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ON THE BASIC REPRODUCTION NUMBER OF REACTION-DIFFUSION EPIDEMIC MODELS*

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Abstract. The basic reproduction number R_0 serves as a threshold parameter of many epidemic models for disease extinction or spread. The purpose of this paper is to investigate R_0 for spatial reaction-diffusion partial differential equation epidemic models. We define R_0 as the spectral radius of a product of a local basic reproduction number R and strongly positive compact linear operators with spectral radii one. This definition of R, viewed as a multiplication operator, is motivated by the definition of basic reproduction numbers for ordinary differential equation epidemic models. We investigate the relation of R_0 and R.

Key words. reaction-diffusion, epidemic models, basic reproduction number

AMS subject classifications. 35K57, 35P05, 92B05, 92D30

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1. Introduction.

For epidemic differential equation models, the basic reproduction number R_0 is a threshold value such that below this value the disease vanishes, while above this value the disease spreads. The calculation of R_0 for ordinary differential equations epidemic models has been developed extensively based on [9, 10]. Many authors have used reaction-diffusion partial differential equation models to study the transmission of diseases in geographical regions (see [1, 5, 6, 7, 8, 11, 12, 16, 19, 20, 22, 23, 27, 29, 30, 32, 33, 34, 35]). The purpose of this paper is to connect basic reproduction numbers for partial differential equations epidemic models to basic reproduction numbers for ordinary differential equation models.

In a recent study, Thieme [28] provided a general theoretical approach to define R_0 as the spectral radius of a resolvent-positive operator for a wide range of epidemic models, which is a generalization of the finite dimensional version in [9, 10]. Another approach to characterize R_0 for reaction-diffusion epidemic models relied on a variational characterization of R_0, which works when the model is relatively simple (the stability of the disease free equilibrium is determined by the sign of an eigenvalue problem consisting of only one equation). For example, Allen et al. [1] characterize R_0 for a simple diffusive SIS model by the formula

$$R_0 = \sup \left\{ \frac{\int_\Omega \beta \varphi^2 dx}{\int_\Omega (d_I |\nabla \varphi|^2 + \gamma \varphi^2) dx} : \quad \varphi \in H^1(\Omega), \varphi \neq 0 \right\},$$

where $\beta = \beta(x)$ is the transmission rate, $\gamma = \gamma(x)$ is the removal rate, and d_I is the diffusion coefficient. This allows the authors to show that R_0 is strictly decreasing in d_I, $R_0 \to \int_\Omega \beta/\gamma dx$ as $d_I \to 0$, and $R_0 \to \int_\Omega \beta / \int_\Omega \gamma$ as $d_I \to \infty$. Here $\beta(x)/\gamma(x)$ is

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the basic reproduction number for the corresponding model without diffusion (which we will call the local basic reproduction number).

For some reaction-diffusion epidemic models, R_0 is related to the principal eigenvalue of an elliptic system, which makes the analysis more difficult. Peng and Zhao [27] write R_0 as the principal eigenvalue of an eigenvalue problem consisting of a single equation. Cui and Lou [6] study the impact of the advection rate on R_0 for a reaction-diffusion-advection SIS model, where they take advantage of the variational characterization of R_0. We note that calculations of R_0 for reaction-diffusion epidemic models have been discussed by Wang and Zhao [31]. We also note the papers [14, 25] for R_0 analysis of stream population models, and [36] for R_0 analysis of time-delayed compartmental population models in periodic environments. Other investigations of R_0 for partial differential equation epidemic models are found in [19, 26, 29, 30, 32], where the computation of R_0 is mostly for constant coefficients in space. Here we explore this question with nonconstant coefficients, which will allow us to explore the impact of the (small and large) diffusion coefficients and spatial heterogeneity.

