R) (resp: M_n (R_+)) the set of $n \times n$-matrices (resp: non-negative $n \times n$-matrices (i.e.: with all the components non-negative)).

Let $A \in M_n$ (R). We denote

$$Sp(A) \text{ the spectrum of A,}$$

and

$$r(A) \text{ the spectral radius of A.}$$

Finally, throughout the paper state, the topology on R_n (resp: $M_n(R)$), is the topology associated to an arbitrary norm on R^n (resp: to the associated operator norm $M_n(R)$).

In section 2, we will show the following theorem.

THEOREM 1: *Let F: $R^n_+ \to R^n_+$ be a continuous map, and let $Q \in M_n$ $(R_+) - \{0\}$ be a projection matrix (i-e: $Q^2 = Q$). Let $\|.\|$ be a norm on R^n_+ in which $\|Q\| = 1$.*

Suppose the following assertions are satisfied:

i) There exists $M > 0$, such that:
the compact and convex set

$$K = \{X \in R^n_+ : \|X\| \leq M\}$$

is positively invariant by F (i.e.: $F(K) \subset K$)

ii) There exists $m > 0$ $(m < M)$ and $\varepsilon > 0$ $(\varepsilon < m)$, such that:
for X: $m - \varepsilon \leq \|X\| \leq m$,

If $QX \neq 0$, and $QF(X) = \alpha_1 QX$, and $(Id_{R^n} - Q)\, F(X) = \alpha_2\, (Id_{R^n} - Q)\, X$ with $\alpha_2 > 1$, then $\alpha_1 \geq 1$.
If $QX = 0$, and $(Id_{R^n} - Q)\, F(X) = \alpha_3\, (Id_{R^n} - Q)\, X$, then $\alpha_3 \leq 1$.

Then, F has a fixed point $\bar{X}$ with $m \leq \|X\| \leq M$.

One has to note that if the assumptions of Krasnoselskii's fixed point theorem are satisfied, then the assumptions of theorem 1 are satisfied with $Q = Id_{R^n}$. This will be shown at the end of section 2.

The authors do not know of any similar result, excepted the Krasnoselskii theorem in the case where $Q = Id_{R^n}$.

In section 3, we will apply our result to equation (5), and we will give a necessary and sufficient condition for existence of a non trivial fixed point. More precisely, we will prove the following results.

We will first prove that, under the assumptions (H1) and (H2), a necessary condition for existence of a non trivial fixed point of equation (5) is:

$$\text{(H3)} \sum_{i=1}^{n} b_i . \tilde{1}_i > 1$$

where

$$\begin{cases} \tilde{1}_i = 1 \\ \text{and} \\ \tilde{1}_i = \prod_{j=1}^{i-1} \exp(-M_j);\ \forall i = 2, \ldots, n. \end{cases}$$

More precisely, when assumption (H3) is not satisfied, we will see that 0 is a global attractor of the non-negative solutions of equation (5). This means that all the non-negative solutions converge to 0 as t goes to infinity. So, in this case, there is no non-trivial fixed point. This is the goal of lemma 5 in section 3.

Finally we will prove the following theorem, which gives an answer to the starting problem.

THEOREM 2 *Under assumptions (H1), (H2), and (H3), and suppose in addition that*

$$b_n > 0.$$

Then, equation (5) has a non trivial fixed point $\bar{X}$.

2 PROOF OF THEOREM 1 AND ITS RELATION WITH A KRASNOSELSKII THEOREM

Before starting the demonstration of theorem 1, we recall the well-known Brouwer fixed point theorem [*BROU*]. We will use this result in the proof of theorem 1.

THEOREM 3: **(BROUWER)** *Let $K \subset R^n$ be a nonempty compact convex set, and $F: K \to K$ be a continuous function. Then, F has a fixed point in K.*

In the sequel, we will define an order on R^n by

$$X \geq 0 \Leftrightarrow X \in R^n_+,$$

and

$$X > 0 \Leftrightarrow X \in Int\,(R^n_+),$$

where $Int\,(R^n_+)$ denotes the interior of R^n_+.

