ON THE BASIC REPRODUCTION NUMBER OF
REACTION-DIFFUSION EPIDEMIC MODELS∗

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Abstract. The basic reproduction number \( R_0 \) serves as a threshold parameter of many epidemic models for disease extinction or spread. The purpose of this paper is to investigate \( R_0 \) for spatial reaction-diffusion partial differential equations epidemic models. We define \( R_0 \) as the spectral radius of a product of a local basic reproduction number \( R \), and strongly positive compact linear operators with spectral radii one. This definition of \( R_0 \), viewed as a multiplication operator, is motivated by the definition of basic reproduction numbers for ordinary differential equations epidemic models. We investigate the relation of \( R_0 \) and \( R \).

Key words. reaction-diffusion, epidemic models, basic reproduction number

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1. Introduction. For epidemic differential equation models, the basic reproduction number \( R_0 \) is a threshold value such that below this value the disease vanishes, while above this value the disease spreads. The calculation of \( R_0 \) for ordinary differential equations models has been developed extensively based on \[9, 10\]. Many authors have used reaction-diffusion partial differential equations models to study the transmission of diseases in geographical regions (see \[1, 5, 6, 7, 8, 11, 12, 16, 19, 20, 22, 23, 27, 29, 30, 32, 33, 35\]). The purpose of this paper is to connect basic reproduction numbers for partial differential equations epidemic models to basic reproduction numbers for ordinary differential equations models.

In a recent study, Thieme \[28\] provided a general theoretical approach to define \( R_0 \) as the spectral radius of a resolvent-positive operator for a wide range of epidemic models, which is a generalization of the finite dimensional version in \[9, 10\]. Another approach to characterize \( R_0 \) for reaction-diffusion epidemic models relied on a variational characterization of \( R_0 \), which works when the model is relatively simple (the stability of the disease free equilibrium is determined by the sign of an eigenvalue problem consisting of only one equation). For example, Allen et al. \[1\] characterize \( R_0 \) for a simple diffusive SIS model by the formula

\[
R_0 = \sup \left\{ \frac{\int_{\Omega} \beta \varphi^2 dx}{\int_{\Omega} (d_I |\nabla \varphi|^2 + \gamma \varphi^2) dx} : \varphi \in H^1(\Omega), \varphi \neq 0 \right\},
\]

where \( \beta = \beta(x) \) is the transmission rate, \( \gamma = \gamma(x) \) is the removal rate, and \( d_I \) is the diffusion coefficient. This allows the authors to show that \( R_0 \) is strictly decreasing in \( d_I \), \( R_0 \to \int_{\Omega} \beta/\gamma dx \) as \( d_I \to 0 \), and \( R_0 \to \int_{\Omega} \beta/\int_{\Omega} \gamma \) as \( d_I \to \infty \). Here \( \beta(x)/\gamma(x) \) is the basic reproduction number for the corresponding model without diffusion (which we will call the local basic reproduction number).

For some reaction-diffusion epidemic models, \( R_0 \) is related to the principal eigenvalue of an elliptic system, which makes the analysis more difficult. Peng and Zhao \[27\] write \( R_0 \) as the principal eigenvalue of an eigenvalue problem consisting of a sin-

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reaction-diffusion system: the model in [12] to study the outbreak of Zika in Rio De Janerio is the following uninfected vectors, and infected vectors at position $H$ and $V_i(x, t)$ and $V_i(x, t)$ be the density of uninfected hosts, infected hosts, uninfected vectors, and infected vectors at position $x$ and time $t$, respectively. Then the model in [12] to study the outbreak of Zika in Rio De Janerio is the following reaction-diffusion system:

$$
\begin{aligned}
\frac{dH_i}{dt} &= -\lambda H_i + \sigma_1 H_u(x)V_i, \\
\frac{dV_i}{dt} &= -\delta_2 V_i + \beta (V_u + V_i) - \mu (V_u + V_i)V_i,
\end{aligned}
$$

(1.1)

where $\delta_1, \delta_2 \in C^{1+\alpha}(\bar{\Omega})$ are strictly positive, and the functions $H_u, \lambda, \beta, \sigma_1, \sigma_2$ and $\mu$ are strictly positive and belong to $C^\alpha(\bar{\Omega})$. It is assumed that uninfected hosts are stationary in space, and the diffusion of infected hosts corresponds indirectly to the movement of the Zika virus in the spatial environment. Both uninfected and infected vectors are assumed to diffuse in the spatial environment.

Following [28, 31], the basic reproduction number $R_0$ for (1.1) is defined as the spectral radius $r(-CB^{-1})$ of $-CB^{-1}$, where $B : D(B) \subset C(\bar{\Omega}; \mathbb{R}^2) \rightarrow C(\bar{\Omega}; \mathbb{R}^2)$ and $C : C(\bar{\Omega}; \mathbb{R}^2) \rightarrow C(\bar{\Omega}; \mathbb{R}^2)$ are the linear operators

$$
B = \begin{pmatrix}
\nabla \cdot \delta_1 \nabla \\
\nabla \cdot \delta_2 \nabla
\end{pmatrix} + \begin{pmatrix}
-\lambda & \sigma_1 H_u \\
0 & -\mu V
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & 0 \\
\sigma_2 \hat{V} & 0
\end{pmatrix},
$$

(1.2)

and $\hat{V}$ is the unique positive solution of the elliptic problem

$$
\begin{aligned}
-\nabla \cdot \delta_2 (x) \nabla \varphi &= \beta (x) \varphi - \mu(x) \varphi^2, & x \in \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0 \text{ on } \partial \Omega \text{ and } B(\varphi, \psi) \in C(\bar{\Omega}; \mathbb{R}^2)
\end{aligned}
$$

(1.3)

The system (1.1) in the case without diffusion, and viewed as an ordinary differential equations system at a specific location $x$, is

$$
\begin{aligned}
dH_i/dt &= -\lambda(x) H_i(t) + \sigma_1(x) H_u(x)V_i(t), \\
dV_i/dt &= -\sigma_2(x)V_u(t) H_i(t) + \beta(x)(V_u(t) + V_i(t)) - \mu(x)(V_u(t) + V_i(t))V_i(t), \\
dV_i/dt &= \sigma_2(x)V_u(t) H_i(t) - \mu(x)(V_u(t) + V_i(t))V_i(t).
\end{aligned}
$$

(1.4)
The basic reproduction number of (1.4) at a specific location $x$, obtained by the next generation method, is

$$ R(x) = R_1(x)R_2(x), \text{ where } R_1(x) = \frac{\sigma_1(x)H_a(x)}{\lambda(x)}, \text{ and } R_2(x) = \frac{\sigma_2(x)}{\mu(x)}. $$

$R_1(x)$ and $R_2(x)$ have their own biological meanings: at a specific location $x$, $R_1(x)$ measures the impact of one infected vector on susceptible hosts while $R_2(x)$ measures the impact of one infected host on the susceptible vectors. Since $R_0$ is difficult to visualize, our main purpose of this research is to study the relation between $R_0$ and $R(x)$, the latter being a function of $x \in \Omega$.

In sections 3 and 4, we study the relation of $R_0$ and $R(x)$, where our approach is based on the formula

$$ R_0 = r(L_1L_2R_2), \quad L_1 := (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \lambda, \text{ and } L_2 := (\mu \nabla - \nabla \cdot \delta_2 \nabla)^{-1} \mu \nabla, $$

where $L_1$ and $L_2$ are viewed as multiplication operators on $C(\bar{\Omega})$, and $L_1$ and $L_2$ are strongly positive compact linear operators on $C(\bar{\Omega})$. This formula establishes an interesting connection between $R_0$ and $R$ as $r(L_1L_2) = r(L_1) = r(L_2) = 1$ (see Lemma 3.4). Consequences of this formula are

- If $R_1$ and $R_2$ are constant, then $R_0 = R$ (see Corollary 3.5);
- $R_0 \geq 1$ if $R_i(x) \geq 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$ and $R_0 \leq 1$ if $R_i(x) \leq 1$, $i = 1, 2$, for all $x \in \Omega$ (see Theorem 3.6).

When the diffusion coefficients $\delta_1$ and $\delta_2$ are constant, we establish a quantitative connection of $R_0$ and $R$. To this end, we prove a result (Theorem 4.1) about the convergence of spectral radii for a sequence of strongly positive compact linear operators in an ordered Banach space. Based on Theorem 4.1, we show

- $\lim_{\delta_1 \to 0} R_0 = \frac{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}$ for $\delta_2 > 0$ and $\lim_{\delta_2 \to 0} R_0 = \frac{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}$ (see Theorem 4.5);
- $\lim_{\delta_1, \delta_2 \to (0, \infty)} R_0 = \frac{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}{\int_0^{\delta_1} \int_0^{\delta_2} dx dx}$ (see Remark 4.4).

In section 5, we conduct numerical simulations to illustrate our results. In section 6, we give concluding remarks and provide two examples about adopting our approach to analyze $R_0$ for reaction-diffusion epidemic models.

