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EVENTUAL COMPACTNESS FOR SEMIFLOWS GENERATED BY NONLINEAR AGE-STRUCTURED MODELS

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Abstract. In this paper we investigate compactness properties for a semiflow generated by a semi-linear equation with non-dense domain. We start with the non-homogeneous linear case, and, and we derive some abstract conditions for non-autonomous semilinear equations. Then we investigate a special situation which is well adapted for age-structured equations. We conclude the paper by applying the abstract results to an age-structured models with an additional structure.

1. **Introduction.** In this paper, we consider an age-structured population model of the form

$$\begin{cases} (\partial_t+\partial_a)u(t,a) = & A(a)u(t,a)+J\big(t,u(t,\cdot)\big)(a) & \begin{cases} t>0, \\ 0<a<c, \end{cases} \\ & +B_2\big(t,a,\overline{u}_2(t)\big)u(t,a), & \\ u(t,0) = & \displaystyle\int_0^c B_1(t,a,\overline{u}_1(t))u(t,a)da, & t>0, \\ \overline{u}_j(t) = & \displaystyle\int_0^c C_j(t,a)u(t,a)da, & t>0, \\ u(0,a) = & u_0(a), & 0\le a<c, \\ u(t,a) = & 0, & t\ge 0, a>c. \end{cases} \tag{1.1}$$

The number c denotes the maximum possible age and $u(t,\cdot)$ is the age distribution of the population. The population may carry an additional structure which is coded in a Banach space Y with $Y \ni u(t,a)$. We refer to the books by Webb [45], Metz and Dieckmann [30], and Iannelli [23], Busenberg and Cooke [15], and Anita [7] for nice surveys on age-structured models.

To investigate such a system, one can use solutions integrated along the characteristics, and derive a nonlinear Volterra equation. One can also use nonlinear semigroup theory. We refer to Webb [45] for more information about these two approaches. Here we use integrated semigroup theory to study equation (1.1). This paper is in the line of the works by Thieme [37, 40, 41, 39], Matsumoto, Oharu,

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Thieme [29], Magal [26], Thieme and Vrabie [42]. One can note that such a technique was also developed in the context of (neutral) delay differential equations. We refer to Adimy [1, 2], Adimy and Arino [3], Adimy and Ezzinbi [4, 5] for more precisions about the delay case.

As we will see in Section 6, under some assumptions on the family of (eventually unbounded) linear operator $A(a)$, one can find a Banach space X, a linear operator $A : D(A) \subseteq X \to X$, and a map $F : [0,T] \times \overline{D(A)} \to X$ such that (1.1) can be written as

$$U(t,s)\,x = x + A\int_s^t U(t,s)\,x ds + \int_s^t F(s, U(t,s)\,x)ds,\ \forall t \geq s \geq 0, \forall x \in \overline{D(A)}. \tag{1.2}$$

The goal of the paper is to investigate compactness properties of the semiflow generated by (1.2). Under some conditions on F one has the uniqueness of the solutions of (1.2), and the family of nonlinear operator $\{U(t,s)\}_{t\geq s\geq 0}$ define a non-autonomous semiflow, that is to say that

$$U(t,t) = Id, \forall t \geq 0, \text{ and } U(t,r)U(r,s) = U(t,s), \forall t \geq r \geq s \geq 0.$$

If we assume that for some bounded set $B \subset \overline{D(A)}$, and some $s \geq 0$, $U(t,s)x$ exists for all $t \geq s$, and for all $x \in B$. Then we look for conditions to verify that

$$\alpha\,(U(t,s)B) \to 0, \text{ as } t \to +\infty,$$

where $\alpha\,(.)$ is the measure of non-compactness of Kuratovski (or ball measure of non-compactness) (see Martin [28], and Deimling [19] for more information about measures of non-compactness). In other words, we want to prove that the semiflow is asymptotically smooth in the sense of Hale [20]. By using this property one is then able to apply attractor theory. We refer to Hale [20], Temam [36], Babin and Vishik [11], Robinson [34], Ladyzhenskaya [25], Sell and You [35], Zhao [46] for nice surveys on global attractor theory.

The compactness properties of semiflows were first investigated in the dense domain case (i.e. $\overline{D(A)} = X$), by Ball [12, 13], Dafermos and Slemrod [17], Pazy [32, 33], Webb [44], Henri [22], Haraux [21], Vrabie [43]. Here, the point is to prove similar results for the non-dense domain case (i.e. $\overline{D(A)} \neq X$). Let $\{T_0(t)\}_{t\geq 0}$ denote the C_0-semigroup of linear operator generated by the part of A in $\overline{D(A)}$. In the case where $\{T_0(t)\}_{t\geq 0}$ is compact (i.e. $T_0(t)$ is compact for each $t > 0$) the problem is well understood (see Bouzahir and Ezzinbi [14], and Theorem 3.9 of this paper). In the case of age-structured models the semigroup $\{T_0(t)\}_{t\geq 0}$ is not compact, and the problem becomes more complicated to study. In fact in the non-compact case, we need some regularity condition for the maps $t \to F(t, U(t,s)x)$ (see Assumption 3.3 b) of Theorem 3.7, and Assumption 4.1 c) of Theorem 4.3). Because of this, the problem becomes completely different compared with the compact case. Also, from the case $Y = \mathbb{R}$ in age-structured model, it is clear that additional conditions like Assumption 3.3 b) are necessary in the non-compact case.

In Thieme [40] this problem is investigated for linear age structured systems, and for a general Banach space Y. In Magal [26] the case $Y = \mathbb{R}^n$, $J = 0$, and $c \in (0, +\infty)$ is investigated. One can also compare Assumption 4.1 b) of this paper, and Assumption 5.3 d) in Magal [26], to see that the condition given here is more general. In particular with the condition given in Magal [26], one needs much stronger assumptions in the applications to obtain some compactness properties (because one cannot use the technics developed in section 5 of this paper). In Thieme and Vrabie [42] this problem is investigated for a general Banach space Y, but they assume that $J = 0$ and $B_2 = 0$. Here we investigate the general situation. In the general case, one needs to work more to derive some time regularity of $t \to F(t, U(t, s)x)$ (compare section 3 in [42], and section 5 of this paper). For age structured problems, we also refer to Webb [45] for a nice treatment of this problem in the case where $Y = \mathbb{R}^n$, $c = +\infty$, and by using solutions integrated along the characteristics.

The plan of the paper is the following. In section 2, we recall some classical results about integrated semigroups. In section 3, we present some general compactness results for an abstract non-homogeneous Cauchy problem. The main novelty in section 3 is Theorem 3.7. Just for comparison between the case where $\{T_0(t)\}_{t\geq 0}$ is compact, and the non-compact case, we also prove Theorem 3.9 which corresponds to the compact case. This will help the reader to compare both situations. In section 4, we derive a general result for non-autonomous semilinear systems. In section 5, we mainly investigate the regularity of $t \to F(t, U(t, s)x)$, and we derive some abstract conditions that will be more applicable in the context of age structured systems. Finally in section 6 we apply the main result of section 5 to equation (1.1).

2. **Preliminaries.**

In this section, we recall some classical results about integrated semigroups. We refer to Arendt [8][9], Kellermann and Hieber [24], Neubrander [31], Arendt et al. [10], and Thieme [38] for nice surveys on the subject. Let Y, Z be two Banach spaces, in the sequel we denote by $\mathcal{L}(Y, Z)$ the space of bounded linear operators from Y to Z.

ASSUMPTION 2.1. *Let $A : D(A) \subseteq X \to X$ be a linear operator. We assume that there exist real constants $M \geq 1$, and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subseteq \rho(A)$, and*

$$\left\|(\lambda - A)^{-n}\right\| \leq \frac{M}{(\lambda-\omega)^n},\ \textit{for all } n \in \mathbb{N}\setminus\{0\},\ \textit{and all } \lambda > \omega.$$

In the sequel, a linear operator $A : D(A) \subseteq X \to X$ satisfying Assumption 2.1 will be called **a Hille-Yosida operator**. We set $X_0 = \overline{D(A)}$, and we denote by A_0 **the part of** A **in** X_0, that is

$$A_0 x = Ax \text{ for all } x \in D(A_0) = \{y \in D(A) : Ay \in X_0\}.$$

Then $D(A_0)$ is dense in X_0, and A_0 generates a strongly continuous semigroup of linear operators on X_0 that is denoted by $\{T_0(t)\}_{t\geq 0}$.

DEFINITION 2.2. *A family of bounded linear operators $S(t)$, $t \geq 0$, on a Banach space X is called an **integrated semigroup** if and only if*
i) $S(0) = 0$.
ii) $S(t)$ is strongly continuous in $t \geq 0$.
iii) $S(r)S(t) = \int_0^r (S(l+t) - S(l))dl = S(t)S(r)$ for all $t, r \geq 0$.

The **generator** A of a non-degenerate integrated semigroup is given by requiring that, for $x, y \in X$,

$$x \in D(A), y = Ax \Leftrightarrow S(t)x - tx = \int_0^t S(s)yds \quad \forall t \geq 0.$$

It follows from this definition that

$$S(t)x = A\int_0^t S(s)xds + tx \quad \forall t \geq 0, \forall x \in X.$$

Notice that the previous formula implies that $\int_0^t S(s)xds \in D(A), \forall t \geq 0, x \in X$. So in particular $S(t)x \in \overline{D(A)}, \forall t \geq 0, x \in X$. It is well known that a Hille-Yosida operator A generates an integrated semigroup $\{S(t)\}_{t\geq 0} \subseteq \mathcal{L}(X, X_0)$. The family $\{S(t)\}_{t\geq 0}$ is locally Lipschitz continuous. More precisely, we have for all t and s such that $t \geq s \geq 0$,

$$\|S(t) - S(s)\|_{\mathcal{L}(X,X_0)} \leq M\int_s^t e^{\omega r}dr.$$

The map $t \to S(t)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$, $\frac{d}{dt}S(t)x = T_0(t)x, \forall t \geq 0, \forall x \in X_0$, and

$$T_0(r)S(t) = S(t+r) - S(r) \quad \forall r, t \geq 0.$$

We also have the following explicit formula (see Magal [27]) for all $x \in X$, and for all $\mu > \omega$,

$$S(t)x = \mu\int_0^t T_0(s)\left(\mu - A\right)^{-1} xds + \left(\mu - A\right)^{-1} x - T_0(t)\left(\mu - A\right)^{-1} x. \tag{2.1}$$

The main tool for nonlinear considerations is the following theorem which was first proved by Da Prato and Sinestrari [18] by using a direct approach, and by Kellermann and Hieber [24] by using integrated semigroups.

THEOREM 2.3. *Assume that A is a Hille-Yosida operator, and $f \in L^1((0,\tau), X)$. We set*

$$(S * f)(t) = \int_0^t S(t-s)f(s)ds, \forall t \in [0,\tau].$$

Then $t \to (S * f)(t)$ is continuously differentiable, $(S * f)(t) \in D(A), \forall t \in [0,\tau]$, $t \to A(S * f)(t)$ is continuous, and if we set $u(t) = \frac{d}{dt}(S * f)(t)$, then

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \forall t \in [0,\tau],$$

and

$$\|u(t)\| \leq M\int_0^t e^{\omega(t-s)}\|f(s)\| ds.$$

From now on, we define

$$\left(S \diamond f\right)(t) := \frac{d}{dt}(S * f)(t).$$

One can prove (see Thieme [37]) the following approximation formula

$$\left(S \diamond f\right)(t) := \lim_{\lambda \to +\infty}\int_0^t T_0(t-s)\lambda\left(\lambda I - A\right)^{-1} f(s)ds, \forall t \in [0,\tau].$$

From this approximation formula, we deduce that for all t and δ such that $0 \leq \delta \leq t \leq \tau$,

$$\left(S \diamond f\right)(t) - T_0(\delta)\left(S \diamond f\right)(t-\delta) = \left(S \diamond f(t-\delta+.)\right)(\delta). \tag{2.2}$$

Consider now the non-homogeneous Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t), t \geq 0; \\ u(0) = x \in X_0. \end{cases} \tag{2.3}$$

DEFINITION 2.4. *: A continuous function $u \in C([0,\tau], X)$ is called an* ***integrated solution*** *of (2.3) if and only if*

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds, \text{ for all } t \in [0,\tau].$$

One can note that in the previous formula we implicitly assume that $\int_0^t u(s)ds \in D(A), \forall t \in [0,\tau]$. So we must have $u(t) \in X_0, \forall t \in [0,\tau]$. From Theorem 2.3, we deduce that

$$u(t) = T_0(t)x + (S \diamond f)(t)$$

is the unique integrated solution of (2.3), and

$$\|u(t)\| \leq M \left[e^{\omega t}\|x\| + \int_0^t e^{\omega(t-s)}\|f(s)\| ds\right], \forall t \in [0,\tau].$$

3. **Compactness for Non-homogeneous Problems.**

In this section we present some compactness results for a non-homogeneous Cauchy problem. More precisely we consider the following Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t), \text{ for } t \in [0,\tau], \text{ and } u(0) = 0,$$

and we investigate the compactness properties of the following set

$$\{(S \diamond f)(t) : t \in [0,\tau], f \in \mathcal{F}\},$$

where $\mathcal{F}$ is a subset of $C([0,\tau], X)$. From now on, we assume that $A : D(A) \subseteq X \to X$ is a Hille-Yosida operator. We start by the classical situation where $f(t)$ belongs to $X_0 = \overline{D(A)}$.

ASSUMPTION 3.1. *Let $\mathcal{F} \subseteq C([0,\tau], X_0)$ be such that the subset*

$$\{f(t) : t \in [0,\tau], f \in \mathcal{F}\}$$

is bounded, and there exists $\delta^* \in (0,\tau)$, such that for each $\delta \in (0,\delta^*)$, the subset

$$\{T_0(\delta)f(s) : s \in [\delta,\tau], f \in \mathcal{F}\}$$

is relatively compact.

The following Theorem summarizes ideas from Webb [44].

THEOREM 3.2. *Let Assumption 3.1 be satisfied. Then*

$$\{(S \diamond f)(t) : t \in [0,\tau], f \in \mathcal{F}\}$$

is relatively compact.

Proof. We set

$$v_f(t) := (S \diamond f)(t) = \int_0^t T_0(t-s)f(s)ds, \qquad \forall t \in [0,\tau], \forall f \in \mathcal{F}.$$

Let $\delta \in (0, \delta^*)$, be fixed. We have for each $t \in [\delta, \tau]$, and each $f \in \mathcal{F}$,

$$\begin{aligned}\int_0^t T_0(t-s)f(s)ds = \int_{t-\delta}^t T_0(t-s)f(s)ds + \int_\delta^{t-\delta} T_0(t-\delta-s)T_0(\delta)f(s)ds \\ + \int_0^\delta T_0(t-s)f(s)ds.\end{aligned} \tag{3.1}$$

By Assumption 3.1, for each $\delta \in (0, \delta^*)$, the set

$$M_{1\delta} := \overline{\{T_0(\delta)f(t) : t \in [\delta, \tau], x \in E\}}$$

is compact. Moreover, as the map $(t, x) \to T_0(t)x$ is continuous from $[0, +\infty) \times X_0$ into X_0, we deduce that for each $\delta \in (0, \tau)$, the set

$$M_{2\delta} := \overline{\{T_0(t)x : t \in [0, \tau], x \in M_{1\delta}\}}$$

is compact. Therefore, for each $\delta \in (0, \tau)$ and each $t \in [\delta, \tau]$,

$$\int_\delta^{t-\delta} T_0(t-\delta-s)T_0(\delta)f(s)ds \in [0, \tau]\,\overline{\text{co}}\,(M_{2\delta}) =: M_{3\delta},$$

where $\overline{\text{co}}(M_{2\delta})$ is the closed convex hull of $M_{2\delta}$. By Mazur's theorem, $M_{3\delta}$ is compact. For each $\delta \in (0, \tau)$, we set

$$M_\delta = M_{3\delta} \cup \{0\},$$

and

$$M_0 = \{v_f(t) : x \in E, t \in [0, \tau]\}.$$

Then by using equation (3.1), the fact that $0 \in M_\delta$, and the fact that $\{f(t) : t \in [0, \tau], f \in \mathcal{F}\}$ is bounded, we deduce that there exists $k > 0$, such that for each $\delta \in (0, \tau)$,

$$\forall x \in M_0, \exists y \in M_\delta, \text{ such that } \|x - y\| \leq k\delta. \tag{3.2}$$

Let $\varepsilon > 0$ be fixed, and let $\delta > 0$ be fixed such that $k\delta \leq \varepsilon/2$. Since M_δ is compact, we can find a finite sequence $\{y_j\}_{j=1,\ldots,p}$ such that

$$M_\delta \subseteq \cup_{j=1,\ldots,p} B\left(y_j, \frac{\varepsilon}{2}\right),$$

and by (3.2) we also have $M_0 \subseteq \cup_{j=1,\ldots,p} B(y_j, \varepsilon)$. So M_0 is relatively compact. □

The following lemma will be useful in section 5.