Note that R^n_+ is a cone, this means

$$R^n_+ + R^n_+ \subset R^n_+$$

and

$$\lambda . R^n_+ \subset R^n_+;\ \forall \lambda \geq 0.$$

Proof of theorem 1 The principle of the proof that we propose here is to derive a new function $\tilde{F}(X)$ from $F(X)$, satisfying:

$$\tilde{F}(\bar{X}) = \bar{X} \Rightarrow F(\bar{X}) = \bar{X} \text{ and } m \leq \|\bar{X}\| \leq M,$$

and to prove that $\tilde{F}(X)$ admits a fixed point $\bar{X}$, by using the Brouwer fixed point theorem.

First step: Transformation of F.

Denote $\forall X \in \mathrm{R}^n_+$

$$\tilde{F}(X) = \varphi_1(\|X\|).F(X) + (1 - \varphi_1(\|X\|)).CQX + \varphi_2(\|X\|).\xi$$

where

$$1 < C < \frac{M}{m}$$

and $\xi \in R^n_+ - \{0\}$ is such that,

$$Q\xi \neq 0$$

and

$$\|\xi\| < M - C.m,$$

and for i = 1, 2, φ_i: $R_+ \to R_+$ is continuous with:

$$\begin{cases} \varphi_1(\alpha) = 1, \forall\alpha \geq m, \\ 0 < \varphi_1(\alpha) < 1, \forall\alpha \in]m - \varepsilon, m[, \\ and \\ \varphi_1(\alpha) = 0, \forall\alpha < m - \varepsilon. \end{cases}$$

Finally,

$$\begin{cases} \varphi_2(\alpha) = 0, \forall\alpha \geq m - \varepsilon, \\ and \\ 0 < \varphi_2(\alpha) < 1, \forall\alpha \in [0, m - \varepsilon[. \end{cases}$$

Second step: Existence of a fixed point of $\tilde{F}$.

Let us prove that

$$K = \{X \in R^n_+ : \|X\| \leq M\}$$

is positively invariant by $\tilde{F}$.

If $\|X\| \in [m, M]$, we have

$$\tilde{F}(X) = F(X)$$

and from i) of Theorem 1, we have

$$F(X) \in K.$$

If $\|X\| \in [m - \varepsilon, m]$, we have

$$\tilde{F}(X) = \varphi_1(\|X\|).F(X) + (1 - \varphi_1(\|X\|)).CQX,$$

thus

$$\begin{aligned}\|\tilde{F}(X)\| &\leq \varphi_1(\|X\|).\|F(X)\| + (1 - \varphi_1(\|X\|)).C\|QX\| \\ &\leq \varphi_1(\|X\|).M + (1 - \varphi_1(\|X\|)).C.m \\ &\leq \varphi_1(\|X\|).M + (1 - \varphi_1(\|X\|)).M = M\end{aligned}$$

(because $C < M / m$)

If $\|X\| \in [0, m - \varepsilon]$, we have

$$\tilde{F}(X) = CQX + \varphi_2(\|X\|).\xi,$$

thus

$$\begin{aligned}\|\tilde{F}(X)\| &\leq C.\|QX\| + \varphi_2(\|X\|).\|\xi\| \\ &\leq C.\|X\| + \|\xi\| \leq C.m + \|\xi\| \leq M\end{aligned}$$

(because $\|\xi\| < M - C.m$).

Finally K is positively invariant by $\tilde{F}$, and we can apply the Brouwer fixed point theorem to $\tilde{F}$. Thus, there exists $\bar{X} \in R^n_+$ such that $\tilde{F}(\bar{X}) = \bar{X}$.

Third step: Let us show that $\tilde{F}$ admits no fixed point $\tilde{X} \in R^n_+$ such that $\|\bar{X}\| < m$.

First case: If $\|\bar{X}\| \in]m - \varepsilon, m[$ then:

$$\bar{X} = \tilde{F}(\bar{X}) = \varphi_1(\|\bar{X}\|).F(\bar{X}) + (1 - \varphi_1(\|\bar{X}\|)).CQ\,\bar{X}$$

(since $\varphi(\alpha) = 0$ for $\alpha \geq m - \varepsilon$).