### 2. Preliminaries

The global dynamics of (1.1) have been analyzed in [24], and we first summarize the results that will be used here. Let $V = V_a + V_i$. Then $V(x, t)$ satisfies

$$ V_t - \nabla \cdot \delta_2(\nabla V) = \beta(x) V - \mu(x) V^2, \quad x \in \Omega, t > 0, $$

$$ \frac{\partial V}{\partial n} = 0, \quad x \in \partial \Omega, t > 0, $$

$$ V(\cdot, 0) = V_0 \in C(\bar{\Omega}). $$

The following result about (2.1) is well-known (see [4, Proposition 3.17] [15, Lemma A.1], and [18, Proposition 2.5]):

**Lemma 2.1.** For any nonnegative nontrivial initial data $V_0 \in C(\bar{\Omega})$, (2.1) has a unique global classic solution $V(x, t)$. Moreover, $V(x, t) > 0$ for all $(x, t) \in \Omega \times (0, \infty)$ and

$$ \lim_{t \to +\infty} \|V(\cdot, t) - \bar{V}\|_\infty = 0, $$

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where $\hat{V}$ is the unique positive solution of the elliptic problem (1.3). Moreover, if $δ_2$ is a constant parameter, then

$$\lim_{δ_2→0} \hat{V} \to \frac{β}{μ} \quad \text{and} \quad \lim_{δ_2→∞} \hat{V} \to \frac{\int_Ω \beta dx}{\int_Ω μ dx} \quad \text{in} \ C(\bar{Ω}).$$

The definition of $R_0$ for (1.1) is closely related to the stability of the semi-trivial equilibrium $E_1 = (0, \hat{V}, 0)$ of (1.1). Linearizing the model at $E_1$, one can see that the stability of $E_1$ is determined by the sign of the principal eigenvalue of the problem:

$$\begin{cases}
κϕ = \nabla \cdot (δ_1 \nabla ϕ) - λϕ + σ_1 H_u ψ, & x ∈ Ω,
κψ = \nabla \cdot (δ_2 \nabla ψ) + σ_2 \hat{V} ϕ - μ \hat{V} ψ, & x ∈ Ω,
∂ϕ/∂n = ∂ψ/∂n = 0, & x ∈ ∂Ω.
\end{cases}$$ (2.3)

Problem (2.3) is cooperative, so it has a principal eigenvalue $κ_0$ associated with a positive eigenvector $(\hat{ϕ}_0, \hat{ψ}_0)$ ([17]). Let $A = B + C$, where $B$ and $C$ are defined in section 1. Then $A$ and $B$ are resolvent positive ([28]), and $A$ is a positive perturbation of $B$. By [28, Theorem 3.5], $κ_0 = s(A)$ and $r(-CB^{-1}) - 1$ have the same sign, where $s(A)$ is the spectral bound of $A$. We then have the following result:

**Theorem 2.2.** $R_0 - 1$ and $κ_0$ have the same sign. Moreover, $E_1$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

The main results proved in [24] about the global dynamics of the model (1.1) are as follows:

**Theorem 2.3.** The following hold:

- If $R_0 ≤ 1$, then for any nonnegative initial data $(H_{ι0}, V_{ι0}, V_0) ∈ C(Ω; \mathbb{R}^3_+)$ with $V_{ι0} + V_0 ≠ 0$, the solution $(H_{ι}, V_{ι}, V_i)$ of (1.1) satisfies

$$\lim_{t→∞} \left\| (H_{ι}(\cdot, t), V_{ι}(\cdot, t), V_i(\cdot, t)) - E_1 \right\|_∞ = 0,$$ (2.4)

where $E_1 = (0, \hat{V}, 0)$.

- If $R_0 > 1$, then for any initial data $(H_{ι0}, V_{ι0}, V_0)$ with $V_{ι0} + V_0 ≠ 0$ and $H_{ι0} + V_0 ≠ 0$, the solution $(H_{ι}, V_{ι}, V_i)$ of (1.1) satisfies

$$\lim_{t→∞} \left\| (H_{ι}(\cdot, t), V_{ι}(\cdot, t), V_i(\cdot, t)) - (\hat{H}_ι, \hat{V}_ι, V_i) \right\|_∞ = 0,$$ (2.5)

where $E_2 = (\hat{H}_ι, \hat{V}_ι, \hat{V}_i)$ is the unique EE of (1.1).

Let $X$ be an ordered Banach space with positive cone $X_+$, and let $L_1, L_2 : X → X$ be two bounded linear operators. Then it is well-known that

$$r(L_1L_2) = r(L_2L_1) ≤ \|L_1\|\|L_2\|,$$ (2.5)

where $r(L_i)$ denotes the spectral radius of $L_i$, $i = 1, 2$. Indeed, this can be derived easily from the Gelfand’s formula

$$r(L_1) = \lim_{n→∞} \|L_1^n\|^{1/n}.$$ (2.6)

**Remark 2.4.** It is very important to note that (2.6) does not imply $r(L_1L_2L_3) = r(L_3L_2L_1)$. 

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Suppose that \( X_+ \) has non-empty interior \( \text{int}(X_+) \). Then \( L_1 \) is strongly positive if \( L_1(X_+ \setminus 0) \subseteq \text{int}(X_+) \). The operator \( L_1 \) is compact if the image of the unit ball is relatively compact in \( X \). We will need the following generalization of Krein-Rutman theorem ([2]).

**Theorem 2.5.** Let \( X \) be an ordered Banach space with positive cone \( X_+ \) such that \( X_+ \) has non-empty interior. Suppose that \( T : X \to X \) is a strongly positive compact linear operator. Then the spectral radius \( r(T) \) is positive and a simple eigenvalue of \( T \) associated with a positive eigenvector, and there is no other eigenvalue with a positive eigenvector. Moreover if \( S : X \to X \) is a linear operator such that \( S \geq T \), i.e. \( S(v) \geq T(v) \) for all \( v \in X_+ \), then \( r(S) \geq r(T) \). If, in addition, \( S - T \) is strongly positive, then \( r(S) > r(T) \).

**3. General diffusion rates.** Our basic result about the basic reproduction number \( R_0 \) of (1.1) is

**Theorem 3.1.** Let \( R_0 = r(-CB^{-1}) \), where \( B \) and \( C \) are defined in (1.2). Then, \( R_0 = r(L_1R_1L_2R_2) \), where \( R_1 \) and \( R_2 \) defined in (1.5) are multiplication operators on \( C(\overline{\Omega}) \), and \( L_1 \) and \( L_2 \) defined in (1.6) are strongly positive compact linear operators on \( C(\overline{\Omega}) \).

**Proof.** It is not hard to compute

\[
B^{-1} = \begin{pmatrix}
(\nabla \cdot \delta_1 \nabla - \lambda)^{-1} & -(\nabla \cdot \delta_1 \nabla - \lambda)^{-1} \sigma_1 H_u (\nabla \cdot \delta_2 \nabla - \mu \hat{V})^{-1} \\
0 & (\nabla \cdot \delta_2 \nabla - \mu \hat{V})^{-1}
\end{pmatrix}.
\]

Therefore,

\[
-CB^{-1} = \begin{pmatrix}
0 & 0 \\
\sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} & \sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u (\mu \hat{V} - \nabla \cdot \delta_2 \nabla)^{-1}
\end{pmatrix}.
\]

It then follows that

\[
R_0 = r(-CB^{-1}) = r \left( \sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u (\mu \hat{V} - \nabla \cdot \delta_2 \nabla)^{-1} \right)
\]

\[
= r \left( \sigma_2 \hat{V} L_1R_1L_2 \frac{1}{\mu \hat{V}} \right).
\]

From (2.5), we have

\[
R_0 = r \left( L_1R_1L_2 \frac{1}{\mu \hat{V}} \sigma_2 \hat{V} \right) = r(L_1R_1L_2R_2).
\]

It is well-known that the elliptic estimates and maximum principles imply that \( L_1 \) and \( L_2 \) are strongly positive compact linear operators on \( C(\overline{\Omega}) \).

**Lemma 3.2.** \( \|L_1\| = 1 \) and \( \|L_2\| = 1 \).

**Proof.** Notice that \( L_i(\pm 1) = \pm 1 \) for \( i = 1, 2 \). For any \( u \in C(\overline{\Omega}) \) with \( \|u\|_\infty \leq 1 \), we have \(-1 \leq u \leq 1\). By the comparison principle, we have

\[
-1 = L_i(-1) \leq L_iu \leq L_i1 = 1, \text{ for } i = 1, 2.
\]

Therefore, \( \|L_iu\|_\infty \leq 1 = \|u\|_\infty \), which implies \( \|L_i\| \leq 1 \) for \( i = 1, 2 \). Moreover, since \( L_11 = 1 \) and \( L_21 = 1 \), we must have \( \|L_1\| = \|L_2\| = 1 \).
We immediately have the following result from (2.5):

**THEOREM 3.3.** If \( R_i(x) < 1 \), \( i = 1, 2 \), for all \( x \in \bar{\Omega} \), then \( R_0 < 1 \).

**Proof.** \( R_0 = r(L_1 R_1 L_2 R_2) \leq \| L_1 \| \| R_1 \| \| L_2 \| \| R_2 \| = \| R_1 \| \| R_2 \| < 1 \). □

We apply the Krein-Rutman theorem to study the spectral radius of \( L_1, L_2 \) and \( L_1 L_2 \).