LEMMA 3.3. *Let Assumption 3.1 be satisfied. Then $t \to (S \diamond f)(t)$ is uniformly right continuous on $[0, \tau)$, uniformly in $f \in \mathcal{F}$.*

Proof. We set

$$v_f(t) := (S \diamond f)(t) = \int_0^t T_0(t-s)f(s)ds, \qquad \forall t \in [0, \tau], \forall f \in \mathcal{F}.$$

Let be $t \in [0, \tau)$, $h \in [0, \tau - t)$, and $f \in \mathcal{F}$. We have

$$\begin{aligned} v_f(t+h) - v_f(t) &= \int_0^{t+h} T_0(t+h-s)f(s)ds - \int_0^t T_0(t-s)f(s)ds \\ &= \int_t^{t+h} T_0(t+h-s)f(s)ds + (T_0(h) - Id)\int_0^t T_0(t-s)f(s)ds. \end{aligned}$$

So
$$\|v_f(t+h) - v_f(t)\| \leq kh + \left\| (T_0(h) - Id) \int_0^t T_0(t-s)f(s)ds \right\|$$
where $k = Me^{\omega^+ \tau} \sup_{f \in \mathcal{F}, t \in [0,\tau]} \|f(t)\|$, and $\omega^+ = \max(0, \omega)$. By Theorem 3.2, the subset
$$C = \overline{\left\{ \int_0^t T_0(t-s)f(s)ds : t \in [0,\tau], f \in \mathcal{F} \right\}}$$
is relatively compact. The map $(t,x) \to T_0(t)x$ is continuous from $[0,+\infty) \times X_0$ into X_0, so this map is uniformly continuous on $[0,\tau] \times C$, and the result follows. □

We now consider the case where $f(t)$ belongs to X.

ASSUMPTION 3.4. *a) Let be $\mathcal{F} \subseteq C([0,\tau], X)$. We assume that there exists $\lambda^* > \omega$, such that the subset*
$$\left\{ (\lambda^* - A)^{-1} f(t) : t \in [0,\tau], f \in \mathcal{F} \right\}$$
is relatively compact.
b) For each $\tau_1 \in (0,\tau)$, we assume that
$$\lim_{h \searrow 0} \sup_{f \in \mathcal{F}} \int_0^{\tau_1} \left\| f(s) - \frac{1}{h} \int_s^{s+h} f(l)dl \right\| ds = 0.$$

LEMMA 3.5. *Let Assumption 3.2 be satisfied. Then, for each $\tau_1 \in (0,\tau)$, the set*
$$\{(S \diamond f)(t) : t \in [0,\tau_1], f \in \mathcal{F}\}$$
is relatively compact.

Proof. We set
$$v_f(t) := (S \diamond f)(t), \qquad \forall t \in [0,\tau], \forall f \in L^1((0,\tau), X).$$
Let $\tau_1 \in (0,\tau)$ be fixed. For each $h \in (0, \tau - \tau_1)$, we define $K_h : L^1((0,\tau), X) \to C([0,\tau_1], X)$ by
$$K_h(f)(t) = \frac{1}{h} \int_t^{t+h} f(s)ds, \quad \forall t \in [0, \tau - h].$$
Then
$$\frac{d}{dt} K_h(f)(t) = \frac{1}{h} [f(t+h) - f(t)], \quad \forall t \in [0, \tau - h], \forall f \in C([0,\tau], X).$$
For all $t \in [0,\tau_1]$, and $f \in C([0,\tau], X)$, we have
$$v_{K_h(f)}(t) = (S \diamond K_h(f))(t) = \frac{d}{dt} \int_0^t S(s) K_h(f)(t-s)ds,$$
so
$$v_{K_h(f)}(t) = S(t)K_h(f)(0) + \int_0^t S(s) \frac{1}{h} [f(t-s+h) - f(t-s)] ds. \tag{3.3}$$
By using equations (2.1), (3.3), Assumption 3.2 a), and by Mazur's theorem, we deduce that
$$A_{\tau_1,h} = \overline{\left\{ v_{K_h(f)}(t) : t \in [0,\tau_1], f \in \mathcal{F} \right\}}$$
is a compact subset. We set
$$A_{\tau_1,0} = \{ v_f(t) : f \in \mathcal{F}, t \in [0,\tau_1] \}.$$

By Theorem 2.3, for each $h \in (0, \tau - \tau_1)$, and each $t \in [0, \tau_1]$, we have

$$\begin{array}{rcl}\left\|v_f(t) - v_{K_h(f)}(t)\right\| & = & \|(S \diamond f - K_h(f))(t)\| \\ & \leq & M \int_0^t e^{\omega(t-s)} \left\| f(s) - \frac{1}{h} \int_s^{s+h} f(l)dl \right\| ds \\ & \leq & M e^{\omega^+ \tau} \int_0^{\tau_1} \left\| f(s) - \frac{1}{h} \int_s^{s+h} f(l)dl \right\| ds,\end{array}$$

where $\omega^+ = \max(0, \omega)$. So by using Assumption 3.2 b), we deduce that, for each $\varepsilon > 0$, there exists $h_\varepsilon \in (0, \tau - \tau_1)$ such that

$$\forall x \in A_{\tau_1,0}, \quad \exists y \in A_{\tau_1,h_\varepsilon} : \quad \|x - y\| \leq \varepsilon.$$

By using the same arguments as in the proof of Theorem 3.2, we conclude that $A_{\tau_1,0}$ is relatively compact. □

ASSUMPTION 3.6. *a) Let $\mathcal{F} \subseteq C([0, \tau], X)$ be such that $\{f(t) : t \in [0, \tau], f \in \mathcal{F}\}$ is a bounded set, and assume that there exists $\lambda^* > \omega$, such that for each $\delta \in (0, \tau)$, the subset*

$$\left\{(\lambda^* - A)^{-1} f(t) : t \in [\delta, \tau], f \in \mathcal{F}\right\}$$

is relatively compact.
b) For each $\tau_1 \in (0, \tau)$, and each $\delta \in (0, \tau_1)$, we assume that

$$\lim_{h \searrow 0} \sup_{f \in \mathcal{F}} \int_\delta^{\tau_1} \left\| f(s) - \frac{1}{h} \int_s^{s+h} f(l)dl \right\| ds = 0.$$

The main result of this section is the following theorem.

THEOREM 3.7. *Let Assumption 3.3 be satisfied. Then, for each $\tau_1 \in (0, \tau)$, the set*

$$\{(S \diamond f)(t) : t \in [0, \tau_1], \quad f \in \mathcal{F}\}$$

is relatively compact.

Remark: In applications, the main difficulty is to verify Assumption 3.3 b). It is clear that if $\mathcal{F}$ is a family of equicontinuous maps, then Assumption 3.3 b) is satisfied. Moreover, if $\mathcal{F}$ is a bounded set in $W^{1,1}((0, \tau), X)$, then one can prove that Assumption 3.3 b) is satisfied. In section 5, we will study this question for a class of semi-linear problem.

Proof. Let $\tau_1 \in (0, \tau)$ be fixed. We set

$$v_{0f}(t) = (S \diamond f)(t), \ \forall t \in [0, \tau_1], \forall f \in \mathcal{F},$$
and
$$\alpha_1 = \sup_{f \in \mathcal{F}, t \in [0,\tau]} \|f(t)\|.$$

For each $\delta \in (0, \tau_1)$, and each $f \in \mathcal{F}$, let $v_{\delta f} : [\delta, \tau] \to X_0$ be the unique solution of

$$v_{\delta f}(t) = A \int_\delta^t v_{\delta f}(s)ds + \int_\delta^t f(s)ds, \ \forall t \in [\delta, \tau].$$

By using Lemma 3.5, we deduce that

$$C_\delta = \overline{\{v_{\delta f}(t) : t \in [\delta, \tau_1], f \in \mathcal{F}\}} \cup \{0\} \text{ is compact.}$$

By using Theorem 2.3, we have

$$\|v_{0f}(t)\| \leq Me^{\omega^+\tau}\alpha_1\delta, \quad \forall t \in [0,\delta].$$

where $\alpha_2 = Me^{\omega^+\tau}\alpha_1$, and $\omega^+ = \max(\omega, 0)$. Moreover, for $t \geq \delta$, we have

$$v_{0f}(t) - v_{\delta f}(t) = v_{0f}(\delta) + A\int_\delta^t v_{0f}(s) - v_{\delta f}(s)ds$$

so by using again Theorem 2.3, we obtain for $t \geq \delta$,

$$\|v_{0f}(t) - v_{\delta f}(t)\| \leq Me^{\omega^+\tau}\|v_{0f}(\delta)\| \leq Me^{\omega^+\tau}\alpha_2\delta.$$

We set

$$C_0 = \{v_{0f}(t) : t \in [0,\tau_1], f \in \mathcal{F}\}.$$

Since $0 \in C_\delta$, we deduce that for $\delta \in (0,\tau)$,

$$\forall x \in C_0, \exists y \in C_\delta, \text{ such that } \|x - y\| \leq \alpha_3\delta,$$

where $\alpha_3 = Me^{\omega^+\tau}\alpha_2$. The relative compactness of C_0 follows. □

Assumption 3.8. *Let $\mathcal{F} \subseteq C([0,\tau], X)$ be such that $\{f(t) : t \in [0,\tau], f \in \mathcal{F}\}$ is a bounded set, and assume that $\{T_0(t)\}_{t\geq 0}$ is compact.*

The proof of the following Theorem is adapted from Bouzahir and Ezzinbi [14].

Theorem 3.9. *Let Assumption 3.4 be satisfied. Then the set*

$$\{(S \diamond f)(t) : t \in [0,\tau], \quad f \in \mathcal{F}\}$$

is relatively compact.

Proof. From equation (2.2) we have for all t and δ such that $0 < \delta \leq t \leq \tau$,

$$(S \diamond f)(t) = T_0(\delta)(S \diamond f)(t-\delta) + (S \diamond f(t-\delta+.))(\delta).$$

We set

$$C_0 = \{(S \diamond f)(t) : t \in [0,\tau], f \in \mathcal{F}\},$$
and
$$C_\delta = \overline{\{T_0(\delta)(S \diamond f)(t-\delta) : t \in [\delta,\tau], f \in \mathcal{F}\}} \cup \{0\}.$$

Then by using Theorem 2.3, we deduce that there exists $k > 0$, such that for all $\delta \in (0,\tau)$,

$$\forall x \in C_0, \quad \exists y \in C_\delta : \quad \|x - y\| \leq k\delta,$$

and as C_δ is compact for all $\delta \in (0,\tau)$, the result follows. □

4. **Compactness for the Semilinear Problem.** From now on, we assume that X is a Banach space, and $A : D(A) \subseteq X \to X$ is a Hille-Yosida operator. We consider

$$u_x(t) = x + A\int_0^t u_x(s)ds + \int_s^t F(s, u_x(s))ds, \text{ for } t \in [0,\tau]. \tag{4.1}$$

where $F : [0,\tau] \times \overline{D(A)} \to X$ is a continuous map. Let Y and Z be two Banach spaces, and let $\Psi : Y \to Z$ be a map. We will say that Ψ is *compact,* if Ψ maps bounded subsets of Y into relatively compact sets of Z.

The following lemma is adapted from Thieme [40], Theorem 7, p:698.

LEMMA 4.1. *Let Y be a Banach space, and $\Psi : [0,\tau] \times \overline{D(A)} \to Y$ be a continuous map satisfying:*
a) For each $0 < s \leq t \leq \tau$, the map $x \to \Psi(t, T_0(s)x)$ is compact, the map $t \to \Psi(t,x)$ is continuous on $[0,\tau]$, uniformly with respect to x on bounded sets of $\overline{D(A)}$, and for each $C > 0$, there exists $K(C) > 0$, such that

$$\|\Psi(t,x) - \Psi(t,y)\| \leq K(C) \|x - y\|,$$

whenever $t \in [0,\tau], \|x\|, \|y\| \leq C$.
b) There exists a bounded subset $E \subseteq \overline{D(A)}$, such that for each $x \in E$, (4.1) has a solution $u_x(t)$ on $[0,\tau]$, and $\{F(t,u_x(t)) : t \in [0,\tau], x \in E\}$ is bounded.
Then for each $\eta \in (0,\tau]$, the subset $\{\Psi(t,u_x(t)) : x \in E, t \in [\eta,\tau]\}$ has a compact closure.