If $Q\bar{X} \neq 0$, then

$$Q\,\bar{X} = \varphi_1(\|\bar{X}\|).QF(\bar{X}) + (1 - \varphi_1(\|\bar{X}\|)).CQ\,\bar{X}$$

and

$$(Id_{R^n} - Q)\,\bar{X} = \varphi_1(\|\bar{X}\|).(Id_{R^n} - Q)\,F(\bar{X})$$

and since $0 < \varphi_1\,(\|\bar{X}\|) < 1$ we have:

$$QF(\bar{X}) = \alpha_1.Q\,\bar{X}$$

and

$$(Id_{R^n} - Q)\,F(\bar{X}) = \alpha_2(Id_{R^n} - Q)\,\bar{X}$$

with

$$\alpha_2 = \frac{1}{\varphi_1(\|\bar{X}\|)} > 1$$

Since $\|\bar{X}\| \in\,]m - \varepsilon, m[$, assumption ii) implies that

$$\alpha_1 \geq 1$$

So, we have:

$$1 = \varphi_1(\|\bar{X}\|).\alpha_1 + (1 - \varphi_1(\|\bar{X}\|)).C$$

with $0 < \varphi_1\,(\|\bar{X}\|) < 1$, $\alpha_1 \geq 1$, and $C > 1$, which is impossible.

If $Q\,\bar{X} = 0$, then:

$$\bar{X} = \varphi_1(\|\bar{X}\|).F(\bar{X})$$

and since $0 < \varphi_1\,(\|\bar{X}\|) < 1$ we have:

$$(Id - Q)\,F(\bar{X}) = \alpha_3(Id - Q)\,\bar{X}$$

with

$$\alpha_3 = \frac{1}{\varphi_1(\|\bar{X}\|)} > 1$$

in contradiction with assumption ii).

So $\|\bar{X}\| \geq m$ or $0 \leq \|\bar{X}\| \leq m - \varepsilon$.

Second case: If $0 \leq \|\bar{X}\| \leq m - \varepsilon$.

If $\|\bar{X}\| = m - \varepsilon$ we have:

$$\bar{X} = CQ\,\bar{X}$$

So, $Q\,\bar{X} \neq 0$ and $Q\,\bar{X} = CQ\,\bar{X}$
and

$$C = 1,$$

which is impossible.
Now, if $0 \leq \|\bar{X}\| < m - \varepsilon$, we have:

$$\bar{X} = C.Q\,\bar{X} + \varphi_2(\|\bar{X}\|)\,.\xi$$

This implies, since $\xi \geq 0$ and $Q\,\bar{X} \geq 0$

$$\bar{X} \geq \varphi_2(\|\bar{X}\|)\,.\xi$$

$$\bar{X} \geq C.Q\,\bar{X}$$

with $\varphi_2\,(\|\bar{X}\|) > 0$ and $Q\xi \neq 0$.
So

$$Q\,\bar{X} \neq 0$$
$$\textit{and}$$
$$Q\,\bar{X} \geq C.Q\,\bar{X}.$$

This gives

$$C \leq 1,$$

which, once again, is impossible.
So, we proved that, if $\bar{X} = \tilde{F}\,(\bar{X})$ then $\|\bar{X}\| \geq m$.
Thus

$$\tilde{F}\,(\bar{X}) = F(\bar{X})$$

and the proof of theorem 1 is complete.□

Our next purpose is to establish a relationship between a Krasnoselskii theorem [*KRA*] and theorem 1. We start by a technical lemma that we will use in the proof of Krasnoselskii's theorem.

LEMMA 4 *Let $F: R^n_+ \to R^n_+$ be a continuous function, and $\|.\|$ a norm on R^n.*
Suppose that there exists $M > 0$ such that:
if $\|X\| = M$ and $F(X) = \alpha X$ then $\alpha > 1$ (resp: $\alpha < 1$).
Then, there exists $\varepsilon > 0$ such that:
if $M - \varepsilon \leq \|X\| \leq M + \varepsilon$ and $F(X) = \alpha X$ then $\alpha > 1$ (resp: $\alpha < 1$).

Proof: This is just a consequence of F being continuous and compactness of any closed bounded subset of R^n.
We will not detail further the proof of this result.□

We now give a statement in finite dimension of the Krasnoselskii fixed point theorem. The following statement is a direct consequence of the fixed point theorem in [*KRA*] theorem 4.14 p: 148.

THEOREM 5: (**KRASNOSELSKII**) *Let $F: R^n_+ \to R^n_+$ be a continuous function, and $\|.\|$ a norm on R^n.*
Suppose the following assertion be satisfied:

i) There exists $M > 0$, such that:
the compact and convex set

$$K = \{X \in R^n_+ : \|X\| \leq M\}$$

is positively invariant by F (i.e.: $F(K) \subset K$)
ii) There exists m: $0 < m < M$ such that:
If $\|X\| = m$ and $F(X) = \alpha X$ then $\alpha > 1$.
Then F has a fixed point $\bar{X} \in R^n_+$ such that: $m < \|X\| < M$.