**LEMMA 3.4.** The spectral radius of \( L_1, L_2 \) and \( L_1 L_2 \) is 1, i.e., \( r(L_1) = r(L_2) = r(L_1 L_2) = 1 \).

**Proof.** Since \( L_1 \) and \( L_2 \) are strongly positive compact operators on \( C(\bar{\Omega}) \), so is \( L_1 L_2 \). By Theorem 2.5, \( r(L_1) \), \( r(L_2) \), and \( r(L_1 L_2) \) are simple positive eigenvalues of \( L_1, L_2 \), and \( L_1 L_2 \), associated with positive eigenvectors, respectively. Moreover, there is no other eigenvalue of \( L_1, L_2 \), or \( L_1 L_2 \) associated with a positive eigenvector. Since \( L_1 1 = L_2 1 = L_1 L_2 1 = 1 \), we must have \( r(L_1) = r(L_2) = r(L_1 L_2) = 1 \). □

Noticing that \( R_0 = r(L_1 R_1 L_2 R_2) \), Lemma 3.4 implies that there is a significant connection between the basic reproduction number \( R_0 \) and the local basic reproduction number \( R(x) \). A consequence of Lemma 3.4 is the following result:

**COROLLARY 3.5.** If \( R_1 \) and \( R_2 \) are constant, then \( R_0 = R \).

Our next result, based on the Krein-Rutman theorem, is stronger than Theorem 3.3.

**THEOREM 3.6.** The following hold:

1. If \( R_i(x) \geq 1 \), \( i = 1, 2 \), for all \( x \in \bar{\Omega} \), then \( R_0 \geq 1 \). If, in addition, \( R_i(x) \neq 1 \) or \( R_2(x) \neq 1 \), then \( R_0 > 1 \).
2. If \( R_i(x) \leq 1 \), \( i = 1, 2 \), for all \( x \in \bar{\Omega} \), then \( R_0 \leq 1 \). If, in addition, \( R_i(x) \neq 1 \) or \( R_2(x) \neq 1 \), then \( R_0 < 1 \).
3. \( R_{1m} R_{2m} \leq R_0 \leq R_{1M} R_{2M} \), where \( R_{im} = \min \{ R_i(x) : x \in \bar{\Omega} \} \) and \( R_{im} = \max \{ R_i(x) : x \in \bar{\Omega} \} \), \( i = 1, 2 \).

**Proof.** We only prove part 1 as the proof of the rest is similar. If \( R_i(x) \geq 1 \) for all \( x \in \bar{\Omega} \), then \( L_1 R_1 L_2 R_2 \geq L_1 L_2 \). By Theorem 2.5 and Lemma 3.4, we have \( R_0 = r(L_1 R_1 L_2 R_2) \geq r(L_1 L_2) = 1 \).

Let \( \phi \) be a positive eigenfunction corresponding to principal eigenvalue \( R_0 \) of \( L_1 R_1 L_2 R_2 \). If, in addition, \( R_1(x) \neq 1 \) or \( R_2(x) \neq 1 \), by the strong positivity of \( L_1 \) and \( L_2 \), we have

\[ R_0 \phi = L_1 R_1 L_2 R_2 \phi \gg L_1 L_2 \phi. \]

Therefore, there exists \( \epsilon > 0 \) such that \( R_0 \phi \geq (1 + \epsilon) L_1 L_2 \phi \). Let \( \phi_m = \min_{x \in \bar{\Omega}} \phi(x) > 0 \). Then, by the positivity of \( L_1 L_2 \) and \( L_1 L_2 1 = 1 \), we have

\[ R_0 \phi \geq (1 + \epsilon) L_1 L_2 \phi \geq (1 + \epsilon) L_1 L_2 \phi_m = (1 + \epsilon) \phi_m. \]

Therefore, \( R_0 \phi \geq (1 + \epsilon) \phi_m \), which implies \( R_0 \geq 1 + \epsilon > 1 \). □

We next study the monotonicity of \( R_0 \). Here, we need the assumption:

(H1) \( \sigma_1 H_u = \sigma_2 \tilde{V} \), or both \( \sigma_1 H_u \) and \( \sigma_2 \tilde{V} \) are constants.

**THEOREM 3.7.** Suppose that (H1) holds. If \( \delta_1 \) is constant, then \( R_0 \) is decreasing in \( \delta_1 \).

**Proof.** Let \( \kappa = 1/R_0 \). By the Krein-Rutman theory, \( \kappa \) is an eigenvalue associated with a positive eigenvector \( \phi \) (we normalize \( \phi \) such that \( \| \phi \|_2 = 1 \)) of the following problem:

\[ \kappa L_1 R_1 L_2 R_2 \phi = \phi. \]
Therefore, we have

\[ (3.2) \quad \kappa \lambda R_1 L_2 R_2 \phi = (\lambda - \delta_1 \Delta) \phi. \]

Differentiating both sides with respect to \( \delta_1 \), we have

\[ (3.3) \quad \kappa \delta_1 \lambda R_1 L_2 R_2 \phi + \kappa \lambda R_1 L_2 R_2 \phi_{\delta_1} = -\Delta \phi + (\lambda - \delta_1 \Delta) \phi_{\delta_1}. \]

Multiplying (3.3) by \( \phi \) and (3.2) by \( \phi_{\delta_1} \), and integrating their difference over \( \Omega \), we obtain

\[ \kappa \delta_1 \int_{\Omega} \phi \lambda R_1 L_2 R_2 \phi \, dx = \int_{\Omega} |\nabla \phi|^2 \, dx, \]

where we used the assumption (H1) to derive

\[ \int_{\Omega} \phi_{\delta_1} \lambda R_1 L_2 R_2 \phi \, dx = \int_{\Omega} \phi \lambda R_1 L_2 R_2 \phi_{\delta_1} \, dx. \]

Since \( \lambda R_1 L_2 R_2 \) is strongly positive, \( \lambda R_1 L_2 R_2 \phi > 0 \). Therefore, \( \kappa \delta_1 \geq 0 \) and \( \kappa \) is increasing in \( \delta_1 \). Hence, \( R_0 \) is decreasing in \( \delta_1 \).

**Remark 3.8.** If \( \beta/\mu \) is constant, \( \bar{V} \) is independent of \( \delta_2 \). Then, similar to Theorem 3.7, \( R_0 = \bar{r}(L_2 R_2 L_1 R_1) \) is decreasing in \( \delta_2 \) if (H1) holds. Moreover, from the proof of Theorem 3.7, \( R_0 \) is strictly decreasing, if the eigenvector is non-constant.

**4. Small or large diffusion rates.** We prove the following result on the convergence of spectral radii for strongly positive compact linear operators, which is essential for our investigation of the role of diffusion rates for the basic reproduction number \( R_0 \).

**Theorem 4.1.** Let \( X \) be an ordered Banach space with positive cone \( X_+ \) such that \( X_+ \) has nonempty interior. Let \( T_n, n \geq 1 \), and \( T \) be strongly positive compact linear operators on \( X \). Suppose \( T_n \xrightarrow{\text{SOT}} T \) (Strong Operator Topology) which means \( T_n(u) \to T(u) \) for any \( u \in X \). If \( \cup_{n \geq 1} T_n(B) \) is precompact, where \( B \) is the closed unit ball of \( X \), and \( r(T_n) \geq r_0 \) for some \( r_0 > 0 \), then \( r(T_n) \to r(T) \).

**Proof.** Since \( T \) and \( T_n \) are compact and strongly positive, by Theorem 2.5, \( r(T) \) and \( r(T_n) \) are positive simple eigenvalues of \( T \) and \( T_n \), respectively. So there exists \( e_n \in \text{int}(X_+) \) with \( \|e_n\| = 1 \) such that \( T_n e_n = r(T_n) e_n \) for all \( n \geq 1 \). Since \( \cup_{n \geq 1} T_n(B) \) is precompact and \( r(T_n) \geq r_0 > 0 \), \( \{e_n\} \) is precompact. So there exists a subsequence \( \{e_{n_k}\} \) of \( \{e_n\} \) such that \( e_{n_k} \to e \) for some \( e \in X \).

We claim \( T_{n_k} e_{n_k} \to Te \). Note that \( \sup_{n \geq 1} \|T_n(u)\| < \infty \) for any \( u \in X \) by the convergence assumption \( T_n \xrightarrow{\text{SOT}} T \). Then by the uniform boundedness principle, there exists \( M > 0 \) such that \( \sup_{n \geq 1} \|T_n\| < M \). Let \( \epsilon > 0 \) be arbitrary. Since \( e_{n_k} \to e \) and \( T_{n_k} e \to Te \), there exists \( N > 0 \) such that \( \|e_{n_k} - e\| < \epsilon \) and \( \|T_{n_k} e - Te\| < \epsilon \) for all \( k > N \). Hence for all \( k > N \), we have

\[ \|T_{n_k} e_{n_k} - Te\| \leq \|T_{n_k} (e_{n_k} - e)\| + \|T_{n_k} e - Te\| \leq M \epsilon + \epsilon. \]

Since \( \epsilon > 0 \) was arbitrary, \( T_{n_k} e_{n_k} \to Te \).