Proof. By Theorem 2.3, the set $\{u_x(t) : x \in E, t \in [0,\tau]\}$ is bounded. We set

$$\alpha_0 = \sup_{t\in[0,\tau], x\in E} \|u_x(t)\|, \text{ and } \alpha_1 = \sup_{t\in[0,\tau], x\in E} \|F(t,u_x(t))\|.$$

Let $\eta \in (0,\tau]$ be fixed. By equation (2.2) we have for all t and δ such that $0 \leq \delta \leq t \leq \tau$,

$$(S \diamond F(.,u(.)))(t) - T_0(\delta)(S \diamond F(.,u(.)))(t-\delta) = (S \diamond F(t-\delta+.,u(t-\delta+.)))(\delta).$$

Then by Theorem 2.3, for each $\delta \in [0,\tau]$, and each $t \in [\delta,\tau]$, we have

$$\begin{aligned}\|(S \diamond F(t-\delta+.,u(t-\delta+.)))(\delta)\| &\leq M\int_0^\delta e^{\omega(\delta-s)} \|F(t-\delta+s,u(t-\delta+s))\| ds \\ &\leq M\alpha_1 \int_0^\delta e^{\omega s} ds.\end{aligned}$$

We set $\gamma = Me^{\omega^+\tau}\alpha_1$, with $\omega^+ = \max(0,\omega)$. Then we obtain

$$\|(S \diamond F(.,u(.)))(t) - T_0(\delta)(S \diamond F(.,u(.)))(t-\delta)\| \leq \delta\gamma, \forall \delta \in [0,\tau], \forall t \in [\delta,\tau].$$

So, for each $0 \leq \delta \leq t \leq \tau$, we have

$$\begin{aligned}&\Psi(t, T_0(t)x + T_0(\delta)(S \diamond F(.,u(.)))(t-\delta)) \\ &= \Psi(t, T_0(\delta)[T_0(t-\delta)x + (S \diamond F(.,u(.)))(t-\delta)]).\end{aligned} \tag{4.2}$$

By using Theorem 2.3, we deduce that for $t \geq \delta$,

$$\begin{aligned}&\|T_0(t-\delta)x + (S \diamond F(.,u(.)))(t-\delta)\| \\ \leq\ & Me^{\omega^+(t-\delta)}[\alpha_0 + (t-\delta)\alpha_1] \leq Me^{\omega^+\tau}[\alpha_0 + \tau\alpha_1] =: \alpha_2.\end{aligned}$$

So

$$\|T_0(t)x + T_0(\delta)(S \diamond F(.,u(.)))(t-\delta)\| \leq Me^{\omega^+\tau}\alpha_2 =: \alpha_3.$$

Moreover, since the map $t \to \Psi(t,x)$ is continuous on $[0,\tau]$, uniformly with respect to x on bounded sets of $\overline{D(A)}$, we deduce that for each $\delta \in (0,\tau)$, the set

$$\widehat{C}_\delta = \left\{\Psi(t, T_0(\delta)x) : x \in \overline{D(A)}, \|x\| \leq \alpha_3, t \in [\delta,\tau]\right\}$$

is relatively compact. So by (4.2), for each $\delta \in (0,\tau)$, the set

$$C_\delta = \overline{\{\Psi(t, T_0(t)x + T_0(\delta)(S \diamond F(.,u(.)))(t-\delta)) : x \in E, t \in [\delta,\tau]\}}$$

is compact. For each $\delta \in (0,\tau)$, let $v_{x,\delta}(t)$ be the unique solution of

$$v_{x,\delta}(t) = T_0(t)x + T_0(\delta)(S \diamond F(.,u(.)))(t-\delta), \text{ for } t \in [\delta,\tau].$$

Then for each $\delta \in (0,\eta)$, and each $t \in [\eta,\tau]$, we have

$$\begin{aligned}\|\Psi(t,u_x(t)) - \Psi(t,v_{x,\delta}(t))\| &\leq K(\alpha_3)\,\|(S \diamond F(t-\delta+.,u(t-\delta+.)))(\delta)\| \\ &\leq K(\alpha_3)\,\gamma\delta.\end{aligned}$$

We set

$$C_{0,\eta} = \{\Psi(t,u_x(t)) : x \in E, t \in [\eta,\tau]\},$$

and

$$\widehat{\gamma} = K(\alpha_3)\,\gamma.$$

We deduce that for each $\delta \in (0,\eta)$,

$$\forall x \in C_{0,\eta}, \exists y \in C_\delta, \text{ such that } \|x-y\| \leq \widehat{\gamma}\delta.$$

So $C_{0,\eta}$ is relatively compact. □

Assumption 4.2. *We assume that $F : [0,\tau] \times \overline{D(A)} \to X$ is a continuous map, which satisfies*

$$F(t,x) = F_1(t,x) + H(t,x)x + \Gamma(t,x),$$

where $F_1 : [0,\tau]\times\overline{D(A)} \to X$, $H : [0,\tau]\times\overline{D(A)} \to \mathcal{L}\left(\overline{D(A)}\right)$ and $\Gamma : [0,\tau]\times\overline{D(A)} \to \overline{D(A)}$ are continuous maps, with the following:

a) *There exists a bounded set $E \subseteq X_0$ such that, for each $x \in E$, there exists a continuous solution $u_x : [0,\tau] \to X_0$ of (4.1) such that*

$$\{F_1(t,u_x(t)) : t \in [0,\tau], x \in E\}, \quad \{\Gamma(t,u_x(t)) : t \in [0,\tau], x \in E\},$$
$$\text{and } \{H(t,u_x(t)) : t \in [0,\tau], x \in E\}$$

are bounded sets.

b) *There exists $\lambda^* > \omega$, such that for each $\delta \in (0,\tau)$, the set*

$$\left\{(\lambda^* - A)^{-1} F_1(t,u_x(t)) : t \in [\delta,\tau], x \in E\right\},$$

is relatively compact.

c) *For each $\tau_1 \in (0,\tau)$, and each $\delta \in (0,\tau_1)$,*

$$\lim_{h\searrow 0}\sup_{x\in E}\int_\delta^{\tau_1}\left\|F_1(s,u_x(s)) - \frac{1}{h}\int_s^{s+h} F_1(l,u_x(l))dl\right\| ds = 0.$$

d) *The map $t \to \Gamma(t,x)$ is continuous from $[0,\tau]$ into $\overline{D(A)}$, uniformly with respect to x in bounded subsets of $\overline{D(A)}$, and for each $t,s \in [0,\tau]$, with $s > 0$, the map $x \to T_0(s)\Gamma(t,x)$ is compact from $\overline{D(A)}$ into $\overline{D(A)}$.*

e) *For each $\delta \in (0,\tau)$, the set*

$$\{H(t,u_x(t)) : t \in [\delta,\tau], x \in E\}$$

is relatively compact.

f) *We assume that there exists $0 < \tau' \leq \tau'' < \tau$, such that for each $x \in E$, if $u_{3x} \in C([0,\tau],X_0)$ is the solution of*

$$u_{3x}(t) = T_0(t)x + \int_0^t T_0(t-s)H(s,u_x(s))(u_{3x}(s))ds, \forall t \in [0,\tau],$$

then the subset $\{u_{3x}(t) : t \in [\tau',\tau''], x \in E\}$ is relatively compact.

From now on, for each $x \in E$, and each $t \in [0, \tau]$, we define

$$\begin{aligned} u_{1x}(t) &= \left(S \diamond F_1\left(., u_x(.)\right)\right)(t) + \int_0^t T_0(t-s)H(s, u_x(s))(u_{1x}(s))ds \\ u_{2x}(t) &= \int_0^t T_0(t-s)\Gamma(s, u_x(s))ds + \int_0^t T_0(t-s)H(s, u_x(s))(u_{2x}(s))ds \\ u_{3x}(t) &= T_0(t)x + \int_0^t T_0(t-s)H(s, u_x(s))(u_{3x}(s))ds. \end{aligned}$$

By uniqueness of the solution of

$$v(t) = x + A\int_0^t v(s)ds + \int_0^t F_1\left(s, u_x(s)\right) + \Gamma(s, u_x(s)) + H(s, u_x(s))(v(s))ds,$$

we deduce that

$$u_x(t) = u_{1x}(t) + u_{2x}(t) + u_{3x}(t), \quad \forall t \in [0, \tau], \forall x \in E.$$

The main result of this section is the following theorem.

Theorem 4.3. *Let Assumptions 4.1 a)-e) be satisfied. Then for each $\tau_1 \in (0, \tau)$, the set*

$$\{u_{1x}(t) + u_{2x}(t) : t \in [0, \tau_1], x \in E\}$$

has compact closure. If in addition Assumption 4.1 f) is satisfied then the set

$$\{u_x(t) : t \in [\tau', \tau''], x \in E\}$$

has compact closure.

Remark: 1) One can use Lemma 4.1 to verify Assumptions 4.1 b) and e). One can also relax the Lipschitz condition in Lemma 4.1 by using similar idea as in Lemma 5.4.
2) In the applications, the main difficulty is to verify Assumption 4.1 c). The section 5 is devoted to this question.
3) The component $u_{3x}(t)$ corresponds to the non-compact part of the semiflow. Also the first part of Theorem 4.3 can be used to prove the existence of a global attractor when the semiflow is not eventually compact, but contracting for some measure of non-compactness (see Sell and You [35] for a definition of contracting semiflows).

Proof. First by using Assumption 4.1 a), Theorem 2.3, and Gronwall's lemma, we deduce that $\{u_x(t) : t \in [0, \tau], x \in E\}$ is a bounded set. Moreover by taking into account Assumption 4.1 f), it remains to prove the compactness of

$$\{u_{ix}(t) : t \in [0, \tau_1], x \in E\}, \qquad \text{for } i = 1, 2, \tau_1 \in (0, \tau).$$

Let $\tau_1 \in (0, \tau)$ be fixed. By Assumptions 4.1 a)-c), and Theorem 3.7, the set

$$\{\left(S \diamond F_1\left(., u_x(.)\right)\right)(t) : t \in [0, \tau_1], x \in E\}$$

is relatively compact. By Assumption 4.1 d), for each $\delta \in (0, \tau)$, the set

$$\{T_0(\delta)\Gamma(t, u_x(t)) : t \in [0, \tau], x \in E\}$$

is relatively compact. So by using Assumption 4.1 a), and Theorem 3.2, we deduce that

$$\left\{\int_0^t T_0(t-s)\Gamma(s, u_x(s))ds : x \in E, t \in [0, \tau]\right\}$$

has a compact closure. We set for each $x \in E$, and each $t \in [0, \tau]$,

$$v_{1x}(t) = \left(S \diamond F_1\left(., u_x(.)\right)\right)(t), \text{ and } v_{2x}(t) = \int_0^t T_0(t-s)\Gamma(s, u_x(s))ds.$$

For $i = 1, 2$, we obtain that

$$u_{ix}(t) = \sum_{k=0}^{\infty} L_x^k(v_{ix}(.))(t),$$

where

$$L_x(\psi(.))(t) = \int_0^t T_0(t-s)H(s, u_x(s))(\psi(s))ds.$$

So for each integer $m \geq 1$,

$$u_{ix}(t) = \sum_{k=0}^{m} L_x^k((v_{ix})(.))(t) + \sum_{k=m+1}^{\infty} L_x^k((v_{ix})(.))(t), \forall t \in [0, \tau].$$

By using Assumption 4.1 e), we deduce that for each $\delta \in (0, \tau)$, the set

$$M_{0\delta} := \{H(s, u_x(s))(v_{ix}(s)) : x \in E, s \in [\delta, \tau_1]\}$$

is relatively compact. So by using Theorem 3.2 we deduce that for each $i = 1, 2$, there exists a compact set $C_0^i \subseteq \overline{D(A)}$ such that

$$L_x(v_{ix}(.))(t) \in C_0^i, \forall t \in [0, \tau_1].$$

By using induction arguments we deduce that for each $m \geq 1$, and each $i = 1, 2$, there exists a compact subset $C_m^i \subseteq X_0$, such that

$$\sum_{k=1}^{m} L_x^k(v_{ix}(.))(t) \in C_m^i, \forall t \in [0, \tau_1].$$

Moreover, we have

$$\|L_x^k((v_{ix})(.))(t)\| \leq \gamma_i e^{\omega^+\tau}\alpha^k \frac{\tau^k}{k!},$$

where $M \geq 1$, $\omega \in I\!R$ the constants from Theorem 2.3,

$$\begin{aligned} \omega^+ \quad &: \quad = \max(0, \omega), \ \gamma_i := \sup_{t \in [0,\tau], x \in E} \|v_{ix}(t)\|, \\ \text{and } \alpha \quad &: \quad = M \sup_{t \in [\delta,\tau], x \in E} \|H(t, u_x(t))\|_{\mathcal{L}(\overline{D(A)})}. \end{aligned}$$

We deduce that $\forall t \in [0, \tau]$,

$$\|\sum_{k=m+1}^{\infty} L_{x_0}^k(v_{ix_0}(.))(t)\| \leq \gamma_i e^{\omega^+\tau} \sum_{k=m+1}^{\infty} \frac{(\alpha\tau)^k}{k!} \leq \gamma_i e^{\omega^+\tau}(e^{\alpha\tau} - \sum_{k=0}^{m} \frac{(\alpha\tau)^k}{k!}) =: \gamma_m,$$

and $\gamma_m \to 0$ as $m \to +\infty$. We let

$$C_\infty^i = \cup_{t \in [0,\tau_1], x \in E} \{u_{ix}(t)\} \text{ for } i = 1, 2.$$

Then for all $x \in C_\infty^i$ there exists $y \in C_m^i$ such that $\|x - y\| \leq \gamma_m$. So for $i = 1, 2$, C_∞^i is relatively compact. $\square$

5. **More about the Semi-linear Case.** In this section we derive abstract conditions which imply in particular Assumptions 4.1 c). Consider the equation

$$u_x(t) = x + A\int_0^t u_x(s)ds + \int_0^t F(s,u_x(s))ds, \text{ for } t \in [0,\tau]. \tag{5.1}$$

ASSUMPTION 5.1. *There exists a bounded set $E \subseteq X_0$ such that, for each $x \in E$, there exists a continuous solution $u_x : [0,\tau] \to X_0$ of (5.1), $F : [0,\tau] \times \overline{D(A)} \to X$ is a continuous map, which satisfies*

$$F(t,x) = F_1(t,x) + H(t,x)x + \Gamma(t,x),$$

where $F_1 : [0,\tau]\times\overline{D(A)} \to X$, $H : [0,\tau]\times\overline{D(A)} \to \mathcal{L}\left(\overline{D(A)}\right)$ and $\Gamma : [0,\tau]\times\overline{D(A)} \to \overline{D(A)}$ are continuous maps, satisfying the following:

a) *The maps $t \to F_1(t,x)$, $t \to \Gamma(t,x)$, $t \to H(t,x)$ are continuous on $[0,\tau]$, uniformly with respect to x in bounded sets of $\overline{D(A)}$.*
b) *There exists a bounded set $E \subseteq X_0$ such that, for each $x \in E$, there exists a continuous solution $u_x : [0,\tau] \to X_0$ of (5.1), and the sets*

$$\begin{array}{l}\{F_1(t,u_x(t)) : t \in [0,\tau], x \in E\}, \ \{\Gamma(t,u_x(t)) : t \in [0,\tau], x \in E\}, \\ \text{and } \{H(t,u_x(t)) : t \in [0,\tau], x \in E\},\end{array}$$

are bounded.

As in section 4, for each $x \in E$, and each $t \in [0,\tau]$, we define

$$\begin{aligned} u_{1x}(t) &= (S \diamond F_1(.,u_x(.)))(t) + \int_0^t T_0(t-s)H(s,u_x(s))(u_{1x}(s))ds \\ u_{2x}(t) &= \int_0^t T_0(t-s)\Gamma(s,u_x(s))ds + \int_0^t T_0(t-s)H(s,u_x(s))(u_{2x}(s))ds \\ u_{3x}(t) &= T_0(t)x + \int_0^t T_0(t-s)H(s,u_x(s))(u_{3x}(s))ds. \end{aligned}$$

LEMMA 5.2. *Let Assumption 5.1 be satisfied. Then $\{u_x(t) : t \in [0,\tau], x \in E\}$, and $\{F(t,u_x(t)) : x \in E, t \in [0,\tau]\}$ are bounded sets.*

Proof. The result follows from Assumption 5.1 b), and Gronwall's lemma. □

ASSUMPTION 5.3. *For all $c, \varepsilon > 0$, and all $t \in [0,\tau)$, there exist $n \in \mathbb{N}$, bounded linear operator H_j from $\overline{D(A)}$ into Banach spaces Z_j, $1 \le j \le n$, and continuous maps G_j from Z_j into a Banach spaces Y_j, such that the following holds:*

$$\begin{array}{r}\|H(t,x) - H(t,\widetilde{x})\| \le \sum_{j=1}^n \|G_j(H_jx) - G_j(H_j\widetilde{x})\| + \varepsilon \\ \text{whenever } x,\widetilde{x} \in \overline{D(A)}, \|x\|, \|\widetilde{x}\| \le c.\end{array}$$

For each $j = 1,...,n$, $H_jT_0(t)$ is compact for $t > 0$.

The following lemma allows to suppress the Lipschitz condition of Lemma 4.1.

LEMMA 5.4. *Let Assumptions 5.1 and 5.2 be satisfied. Then, for each $\delta \in (0,\tau]$, the set*

$$\{H(t,u_x(t)) : t \in [\delta,\tau], x \in E\},$$

is a relatively compact set.