Proof: From Lemma 3, there exists $\varepsilon > 0$ such that:
if $m - \varepsilon \leq \|X\| \leq m + \varepsilon$ and $F(X) = \alpha X$ then $\alpha > 1$.
And one can always choose ε small enough so that

$$m - \varepsilon > 0$$

and

$$m + \varepsilon < M$$

So, as a consequence of theorem 1, for $Q = Id_{R^n}$, we deduce that: there exists $\bar{X} \in R^n_+$ such that: $F(\bar{X}) = \bar{X}$ and $m < \|X\| < M$.□

Finally, the above result shows that the Krasnoselskii fixed point theorem can be obtained as a consequence of Theorem 1.

3 APPLICATION TO A MODEL OF POPULATION DYNAMICS

In this section we give a proof of theorem 2. To do it, we first need to prove some intermediate results, and theorem 2 will follow from this. On the other hand, in this section, we will use some general properties of non negative matrices. We refer to the book by Horn and Johnson [*HJ*] or Gantmacher [*GA*] for more details.

In the sequel, if $X \in R^n$ and $i \in \{1, \ldots, n]$ we denote by

$$X_i \text{ or } (X)_i$$

the i^{th} component of X.

In particular if $A \in M_n(R_+)$ we denote by

$$(AX)_i$$

the i^{th} component of AX.

We denote $F\colon R_+^{n+1} \to R_+^{n+1}$ the map associated to equation (5).

Equation (5) can be written as follows:

$\forall t = 0, 1, 2, \ldots$

$$\begin{aligned} X(t+1) &= F(X(t)) \\ X(0) &= X_0 \in R_+^{n+1} \end{aligned} \tag{8}$$

where

$\forall t = 0, 1, 2, \ldots$

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t), E(t)))^T \tag{9}$$

We will first prove that, when $\sum_{i=1}^{n} b_i . \tilde{1}_i \leq 1$, 0 is a global attractor of the non-negative solutions of equation (5). This means that all the non-negative solutions converge to 0 as t goes to infinity. So in this case, we can not have a non-trivial fixed point.

LEMMA 5 *Under the assumptions (H1) and (H2), and if* $\sum_{i=1}^{n} b_i . \tilde{1}_i \leq 1$, *then for all* $X_0 \in R_-^{n-1}$. *the solution of equation (5) with* X_0 *as initial value. converges to 0 as t goes to infinity.*

Proof Equation (5) reads
$\forall t = 0, 1, 2, \ldots$

$$
\begin{cases}
x_1(t+1) = f_1\left(\sum_{i=1}^{n} b_i . x_i(t)\right) \\
x_2(t+1) = x_1(t).\exp(-(M_1 + q_1 E(t))) \\
x_3(t+1) = x_2(t).\exp(-(M_2 + q_2 E(t))) \\
\vdots \\
x_n(t+1) = x_{n-1}(t).\exp(-(M_{n-1} + q_{n-1} E(t))) \\
E(t+1) = f_2\left(\sum_{i=1}^{n-1} W_i . x_i(t) . \dfrac{q_i}{(q_i E(t) + M_i)} (1 - \exp(-(M_i + q_i . E(t))))\right)
\end{cases}
$$

Now, make the following change of variable:
$\forall t = 0, 1, 2, \ldots$

$$
\begin{aligned}
y_1(t) &= x_1(t) \\
y_2(t) &= x_2(t) \exp(M_1) \\
&\vdots \\
y_j(t) &= x_j(t) \prod_{i=1}^{j-1} \exp(M_i) \\
&\vdots \\
y_n(t) &= x_n(t) \prod_{i=1}^{n-1} \exp(M_i)
\end{aligned}
$$

Then under this change of variables, equation (5) becomes
$\forall t = 0, 1, 2, \ldots$

$$
\begin{cases}
y_1(t+1) = f_1\left(\sum_{i=1}^{n} b_i \tilde{l}_i y_i(t)\right) \\
y_2(t+1) = y_1(t).\exp(-q_1 E(t)) \\
y_3(t+1) = y_2(t).\exp(-q_2 E(t)) \\
\vdots \\
y_n(t+1) = y_{n-1}(t).\exp(-q_{n-1} E(t))) \\
E(t+1) = f_2\left(\sum_{i=1}^{n-1} W_i \tilde{1}_i y_i(t) \dfrac{q_i}{(q_i E(t) + M_i)} (1 - \exp(-(M_i + q_i . E(t))))\right)
\end{cases}
\tag{10}
$$

Now, we will use the Liapunov direct method to show the result. On this subject, we refer to the monograph by La Salle [*LA*].