Since \( T_{n_k} e_{n_k} = r(T_{n_k}) e_{n_k}, \ T_{n_k} e_{n_k} \to Te \) and \( e_{n_k} \to e \), we have \( r(T_{n_k}) = \|T_{n_k} e_{n_k}\| \to \|Te\| \) and \( Te = \|Te\| e \). Since \( e_n \in X_+ \) and \( \|e_n\| = 1, e \in X_+ \and \)

\[ \|e\| = 1. \] Thus \( e \) is a positive eigenvector of \( T \) corresponding to eigenvalue \( \|Te\| \).

Again by Theorem 2.5, we have \( r(T) = \|Te\| \). Hence \( r(T_{n_k}) \to r(T) \) and \( r(T_n) \to r(T) \)

(Here we use a well-known result: if every subsequence of the sequence \( \{a_n\} \) has a convergent subsequence with limit \( a \), then \( a_n \to a \)).
The convergence of a sequence of compact operators in the strong operator topology is not sufficient to guarantee the convergence of their spectral radii. We use the following simple example to illustrate this fact:

**Example 4.2.** Let $H$ be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^\infty$. For $n \geq 1$, define $T_n : H \to H$ by

$$T_n(a) = \sum_{i=1}^\infty a_i e_i \text{ for any } a = \sum_{i=1}^\infty a_i e_i \in H.$$ 

Then $\{T_n\}$ is a sequence of compact operators with $r(T_n) = 1$, and $T_n \overset{\text{SOT}}{\to} 0$. Since $r(T_n) = 1$ and $r(T) = 0$, $r(T_n) \not\to r(T)$.

It is interesting to see whether some of the hypotheses in Theorem 4.1 can be dropped. We leave this as an open problem.

### 4.1. Large diffusion rates

In the following two subsections, we investigate $R_0$ quantitatively when the diffusion rates are large or small. To this end, we assume that $\delta_1$ and $\delta_2$ are constants. Define two integral operators $L_{1,\infty}, L_{2,\infty} : C(\Omega) \to C(\Omega)$ by

$$L_{1,\infty}(\phi) = \frac{\int_\Omega \lambda(x) \phi(x) \, dx}{\int_\Omega \lambda(x) \, dx} \quad \text{and} \quad L_{2,\infty}(\phi) = \frac{\int_\Omega \mu(x) \phi(x) \, dx}{\int_\Omega \mu(x) \, dx} \text{ for any } \phi \in C(\bar{\Omega}).$$

**Lemma 4.3.** $L_1 \overset{\text{SOT}}{\to} L_{1,\infty}$ in $C(\bar{\Omega})$ as $\delta_1 \to \infty$.

*Proof.* Let $u \in C(\bar{\Omega})$ be given. We need to prove that $L_1(u) \to L_{1,\infty}(u)$ in $C(\bar{\Omega})$ as $\delta_1 \to \infty$. For any $\delta_1 > 0$, let $v_{\delta_1} = L_1(u)$. Then $v_{\delta_1}$ is the solution of the problem

$$\begin{cases} 
\lambda v_{\delta_1} - \delta_1 \Delta v_{\delta_1} = \lambda u, & x \in \Omega, \\
\frac{\partial}{\partial n} v_{\delta_1} = 0, & x \in \partial \Omega.
\end{cases} \tag{4.1}$$

By the comparison principle, we have $-\|u\|_\infty \leq v_{\delta_1} \leq \|u\|_\infty$ for all $\delta_1 > 1$. Hence by the $L^p$ estimate, $\{v_{\delta_1}\}_{\delta_1 > 1}$ is uniformly bounded in $W^{2,p}(\Omega)$ for any $p > 1$. Since the embedding $W^{2,p}(\Omega) \subseteq C(\bar{\Omega})$ is compact for $p > n$, up to a subsequence, $v_{\delta_1} \to v$ weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ for some $v \in W^{2,p}(\Omega)$ as $\delta_1 \to \infty$. Moreover, $v$ satisfies

$$\begin{cases} 
-\Delta v = 0, & x \in \Omega, \\
\frac{\partial}{\partial n} v = 0, & x \in \partial \Omega.
\end{cases} \tag{4.1}$$

By the maximum principle, $v$ is a constant. Integrating both sides of the first equation of (4.1) and taking $\delta_1 \to \infty$, we find $v = \int_\Omega \frac{\lambda u \, dx}{\int_\Omega \lambda \, dx}$.

**Lemma 4.4.** $L_2 \overset{\text{SOT}}{\to} L_{2,\infty}$ in $C(\bar{\Omega})$ as $\delta_2 \to \infty$.

*Proof.* Let $u \in C(\bar{\Omega})$ be given. We need to prove that $L_2(u) \to L_{2,\infty}(u)$ in $C(\bar{\Omega})$ as $\delta_2 \to \infty$. For any $\delta_2 > 0$, let $v_{\delta_2} = L_2(u)$. Then $v_{\delta_2}$ is the solution of the problem

$$\begin{cases} 
\mu \hat{V} v_{\delta_2} - \delta_2 \Delta v_{\delta_2} = \mu \hat{V} u, & x \in \Omega, \\
\frac{\partial}{\partial n} v_{\delta_2} = 0, & x \in \partial \Omega.
\end{cases} \tag{4.2}$$

Noticing that $\hat{V}$ is the positive solution of

$$\begin{cases} 
-\delta_2 \Delta V = \beta V - \mu V^2, & x \in \Omega, \\
\frac{\partial}{\partial n} V = 0, & x \in \partial \Omega,
\end{cases}$$
it satisfies
\[ \hat{V} \to \frac{\int_{\Omega} \beta dx}{\int_{\Omega} \mu dx}, \quad \text{as} \; \delta_2 \to \infty. \] (4.3)

(see [4, Proposition 3.17] and [18, Proposition 2.5]). The rest of the proof is essentially the same as the proof of Lemma 4.3.

We now investigate \( R_0 \) for large diffusion rates by Theorem 4.1.

**Theorem 4.5.** The following statements hold:

1. For fixed \( \delta_2 > 0 \),

\[ R_0 \to r(L_{1,\infty} R_1 L_2 R_2) = \frac{\int_{\Omega} \lambda R_1(L_2 R_2) dx}{\int_{\Omega} \lambda dx} \quad \text{as} \; \delta_1 \to \infty; \]

2. For fixed \( \delta_1 > 0 \),

\[ R_0 \to r(L_{2,\infty} R_2 L_1 R_1) = \frac{\int_{\Omega} \mu R_2(L_1 R_1) dx}{\int_{\Omega} \mu dx} \quad \text{as} \; \delta_2 \to \infty. \]

**Proof.** For \( i = 1, 2 \), define two bounded linear operators \( H_{i,\infty} : C(\bar{\Omega}) \to C(\bar{\Omega}) \) by

\[ H_{1,\infty}(\phi) = \frac{\int_{\Omega} \lambda R_1 L_2 R_2 \phi dx}{\int_{\Omega} \lambda dx} \quad \text{and} \quad H_{2,\infty}(\phi) = \frac{\int_{\Omega} \mu R_2 L_1 R_1 \phi dx}{\int_{\Omega} \mu dx} \]

for any \( \phi \in C(\bar{\Omega}) \).

Then \( H_{1,\infty} = L_{1,\infty} R_1 L_2 R_2 \) and \( H_{2,\infty} = L_{2,\infty} R_2 L_1 R_1 \). By Lemmas 4.3-4.4, we have

\[ L_1 R_1 L_2 R_2 \xrightarrow{\text{SOT}} H_{1,\infty} \quad \text{as} \; \delta_1 \to \infty \quad \text{and} \quad L_2 R_2 L_1 R_1 \xrightarrow{\text{SOT}} H_{2,\infty} \quad \text{as} \; \delta_2 \to \infty. \]

Clearly, \( L_1 R_1 L_2 R_2, L_2 R_2 L_1 R_1, H_{1,\infty} \) and \( H_{2,\infty} \) are strongly positive compact operators on \( C(\bar{\Omega}) \). In the proof of Lemma 3.2, we have shown that \( L_i(\mathcal{B}) \subset \mathcal{B}, \; i = 1, 2 \). This implies that \( \bigcup_{i_1 > 1} L_{i_1} R_{i_2} R_2(B) \subset L_1 R_1(R_2 M B) \) and \( \bigcup_{i_2 > 1} L_2 R_{i_2} R_2(B) \subset L_2 R_2(R_1 M B) \) are precompact in \( C(\Omega) \), where \( R_{1 M} \) and \( R_{2 M} \) are defined in Theorem 3.6. By Theorem 3.6, we have \( r(L_{1,\infty} R_1 L_2 R_2) = r(L_2 R_2 L_1 R_1) \geq R_{1 M} R_{2 M} > 0 \).