Proof. Let $\delta \in (0,\tau]$ be fixed. Let $\{t_k\}_{k\geq 0} \subseteq [\delta,\tau]$ and $\{x_k\}_{k\geq 0} \subseteq E$ be two sequences. We define $y_k = H(t_k, u_{x_k}(t_k)), \forall k \geq 0$. It is sufficient to show that for each $\varepsilon > 0$, we can extract a sequence $\left\{y_{k_p}\right\}_{p\geq 0}$, such that there exists $p_0 \geq 0$, such that $\left\|y_{k_p} - y_{k_l}\right\| \leq \varepsilon, \forall p,l \geq p_0$. Then by setting $\varepsilon = \frac{1}{j+1}, j \in \mathbb{N}$, and by diagonalization procedure, we can extract a converging subsequence.

Let $\varepsilon > 0$ be fixed. Since $\{t_k\}_{k\geq 0} \subseteq [\delta,\tau]$, we can extract a subsequence (for which we use the same index) such that $t_k \to \widetilde{t} \in [\delta,\tau]$, as $k \to +\infty$. Since $\{F(t,u_x(t)) : x \in E, t \in [0,\tau]\}$ is bounded, there exists $\alpha_0 > 0$, such that

$$\|u_x(t)\| \leq \alpha_0, \forall t \in [0,\tau], \forall x \in E.$$

We have $\forall k,l \geq 0$,

$$\begin{aligned}\|H(t_k,u_{x_k}(t_k)) - H(t_l,u_{x_l}(t_l))\| &\leq \left\|H(t_k,u_{x_k}(t_k)) - H(\widetilde{t},u_{x_k}(t_k))\right\| \\ &+ \left\|H(\widetilde{t},u_{x_k}(t_k)) - H(\widetilde{t},u_{x_l}(t_l))\right\| + \left\|H(\widetilde{t},u_{x_l}(t_l)) - H(t_l,u_{x_l}(t_l))\right\|,\end{aligned}$$

and since $t \to H(t,x)$ is continuous on $[0,\tau]$, uniformly with respect to x in bounded sets of $\overline{D(A)}$, we can find $k_0 \geq 0$, such that for all $k,l \geq k_0$,

$$\|H(t_k,u_{x_k}(t_k)) - H(t_l,u_{x_k}(t_l))\| \leq \frac{\varepsilon}{2} + \left\|H(\widetilde{t},u_{x_k}(t_k)) - H(\widetilde{t},u_{x_l}(t_l))\right\|.$$

Choose $n \in \mathbb{N}$, operators H_j, and maps G_j according to Assumption 5.2, for $\dfrac{\varepsilon}{4}$ rather than ε, $t = \widetilde{t} \in (0,\tau)$, and $c = \alpha_0$. Then

$$\left\|H(\widetilde{t},u_{x_k}(t_k)) - H(\widetilde{t},u_{x_l}(t_l))\right\| \leq \sum_{j=1}^{n} \|G_j\left(H_j u_{x_k}(t_k)\right) - G_j\left(H_j u_{x_l}(t_l)\right)\| + \frac{\varepsilon}{4}.$$

By Lemma 4.1 (with $Y = Z_j$, and $\Psi(t,x) = H_j x$), we deduce that for each $j = 1,...,n$,

$$\{H_j u_x(t) : t \in [\delta,\widehat{\tau}], x \in E\}$$

is relatively compact. So, we can find a converging subsequence of $\{H_j u_{x_k}(t_k)\}_{k\geq 0}$ (that is denoted with the same index), and $H_j u_{x_k}(t_k) \to z_j$ as $k \to +\infty$, for each $j = 1,...,n$. For each $j = 1,...,n$, we have

$$\begin{aligned}&\|G_j\left(H_j u_{x_k}(t_k)\right) - G_j\left(H_j u_{x_l}(t_l)\right)\| \\ \leq\quad &\|G_j\left(H_j u_{x_k}(t_k)\right) - G_j\left(z_j\right)\| + \|G_j\left(H_j u_{x_l}(t_l)\right) - G_j\left(z_j\right)\|\end{aligned}$$

and since G_j is continuous, we can find $k_1 \geq 0$, such that for each $k,l \geq k_1$, and each $j = 1,...,n$,

$$\|G_j\left(H_j u_{x_k}(t_k)\right) - G_j\left(H_j u_{x_l}(t_l)\right)\| \leq \frac{\varepsilon}{4n}.$$

Finally, for all $k,l \geq k_1$, we obtain

$$\|H(t_k,u_{x_k}(t_k)) - H(t_l,u_{x_l}(t_l))\| \leq \varepsilon.$$

□

ASSUMPTION 5.5. *For all $t,s \in [0,\tau]$, $s > 0$, the map $x \to T_0(s)\Gamma(t,x)$ is compact from $\overline{D(A)}$ into $\overline{D(A)}$.*

LEMMA 5.6. *Let the Assumptions 5.1-5.3 be satisfied. Then $\{u_{2x}(t) : t \in [0,\tau], x \in E\}$ is a relatively compact set, and the map $t \to u_{2x}(t)$ is right continuous, uniformly with respect to $x \in E$.*

Proof. By using Lemma 5.4, and the same arguments as in the proof of Theorem 4.3, we deduce that

$$\{u_{2x}(t) : t \in [0,\tau], x \in E\}$$

is a relatively compact set. By using again Lemma 5.4, we deduce that for each $\delta \in (0,\tau]$, the set

$$\{H(t,u_x(t))u_{2x}(t) : t \in [\delta,\tau], x \in E\}$$

is a relatively compact set. Since

$$\Gamma(t,u_x(t)) + H(t,u_x(t))(u_{2x}(t)) \in X_0, \forall t \in [0,\tau], \forall x \in E,$$

we have

$$u_{2x}(t) = \int_0^t T_0(t-s)\left[\Gamma(s,u_x(s)) + H(s,u_x(s))(u_{2x}(s))\right] ds, \forall t \in [0,\tau],$$

and the result follows from Lemma 3.3. □

Assumption 5.7. *$F_1(t,x)$ is of the form $F_1(t,x) = K_1G(t,x)$, with X_1 a Banach space, $G : [0,\tau] \times \overline{D(A)} \to X_1$ a continuous map, $K_1 : X_1 \to X$ a bounded linear operator, with the following:*

c) *There exists $\lambda^* > \omega$, such that for each $\delta \in (0,\tau)$, the set*

$$\left\{(\lambda^* - A)^{-1} F_1(t,u_x(t)) : t \in [\delta,\tau], x \in E\right\},$$

is relatively compact.

d) *There exists a Banach space $\left(\widehat{X}, \|.\|_{\widehat{X}}\right)$, with $\widehat{X} \subseteq \overline{D(A)}$ such that, for all $c,\varepsilon > 0$ and all $t \in [0,\tau)$, there exist $n \in N$, bounded linear operators H_j from $\overline{D(A)}$ into Banach spaces Y_{H_j}, $1 \le j \le n$, and continuous maps L_j from Y_{H_j} into $\mathcal{L}\left(\overline{D(A)}, Y_{L_j}\right)$, the space of bounded linear operators from $\overline{D(A)}$ into a Banach spaces Y_{L_j}, such that the following holds:*

1. *$\|F_1(t,x) - F_1(t,\widetilde{x})\| \le \sum_{j=1}^n \|L_j(H_jx)x - L_j(H_j\widetilde{x})\widetilde{x}\| + \varepsilon$ whenever $x,\widetilde{x} \in \overline{D(A)}, \|x\|, \|\widetilde{x}\| \le c$.*
2. *$H_jT_0(t)$ is compact from $\overline{D(A)}$ to Y_{H_j} for all $j = 1,...,n$, $t > 0$.*
3. *If $\widetilde{K} \in \{H_j : j = 1,...,n\} \cup \left\{L_j(z) : z \in Y_{H_j}, j = 1,...,n\right\}$, then $\widetilde{K}T_0(t) \in L(\widehat{X}, Y_{\widetilde{K}})$ for all $t > 0$, and $t \to \widetilde{K}T_0(t)$ is operator norm continuous from $(0,+\infty)$ into $\mathcal{L}(\widehat{X}, Y_{\widetilde{K}})$, and we have for all sufficiently large $\lambda > 0$ that*

$$\widetilde{K}(\lambda - A)^{-1} K_1 = \int_0^\infty e^{-\lambda s} W(s) ds \text{ on } X,$$

with $W(t)$ forming an exponentially bounded operator-norm Borel measurable family of bounded linear operators.

e) *There exists $\tau^* \in [0,\tau)$ such that $\forall x \in E, \forall t \in [\tau^*,\tau]$,*

$$(u_{1x} + u_{3x})(\tau^*) \in \widehat{X}, \text{ and } H(t,u_x(t))(u_{1x} + u_{3x})(t) \in \widehat{X},$$

and

$$\sup_{x\in E} \|(u_{1x} + u_{3x})(\tau^*)\|_{\widehat{X}} < +\infty,$$

$$\sup_{x\in E, t\in[\tau^*,\tau]} \|H(t,u_x(t))(u_{1x} + u_{3x})(t)\|_{\widehat{X}} < +\infty.$$

Remark: If for each $\widetilde{K} \in \{H_j : j = 1, ..., n\} \cup \left\{L_j(z) : z \in Y_{H_j}, j = 1, ..., n\right\}$, $t \to \widetilde{K}T_0(t)$ is operator norm continuous from $(0, +\infty)$ into $L(\overline{D(A)}, Y_{\widetilde{K}})$, then we can choose $\widehat{X} = \overline{D(A)}$ and $\tau^* = 0$. But in the example of section 6, we will need to choose a Banach $\widehat{X} \neq \overline{D(A)}$, to obtain such operator norm continuity.

We now prove that Assumption 4.1 c) is satisfied.

LEMMA 5.8. *Let Assumptions 5.1 and 5.4 be satisfied. Further let K_2 be a bounded linear operator from X into a Banach X_2 such that, for sufficiently large $\lambda > 0$,*

$$K_2 (\lambda - A)^{-1} K_1 = \int_0^{+\infty} e^{-\lambda t} W(t) dt \text{ on } X_1,$$

with $W(t)$ forming an exponentially bounded operator-norm Borel measurable family of bounded linear operators. Assume that $K_2T_0(t) \in \mathcal{L}(\widehat{X}, X_2), \forall t > 0$, and $t \to K_2T_0(t)$ is operator norm continuous from $(0, +\infty)$ into $\mathcal{L}(\widehat{X}, X_2)$. Then for each $\delta \in (0, \tau - \tau^*)$, the map $t \to K_2 (u_x(t))$ is uniformly right-continuous on $[\tau^* + \delta, \tau)$, uniformly in $x \in E$.

Proof. Let $\delta \in (0, \tau - \tau^*)$ be fixed. We set

$$v_x(t) = u_{1x}(t) + u_{3x}(t), \forall x \in E, \forall t \in [0, \tau].$$

By taking into account Lemma 5.6, it is sufficient to show that $t \to K_2 (v_x(t))$ is uniformly right-continuous on $[\tau^* + \delta, \tau)$, uniformly in $x \in E$. We have for each $t \in [0, \tau - \tau^*]$, and each $x \in E$,

$$\begin{aligned} v_x(t + \tau^*) &= T_0(t)v_x(\tau^*) + S \diamond F_1\big(\tau^* + \cdot, u_x(\tau^* + \cdot)\big)(t) \\ &\quad + \int_0^t T_0(t - s)H(\tau^* + s, u_x(\tau^* + s))(v_x(\tau^* + s))ds \end{aligned}$$

The uniqueness property of the Laplace transform implies that for all $t \geq 0$,

$$K_2S(t)K_1 = \int_0^t W(s)ds \text{ on } X_1.$$

So for each $t \in [0, \tau - \tau^*]$, and each $x \in E$,

$$\begin{aligned} K_2v_x(t + \tau^*) &= V(t)v_x(\tau^*) + \int_0^t W(t - s)G(\tau^* + s, u_x(\tau^* + s))ds \\ &\quad + \int_0^t V(t - s)f_x(s)ds, \end{aligned}$$

where $f_x(t) = H(\tau^* + t, u_x(\tau^* + t))(v_x(\tau^* + t))$, $V(t) = K_2T_0(t)$ for all $t \geq 0$. Since $V(t)$ is operator-norm continuous from $(0, +\infty)$ into $\mathcal{L}(\widehat{X}, X_2)$, and $\left\{\|v_x(\tau^*)\|_{\widehat{X}} : x \in E\right\}$ is bounded, we deduce that

$$t \to V(t)v_x(\tau^*)$$

is uniformly continuous in t on $[\delta, \tau - \tau^*]$, uniformly in $x \in E$. So it remains to consider $\forall t \in [0, \tau - \tau^*]$,

$$\begin{aligned} l_x(t) &= \int_0^t V(t - s)f_x(s)ds, \\ \text{and } k_x(t) &= \int_0^t W(t - s)G(\tau^* + s, u_x(\tau^* + s))ds. \end{aligned}$$

For $Y = X$ or $Y = \widehat{X}$, we set

$$C_Y = \sup \left\{ \|f_x(t)\|_Y : x \in E, t \in [0, \tau - \tau^*] \right\}.$$

Let $\varepsilon > 0$ be fixed, let be $t \in [\delta, \tau - \tau^*)$, and $h \in (0, \tau - \tau^* - t)$. Then

$$\begin{aligned} \|l_x(t) - l_x(t+h)\|_{X_2} &\leq \left\| \int_0^t V(t-s) f_x(s) ds - \int_0^{t+h} V(t+h-s) f_x(s) ds \right\|_{X_2} \\ &\leq \left\| \int_t^{t+h} V(t+h-s) f_x(s) ds \right\|_{X_2} + \left\| \int_0^t \left[V(t-s) - V(t+h-s) \right] f_x(s) ds \right\|_{X_2}. \end{aligned}$$

In addition, let $\gamma \in (0, \delta)$. Then

$$\begin{aligned} &\|l_x(t) - l_x(t+h)\|_{X_2} \leq C_X h \|K_2\| M e^{\omega\tau} + \left\| \int_0^t \left[V(t-s) - V(t+h-s) \right] f_x(s) ds \right\|_{X_2} \\ &\leq C_X h \|K_2\| M e^{\omega\tau} + \left\| \int_{t-\gamma}^t \left[V(t-s) - V(t+h-s) \right] f_x(s) ds \right\|_{X_2} \\ &\quad + \left\| \int_0^{t-\gamma} \left[V(t-s) - V(t+h-s) \right] f_x(s) ds \right\|_{X_2} \\ &\leq C_X h \|K_2\| M e^{\omega\tau} + 2 C_X \gamma \|K_2\| M e^{\omega\tau} \\ &\quad + \int_0^{t-\gamma} \left\| \left[V(t-s) - V(t+h-s) \right] f_x(s) \right\|_{X_2} ds. \\ &\leq C_X h \|K_2\| M e^{\omega\tau} + 2 C_X \gamma \|K_2\| M e^{\omega\tau} \\ &\quad + (t-\gamma) \sup\nolimits_{s \in [0, t-\gamma]} \|V(t-s) - V(t+h-s)\|_{\mathcal{L}(\widehat{X}, X_2)} \sup\nolimits_{s \in [0, t-\gamma]} \|f_x(s)\|_{\widehat{X}} \end{aligned}$$

If we choose $\gamma > 0$ small enough, we have

$$\|l_x(t) - l_x(t+h)\| \leq \frac{\varepsilon}{2} + C_X h \|K_2\| M e^{\omega\tau} + t C_{\widehat{X}} \sup_{\gamma \leq l \leq t} \|V(l) - V(l+h)\|_{\mathcal{L}(\widehat{X}, X_2)}.$$

But $t \to V(t)$ is operator norm continuous from $[\gamma, \tau - \tau^*]$ into $\mathcal{L}(\widehat{X}, X_2)$, so it is uniformly operator norm continuous on $[\gamma, \tau - \tau^*]$. We deduce that

$$\|l_x(t+h) - l_x(t)\| \to 0 \text{ as } h \to 0, \text{ uniformly in } t \in [\delta, \tau - \tau^*), x \in E.$$

We now consider $k_x(t)$. For $t \in [0, \tau)$, and $0 < h \leq \tau - t$,

$$\begin{aligned} k_x(t+h) &= \int_0^t W(t+h-s) G(s, u_x(s)) ds + \int_t^{t+h} W(t+h-s) G(s, u_x(s)) ds \\ &= \int_0^t W(h+s) G(t-s, u_x(t-s)) ds + \int_0^h W(h-s) G(t+s, u_x(t+s)) ds. \end{aligned}$$

So for some $\widetilde{M} > 0$,

$$\|k_x(t+h) - k_x(t)\| \leq \widetilde{M} \int_0^\tau \|W(h+s) - W(s)\| ds + \widetilde{M} \int_0^h \|W(s)\| ds.$$

Since $W(t)$ is Borel measurable with respect to operator-norm, we have

$$\|k_x(t+h) - k_x(t)\| \to 0 \text{ as } h \to 0, \text{ uniformly in } t \in [0, \tau), x \in E.$$

□

PROPOSITION 5.9. *Let Assumptions 5.1-5.4 be satisfied. Then, for each $t \in [0, \tau]$ and each $\delta \in (0, \tau - \tau^*)$, $s \to F_1(t, u_x(s))$ is uniformly right-continuous on $[\tau^* + \delta, \tau]$, uniformly in $x \in E$.*

Proof. Let $\alpha_0 > 0$ such that

$$\|u_x(t)\| \leq \alpha_0, \qquad \forall t \in [0,\tau], \forall x \in E.$$

Let $\varepsilon > 0$, $t \in [0,\tau]$. Choose $n \in \mathbb{N}$ and operators H_j and maps L_j according to Assumption 5.4 d) for $\varepsilon/2$ rather than ε, and $c = \alpha_0$. Set $G_j(x) = L_j(H_j x)x$. Obviously it is sufficient to show that, for any $\delta \in (0, \tau - \tau^*)$, $G_j \circ u_x$ is uniformly right continuous on $[\tau^* + \delta, \tau]$, uniformly with respect to $x \in E$.