Denote

$\forall t=0, 1, 2, \ldots$

$$Y(t) = {}^T(y_1(t), y_2(t), \ldots, y_n(t), E(t)))$$

and

$$V(Y(t)) = \max_{i=1,\ldots,n} (y_i(t))$$

We have:

$$y_1(t+1) = \left(\sum_{i=1}^{n} b_i \tilde{l}_i y_i(t)\right) h_1 \left(\sum_{i=1}^{n} b_i \tilde{l}_i y_i(t)\right)$$
$$\leq \left(\sum_{i=1}^{n} b_i \tilde{l}_i y_i(t)\right)$$

(since $h_1\ (x) \leq 1;\ \forall x \geq 0$)

so

$$y_1(t+1) \leq \left(\sum_{i=1}^{n} b_i \tilde{l}_i\right) V(Y(t)) \leq V(Y(t)),$$

(since $(\sum_{i=1}^{n} b_i \tilde{l}_i) \leq 1$).

In addition, one can see that, under the assumption made on $h_1\ (\ )$ we have:

$$y_1(t+1) = V(Y(t)) \Leftrightarrow \left(\sum_{i=1}^{n} b_i \tilde{l}_i y_i(t)\right) = 0$$

(because $0 < h_1\ (x) < 1;\ \forall x \geq 0$)

So

$$y_1(t+1) = V(Y(t)) \Leftrightarrow y_1(t+1) = 0 \tag{11}$$

and for i = 1, 2,..., n-1, we have:

$$y_{i+1}(t+1) = y_i(t).\exp(-q_i E(t))) \leq y_i(t) \leq V(Y(t))$$

Finally, this yields

$$V(Y(t+1)) \leq V(Y(t)), \forall t = 0, 1, 2, \ldots$$

So, V(.) is a Liapunov function for equation (10). From (11) we deduce that:

$$\begin{cases} V(Y(t+1)) = V(Y(t)), \forall t = 0, 1, 2, \ldots \\ \Rightarrow \\ Y((n+1)) = 0 \end{cases} \tag{12}$$

Now,

$$V(Y(t)) \geq 0; \forall t = 0, 1, 2, \ldots$$

and the sequence $\{V(Y(t))\}_{t=0,1,2,\ldots}$ is decreasing. This implies $V(Y(t)) \to \alpha$ as t goes to infinity.

Now, denote

$$\omega(Y(0)) = \bigcap_{p=1}^{+\infty} clos(\{Y(t)\}_{t \geq p}),$$

the omega-limit set of $\{Y(t)\}_{t=0,1,2,\ldots}$. We remark that this set is nonempty, because the solution $\{Y(t)\}_{t=0,1,2,\ldots}$ is bounded.

Let G: $R_+^{n+1} \to R_+^{n+1}$ the map associated to equation (10). We have:

$$G(\omega(Y(0))) = \omega(Y(0))$$

and $\forall Z \in \omega(Y(0))$

$$V(Z) = \alpha.$$

We conclude by using (12) that:

$$\omega(Y(0)) = \{0\},$$

and the lemma is proved.□

Now we show that the assumption ii) of theorem 1 is satisfied by F, the associated map of system (5), under the assumptions (H1), (H2), and (H3).

LEMMA 6: *Let* $\|.\|$ *be a norm on* R^{n+1}.

Under the assumptions (H1), (H2), (H3), and suppose:

$$b_n > 0.$$

Then, if we denote:

$$Q = \begin{pmatrix} & & 0 \\ & Id_{R^n} & \vdots \\ & & 0 \\ 0 \cdots & 0 & 0 \end{pmatrix} \in M_{n+1}(R),$$

there exists $m > 0$, such that:
for all $X \in R_+^{n-1}$, with $0 < \|X\| < m$,
If $QX \neq 0$ and $QF(X) = \alpha QX$ then $\alpha > 1$,
If $QX = 0$ then $F(X) = 0$.