Then by Theorem 4.1, we have \( R_0 = r(L_1 R_1 L_2 R_2) \to r(H_{1,\infty}) \) as \( \delta_1 \to \infty \) and \( R_0 = r(L_2 R_2 L_1 R_1) \to r(H_{2,\infty}) \) as \( \delta_2 \to \infty \). Finally, we observe that the eigenfunctions of \( H_{1,\infty} \) and \( H_{2,\infty} \) must be constants, and

\[ r(H_{1,\infty}) = \frac{\int_{\Omega} \lambda R_1(L_2 R_2) dx}{\int_{\Omega} \lambda dx} \quad \text{and} \quad r(H_{2,\infty}) = \frac{\int_{\Omega} \mu R_2(L_1 R_1) dx}{\int_{\Omega} \mu dx}. \]

**Remark 4.6.** If \( R_2 \) is constant, then \( L_2 R_2 = R_2 \) and

\[ R_0 \to \frac{\int_{\Omega} \lambda R_1(L_2 R_2) dx}{\int_{\Omega} \lambda dx} = \frac{\int_{\Omega} \lambda R dx}{\int_{\Omega} \lambda dx} \quad \text{as} \; \delta_1 \to \infty, \]

which is independent of \( \delta_2 \). Similarly, if \( R_1 \) is constant, then

\[ R_0 \to \frac{\int_{\Omega} \mu R_2(L_1 R_1) dx}{\int_{\Omega} \mu dx} = \frac{\int_{\Omega} \mu R dx}{\int_{\Omega} \mu dx} \quad \text{as} \; \delta_2 \to \infty, \]

which is independent of \( \delta_1 \).
Define
\[
\hat{R}_1 := \frac{\int_{\Omega} \lambda R_1 \, dx}{\int_{\Omega} \lambda \, dx} = \frac{\int_{\Omega} \sigma_1 \mu_1 \, dx}{\int_{\Omega} \sigma_1 \, dx} \quad \text{and} \quad \hat{R}_2 := \frac{\int_{\Omega} \lambda R_2 \, dx}{\int_{\Omega} \lambda \, dx} = \frac{\int_{\Omega} \sigma_2 \, dx}{\int_{\Omega} \mu_1 \, dx}.
\]

**Theorem 4.7.** The following statements hold:
1. \( r(L_{1,\infty} R_1 L_2 R_2) \rightarrow \hat{R}_1 \hat{R}_2 \) as \( \delta_2 \rightarrow \infty \);
2. \( r(L_{2,\infty} R_2 L_1 R_1) \rightarrow \hat{R}_1 \hat{R}_2 \) as \( \delta_1 \rightarrow \infty \).

**Proof.** By Lemmas 4.3–4.4, we have
\[
L_2 R_2 \rightarrow \frac{\int_{\Omega} \lambda R_2 \, dx}{\int_{\Omega} \lambda \, dx} \quad \text{and} \quad L_1 R_1 \rightarrow \frac{\int_{\Omega} \lambda R_1 \, dx}{\int_{\Omega} \lambda \, dx} \quad \text{in} \ C(\tilde{\Omega}).
\]

Our claim now just follows from Theorem 4.5.

**Remark 4.8.** By Theorems 4.5–4.7, we have
\[
\lim_{\delta_1 \rightarrow \infty} \lim_{\delta_2 \rightarrow \infty} R_0 = \lim_{\delta_2 \rightarrow \infty} \lim_{\delta_1 \rightarrow \infty} R_0 = \hat{R}_1 \hat{R}_2.
\]

We can actually prove
\[
(4.4) \quad \lim_{(\delta_1, \delta_2) \rightarrow (\infty, \infty)} R_0 = \hat{R}_1 \hat{R}_2,
\]

by making use of \( L_1 R_1 L_2 R_2 \xrightarrow{\text{SOT}} L_{1,\infty} R_1 L_{2,\infty} R_2 \) and Theorem 4.1.

**4.2. Small diffusion rates.** We next study \( R_0 \) when the diffusion rates are small.

**Theorem 4.9.** The following statements hold:
1. For fixed \( \delta_2 > 0 \), \( R_0 \rightarrow r(RL_2) \) as \( \delta_1 \rightarrow 0 \);
2. For fixed \( \delta_1 > 0 \), \( R_0 \rightarrow r(RL_1) \) as \( \delta_2 \rightarrow 0 \).

**Proof.** 1. It is well-known that, for each \( \phi \in C(\tilde{\Omega}) \), \( L_1 \phi \rightarrow \phi \) in \( C(\tilde{\Omega}) \) as \( \delta_1 \rightarrow 0 \). So we have \( R_1 L_2 R_2 L_1 \xrightarrow{\text{SOT}} R_1 L_2 R_2 \) as \( \delta_1 \rightarrow 0 \). Let \( B \) be the closed unit ball in \( C(\tilde{\Omega}) \). Since \( L_1(B) \subseteq B \), we have \( \cup_{\delta_1 < 1} R_1 L_2 R_2 L_1(B) \subseteq R_1 L_2 R_2(B) \). By the compactness of \( L_2 \), \( \cup_{\delta_1 < 1} R_1 L_2 R_2 L_1(B) \) is precompact in \( C(\tilde{\Omega}) \). By Theorem 3.6, we have \( r(R_1 L_2 R_2 L_1) \geq R_1 \delta_2 \delta_2 > 0 \). Noticing that \( R_1 L_2 R_2 L_1 \) and \( R_1 L_2 R_2 \) are strongly positive compact operators on \( C(\tilde{\Omega}) \), by Theorem 4.1, we have
\[
R_0 = r(R_1 L_2 R_2 L_1) \rightarrow r(R_1 L_2 R_2) = r(R_2 R_1 L_2) = r(RL_2), \quad \text{as} \quad \delta_1 \rightarrow 0.
\]

2. By [15, Lemma A.1], \( \hat{\nu} \rightarrow \beta/\mu \) in \( C(\tilde{\Omega}) \) and \( L_2 \phi \rightarrow \phi \) for any \( \phi \in C(\tilde{\Omega}) \) as \( \delta_2 \rightarrow 0 \). Hence \( R_2 L_1 R_2 L_1 \xrightarrow{\text{SOT}} R_2 L_1 R_1 \) as \( \delta_2 \rightarrow 0 \). The rest of the proof is similar to part 1.

Let \( R_M = \max \{ R(x) : x \in \tilde{\Omega} \} \). The proof of the following result is similar to [21, Lemma 3.1], and we attach it in the appendix for readers’s convenience. Unfortunately, we can not apply Theorem 4.1, since \( R \) is not compact. Can we generalize Theorem 4.1 so that it can be used to prove the following result? We leave this as an open question.

**Theorem 4.10.** The following statements hold:
1. \( r(RL_2) \rightarrow R_M \) as \( \delta_2 \rightarrow 0 \);
2. \( r(RL_1) \rightarrow R_M \) as \( \delta_1 \rightarrow 0 \).
Combining Theorems 4.9-4.10, we actually have

\[ \lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} R_0 = \lim_{\delta_2 \to 0} \lim_{\delta_1 \to 0} R_0 = \max \{ R(x) : x \in \bar{\Omega} \}. \]

We can apply [17] to prove the following result.

**Theorem 4.11.** The following statement holds:

\[ \lim_{(\delta_1, \delta_2) \to (0,0)} R_0 = \max \{ R(x) : x \in \bar{\Omega} \}. \]

**Proof.** Let \( R_M = \max \{ R(x) : x \in \bar{\Omega} \}. \) Firstly, suppose \( R_M = 1 \) and \( \hat{V} \) is independent of \( \delta_2 \). We need to show that \( R_0 \to 1 \) as \( (\delta_1, \delta_2) \to (0,0) \). Let \( \kappa = 1/R_0 \)
and view it as a function of \((\delta_1, \delta_2)\). Since \( R_0 \) is the principal eigenvalue of \( L_1 R_1 L_2 R_2 \), there exists a positive \( \Phi_0 = (\varphi_0, \psi_0)^T \) (satisfying homogeneous Neumann boundary conditions) such that \( \kappa \) satisfies

\[ A\Phi_0 + \kappa B\Phi_0 = 0, \]

where

\[ A = \begin{pmatrix} \delta_1 \Delta - \lambda & 0 \\ \mu \hat{V} R_2 & \delta_2 \Delta - \mu \hat{V} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \lambda R_1 \\ 0 & 0 \end{pmatrix}. \]

For any positive \( a, \delta_1, \) and \( \delta_2 \), let \( e = e(a, \delta_1, \delta_2) \) be the principal eigenvalue of the following eigenvalue problem (with homogeneous Neumann boundary conditions)

\[ A\Phi + aB\Phi = e\Phi. \]

Then, we have \( e(\kappa, \delta_1, \delta_2) = 0. \)

It has been shown in [17, Theorem 1.4] that

\[ \lim_{(\delta_1, \delta_2) \to (0,0)} e = \max_{x \in \bar{\Omega}} \hat{e}(C_a(x)), \]

where \( \hat{e}(C_a(x)) \) denotes the eigenvalue of the matrix \( C_a(x) \) with greater real part for each \( x \in \bar{\Omega} \) (By the Perron-Frobenius Theorem, the eigenvalues of \( C_a(x) \) are real), and

\[ C_a = \begin{pmatrix} -\lambda & a\lambda R_1 \\ \mu \hat{V} R_2 & -\mu \hat{V} \end{pmatrix}. \]

Therefore, for each \( a, e = e(a, \delta_1, \delta_2) \) can be extended to be a continuous function of \((\delta_1, \delta_2)\) on \((0, \infty) \times (0, \infty) \cup \{ (0,0) \} \) by \( e(a, 0, 0) := \max_{x \in \bar{\Omega}} \hat{e}(C_a(x)) \).