Let $\widetilde{K}$ be an operator in $\{H_j : j = 1, ..., n\} \cup \{L_j(z) : z \in Y_{H_j}, j = 1, ..., n\}$. By Lemma 5.8, for each $\delta \in (0, \tau - \tau^*)$, $t \to \widetilde{K} u_x(t)$ is uniformly right-continuous on $[\tau^* + \delta, \tau]$, uniformly with respect to x in E.

Let $\delta \in (0, \tau - \tau^*)$, and $j \in \{1, ..., n\}$ be fixed. By Assumption 5.4 d)-2), $H_j T_0(t)$ is a compact operator for $t > 0$. So by Lemma 4.1 (with $Y = Y_{H_j}$, and $\Psi(t,x) = H_j x$), the subset

$$M_{\delta j} = \overline{\{H_j u_x(s) : s \in [\tau^* + \delta, \tau], x \in E\}}$$

is compact.

Suppose that $G_j \circ u_x$ is not uniformly continuous on $[\tau^* + \delta, \tau)$, uniformly with respect to $x \in E$. Then there exist sequences (s_k) in $[\tau^* + \delta, \tau)$, (x_j) in E and (h_k) in $(0,1)$ such that $s_k + h_k < \tau$, $h_k \searrow 0$ as $k \to \infty$, and

$$\liminf_{k\to\infty} \left\| G_j(u_{x_k}(s_k + h_k)) - G_j(u_{x_k}(s_k)) \right\| > 0.$$

After choosing a subsequence, $s_k \to s \in [\tau^* + \delta, \tau]$. Since $M_{\delta,j}$ is compact, we can choose another subsequence such that $H_j u_{x_k}(s) \to y \in Y_{H_j}$. Since $H_j u_{x_k}$ is uniformly continuous on $[\tau^* + \delta, \tau]$ uniformly in k,

$$H_j u_{x_k}(s_k) \to y, \quad H_j u_{x_k}(s_k + h_k) \to y, \qquad k \to \infty.$$

Since L_j is continuous from Z_j to $\mathcal{L}(\overline{D(A)}, Y_j)$,

$$L_j(H_j u_{x_k}(s_k)) \to L_j(y), \quad L(H_j u_{x_k}(s_k + h_k)) \to L_j(y), \qquad k \to \infty,$$

with the convergence holding in operator norm. Since $t \to L_j(y) u_x(t)$ is uniformly right-continuous on $[\tau^* + \delta, \tau]$, uniformly with respect to x in E, we have

$$L_j(y) u_{x_k}(s_k + h_k) - L_j(y) u_{x_k}(s_k) \to 0, \qquad k \to \infty.$$

By definition of G_j,

$$\begin{aligned}
&\|G_j(u_{x_k}(s_k + h_k)) - G_j(u_{x_k}(s_k))\| \leq \|L_j(H_j u_{x_k}) u_{x_k}(s_k + h_k) - L_j(y) u_{x_k}(s_k + h_k)\| \\
&\quad + \|L_j(y) u_{x_k}(s_k + h_k) - L_j(y) u_{x_k}(s_k)\| + \|L_j(y) u_{x_k}(s_k) - L_j(H_j u_{x_k}) u_{x_k}(s_k)\| \\
&\quad \leq \|L_j(H_j u_{x_k}) - L_j(y)\| \alpha_0 + \|L_j(y) u_{x_k}(s_k + h_k) - L_j(y) u_{x_k}(s_k)\| \\
&\quad\quad + \|L_j(y) - L_j(H_j u_{x_k})\| \longrightarrow 0, \qquad k \to \infty,
\end{aligned}$$

and we obtain a contradiction. □

COROLLARY 5.10. *Let Assumptions 5.1-5.4 be satisfied. Then, if* $\delta \in (0, \tau - \tau^*)$, $t \to F_1(t, u_x(t))$ *is uniformly right-continuous on* $[\tau^* + \delta, \tau)$, *uniformly in* $x \in E$.

Proof. Let $\delta \in (0, \tau - \tau^*)$, and $\varepsilon > 0$ be fixed. By Assumption 5.1, the map $t \to F_1(t,x)$ is uniformly continuous on $[0,\tau]$, uniformly with respect to x in bounded sets. So, we can choose some $\eta_0 > 0$ such that

$$\|F_1(t, u_x(s)) - F_1(r, u_x(s))\| < \varepsilon/4,$$

whenever $|t-r| < \eta_0, r, s, t \in [0, \tau], x \in E$. Further we choose a partition $\tau^* + \delta = t_0 \leq \cdots \leq t_{n+1} = \tau$ such that $t_j - t_{j-1} < \eta_0,\ j = 1, ..., n+1$. Then, if $0 < h < \eta$ and $s \in [\tau^* + \delta, \tau)$, with $s + h \leq \tau$,

$$\|F_1(s+h, u_x(s+h)) - F_1(s, u_x(s))\| < \varepsilon/4 + \|F_1(s, u_x(s+h)) - F_1(s, u_x(s))\|.$$

For each $s \in [\tau^* + \delta, \tau)$ we find some $j \in \{1, ..., n\}$ such that $|s - t_j| < \eta$. Hence

$$\|F_1(s+h, u_x(s+h)) - F_1(s, u_x(s))\| \leq \frac{3\varepsilon}{4} + \|F_1(t_j, u_x(s+h)) - F_1(t_j, u_x(s))\|.$$

By Proposition 5.9, for each $j = 1, ..., n$, there exists some $\eta_j > 0$ such that

$$\|F_1(t_j, u_x(s+h)) - F_1(t_j, u_x(s))\| \leq \varepsilon/4$$

whenever $0 < h < \eta_j, s \in [\tau^* + \delta, \tau)$, with $s + h < \tau, x \in E$. Set $\eta = \min_{j=0,...,n} \eta_j$, then

$$\|F_1(t_j, u_x(s+h)) - F_1(t_j, u_x(s))\| \leq \varepsilon$$

whenever $0 < h < \delta, s \in [\tau^* + \delta, \tau), x \in E, s + h < \tau$. □

Assumption 5.11. *We assume that there exist $\tau', \tau'' : 0 \leq \tau^* < \tau' \leq \tau'' < \tau$, such that for each $x \in E$, if $u_{3x,\tau^*} \in C([\tau^*, \tau], X_0)$ is the solution of*

$$\begin{aligned} u_{3x,\tau^*}(t+\tau^*) &= T_0(t)u_x(\tau^*) \\ &+ \int_0^t T_0(t-s)H(\tau^*+s, u_x(\tau^*+s))(u_{3x,\tau^*}(s+\tau^*))ds, \forall t \in [0, \tau-\tau^*], \end{aligned}$$

then the subset $\{u_{3x,\tau^*}(t) : t \in [\tau', \tau''], x \in E\}$ is relatively compact.

Theorem 5.12. *Let the Assumptions 5.1-5.4 be satisfied. Then for each $\tau_1 \in (\tau^*, \tau)$, the set*

$$\{u_{1x}(t) + u_{2x}(t) : t \in [\tau^*, \tau_1], x \in E\}$$

has compact closure. If in addition Assumption 5.5 is satisfied then the set

$$\{u_x(t) : t \in [\tau', \tau''], x \in E\}$$

has compact closure.

Proof. We have for each $x \in E$, and each $t \in [\tau^*, \tau]$,

$$u_x(t) = u_x(\tau^*) + A\int_{\tau^*}^t u_x(s)ds + \int_{\tau^*}^t F(s, u_x(s))ds,$$

so by replacing the initial time 0 by τ^*, the result follows from Theorem 4.3. □

6. **Application to an Age-structured Population Model with an Additional Structure.** Let Y be a Banach space that represents the distribution of a population with respect to a structure different from age, e.g., induced by space or body size. It can also represent the distributions of several populations with or without a structure different from age. The additional structure would then come from the multi-species composition. Let $u(t, a)$ denote the structural distribution (with respect to this structure) of the individuals with age a at time t. More precisely $u(t, \cdot) \in L^1(0, c, Y)$ where the latter denotes the space of integrable functions

on $(0,c)$ with values in Y. We consider the following model:

$$\begin{cases} (\partial_t + \partial_a)u(t,a) = & \begin{aligned} A(a)u(t,a) + J\big(t, u(t,\cdot)\big)(a) \\ +B_2\big(t,a,\overline{u}_2(t)\big)u(t,a), \end{aligned} & \begin{cases} t > 0, \\ 0 < a < c, \end{cases} \\ u(t,0) = & \int_0^c B_1(t,a,\overline{u}_1(t))u(t,a)da, & t > 0, \\ \overline{u}_j(t) = & \int_0^c C_j(t,a)u(t,a)da, & t > 0, \\ u(0,a) = & u_0(a), & 0 \le a < c, \\ u(t,a) = & 0, & t \ge 0, a > c. \end{cases} \tag{6.1}$$

The number $c \in (0,\infty)$ denotes the maximum age of an individual. The operators $A(a)$ describe how individuals of age a change with respect to the other structure and also to what extent they die a natural death (i.e. a death not depending on the distribution of the population(s)) or emigrate. $J(t,x)$ can be interpreted as an immigration rate which depends on the number and age distribution of the resident population described by $x \in L^1(0,c,Y)$. The operators $B_2(t,a,\overline{u}_1(t))$ may represent additional mortality factors that depend on the density (or densities) of the population(s). If the populations are counted as biomasses, also biomass gains through predation may be incorporated here. The boundary condition describes the birth of individuals. The operators $B_1(t,a,z)$ represent the per capita birth rates of individuals with age a where z is the competition by other individuals. The operators $C_1(t,a)$ may describe to what degree individuals compete or cooperate for the resources necessary for reproduction.

Assumption 6.1. a) *For $j = 1,2$, let $C_j : [0,\tau] \times [0,c) \to L(Y,Z_j)$ have the following properties:*

i) *For every $t \in [0,\tau]$, $y \in Y$, $C_j(t,a)y$ is Borel measurable in $a \in [0,c)$.*

ii) *For every $t_0 \in [0,\tau]$, $ess - \sup_{0<a<c} \|C(t_0,a)\| < \infty$ and*

$$ess - \sup_{0<a<c} \|C(t,a) - C(t_0,a)\| \to 0, \qquad t \to t_0.$$

iii) *For every $t \in [0,\tau]$, $r \in [0,c)$, and $y \in Y$, there exists a subset $N = N_{t,r,y}$ of $[0,c)$ with Lebesgue measure 0 such that $\{C_j(t,a)y; a \in [r,c) \setminus N\}$ has compact closure in Z_j.*

b) *The map $J : [0,\tau] \times L^1(0,c,Y) \to L^1(0,c,Y)$ is continuous. Further, if $\tilde{E}$ is a bounded subset of $L^1(0,c,Y)$ and $t > 0$, $J(t,v)$ is continuous in $t \in [0,\tau]$, uniformly in $v \in \tilde{E}$, and the following hold:*

i) *If $b_0, b_1, b_2 \in [0,c)$, $b_2 > b_1$, and $b_2 \to b_0$, $b_1 \to b_0$, then $\int_{b_1}^{b_2} \|J(t,v)(a)\| da \to 0$ uniformly in $v \in \tilde{E}$.*

ii) *$\int_{\{a \in [0,c); \|J(t,v)(a)\| > m\}} \|J(t,v)(a)\|\, da \to 0$ as $m \to \infty$, uniformly for $v \in \tilde{E}$.*

iii) *For any $\eta \in (0, c-s)$,*

$$\int_0^\eta \|U(s+a+h,a+h)J(t,v)(a+h) - U(s+a,a)J(t,v)(a)\|\, da \to 0,$$

as $h \searrow 0$, uniformly in $v \in \tilde{E}$.

c) *For $j = 1,2$, let $B_j : [0,\tau] \times [0,c) \times Z_j \to L(Y,Y)$ have the following properties:*

i) *For every $t \in [0,\tau]$, $y \in Y$, $z \in Z_j$, $B_j(t,a,z)y$ is Borel measurable in $a \in [0,c)$, and*

$$ess - \sup_{0<a<c} \|B_j(t_0,a,z)\| < \infty.$$

ii) *For every* $t_0 \in [0,\tau]$,

$$ess - \sup_{0<a<c} \|B_j(t_0,a,z) - B_j(t_0,a,z_0)\| \to 0, \qquad z \to z_0$$

and for every $\delta > 0$,

$$\sup_{\|z\|\leq\delta} ess - \sup_{0<a<c} \|B_j(t,a,z) - B_j(t_0,a,z)\| \to 0, \qquad t \to t_0.$$

iii) *For every* $t \in [0,\tau]$, $z \in Z_j$, $r \in [0,c)$, *and* $y \in Y$, *there exists a subset* $N = N_{t,z,r,y}$ *of* $[0,c)$ *with Lebesgue measure 0 such that* $\{B_j(t,a,z)y; a \in [r,c) \setminus N\}$ *has compact closure in* Z_j.

There are various assumptions and procedures under which the operators $A(a)$ can be generators of an evolutionary system $U(t,r) : Y \to Y$, $t \geq r \geq 0$ such that

$$\lim_{h\downarrow 0} \frac{1}{h}(U(t,t-h)x - x) = A(t)x, \qquad t > 0, x \in D(A(t)).$$

We will take the point of view that an evolutionary system U is given and interpret the operators $A(a)$ in a generalized sense by looking at the associated evolution semigroup.