Proof: Recall that in accordance to formula (9),

$$X = (x_1, x_2, \ldots, x_n, E)^T;$$

We will denote

$$\tilde{X} = (x_1, x_2, \ldots, x_n)^T.$$

The case $QX = 0$ is trivial.
Since $QX = 0 \Rightarrow x_1 = x_2 = \ldots = x_n = 0$
And from assumption (H2), we have $f_2(0) = 0$.
We have:

$$QX = 0 \Rightarrow F(X) = 0$$

Now, consider the $QX \neq 0$ and $QF(X) = \alpha QX$.
We have:

$$\begin{cases} \alpha x_1 = f_1(\sum_{i=1}^{n} b_i . x_i) \\ \alpha x_2 = x_1 . \exp(-(M_1 + q_1 E)) \\ \alpha x_3 = x_2 . \exp(-(M_2 + q_2 E)) \\ \vdots \\ \alpha x_n = x_{n-1} . \exp(-(M_{n-1} + q_{n-1} E)) \end{cases} \tag{13}$$

Introducing the matrix

$$L(X) = \begin{pmatrix} b_1 h_1 \left(\sum_{i=1}^{n} b_i x_i \right) & b_2 h_1 \left(\sum_{i=1}^{n} b_i x_i \right) & \cdot\ \cdot & & & b_n h_1 \left(\sum_{i=1}^{n} b_i x_i \right) \\ \exp(-(M_1 + q_1 E)) & 0 & \cdot & & \cdot & 0 \\ 0 & \exp(-(M_2 + q_2 E)) & & & & \\ & \cdot & & & & \\ & & & & & \cdot \\ 0 & \cdot & & 0 & \exp(-(M_{n-1} + q_{n-1} E)) & 0 \end{pmatrix}$$

equation (13) can be rewritten as:

$$\alpha \tilde{X} = L(X) \tilde{X} \tag{14}$$

Now, we remark that, since $b_n > 0$, for all $X \in R_+^{n+1}$ L(X) is an irreducible non-negative matrix, because the oriented graph of L(X) is strongly connected if and only if $b_n > 0$.

So, by using the property of irreducible non-negative matrices see [*HJ*],
$\forall V \in R_+^n - \{0\}$ such that $L(X)V = \alpha V$ with $\alpha \geq 0$
then $\alpha = r(L(X))$ and $V_i > 0$ $\forall i = 1, \dots, n$.

In addition, the map L: $R_+^{n+1} \to M_n(R_+)$ is continuous and in view of (H3) we have:

$$r(L(0)) > 1.$$

Indeed, the characteristic equation of $L(0)$ is:

$$\lambda^n = b_1 \tilde{l}_1 \lambda^{n-1} + b_2 \tilde{l}_2 \lambda^{n-2} + \ldots + b_n \tilde{l}_n,$$

and, so:

$$r(L(0))^n = b_1 \tilde{l}_1^{n-1} r(L(0))^{n-1} + b_2 \tilde{l}_2 r(L(0))^{n-2} + \ldots + b_n \tilde{l}_n$$
$$\text{and}$$
$$r(L(0)) > 0$$

One can rewrite this equation as:

$$P(r(L(0))) = 1$$
$$\text{and}$$
$$r(L(0)) > 0$$

with

$$P(x) = \sum_{i=1}^{n} b_i \tilde{l}_i x^{-i}, \forall x > 0$$

But, the derivative of P(x) satisfies

$$P'(x) < 0, \forall x > 0$$

and

$$P(1) = \sum_{i=1}^{n} b_i \tilde{l}_i > 1$$

So

$$r(L(0)) > 1.$$

To conclude, remark that $r(L(X))$ is a continuous function of X, because L(X) is a continuous function of X.

So, there exists $m > 0$ such that:

for all $X \in R_+^{n+1}$, with $0 \leq \|X\| \leq m$, $r(L(X)) > 1$.

Now, return to the initial problem.

For all $X \in R_+^{n+1}$, with $0 < \|X\| < m$, $QF(X) = \alpha QX$ and $QX \neq 0 \Leftrightarrow L(X)\tilde{X} = \alpha \tilde{X}$

So $\alpha = r(L(X)) > 1$

and lemma 6 is proved.□

To apply theorem 1, it remains to be shown that assertion i) holds, for equation (5).