We claim that \( e \) is increasing in \( a \) for each \((\delta_1, \delta_2) \in (0, \infty) \times (0, \infty) \). To see this, we can choose \( \Phi = (\varphi, \psi) \) to be a positive eigenvector with \( \| \varphi \|_2 + \| \psi \|_2 = 1 \) of (4.8).

Then differentiate both sides of (4.8) with respect to \( a \), we obtain

\[ A\Phi_a + aB\Phi_a + B\Phi = e_a\Phi + e\Phi_a. \]

Multiplying (4.9) by \( \Phi^T \) to the left and (4.8) by \( \Phi_a^T \) to the left, and integrating their difference over \( \Omega \), we obtain \( \Phi^T B\Phi = e_a\Phi^T \Phi. \) Therefore, \( e_a = \int_{\Omega} \lambda R_1 \varphi \psi dx > 0 \) and \( e \) is strictly increasing in \( a \).

Noticing \( \max \{ R(x) : x \in \bar{\Omega} \} = 1 \), it is not hard to check that \( e(a, 0, 0) = \max_{x \in \bar{\Omega}} \hat{e}(C_a(x)) = 0 \) if and only if \( a = 1 \). Moreover, \( e(a, 0, 0) \) is strictly increasing in \( a \). Assume to the contrary that \( \kappa(\delta_1, \delta_2) \not\to 1 \) as \((\delta_1, \delta_2) \to (0,0) \). Then there
exists a sequence \( \{ (\delta_{1n}, \delta_{2n}) \}_{n=1}^{\infty} \) and \( a_0 \neq 1 \) such that \( \kappa_n := \kappa(\delta_{1n}, \delta_{2n}) \to a_0 \) as \( n \to \infty \). Without loss of generality, we may assume \( a_0 > 1 \). Choose \( \epsilon_0 > 0 \) such that
\[
0 < a_0 - \epsilon_0 < 1,
\]
which implies \( \kappa(a_0 - \epsilon_0, 0, 0) > \kappa(1, 0, 0) = 0 \). Then there exists \( N > 0 \) such that \( \kappa_n > a_0 - \epsilon_0 \) for all \( n \geq N \). By the monotonicity of \( \kappa \), we have
\[
0 = \epsilon(\kappa_n, \delta_{1n}, \delta_{2n}) > \epsilon(a_0 - \epsilon_0, \delta_{1n}, \delta_{2n}) \quad \text{for all } n \geq N.
\]
Taking \( n \to \infty \) and by the continuity of \( \epsilon(a_0 - \epsilon_0, \cdot, \cdot) \), we have
\[
0 \geq \lim_{n \to \infty} \epsilon(a_0 - \epsilon_0, \delta_{1n}, \delta_{2n}) = \epsilon(a_0 - \epsilon_0, 0, 0) > 0,
\]
which is a contradiction. Therefore, \( \kappa(\delta_1, \delta_2) \to 1 \) as \( (\delta_1, \delta_2) \to (0, 0) \). This proves the case \( \max\{R(x) : x \in \bar{\Omega}\} = 1 \).

Finally, we drop the assumption \( R_M = 1 \) but still suppose that \( \hat{V} \) is independent of \( \delta_2 \). We have
\[
\frac{R_0}{R_M} = r \left( L_1 R_1 L_2 \frac{R_2}{R_M} \right) \to \max \left\{ \frac{R_1(x) R_2(x)}{R_M} : x \in \bar{\Omega} \right\} = 1 \quad \text{as } (\delta_1, \delta_2) \to (0, 0).
\]
This means \( R_0 \to R_M \) as \( (\delta_1, \delta_2) \to (0, 0) \).

By Lemma 2.1, there exists \( \delta > 0 \) such that \( \|\hat{V} - \beta/\mu\|_{\infty} < \epsilon \) for all \( \delta_2 < \delta \). By the comparison principle, for \( \delta_2 < \delta \), we have
\[
(\mu(\frac{\beta}{\mu} + \epsilon) - \delta_2 \Delta)^{-1} \mu(\frac{\beta}{\mu} - \epsilon) \leq L_2 = (\mu \hat{V} - \delta_2 \Delta)^{-1} \mu \hat{V} \leq (\mu(\frac{\beta}{\mu} - \epsilon) - \delta_2 \Delta)^{-1} \mu(\frac{\beta}{\mu} + \epsilon).
\]
Define
\[
\hat{L}_{2\epsilon} = (\mu(\frac{\beta}{\mu} - \epsilon) - \delta_2 \Delta)^{-1} \mu(\frac{\beta}{\mu} - \epsilon)
\]
and
\[
\hat{R}_{2\epsilon} = \frac{\frac{\beta}{\mu} + \epsilon}{\frac{\beta}{\mu} - \epsilon} R_2.
\]
Similarly, we define \( \hat{L}_{2\epsilon} \) and \( \hat{R}_{2\epsilon} \) only with \( \epsilon \) replaced by \( -\epsilon \) in (4.10)-(4.11). Then, we have
\[
L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon} \leq L_1 R_1 L_2 R_2 \leq L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}, \quad \text{for } \delta_2 < \delta.
\]
It follows from Theorem 2.5 that
\[
r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}) \leq R_0 \leq r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}), \quad \text{for } \delta_2 < \delta.
\]
By the previous step,
\[
\lim_{(\delta_1, \delta_2) \to (0, 0)} r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}) = \max \{ R_1(x) \hat{R}_{2\epsilon}(x) : x \in \bar{\Omega} \} := \hat{R}_{M\epsilon}
\]
and
\[
\lim_{(\delta_1, \delta_2) \to (0, 0)} r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}) = \max \{ R_1(x) \hat{R}_{2\epsilon}(x) : x \in \bar{\Omega} \} := \hat{R}_{M\epsilon}.
\]
Taking \((\delta_1, \delta_2) \to (0, 0)\) in (4.12), we obtain
\[
\bar{R}_M \leq \lim \inf_{(\delta_1, \delta_2) \to (0, 0)} R_0 \leq \lim \sup_{(\delta_1, \delta_2) \to (0, 0)} R_0 \leq \hat{R}_M.
\]
Taking \(\epsilon \to 0\), we have
\[
\lim \inf_{(\delta_1, \delta_2) \to (0, 0)} R_0 = \lim \sup_{(\delta_1, \delta_2) \to (0, 0)} R_0 = R_M.
\]

By Theorem 4.11, we have the following result.

**Proposition 4.12.** The following statements hold:

1. If \(R(x) < 1\) for all \(x \in \Omega\), then there exists \(\hat{\delta} > 0\) such that \(R_0 < 1\) for all \((\delta_1, \delta_2)\) with \(\delta_1, \delta_2 \leq \hat{\delta}\);
2. If \(R(x) > 1\) for some \(x \in \Omega\), then there exists \(\hat{\delta} > 0\) such that \(R_0 > 1\) for all \((\delta_1, \delta_2)\) with \(\delta_1, \delta_2 \leq \hat{\delta}\).

5. Simulations.

5.1. Dependence on \(\delta_1\). In this section, we investigate the dependence of \(R_0\) on \(\delta_1\). Let \(\Omega = [0, 1] \times [0, 1]\). We fix all the coefficients except for \(\delta_1\): \(\delta_2 = 4, \sigma_1 = 5\sin(x) + 3, \sigma_2 = \mu = \beta = (x + 1)^2 + 0.1, H_u = \cos(y) + 1.5, \lambda = 12\). Since \(\beta/\mu = 1\), the unique positive solution of (1.3) is \(\hat{V} = 1\). By Theorem 3.6, \(R_0 \leq \max \{R(x) : x \in \Omega\} = 1.5015\). Noticing that \(R_2 = \sigma_2/\mu = 1\) and \(\lambda\) are constant, by Remark 4.6,

\[
R_0 \to \int_{\Omega} \lambda R dx = \int_{\Omega} R dx = 0.5854 \quad \text{as} \quad \delta_1 \to \infty.
\]

We then find \(r(\mathcal{R}_L)\). Using the fact that \(\kappa' = 1/r(\mathcal{R}_L)\) is the principal eigenvalue of the following problem (with homogeneous Neumann boundary conditions):

\[
(\mu \hat{V} - \delta_2 \Delta) \phi = \kappa \mu \hat{V} R \phi,
\]
we can compute \(r(\mathcal{R}_L) = 1.0075\) numerically. By Theorem 4.9, we expect

\[
R_0 \to r(\mathcal{R}_L) = 1.0075 \quad \text{as} \quad \delta_1 \to 0.
\]

We now compute \(R_0\). By the definition, \(\kappa = 1/R_0\) is the principal eigenvalue of the following problem (with homogeneous Neumann boundary conditions):

\[
\begin{pmatrix}
-\nabla \cdot \delta_1 \nabla \varphi + \lambda & -\sigma_1 H_u \\
-\nabla \cdot \delta_2 \nabla \psi & \mu \hat{V}
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix}
= \kappa
\begin{pmatrix}
0 & 0 \\
0 & \sigma_2 \hat{V}
\end{pmatrix}
\begin{pmatrix}
\varphi \\
\psi
\end{pmatrix}.
\]

For different values of \(\delta_1 \in [0.001, 400]\), we solve the eigenvalue problem numerically and plot \(R_0\) in Figure 1. In particular, \(R_0 = 1.0074\) when \(\delta_1 = 0.001\) and \(R_0 = 0.5904\) when \(\delta_1 = 400\), which agrees with (5.1)-(5.2). Moreover, we observe that \(R_0\) is decreasing in \(\sigma_1\). We conjecture that this is true in general.