Assumption 6.2. *Let* $U(t,s)$, $0 \leq s \leq t < c$, *be a family of bounded linear operators on* Y *with the following properties:*
a) $U(t,s)U(s,r) = U(t,r), \qquad 0 \leq r \leq s \leq t < c.$
b) $U(s,s)x = x, \qquad x \in Y, 0 \leq s < c.$
c) $U(t,s)x$ *is a continuous function of* s *and* t *for* $0 \leq s \leq t < c$ *and* $x \in Y$.
d) $\sup_{0\leq s\leq t<c} \|U(t,s)\| < \infty$.
e) $U(t,s)$ *is compact for each* $0 < s < t$.
f) For all $r \in [0,c)$, $y \in Y$, $\{U(a,r)y; a \in [r,c)\}$ *has compact closure in* Y.

Here

$$X = Y \times L^1((0,c),Y), \text{ and } X_0 = \{0\} \times L^1((0,c),Y).$$

Let

$$T_0(t)x = \left(0, \widehat{T}_0(t)v\right), \ \forall x = (0,v) \in X_0,$$

where $\left\{\widehat{T}_0(t)\right\}_{t\geq 0}$ is the evolutionary semigroup on $L^1(0,c,Y)$ associated with U, defined for each $v \in L^1(0,c,Y)$, and almost every $a \in (0,c)$, by

$$(\widehat{T}_0(t)v)(a) = \begin{cases} U(a,a-t)v(a-t), \text{ if } a \geq t \\ 0, \text{ if } a < t. \end{cases}$$

We refer to Chicone and Latushkin [16](and the references therein) for a nice overview about this type of semigroups. It is readily checked that $\widehat{T}_0$ is a C_0-semigroup on $L^1(0,c,Y)$.

By Assumption 6.2 d) there exists $\Lambda \geq 1$, such that

$$\|U(t,s)\| \leq \Lambda, \forall 0 \leq s \leq t < c.$$

It follows from the considerations in [41], Sec. 5, that there exists a Hille-Yosida operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ such that

$$\begin{aligned} &(\lambda - \mathcal{A})^{-1}(x,v) = (0,w), \\ &w(a) = e^{-\lambda a}U(a,0)x + \int_0^a e^{-\lambda s}U(a,a-s)v(a-s)ds, \end{aligned}$$

and there exists $\widetilde{M} \geq 1$, such that

$$\left\| (\lambda - \mathcal{A})^{-n} \right\| \leq \widetilde{M} \lambda^{-n}, \qquad \forall \lambda > 0, n \in I\!N.$$

Then $\overline{D(\mathcal{A})} = X_0$, and $\mathcal{A}$ generates an integrated semigroup $S(.)$.

We define $H_j : [0, \tau] \times X_0 \to Z_j$, for $j = 1, 2$, for each $x = (0, v) \in X_0$, and each $t \in [0, \tau]$, by

$$H_j(t)x = \int_0^c C_j(t, a) v(a) da.$$

We define $F : [0, \tau] \times X_0 \to X$, $F_1 : [0, \tau] \times X_0 \to X$, and $H : [0, \tau] \times X_0 \to \mathcal{L}(X_0)$, for each $x = (0, v), y = (0, w) \in X_0$, and each $t \in [0, \tau]$, by

$$\begin{aligned} F_1(t, x) &= \left(\int_0^c B_1(t, a, H_1(t)(0, v)) v(a) da, 0 \right), \\ H(t, x) y &= (0, B_2(t, a, H_2(t)(0, v)) w(a)), \\ \Gamma(t, x) &= (0, J(t, v)), \end{aligned}$$

and

$$F(t, x) = F_1(t, x) + H(t, x)x + \Gamma(t, x).$$

We now consider equation (6.1) under the following abstract form

$$\frac{du_x(t)}{dt} = \mathcal{A} u_x(t) + F(t, u_x(t)), \quad t \geq 0, \qquad u_x(0) = x. \tag{6.2}$$

The main result of this section is the following theorem.

THEOREM 6.3. *Let the Assumptions 6.1-6.2 be satisfied and $\tau > 2c$. Assume that there exists a bounded set $E \subseteq X_0$ such that for each $x \in E$, (6.2) has an integrated solution $u_x(t)$ on $[0, \tau]$, the subset $\{F_1(t, u_x(t)) : t \in [0, \tau], x \in E\}$ is bounded,*

$$\sup_{0 \leq t \leq \tau, x \in E} ess - \sup_{0 < a < c} \| B_2(t, a, H_2(t)(0, u_x(t))) \| < \infty,$$
$$\sup_{0 \leq t \leq \tau, x \in E} \int_0^c \| J(t, u_x(t))(a) \| da < \infty.$$

Then for each $\widehat{\tau} \in [2c, \tau)$, the set $\{u_x(t) : x \in E, t \in [2c, \widehat{\tau}]\}$ has a compact closure.

To prove the previous theorem we now apply Theorem 5.12. In order to apply Theorem 5.12, we need some preliminary results. The following result will be used to verify Assumption 5.3.

LEMMA 6.4. *Let $\mathcal{M} \subseteq L^1(0, c, Y)$, $s > 0$. Then $\widehat{T}_0(s)\mathcal{M}$ is compact if the following holds:*

i) *If $b_0, b_1, b_2 \in [0, c)$, $b_2 > b_1$, and $b_2 \to b_0$, $b_1 \to b_0$, then $\int_{b_1}^{b_2} \|f(a)\| da \to 0$ uniformly in $f \in \mathcal{M}$.*
ii) *$\int_{\{a \in [0,c); \|f(a)\| > m\}} \|f(a)\| da \to 0$ as $m \to \infty$, uniformly for $f \in \mathcal{M}$.*
iii) *For any $\eta \in (0, c - s)$, $\int_0^\eta \big\| U(s+a+h, a+h) f(a+h) - U(s+a, a) f(a) \big\| da \to 0$ as $h \searrow 0$, uniformly in $f \in \mathcal{M}$.*

Proof. For $f \in L^1(0, c; Y)$, we define

$$\Xi_i(f)(a) = i \int_a^{a + \frac{1}{i}} U(s + r, r) f(r) dr, \quad a \geq 0, i \in I\!N,$$

where $U(s + r, r) f(r) := 0$ for $r \geq c - s$, further

$$f_{[m]}(a) = \left\{ \begin{array}{cc} f(a), & \|f(a)\| \leq m \\ 0, & \|f(a)\| > m \end{array} \right\}, m \in I\!N.$$

Let $i \in I\!N, \frac{1}{i} < \frac{s}{4}$. We want to apply the Arzela-Ascoli theorem to the sets $\{\Xi_i(f) : f \in \mathcal{M}\}$. By i) and by Assumption 6.2 d), $\{\Xi_i(f) : f \in \mathcal{M}\}$ is an equicontinuous set.

We claim that, for all $a \in [0, c)$, $\{\Xi_i(f)(a) : f \in \mathcal{M}\}$ is a compact set in Y.

To prove this claim, let $a \in [0, c)$. After a substitution,

$$\Xi_i(f_{[m]})(a) = \int_0^1 U(s + a + \frac{r}{i}, a + \frac{r}{i}) f_{[m]}(a + \frac{r}{i}) dr.$$

$\left\{\Xi_i(f_{[m]})(a) : f \in \mathcal{M}\right\}$ is contained in the closed convex hull of the set

$$\begin{aligned}\mathcal{M}_0 &= \left\{U(s + a + \frac{r}{i}, a + \frac{r}{i}) f_{[m]}(a + \frac{r}{i}) : r \in [0, 1], f \in \mathcal{M}\right\} \\ &\subset \left\{U(s + a + \frac{r}{i}, s + a) U(s + a, \frac{s}{2} + a) y : y \in \mathcal{N}, r \in [0, 1]\right\}\end{aligned}$$

where

$$\mathcal{N} = \left\{U(\frac{s}{2} + a, a + \frac{r}{i}) f_{[m]}(a + \frac{r}{i}) : r \in [0, 1], f \in \mathcal{M}\right\}.$$

Notice that $\frac{s}{2} > \frac{r}{i}$ for all $r \in [0, 1]$ by our choice of i. By Assumption 6.2 d) and the definition of $f_{[m]}$, $\mathcal{N}$ is a bounded set. By Assumption 6.2 e), $U(s + a, \frac{s}{2} + a)$ is a compact operator and so $\mathcal{N}_0 = U(s + a, \frac{s}{2} + a)\mathcal{N}$ has compact closure. Since the map $(r, y) \to U(s + a + \frac{r}{i}, s + a)y$ is continuous,

$$\mathcal{M}_0 \subset \left\{U(s + a + \frac{r}{i}, s + a) y : y \in \mathcal{N}_0, r \in [0, 1]\right\}$$

has compact closure. By a theorem by Mazur, the closed convex hull of $\mathcal{M}_0$ is compact and so is $\left\{\Xi_i(f_{[m]})(a) : f \in \mathcal{M}\right\}$. By definition of $f_{[m]}$,

$$\left\|\Xi_i(f)(a) - \Xi_i(f_{[m]})(a)\right\| = \left\| i \int_{\left[a, a + \frac{1}{i}\right] \cap \{r \in (0,c) : \|f(r)\| > m\}} U(s + r, r) f(r) dr \right\|.$$

By Assumption 6.2 d), there exists some constant $\Lambda > 0$, such that

$$\begin{aligned}&\left\|\Xi_i(f)(a) - \Xi_i(f_{[m]})(a)\right\| \leq i\Lambda \textstyle\int_{\{r \in (0,c) : \|f(r)\| > m\}} \|f(r)\| \, dr \\ &\to 0, \qquad m \to \infty, \text{ uniformly for } f \in \mathcal{M}.\end{aligned}$$

Here we have used ii). Since $\left\{\Xi_i(f_{[m]})(a) : f \in \mathcal{M}\right\}$ has compact closure, so has $\{\Xi_i(f)(a) : f \in \mathcal{M}\}$.

In order to show that $\widehat{T}_0(s)\mathcal{M}$ has compact closure, let (f_k) be a sequence in $\mathcal{M}$. It follows from our previous considerations and Arzela-Ascoli theorem that, for any $i \in I\!N$, there exists a subsequence of $(\Xi_i(f_k))_{k \in I\!N}$ which converges locally uniformly on $[0, c - s)$, i.e. uniformly on $[0, c - s - \eta]$ for every sufficiently small $\eta > 0$. By a diagonalization procedure, we have, after choosing a subsequence, that $(\Xi_i(f_k))_{k \in I\!N}$ converges locally uniformly on $[0, c - s)$ for every $i \in I\!N$. It follows from ii) and Assumption 6.2 d) that the functions $\Xi_i(f_k)$ are bounded on $[0, c - s)$, with a bound that does not depend on k or i. Thus we have for every $i \in I\!N$ that

$$\int_0^{c-s} \|\Xi_i(f_k)(a) - \Xi_i(f_l)(a)\| \, da \to 0, \quad k, l \to \infty.$$

Now

$$\begin{aligned}\left\|\widehat{T}_0(s)f_k - \widehat{T}_0(s)f_l\right\| &= \int_0^{c-s} \|U(s+a,a)f_k(a) - U(s+a,a)f_l(a)\|\,da \\ &\leq \int_0^{c-s} \|U(s+a,a)f_k(a) - \Xi_i(f_k)(a)\|\,da \\ &\quad + \int_0^{c-s} \|\Xi_i(f_k)(a) - \Xi_i(f_l)(a)\|\,da \\ &\quad + \int_0^{c-s} \|\Xi_i(f_l)(a) - U(s+a,a)f_l(a)\|\,da.\end{aligned}$$

Hence, for any $i \in I\!N$,

$$\begin{aligned}\lim_{k,l\to\infty}\left\|\widehat{T}_0(s)f_k - \widehat{T}_0(s)f_l\right\| \leq \lim_{k\to\infty}\sup \int_0^{c-s} \|U(s+a,a)f_k(a) - \Xi_i(f_k)(a)\|\,da \\ + \lim_{l\to\infty}\sup \int_0^{c-s} \|U(s+a,a)f_l(a) - \Xi_i(f_l)(a)\|\,da.\end{aligned} \tag{6.3}$$

By definition of $\Xi_i(f_k)$ and Assumption 6.2 d), there exists some constant $\Lambda > 0$, such that

$$\begin{aligned}&\int_0^{c-s} \|U(s+a,a)f_k(a) - \Xi_i(f_k)(a)\|\,da \\ &\leq \Lambda \int_0^{c-s} \left(\int_0^1 \left\|U(s+a,a)f_k(a) - U(s+a+\frac{r}{i}, a+\frac{r}{i})f_k(a+\frac{r}{i})\right\| dr\right) da \\ &= \Lambda \int_0^1 \left(\int_0^{c-s} \left\|U(s+a,a)f_k(a) - U(s+a+\frac{r}{i}, a+\frac{r}{i})f_k(a+\frac{r}{i})\right\| da\right) dr.\end{aligned}$$

By iii), $\int_0^{c-s} \|U(s+a,a)f_k(a) - \Xi_i(f_k)(a)\|\,da \to 0$, as $i \to \infty$, uniformly in $k \in I\!N$. Taking the limit $i \to \infty$ in (6.3), we have

$$\lim_{k,l\to\infty}\sup \left\|\widehat{T}_0(s)f_k - \widehat{T}_0(s)f_l\right\| = 0.$$

□

We now prove some results that will be used to verify Assumptions 5.4 d)-2) and d)-3.

PROPOSITION 6.5. *Let Ω be a set and μ a non-negative measure on a σ-algebra of measurable subsets of Ω. Further let Z, Y, Z_0 be normed vector spaces, $K : Z \to Y$ a compact linear operator, and V a bounded linear operator from Y to $L^\infty(\Omega, Z_0)$. Assume that, for every $y \in Y$, there exists a subset N_y of Ω such that $\mu(N_y) = 0$ and $(Vy)(\Omega \setminus N_y)$ has compact closure in X.*

Then there exists a bounded measurable map $W : \Omega \to \mathcal{L}(Z, Z_0)$ and a measurable subset N of Ω such that $\mu(N) = 0$ and $W(a)z = (VKz)(a)$ for all $z \in Z$ and all $a \in \Omega \setminus N$. Further, for every $\epsilon > 0$, there exist $n \in \mathbb{N}$ and measurable sets $\Omega_1,\ldots, \Omega_n \subseteq \Omega$ and elements $a_1 \in \Omega_1, \ldots, a_n \in \Omega_n$ such that $\Omega = \biguplus_{j=1}^{n} \Omega_j \uplus N$ and

$$\|W(a) - W(a_j)\| < \epsilon \qquad \forall a \in \Omega_j, j = 1, \ldots, n.$$

We write $\uplus$ for a disjoint union.