LEMMA 7: *Under the assumptions (H1), (H2), and (H3), there exists* $\|.\|$ *a norm on* R^{n+1} *such that*

$$K = \{X \in R_+^{n+1}: \|X\| \leq 1\}$$

is positively invariant by F(X), the associated map to system (5). Moreover, for this norm we have

$$\|Q\| = 1.$$

Proof: Let $X \in R_+^{n+1} - \{0\}$ $X = (x_1, x_2, \ldots, x_n, E)^T$.

$$F(X) = \begin{pmatrix} f_1(\sum_{i=1}^{n} b_i.x_i) \\ x_1.\exp(-(M_1 + q_1E)) \\ x_2.\exp(-(M_2 + q_2E)) \\ \vdots \\ x_{n-1}.\exp(-(M_{n-1} + q_{n-1}E)) \\ f_2\left(\sum_{i=1}^{n-1} W_i.x_i \frac{q_i}{(q_iE + M_i)}(1 - \exp(-(M_i + q_i.E)))\right) \end{pmatrix}$$

Denote by

$$\breve{f}_1(x) = f_1\left(\left(\sum_{i=1}^{n} b_i\right)x\right); \forall x \in R_+$$

From (H1), there exists $\gamma_1 > 0$ such that:

$$\tilde{f}_1(x) < \gamma_1, \forall x \in [0, \gamma_1].$$

So, if

$$x_i \in [0, \gamma_1]; \forall i = 1,\ldots, n$$

we must have

$$f_1(\sum_{i=1}^{n} b_i.x_i) < \gamma_1$$

(since $\alpha \geq 1$).

$$\begin{gathered} x_1.\exp(-(M_1 + q_1E)) \leq x_1 < \gamma_1 \\ x_2.\exp(-(M_2 + q_2E)) \leq x_2 < \gamma_1 \\ \vdots \\ x_{n-1}.\exp(-(M_{n-1} + q_{n-1}E)) \leq x_{n-1} < \gamma_1 \end{gathered}$$

and

$$f_2\left(\sum_{i=1}^{n-1} W_i x_i \frac{q_i}{(q_i E + M_i)}(1 - \exp(-(M_i + q_i.E)))\right)$$
$$\leq \max_{0\leq\beta\leq\sum_{i=1}^{n-1} W_i \frac{q_i}{M_i}\gamma_1} f_2(\beta)$$

Let $\gamma_2 > 0$ be a positive constant, such that

$$\max_{0\leq\beta\leq\sum_{i=1}^{n-1} W_i \frac{q_i}{M_i}\gamma_1} f_2(\beta) \leq \gamma_2$$

Finally, if we denote, for all $X \in R^{n+1}$, $X = (x_1, x_2, \ldots, x_{n+1})^T$

$$\|X\| = \max\left(\max_{i=1,\ldots,n}\left(\frac{1}{\gamma_1}|x_i|\right), \frac{1}{\gamma_2}|x_{n+1}|\right)$$

we have

$$F(K) \subset K$$

where $K = \{X \in R^{n+1}: \|X\| \leq 1\}$.

On the other hand, by a simple computation obtain

$$\|Q\| = \sup\{\|QX\|: X \in R^{n+1}, \|X\| = 1\} = 1.$$

Lemma 7 is proved.□

Proof of theorem 2: It is a direct application of theorem 1, by using lemma 6 and lemma 7, and the fact that all norms are equivalent in finite dimension space.□

CONCLUSION

Once the fixed point problem has been solved and non trivial steady states have been determined a fundamental question remains, that is, the stability of those points.

A related question is to look at the particular role of the various parameters entering the equation on the appearance of new equilibria, and possibly the exchange of stability. Regarding the exploitation parameters, it does not seem to play any role in the onset of new equilibria.

The condition (H3), that is

$$\sum_{i=1}^{n} b_i . \tilde{l}_i > 1$$

which yields such equilibria does not involve the function f_2 and holds even in the case when $f_2 = 0$, that is, without exploitation.

This apparent independence can be explained by the fact that the exploitation we are considering here depends on the yield of catch by fishing effort unit, it is "adapted" to the abundance of the previous year.

The situation would probably be different if for example we assumed that:

$$f_2(x) > E_{min} > 0, \forall x \geq 0.$$

In this case, we conjecture that the condition for non extinction of the population (or, existence of a non trivial equilibrium) will involve the parameter $E_{\min}$.

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