5.2. Simulations in a realistic situation. In this section, we will simulate the model using geometric and population data of Puerto Rico. The domain \(\Omega\) is taken as the geometric boundary of Puerto Rico, which can be obtained from Mathematica as a polygon. The population density data of the 76 districts of Puerto Rico can also be found in Mathematica, which can be used to construct the susceptible human distribution, i.e. \(H_u(x)\), by interpolation. \(H_{i0}\) is assumed to be 100
people, distributed normally, centered at (0, -20). Set $V_{i0} = 10H_{i0}$, $V_{u0} = 150$, $\sigma_1 = 0.000001$, $\sigma_2 = 0.7$, $\lambda = 1$, $\beta = 5$, and $\mu = 0.0005$. The local basic reproduction number $R(x) = \sigma_1 \sigma_2 H_u/\lambda \mu$ is shown in Figure 2. Then we compute $\max\{R(x) : x \in \Omega\} = 4.3167$ and $\int_{\Omega} \frac{\lambda R(L_1 R_2) dx}{\lambda} = \int_{\Omega} \frac{R dx}{\|\Omega\|} = 0.6513$. By Theorems 2.3, (4.5)-(4.7), and (4.9)-(4.10), we expect that the solution of (1.1) converges to a positive steady state when the diffusion rates are small and to the semitrivial equilibrium $(0, V, 0)$ when $\delta_2$ is large. For verification, we choose different diffusion rates and use finite element method in Matlab to solve (1.1).

Case 1. Set $\delta_1 = \delta_2 = 4$. We plot the total infected host cases in Figure 3 and the density of infected hosts for $t = 4, 8, 12, 16$ in Figure 4. In this case, the solution converges to the positive steady state and the disease persists.

Case 2. Set $\delta_1 = 4$ and $\delta_2 = 4000$. We plot the total infected host cases in Figure 5 and the density of infected hosts in Figure 6. In this case, the density of infected hosts converges to zero and the disease dies out.

6. Discussion. In this paper, we have shown that the basic reproduction number $R_0$ of the reaction-diffusion model (1.1) can be written as $R_0 = r(L_1 R_1 L_2 R_2)$, where the local basic reproduction number $R(x) = R_1(x) R_2(x)$ is a multiplication operator on $C(\bar{\Omega})$, and $L_1$ and $L_2$ are strongly positive compact linear operators with spectral radii one. We are then able to study the relation of $R_0$ and $R(x)$. We prove that
\(R_0 \geq 1\) if \(R_1(x) \geq 1\) and \(R_2(x) \geq 1\) for all \(x \in \Omega\), and \(R_0 \leq 1\) if \(R_1(x) \leq 1\) and \(R_2(x) \leq 1\). Actually, \(R_0\) is bounded below and above by the products of the minimum and maximum of \(R_1\) and \(R_2\). When the diffusion rates are small, \(R_0 > 1\) provided that \(R(x) > 1\) for some \(x \in \bar{\Omega}\). When the diffusion rates are large, \(R_0\) approximates \(\hat{R}_1 \hat{R}_2\). Moreover, our numerical simulations suggest that \(R_0\) is decreasing in \(\delta_1\), however we are only able to prove it under the assumption (H1). The dependence of \(R_0\) on \(\delta_2\) is more difficult to study since \(\hat{V}\) is also dependent on \(\delta_2\). We only know that if \(\beta/\mu\) is constant, then \(\hat{V}\) is independent of \(\delta_2\) and \(R_0\) is decreasing in \(\delta_2\) under the assumption (H1).

We remark that our approach can be applied to many other reaction-diffusion epidemic models. For example, if we adopt our approach to analyze \(R_0\) for the diffusive SIS model in Allen et al. [1], we will compute \(R_0 = r(-CB^{-1}) = r(\beta(\gamma - d_I \Delta)^{-1})\).

Then we can write \(R_0\) as \(R_0 = r(RL)\), where \(R(x) = \beta(x)/\gamma(x)\) is the local basic reproduction number and \(L = (\gamma - d_I \Delta)^{-1}\) is a strongly positive compact linear operator in \(C(\bar{\Omega})\) with spectral radius one. To further illustrate this, we briefly adopt this approach to study the basic reproduction number of some other models in the following two subsections.
6.1. A within-host model on viral dynamics. Suppose that $T(x,t)$, $I(x,t)$, and $V(x,t)$ are the density of target cells, infected cells and free virus particles at position $x$ and time $t$, respectively. The model proposed in [19] to study the repulsion effect of superinfecting virion by infected cells is the following:

\[
\begin{aligned}
\frac{\partial T}{\partial t} &= D_T \Delta T + h(x) - d_T T - \beta(x)TV, \\
\frac{\partial I}{\partial t} &= D_I \Delta I + \beta(x)TV - d_I I, \\
\frac{\partial V}{\partial t} &= \nabla \cdot (D_V(I)\nabla V) + \gamma(x)I - d_V V,
\end{aligned}
\]

subject to homogeneous Neumann boundary conditions and nonnegative initial conditions.

Let $\hat{T}(x)$ be the unique positive solution of

\[D_T \Delta T + h(x) - d_T T = 0.\]

Linearizing (6.1) at the equilibrium $(\hat{T},0,0)$, the stability of it is related to the following eigenvalue problem

\[
\begin{aligned}
\kappa \varphi &= D_I \Delta \varphi - d_I \varphi + \beta \hat{T} \psi, \\
\kappa \psi &= D_0 \Delta \psi + \gamma \varphi - d_V \psi,
\end{aligned}
\]
where $D_0 = D_V(0)$. As before, we define

$$B = \begin{pmatrix} D_I \Delta & 0 \\ 0 & D_0 \Delta \end{pmatrix} + \begin{pmatrix} -d_I & \beta \hat{T} \\ 0 & -d_V V \end{pmatrix}$$

and $C = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$,

and the basic reproduction number

$$R_0 = r(-CB^{-1})$$

Similar to Theorem 3.1, we write $R_0$ as

$$R_0 = r\left(\frac{\gamma(d_I - D_I \Delta)}{d_I} - 1\beta \hat{T} (d_V - D_0 \Delta) - 1\right).$$

We have

$$R_0 = r(L_1 R_1 L_2 R_2),$$

with

$$L_1 = (d_I - D_I \Delta)^{-1} d_I, \quad L_2 = (d_V - D_0 \Delta)^{-1} d_V,$$

and

$$R_1 = \frac{\beta \hat{T}}{d_I}, \quad R_2 = \frac{\gamma}{d_V}.$$

The local basic reproduction number is defined as

$$R = R_1 R_2 = \frac{\gamma \beta \hat{T}}{d_I d_V}.$$

Here, $L_1$ and $L_2$ are strongly positive compact linear operators on $C(\bar{\Omega})$ with spectral radius one, and $\hat{T} = (d_T - D_T \Delta)^{-1} h$ satisfies

$$\lim_{D_T \to 0} \hat{T} = R_3, \quad \lim_{D_T \to \infty} \hat{T} = \frac{\int_{\bar{\Omega}} d_T R_3 dx}{\int_{\bar{\Omega}} d_T dx},$$

and

$$\min\{R_3(x) : x \in \bar{\Omega}\} \leq \hat{T} \leq \max\{R_3(x) : x \in \bar{\Omega}\},$$

with

$$R_3 = \frac{h}{d_T}.$$

An immediate consequence of (6.2) is the following result.

**Theorem 6.1.** The following statements hold:

- If $R_1$ and $R_2$ are constant, then $R_0 = R$;
- Let $R_{1m} = \min\{R_i(x) : x \in \bar{\Omega}\}$ and $R_{1M} = \max\{R_i(x) : x \in \bar{\Omega}\}$ for $i = 1, 2$, then
  $$R_{1m} R_{2m} \leq R_0 \leq R_{1M} R_{2M},$$
- \[\lim_{(D_I, D_T, D_V) \to (\infty, \infty, \infty)} R_0 = \frac{\beta \gamma h}{d_I d_V d_T},\]

where $\bar{f}$ denotes the average of $f$, i.e. $\bar{f} = \int_{\bar{\Omega}} f dx / |\bar{\Omega}|$ for $f = \beta, \gamma, h, d_I, d_V, d_T$. 

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\[
\lim_{D_I \to 0} \lim_{D_V \to 0} R_0 = \lim_{D_V \to 0} \lim_{D_I \to 0} R_0 = \lim_{(D_I, D_V) \to (0, 0)} R_0 = \max\{R(x) : x \in \Omega\}.
\]

We notice that \( R \) is consistent with the basic reproduction number defined using [13] \((R \text{ can be viewed as the total number of newly infected cells produced by one infected cell})\) for the corresponding ordinary differential equation model. We will leave the interested readers to investigate the monotonicity of \( R_0 \) with respect the diffusion rates.