Proof. Let $\ell \in I\!N$. Since K is a compact operator, there exist $m_\ell \in I\!N$ and $\tilde{z}_1, \dots, \tilde{z}_{m_\ell} \in Z$ with $\|\tilde{z}_k\| \leq 1$ such that, for all $z \in Z$ with $\|z\| \leq 1$, there exists some $k \in \{1, \dots, m_\ell\}$ such that $\|Kz - K\tilde{z}_k\| < \frac{1}{\ell+1}$. By assumption, for every $k = 1, \dots, m_\ell$, there exists a measurable set $N_{\ell,k}$ in Ω such that $[VKz_k](\Omega \setminus N_{\ell,k})$ has compact closure in Z_0. Set $N_\ell = \bigcup_{k=1}^{m_\ell} N_{\ell,k}$, $N = \bigcup_{\ell \in I\!N} N_\ell$. Then $\mu(N) = 0$ and $[VKz_k](\Omega \setminus N)$ has compact closure for all $k = 1, \dots, m_\ell$. Let us fix ℓ for a moment and set $m = m_\ell$. We define

$$\tilde{W}(a) = \big(VKz_1(a), \dots, VKz_m(a)\big), \qquad a \in \Omega \setminus N.$$

Then $\tilde{W} : \Omega \to Z_0^m$, where Z_0^m is the m-fold Cartesian product of Z_0 with itself, endowed with the maximum norm. Further

$$\tilde{W}(\Omega \setminus N) \subseteq [VKz_1](\Omega \setminus N) \times \ldots \times [VKz_m](\Omega \setminus N).$$

The latter set has compact closure as the Cartesian product of sets with compact closure. Hence $\tilde{W}(\Omega \setminus N)$ has compact closure itself. Thus there exist $n \in I\!N$ and elements $a_1, \dots, a_n \in \Omega \setminus N$ such that for every $a \in \Omega \setminus N$ there exists some $j \in \{1, \dots, n\}$ with $\|\tilde{W}(a) - \tilde{W}(a_j)\| < \frac{1}{\ell+1}$, i.e.

$$\big\|[VKz_k](a) - [VKz_k](a_j)\big\| < \frac{1}{\ell+1} \qquad k = 1, \dots, m.$$

We set

$$\widetilde{\Omega}_j = \bigcap_{k=1}^{m} \{a \in \Omega \setminus N;\ \big\|[VKz_k](a) - [VKz_k](a_j)\big\| < \frac{1}{\ell+1}\}.$$

Then $\Omega = \bigcup_{j=1}^{n} \widetilde{\Omega}_j$ and $a_j \in \widetilde{\Omega}_j$. Since $VKz_k \in L^\infty(\Omega, Z_0)$, $\widetilde{\Omega}_j$ is a measurable set.

Now let $a \in \widetilde{\Omega}_j$, $j \in \{1, \dots, n\}$, $z \in Z$, $\|z\| \leq 1$. We choose $k \in \{1, \dots, m\}$ such that $\|Kz - Kz_k\| < \frac{1}{\ell+1}$. Then

$$\begin{aligned}\big\|[VKz](a) - [VKz](a_j)\big\| \leq& \big\|[VKz](a) - [VKz_k](a)\big\| + \big\|[VKz_k(a) - [VKz_k](a_j)\big\| \\ &+ \big\|[VKz_k](a_j) - [VKz_k](a)\big\| \\ <& 2\|V\|\ \|Kz - Kz_k\| + \frac{1}{\ell+1} < \frac{2\|V\|+1}{\ell+1}.\end{aligned}$$

We set $W(a)z = [VKz](a)$ for $a \in \Omega \setminus N$ and $W(a) = 0$ for $a \in N$. Notice that this definition does not depend on ℓ. For every $j = 1, \dots, n$, $\|W(a) - W(a_j)\| < (2\|V\| + 1)\frac{1}{\ell+1}$ for all $a \in \widetilde{\Omega}_j$. We set $\Omega_1 = \widetilde{\Omega}_1 \setminus \bigcup_{i>1} \{a_i\}$ and

$$\Omega_{j+1} = \widetilde{\Omega}_{j+1} \setminus \left(\bigcup_{i=1}^{j} \Omega_j \cup \bigcup_{i>j+1} \{a_i\} \right), \qquad j = 1, \dots, n-1.$$

Then $\Omega = \biguplus_{j=1}^{n} \Omega_j \uplus N$, $a_j \in \Omega_j$, for $j = 1, ..., n$, (since $a_j \in \widetilde{\Omega}_j$ for $j = 1, ..., n$), and

$$\|W(a) - W(a_j)\| < \frac{2\|V\|+1}{\ell+1} \qquad \forall a \in \Omega_j.$$

$W : \Omega \to \mathcal{L}(Z, Z_0)$ is measurable because for every $\ell \in I\!N$ we find a finite-valued measurable function $W_\ell : \Omega \to \mathcal{L}(Z, Z_0)$, $W_\ell(a) = W(a_j)$ for all $a \in \widehat{\Omega}_j$, $W_\ell(a) = 0$ for all $a \in N$, such that

$$\|W(a) - W_\ell(a)\| \leq \frac{2\|V\|+1}{\ell+1} \qquad \forall a \in \Omega.$$

$\square$

LEMMA 6.6. *Let Assumption 6.2 be satisfied. Let Z_0 be a Banach space, and $C : [0,c) \to \mathcal{L}(Y, Z_0)$ be strongly measurable,*

$$ess - \sup_{0<a<c} \|C(a)\| < \infty.$$

Let $r \in (0,c)$. Assume that for every $y \in Y$, there exists a subset N_y of $[0,c)$ with Lebesgue measure 0 such that $\{C(a)y; a \in [r,c) \setminus N_y\}$ has compact closure in Z_0. Then, for every $y \in Y$, there exists a subset $\widetilde{N}_y$ of $[0,c)$ with Lebesgue measure 0 such that $\{C(a)U(a,r)y; a \in [r,c) \setminus \widetilde{N}_y\}$ has compact closure in Z_0.

Proof. We choose a Borel set $\tilde{N} \subseteq [0,c)$ with Lebesgue measure 0 and a number $\Lambda_C > 0$ such that

$$\|C(a)\| \le \Lambda_C \qquad \forall a \in [0,c) \setminus \tilde{N}.$$

Let $r \in [0,c)$ and $y \in Y$. By Assumption 6.2 f) the set $Y_r = \{U(a,r)y; r \le a < c\}$ is separable and we choose a countable dense subset $\tilde{Y}$. By assumption, for every $\tilde{y} \in \tilde{Y}$, there exists a subset $N_{\tilde{y}}$ of $[r,c)$ such that the set $\{C(a)\tilde{y}; a \in [r,c) \setminus N_{\tilde{y}}\}$ has compact closure. We set $\widetilde{N}_y = \tilde{N} \cup \bigcup_{\tilde{y} \in \tilde{Y}} N_{\tilde{y}}$. Since $\tilde{Y}$ is countable, $\widetilde{N}_y$ is a Borel set with Lebesgue measure 0. Moreover $\{C(a)\tilde{y}; a \in [r,c) \setminus \widetilde{N}_y\}$ has compact closure for every $\tilde{y} \in \tilde{Y}$. Let (a_j) be a sequence in $[r,c) \setminus \widetilde{N}_y$. By Assumption 6.2 f), there exists some $\hat{y} \in Y$ such that $U(a_j,r)y \to \hat{y}$, after choosing a subsequence. We choose a sequence $(\tilde{y}_k)$ in $\tilde{Y}$ such that $\|\tilde{y}_k - \hat{y}\| < \frac{1}{k}$. By assumption, for each $k \in \mathbb{N}$, the sequence $(C(a_j)\tilde{y}_k)_{j \in \mathbb{N}}$ has a converging subsequence. Using a diagonalization procedure we can assume that, after choosing a subsequence, $C(a_j)\tilde{y}_k \to z_k$ as $j \to \infty$ for every $k \in \mathbb{N}$. Further,

$$\begin{aligned}\|z_k - z_\ell\| \le &\|z_k - C(a_j)\tilde{y}_k\| + \|C(a_j)\tilde{y}_k - C(a_j)\tilde{y}_l\| + \|C(a_j)\tilde{y}_l - z_l\| \\ \le &\|z_k - C(a_j)\tilde{y}_k\| + \Lambda_C\|\tilde{y}_k - \tilde{y}_l\| + \|C(a_j)\tilde{y}_l - z_l\|.\end{aligned}$$

Taking the limit $j \to \infty$ in this inequality we have

$$\|z_k - z_l\| \le \Lambda_C \|\tilde{y}_k - \tilde{y}_l\|.$$

Since $(\tilde{y}_k)$ is a Cauchy sequence, (z_k) is a Cauchy sequence and has a limit z. Now

$$\begin{aligned}&\|C(a_j)U(a_j,r)y - z\| \\ \le &\|C(a_j)U(a_j,r)y - C(a_j)\hat{y}\| + \|C(a_j)\hat{y} - C(a_j)\tilde{y}_k\| + \|C(a_j)\tilde{y}_k - z_k\| + \|z_k - z\| \\ \le &\Lambda_C\|U(a_j,r)y - \hat{y}\| + \Lambda_C\|\hat{y} - \tilde{y}_k\| + \|C(a_j)\tilde{y}_k - z_k\| + \|z_k - z\|.\end{aligned}$$

For every $k \in \mathbb{N}$,

$$\limsup_{j\to\infty} \|C(a_j)U(a_j,r)y - z\| \le \Lambda_C\|\hat{y} - \tilde{y}_k\| + \|z_k - z\|.$$

Taking the limit for $k \to \infty$,

$$\limsup_{j\to\infty} \|C(a_j)U(a_j,r)y - z\| = 0.$$

$\square$

PROPOSITION 6.7. *Let Assumption 6.2 be satisfied. Let Z_0 be a Banach space, and $C : [0,c) \to \mathcal{L}(Y, Z_0)$ be strongly measurable,*

$$ess - \sup_{0<a<c} \|C(a)\| < \infty.$$

Assume that, for every $r \in [0,c)$ and $y \in Y$, $\{U(a,r)y; a \in [r,c)\}$ has compact closure in Y and that there exists a subset $N_{y,r}$ of $[0,c)$ with Lebesgue measure 0 such that $\{C(a)y; a \in [r,c) \setminus N_{y,r}\}$ has compact closure in Z_0.

Then the operators $Q(s) : L^1(0,c,Y) \to Z_0$ with

$$Q(s)v = \begin{cases} \displaystyle\int_s^c C(a)U(a,a-s)v(a-s)da, & 0 \le s < c, \\ 0, & s \ge c, \end{cases}$$

are compact for every $s > 0$. Moreover the restrictions of $Q(s)$ from $L^\infty(0,c,Y)$ to Z_0 are operator-norm continuous in $s > 0$.

Proof. To show the compactness of $Q(s)$ for $s > 0$ we can assume that $s < c$, otherwise $Q(s) = 0$. So let $s \in (0,c)$ be fixed but arbitrary. After a substitution,

$$Q(s)v = \int_0^{c-s} C(a+s)U(a+s,a)v(a)da.$$

We combine Proposition 6.5 with the approach in [40], pp. 709-710. We choose a partition of $0 = b_0 < \cdots < b_m = c - s$ such that $b_k - b_{k-1} < \frac{s}{4}$ for $k = 1,\ldots,m$. Obviously it is sufficient to show that each operator

$$Q_k v = \int_{b_{k-1}}^{b_k} C(a+s)U(a+s,a)v(a)da$$

is compact. Set $\Omega = [b_{k-1}, b_k)$. Since U is an evolutionary system we can write $Q_k v$ as

$$Q_k v = \int_\Omega C(a+s)U\left(a+s,b_k\right)U(b_k,a)v(a)da.$$

Set $(Vy)(a) = C(a+s)U\left(a+s, b_k + \frac{s}{2}\right)y$ for $y \in Y$ and $a \in \Omega$, and $K = U(b_k + \frac{s}{2}, b_k)$. By Lemma 6.6 and Assumption 6.2 e), V and K satisfy the assumptions of Proposition 6.5 and $(VKy)(a) = C(a+s)U(a+s,b_k)y$ for all $y \in Y$, $a \in \Omega$. So there exist a bounded Borel measurable function $W : \Omega \to \mathcal{L}(Y, Z_0)$ and a Borel set $N \subseteq \Omega$ of Lebesgue measure 0 such that

$$W(a)y = C(a+s)U\left((a+s,b_k\right)y \qquad \forall a \in \Omega \setminus N, y \in Y.$$

Hence

$$Q_k v = \int_{\Omega\setminus N} W(a)U(b_k,a)v(a)da.$$

Moreover, for every $\epsilon > 0$, there exist Borel sets $\Omega_1,\ldots,\Omega_n$ in $\Omega \setminus N$ and elements $a_1 \in \Omega_1,\ldots,a_n \in \Omega_n$ such that $\|W(a) - W(a_j)\| \le \epsilon$ for all $a \in \Omega_j$. Define

$$\widetilde{Q}_K v = \sum_{j=1}^n W(a_j)\int_{\Omega_j} U(b_k,a)v(a)da,$$

with $W(a_j) = C(a_j+s)U(a_j+s,b_k)$ being compact operators. By Assumption 6.2 d), the set $\{\int_{\Omega_j} U(b_k,a)v(a)da; v \in L^1(0,c,Y), \int_0^c \|v(a)\|da \le 1\}$ is bounded. So the operators $\widetilde{Q}_k : L^1(0,c,Y) \to Z_0$ are compact. Moreover

$$\begin{aligned}\left\|Q_k v - \widetilde{Q}_k v\right\| \le& \sum_{j=1}^n \int_{\Omega_j} \|W(a) - W(a_j)\|\|U(b_k,a)v(a)\|da \\ \le& \epsilon \int_{\Omega\setminus N} \|U(b_k,a)\|\ \|v(a)\|da.\end{aligned}$$

By Assumption 6.2 d),

$$\Lambda_U = \sup\{\|U(a,r)\|; 0 \le r \le a < c\} < \infty,$$

and

$$\|Q_k - \widetilde{Q}_k\| \le \epsilon \Lambda_U.$$

So Q_k can be approximated by compact operators in the operator norm and is compact itself.

We next show that, for arbitrary $\eta > 0$, $Q : [\eta, \infty) \to \mathcal{L}(L^\infty(0,c,Y), Z_0)$ is continuous. We show that Q is continuous at every $s \in [\eta, c)$, with $\eta \in (0,c)$. The other cases can then be done easily. By the boundedness properties of C and U, it is sufficient to consider operators

$$\tilde{Q}(s)v = \int_0^b C(a+s)U(a+s,a)v(a)da$$

with a fixed $b \in [0,c)$, $b+s < c$ and to show that $\|\tilde{Q}(s) - \tilde{Q}(r)\| \to 0$ if $b + r < c$, $r \ge \eta$ and $r \to s$. Choosing appropriate partitions, it is enough to show this for operator families $\widetilde{Q}_k(\cdot)$,

$$\widetilde{Q}_k(s)v = \int_{b_{k-1}}^{b_k} C(a+s)U(a+s,a)v(a)da$$

with $0 < b_k - b_{k-1} < \frac{\eta}{4}$. Since U is an evolutionary system, we can write $\widetilde{Q}_k$ as

$$\widetilde{Q}_k(s)v = \int_{b_{k-1}}^{b_k} C(a+s)U(a+s,b_k)U(b_k,a)v(a)da.$$

For $a \ge [b_{k-1} + \eta, c)$ we have

$$C(a)U(a,b_k) = C(a)U\left(a, b_k + \frac{\eta}{2}\right)U\left(b_k + \frac{\eta}{2}, b_k\right).$$

Applying Proposition 6.5 as before, we find a bounded Borel measurable function $W : [b_{k-1} + \eta, c) \to \mathcal{L}(Y, Z_0)$ such that $W(a) = C(a)U(a,b_k)$ for almost all $a \in [b_{k-1} + \eta, c)$. Hence, if $\|v\|_\infty \le 1$,

$$\begin{aligned}\left\|\widetilde{Q}_k(s)v - \widetilde{Q}_k(r)v\right\| &\le \int_{b_{k-1}}^{b_k} \|W(a+s) - W(a+r)\| \, \|U(b_k,a)v(a)\| da \\ &\le \Lambda_U \int_{b_{k-1}}^{b_k} \|W(a+s) - W(a+r)\| da \longrightarrow 0, \qquad r \to s.\end{aligned}$$

□

We now apply Theorem 5.12.

Lemma 6.8. *Under the assumptions of Theorem 6.3, Assumptions 5.1-5.3, and Assumptions 5.4 c) and d) are satisfied, with*

$$\widehat{X} = \{0\} \times L^\infty(0,c,Y)$$

endowed with the usual norm $\|(0,v)\|_{\widehat{X}} = \|v\|_{L^\infty(0,c,Y)}$.