6.2. An HIV model with cell-to-cell transmission. Let \( T(x, t), T^*(x, t), \) and \( V(x, t) \) be the density of healthy T cells, infected T cells and virions at position \( x \) and time \( t \), respectively. The model proposed in [26] to describe the cell-to-cell HIV transmission is the following:

\[
\begin{align*}
\frac{\partial T}{\partial t} & = d_1 \Delta T + \lambda(x) - d(x)T - \beta_1(x)TV - \beta_2(x)TT^*, \\
\frac{\partial T^*}{\partial t} & = d_2 \Delta T^* + \beta_1(x)TV + \beta_2(x)TT^* - \gamma(x)T^*, \\
\frac{\partial V}{\partial t} & = d_3 \Delta V + N(x)T^* - e(x)V,
\end{align*}
\]

subject to homogeneous Neumann boundary conditions and nonnegative initial conditions.

Let \( T_0(x) \) be the unique positive solution of

\[
d_1 \Delta T + \lambda(x) - d(x)T = 0.
\]

Linearizing (6.1) at the equilibrium \((T_0, 0, 0)\), we obtain the following eigenvalue problem

\[
\begin{align*}
\kappa \varphi & = d_2 \Delta \varphi + (\beta_2 T_0 - \gamma) \varphi + \beta_1 T_0 \psi, \\
\kappa \psi & = d_3 \Delta \psi + N \varphi - e \psi,
\end{align*}
\]

We define

\[
B = \begin{pmatrix} d_2 \Delta & 0 \\ 0 & d_3 \Delta \end{pmatrix} + \begin{pmatrix} -\gamma & 0 \\ N & -e \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \beta_2 T_0 & \beta_1 T_0 \\ 0 & 0 \end{pmatrix},
\]

and the basic reproduction number

\[
R_0 = r(-CB^{-1}).
\]

Similar to Theorem 3.1, we compute \( R_0 \) as

\[
R_0 = r \left( \beta_2 T_0 (\gamma - d_2 \Delta)^{-1} + \beta_1 T_0 (e - d_3 \Delta)^{-1} N (\gamma - d_2 \Delta)^{-1} \right).
\]

So we have

\[
R_0 = r(L_2(R_2^2 + R_1^2 L_3 R_3)),
\]

with

\[
L_2 = (\gamma - d_2 \Delta)^{-1} \gamma, \quad L_3 = (e - d_3 \Delta)^{-1} e,
\]nand

\[
R_1 = \frac{\beta_1 T_0}{\gamma}, \quad R_2 = \frac{\beta_2 T_0}{\gamma}, \quad R_3 = \frac{N}{e}.
\]
Here $L_1$ and $L_2$ are strongly positive compact linear operator on $C(\bar{\Omega})$ with spectral radius one, and $L_i 1 = 1$ for $i = 1, 2$. The local basic reproduction number $R$ is defined as

$$R = R^2_2 + R^1_2 R_3 = \frac{(\beta_1 N + \beta_2 e)T_0}{er},$$

where $T_0 = (d - d_1 \Delta)^{-1} \lambda$ satisfies

$$\lim_{d_1 \to 0} T_0 = R_1, \quad \lim_{d_1 \to \infty} T_0 = \int_0^1 dR_1,$$

and

$$\min \{ R_1(x) : x \in \Omega \} \leq T_0 \leq \max \{ R_1(x) : x \in \Omega \},$$

with

$$R_1 = \frac{\lambda}{d}.$$

We can also prove:

**Theorem 6.2.** The following statements hold:

- If $R^2_1, R^2_2$ and $R_3$ are constant, then $R_0 = R$;
- Let $S_m = \min \{ S(x) : x \in \Omega \}$ and $S_M = \max \{ S(x) : x \in \Omega \}$ for $S = R^2_1, R^2_2, R_3$, then
  $$R^2_1 + R^2_2 + R_3 \leq R_0 \leq R^{1,2}_1 + R^2_1 R_3 M.$$
- \( \lim_{(d_1, d_2, d_3) \to (\infty, \infty, \infty)} R_0 = \frac{\beta_1 N + \beta_2 e \lambda}{e r d}, \)

where $\bar{f}$ denotes the average of $f$ over $\Omega$, i.e. $\bar{f} = \int_\Omega f dx/|\Omega|$ for $f = \beta_1, \beta_2, e, r, d, \lambda$.

- $\lim_{d_2 \to 0} \lim_{d_3 \to 0} R_0 = \max \{ R(x) : x \in \Omega \}.$

**Proof.** We will only sketch the proof of the last part. Noticing that $L_3 \phi \to \phi$ in $C(\bar{\Omega})$, we have $L_2(R^2_2 + R^1_2 L_3 R_3) \xrightarrow{\text{SOT}} L_2(R^2_2 + R^1_2 R_3) = L_2 R$ as $d_3 \to 0$. Let $B \subset C(\bar{\Omega})$ be the closed unit ball, then

$$\cup_{d_3 > 0} L_2(R^2_2 + R^1_2 L_3 R_3)(B) \subset L_2((R^1_2 + R^2_1 R_3 M)B),$$

which is compact. By Theorem 4.1, we have $R_0 = r(L_2(R^2_2 + R^1_2 L_3 R_3)) \to r(L_2 R)$ as $d_3 \to 0$. The proof of $r(L_2 R) \to \max \{ R(x) : x \in \Omega \}$ as $d_2 \to 0$ is the same with Theorem 4.10.

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**Appendix A. Appendix - Proof of Theorem 4.10.**

**Proof.** We only prove part 1. Define $r_{\delta_2} := r(RL_2) = r(L_2 R)$. Then $\kappa_{\delta_2} = 1/r_{\delta_2}$ is the principal eigenvalue of the problem

$$\begin{cases} (\mu V - \delta_2 \Delta) v = \kappa \mu V R v, & x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega. \end{cases}$$

(A.1)
By (A.1),
\[
\kappa_{\delta_2} = \frac{1}{r_{\delta_2}} = \min \left\{ \frac{\delta_2 \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} \mu \hat{V} v^2 \, dx}{\int_{\Omega} R \hat{V} v^2 \, dx} : v \in H^1(\Omega) \text{ and } v \neq 0 \right\}
\]
\[
\geq \frac{1}{R_M} \min \left\{ \frac{\delta_2 \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} \mu \hat{V} v^2 \, dx}{\int_{\Omega} \mu \hat{V} v^2 \, dx} : v \in H^1(\Omega) \text{ and } v \neq 0 \right\} = \frac{1}{R_M}.
\]

It then follows that \( \liminf_{\delta_2 \to 0} \kappa_{\delta_2} \geq 1/R_M. \)

We only need to show \( \limsup_{\delta_2 \to 0} \kappa_{\delta_2} \leq 1/R_M. \) Assume to the contrary that the statement does not hold, i.e. \( \limsup_{\delta_2 \to 0} \kappa_{\delta_2} > 1/R_M. \) Then there exists \( \epsilon_0 > 0 \) and a sequence \( \{\delta_{2,n}\} \) with \( \delta_{2,n} \to 0 \) such that \( \kappa_{\delta_{2,n}} > 1/(R_M - \epsilon_0). \) Let \( x_0 \in \Omega \) and \( a > 0 \) such that \( R(x) > R_M - \epsilon_0/2 \) in \( B(x_0, a). \) Let \( v_{\delta_{2,n}} \) be a positive eigenvector of (A.1) associated with the principal eigenvalue \( \kappa_{\delta_{2,n}}. \) Then in \( B(x_0, a), \) we have
\[
(\mu \hat{V} - \kappa_{\delta_{2,n}} \Delta) v_{\delta_{2,n}} = \kappa_{\delta_{2,n}} R v_{\delta_{2,n}} \geq \frac{(R_M - \epsilon_0/2) \mu \hat{V} v_{\delta_{2,n}}}{R_M - \epsilon_0}.
\]

It follows that, in \( B(x_0, a), \)
\[
-\Delta v_{\delta_{2,n}} > \frac{\epsilon_0}{2\delta_{2,n}(R_M - \epsilon_0)} \mu \hat{V}.
\]

Let \( \kappa' \) be the principal eigenvalue of \( -\Delta \) in domain \( B(x_0, a) \) with Dirichlet boundary condition. By a minimax formulation of \( \kappa' \) ([3]), we have
\[
(A.2) \quad \kappa' = \sup_{u \in W^{2,\infty}(B(x_0, a)), u \geq 0} \inf_{x \in B(x_0, a)} \frac{-\Delta u}{u} > \frac{\epsilon_0}{2\delta_{2,n}(R_M - \epsilon_0)} \inf_{x \in B(x_0, a)} \{ \mu \hat{V} \}.
\]

Noticing that \( \hat{V} \geq \min\{\beta(x) : x \in \Omega\}/\max\{\mu(x) : x \in \Omega\}, \) the right hand side of (A.2) tends to \( \infty \) as \( \delta_{2,n} \to 0. \) This is a contradiction. Hence, \( \kappa_{\delta_2} \to 1/R_M \) and \( r_{\delta_2} \to R_M \) as \( \delta_2 \to 0. \)

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