Proof. Assumption 5.1 is trivially satisfied. To verify Assumption 5.2 it is sufficient to show that $x \to H_2(t)T_0(s)x$ is compact. But

$$H_2(t)T_0(s)(0,v) = \begin{cases} \int_s^c C_2(t,a)U(a,a-s)v(a-s)da, \; 0 \le s \le c \\ 0, \; s \ge c. \end{cases}$$

So by using Assumption 6.1 a), and Assumption 6.2, the result follows from Proposition 6.7. Assumption 5.3 follows from Lemma 6.4, and the Assumptions 6.1-b). We now prove Assumption 5.4 c) and d). We set

$$\begin{aligned}
&L_1(t,z)(0,v) = \int_0^c B_1(t,a,z)v(a)da,\\
&G(t,(0,v)) = L_1(t,H_1(t,(0,v)))(0,v)\\
&K_1x = (x,0),\ x \in Y =: X_1,
\end{aligned}$$

Then

$$F_1(t,x) = K_1 G(t,x).$$

We have $(\lambda - \mathcal{A})^{-1}(K_1 x) = (0,y)$ with $y(a) = e^{-\lambda a}U(.,0)x$. Hence $(\lambda - \mathcal{A})^{-1} K_1$ is a compact operator from Y into $\overline{D(\mathcal{A})} = \{0\} \times L^1(0,c,Y)$ by the Arzela-Ascoli theorem, because all operators $U(a,0)$ with $a > 0$ are compact. We have

$$H_1(t)\left(\lambda - \mathcal{A}\right)^{-1} K_1 x = \int_0^c e^{-\lambda a} C_j(t,a)U(a,0)x da = \int_0^t e^{-\lambda a} W(s)x ds$$

with

$$W(s) = C_j(t,s)U(s,0),\ 0 \le s < c,\ W(s) = 0,\ s > c.$$

Since $U(t,0)$ is compact for all $t > 0$ and strongly continuous, it continuously depends on $t > 0$ in operator norm as $U(t,0) = U(t,s)U(s,0)$ for all $s \in (0,t)$. Since $C_j(t,s)$ is strongly Borel measurable in s, $W(s)$ is Borel measurable in s with respect to the operator norm. A similar representation can be found for $L_1(t,z)\left(\lambda - \mathcal{A}\right)^{-1} K_1, z \in Z$. Further

$$H_1(t)T_0(s)(0,v) = \begin{cases} \int_s^c C_1(t,a)U(a,a-s)v(a-s)da,\ 0 \le s \le c \\ 0,\ s \ge c. \end{cases}$$

After a substitution,

$$H_1(t)T_0(s)(0,v) = \begin{cases} \int_0^{c-s} C_1(t,s+a)U(s+a,a)v(a)da,\ 0 \le s \le c \\ 0,\ s \ge c. \end{cases}$$

Clearly $H_1(t)T_0(s) \in \mathcal{L}(\widehat{X}, Z_1)$ for $t \ge 0$. By Assumption 6.1 a) and c), and by Proposition 6.7, we deduce that Assumption 5.4 d) is satisfied. □

The proof of Theorem 6.3 will be complete with the following lemma.

LEMMA 6.9. *Under the assumptions of Theorem 6.3, Assumptions 5.4 e) and 5.5 are satisfied with $\tau^* = c$, $\tau' = 2c$.*

Proof. Let us first consider

$$u_{1,x}(t) = (S \diamond f_x)(t) + \int_0^t \widehat{T}_0(s)K(t-s)u_{1,x}(t-s)ds,\ \text{for } x \in E, t \in [0,\tau],$$

with $f_x(t) = F_1(t, u_x(t))$, and $K(t)$ the operators on X_0 defined by

$$\begin{aligned}
\left[K(t)v\right](a) &= \widetilde{B}(t,a)v(a)\\
\widetilde{B}(t,a) &= B_2(t,a,\overline{u}_2(t)), \qquad \overline{u}_2(t) = \int_0^c C_2(t,a)u_x(t,a)da.
\end{aligned}$$

Then one has

$$u_{1,x}(t)(a) = U(a,0)f_x(t-a) + \int_0^a U(a,a-s)\widetilde{B}(t-s,a-s)\left[u_{1,x}(t-s)\right](a-s)ds,$$
$$0 \le a \le t \le \tau, a \le c,$$
$$u_{1,x}(t)(a) = \int_0^t U(a,a-s)\widetilde{B}(t-s,a-s)\left[u_{1,x}(t-s)\right](a-s)ds,$$
$$0 \le t \le a \le c.$$

Using a fixed point argument, we find a bounded function $\psi : [0,\tau]\times[0,c] \to Y$ with $\psi(t,a) = 0$, for all $0 \le t < a \le c$, ψ restricted to $\Delta = \{(t,a) : 0 \le a \le t \le \tau, a \le c\}$ is continuous, and ψ satisfies the following fixed point problem

$$\psi(t,a) = U(a,0)f_x(t-a) + \int_0^a U(a,a-s)\widetilde{B}(t-s,a-s)\psi(t-s,a-s)ds,$$
$$0 \le a \le t \le \tau, a \le c,$$
$$\psi(t,a) = \int_0^t U(a,a-s)\widetilde{B}(t-s,a-s)\psi(t-s,a-s)ds,$$
$$0 \le t < a \le c.$$

By uniqueness, $[u_{1,x}(t)](a) = \psi(t,a)$ for all $t \in [0,\tau]$ and a.e. $a \in [0,c]$. In particular, $u_{1,x}(t) \in L^\infty(0,c,Y)$ for $t \in [0,\tau]$. Moreover, if we set

$$\zeta(t) = \sup_{0\le a\le c} \|\psi(t,a)\|,$$

ζ is Borel measurable and satisfies

$$\zeta(t) \le M_1 + \widetilde{M}\int_0^t \zeta(s)ds, \qquad t \in [0,\tau]$$

with a constant $M_1 > 0$ that does not depend on $t \in [0,\tau]$ and $x \in E$. A Gronwall argument implies that $\sup_{0\le t\le\tau} \|u_{1,x}(t)\|_\infty < \infty$. We conclude from the assumption concerning B_2 in Theorem 6.3 that $H\left(t,(0,u_{1,x}(t))\right)(0,u_{1,x}(t)) \in \hat{X}$ and

$$\sup_{0\le t\le\tau, x\in E} \left\|H\left(t,(0,u_{1,x}(t))\right)(0,u_{1,x}(t))\right\|_{\widehat{X}} < \infty.$$

We now prove the part of Assumption 5.4 e) concerning $u_{3,x}(t)$. Let $x = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in E$ be fixed, let $u_{3,x} : [0,\tau] \to L^1((0,c),Y)$ be a continuous solution of

$$u_{3,x}(t) = \widehat{T}_0(t)\varphi + \int_0^t \widehat{T}_0(s)K(t-s)u_{3,x}(t-s)ds.$$

It is sufficient to show that $u_{3,x}(t)(a) = 0$ for $a < t, a \le c$.

Then, for $a < t$,

$$\left[u_{3,x}(t)\right](a) = \int_0^a U(a,a-s)\widetilde{B}(t-s,a-s)\left[u_{3,x}(t-s)\right](a-s)ds.$$

Let $0 \le t \le \tau$. We define

$$\phi_\lambda(t) = \int_0^{t\wedge c} e^{-\lambda a}\left\|[u_{3,x}(t)](a)\right\| da,$$

with $t \wedge c = \min\{t, c\}$. Changing the order of integration

$$\begin{aligned}\phi_\lambda(t) &\leq \widetilde{M} \int_0^{t\wedge c} \left(\int_s^{t\wedge c} \left\| \left[u_{3,x}(t-s)\right](a-s) \right\| e^{-\lambda(a-s)} da \right) e^{-\lambda s} ds \\ &\leq \widetilde{M} \int_0^{t\wedge c} \left(\int_0^{(t\wedge c)-s} \left\| \left[u_{3,x}(t-s)\right](a) \right\| e^{-\lambda a} da \right) e^{-\lambda s} ds \\ &\leq \widetilde{M} \int_0^{t\wedge c} \phi_\lambda(t-s) e^{-\lambda s} ds.\end{aligned}$$

Define $\alpha_\lambda := \sup_{0\leq t\leq \tau} \phi_\lambda(t)$. Then $\alpha_\lambda \leq \frac{\widetilde{M}}{\lambda}\alpha_\lambda$. Choosing $\lambda > \widetilde{M}$, we find that $\alpha_\lambda = 0$. Hence $\left[u_{3,x}(t)\right](a) = 0$ for $t \in [0, \tau]$ and almost all $a \in [0, t \wedge c]$.

The proof of Assumption 5.5 is similar. □

REFERENCES

[1] M. Adimy, *Bifurcation de Hopf locale par les semi-groupes*, intégrés. C. R. Acad. Sci., Paris, 311 (1990), 423–428 .
[2] M. Adimy, *Integrated semigroups and delay differential equations*, J. Math. Anal. Appl., 177 (1990), 125–134.
[3] M. Adimy and O. Arino, *Bifurcation de Hopf globale pour des équations à retard par des semi-groupes intégrés*, C. R. Acad. Sci. Paris, 317 (1993), 767–772.
[4] M. Adimy and K. Ezzinbi, *Equations de type neutre et semi-groupes intégrés*, C. R. Acad. Sci. Paris, 318 (1994), 529–534 .
[5] M. Adimy and K. Ezzinbi, *Semi-groupes intégrés et équations différentielles à retard en dimention infinie*, C. R. Acad. Sci. Paris, 323 (1996), 481–486.
[6] M. Adimy and K. Ezzinbi, *A class of Linear Partial Neutral Functional differential Equations with Non-Dense Domain*, J. Diff. Equ., 147 (1998), 285–332.
[7] S. Anita, *Analysis and control of age-dependent population dynamics. Mathematical Modelling, Theory and Applications*, 11, Kluwer Academic Publishers, Dordrecht, (2000).
[8] W. Arendt, *Resolvent Positive Operators and Integrated Semigroups*, Proc. London Math. Soc., 54 (1987), 321–349.
[9] W. Arendt, *Vector Valued Laplace Transforms and Cauchy Problems*, Israel J. Math., 59 (1987), 327–352.
[10] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, "Vector-valued Laplace Transforms and Cauchy Problems", Birkhäuser, Basel 2001.
[11] A.V. Babin and M.I. Vishik, "Attractors of Evolutions Equations", North-Holland (1992).
[12] J. Ball, *Initial-boundary value problems for an extensible beam*, J. Math. Anal. Appl., 42 (1973), 61–90.
[13] J. Ball, *On the asymptotic behavior of generalized process with applications to nonlinear equation equations*, J. Diff. Equ. , 27 (1978), 224–265.
[14] H. Bouzahir and K. Ezzinbi, *Global Attractor for a Class of Partial Differential Equations with Infinite Delay*, Fields Institute Communications, 29 (2001), 63–71.
[15] S. Busenberg, and K.L. Cooke, "Vertically Transmitted Diseases", Biomathematics Vol.23. Springer-Verlag, 1993.
[16] C. Chicone, Y. Latushkin, "Evolution Semigroups in Dynamical Systems and Differential Equations", AMS, Providence 1999.
[17] C. Dafermos and M. Slemrod, *Asymptotic behavior of nonlinear contraction semigroups*, J. Funct. Anal., (1973), 97–106.
[18] G. Da Prato and E. Sinestrari, *Differential Operators with Non-Dense Domain*, Ann. Sc. Norm. Pisa, 14 (1987), 285–344.
[19] K. Deimling, "Nonlinear Functional Analysis", Springer-Verlag (1985).
[20] J.K. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs (1988).
[21] A. Haraux, "Nonlinear Evolution Equations - Global Behavior of Solutions", Lecture Notes in Mathematics 841, Springer-Verlag (1981).

[22] D. Henri, “Geometric Theory of Semilinear Parabolic Equations”, Lecture Notes in Math. 840, Springer-Verlag, (1981).
[23] M. Iannelli, “Mathematical Theory of Age-Structured Population Dynamics”, Giadini Editori e Stampatori in Pisa, 1994.
[24] H. Kellermann, and M. Hieber, *Integrated Semigroups*, J. Funct. Anal., 84 (1989), 160–180.
[25] O. Ladyzhenskaya, “Attractors for Semigroups and Evolution Equations”, Cambridge University Press (1991).
[26] P. Magal, *Compact attractors for time periodic age-structured population models. Electronic Journal of Differential Equations*, 65 (2001), 1–35.
[27] P. Magal, *Asymptotic behavior of age structured model*, Habilitation thesis. (2003)
[28] R.H. Martin, “Nonlinear Operators and Differential Equations in Banach spaces”, John Wiley & Sons (1976).
[29] T. Matsumoto, S.Oharu, H.R. Thieme, *Nonlinear Perturbations of a Class of Integrated Semigroups*, Hiroshima Mathematical Journal 26 (1996), 433–473.
[30] J.A.J. Metz, and O. Diekmann, “The Dynamics of Physiologically Structured Populations”, Lecture Notes in Biomathematics 68. Springer-Verlag, 1986.
[31] F. Neubrander, *Integrated Semigroups and their Application to the Abstract Cauchy Problem*, Pac. J. Math., 135 (1988), 111–155.
[32] A. Pazy, *A class of semi-linear equations of evolution*, Israel J. Math. 20 (1975), 23–36.
[33] A. Pazy, “Semigroups of Linear Operator and Applications to Partial Differential Equations”, Springer, 1983.
[34] J.C. Robinson, “Infinite-Dimensional Dynamical Systems”, An introduction to Dissipative Parabolic PDE’s and the Theory of Global Attractors. Cambridge Texts in Applied Math. (2001).
[35] G.R. Sell and Y. You, “Dynamics of Evolutionary Equations”, Sringer-Verlag (2002).
[36] R. Temam, “Infinite-Dimensional Dynamical Systems in Mechanics and Physics”, Springer-Verlag (1988).
[37] H.R. Thieme, “Semiflows Generated by Lipschitz Perturbations of Non-Densely Defined Operators”, Differential and Integral Equations 3, 1035–1066 (1990).
[38] H.R. Thieme, ”Integrated Semigroups” and Integrated Solutions to Abstract Cauchy Problems. J. Math. Anal. Appl. 152, 416–447 (1990).
[39] H.R. Thieme, *Analysis of Age-Structured Population Models with an Additional Structure*, Proceedings of the second international conference (O.Arino, D.E.Axelrod, M.Kimmel, eds.), 115–125, Marcel Dekker, 1991.
[40] H.R. Thieme, “Quasi-Compact Semigroups via Bounded Perturbation”, Advances in Mathematical Population Dynamics-Molecules, Cells and Man. (O.Arino, D.Axelrod, and M.Kimmel, eds.). 691–713, Worlds Scientific, 1997.
[41] H.R. Thieme, *Positive Perturbation of Operator Semigroups, Growth Bounds, Essential Compactness, and Asymchronous Exponential Growth* Discrete and Continuous Dynamical Systems, 4 (1998), 735–764.
[42] H.R. Thieme and I.I. Vrabie, *Relatively compact orbits and compact attractors for a class of nonlinear evolution equations*, Submitted (2002).
[43] I.I. Vrabie, *Compactness methods for nonlinear evolutions*, Ptiman monographs and surveys in pure and applied mathematics. (1987)
[44] G.F. Webb, *Compactness of Bounded Trajectories of Dynamical Systems in Infinite Dimensional Spaces*, Proc. Roy. Soc. Edinburgh. 84A, 19–33 (1979).
[45] G.F. Webb, “Theory of Nonlinear Age-Dependent Population Dynamics”, Marcel Dekker, (1985).
[46] X.Q. Zhao, “Dynamical Systems in Population Biology”, Springer (2003)

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