Although our approach is applicable to a wide range of reaction-diffusion epidemic models, we will focus on the vector-host model in [12] (see also [24]). Suppose that individuals are living in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $H_u(x), H_i(x,t), V_u(x,t)$, and $V_i(x,t)$ be the density of uninfected hosts, infected hosts, uninfected vectors, and infected vectors at position x and time t, respectively. Then the model in [12] to study the outbreak of Zika in Rio De Janerio is the following reaction-diffusion system:

$$\begin{cases} \partial H_i/\partial t - \nabla \cdot \delta_1 \nabla H_i = -\lambda H_i + \sigma_1 H_u(x) V_i, \\ \partial V_u/\partial t - \nabla \cdot \delta_2 \nabla V_u = -\sigma_2 V_u H_i + \beta(V_u + V_i) - \mu(V_u + V_i)V_u, \\ \partial V_i/\partial t - \nabla \cdot \delta_2 \nabla V_i = \sigma_2 V_u H_i - \mu(V_u + V_i)V_i, \\ \partial H_i/\partial n = \partial V_u/\partial n = \partial V_i/\partial n = 0, \\ (H_i(.,0), V_u(.,0), V_i(x,0)) = (H_{i0}, V_{u0}, V_{i0}) \in C(\bar{\Omega}; \mathbb{R}^3_+), \end{cases} \tag{1.1}$$

where $\delta_1, \delta_2 \in C^{1+\alpha}(\bar{\Omega})$ are strictly positive, and the functions $H_u, \lambda, \beta, \sigma_1, \sigma_2$, and μ are strictly positive and belong to $C^{\alpha}(\bar{\Omega})$. It is assumed that uninfected hosts are stationary in space, and the diffusion of infected hosts corresponds indirectly to the movement of the Zika virus in the spatial environment. Both uninfected and infected vectors are assumed to diffuse in the spatial environment.

Following [28, 31], the basic reproduction number R_0 for (1.1) is defined as the spectral radius $r(-CB^{-1})$ of $-CB^{-1}$, where $B : D(B) \subset C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ and $C : C(\bar{\Omega}; \mathbb{R}^2) \to C(\bar{\Omega}; \mathbb{R}^2)$ are the linear operators

$$B = \begin{pmatrix} \nabla \cdot \delta_1 \nabla \\ \nabla \cdot \delta_2 \nabla \end{pmatrix} + \begin{pmatrix} -\lambda & \sigma_1 H_u \\ 0 & -\mu \hat{V} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ \sigma_2 \hat{V} & 0 \end{pmatrix}, \tag{1.2}$$

$$D(B) = \left\{ (\varphi, \psi) \in \bigcap_{p \geq 1} W^{2,p}(\Omega; \mathbb{R}^2) : \frac{\partial \varphi}{\partial n} = \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega \text{ and } B(\varphi, \psi) \in C(\bar{\Omega}; \mathbb{R}^2) \right\},$$

and $\hat{V}$ is the unique positive solution of the elliptic problem

$$\begin{cases} -\nabla \cdot \delta_2(x) \nabla V = \beta(x) V - \mu(x) V^2, & x \in \Omega, \\ \frac{\partial}{\partial n} V = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The system (1.1) in the case without diffusion, and viewed as an ordinary differential equation system at a specific location x is

(1.4)
$$\begin{cases} dH_i/dt = -\lambda(x)H_i(t) + \sigma_1(x)H_u(x)V_i(t), \\ dV_u/dt = -\sigma_2(x)V_u(t)H_i(t) + \beta(x)(V_u(t) + V_i(t)) - \mu(x)(V_u(t) + V_i(t))V_u(t), \\ dV_i/dt = \sigma_2(x)V_u(t)H_i(t) - \mu(x)(V_u(t) + V_i(t))V_i(t). \end{cases}$$

The basic reproduction number of (1.4) at a specific location x, obtained by the next generation method, is

$$R(x) = R_1(x)R_2(x), \quad \text{where } R_1(x) = \frac{\sigma_1(x)H_u(x)}{\lambda(x)} \quad \text{and } R_2(x) = \frac{\sigma_2(x)}{\mu(x)}. \tag{1.5}$$

$R_1(x)$ and $R_2(x)$ have their own biological meanings: at a specific location x, $R_1(x)$ measures the impact of one infected vector on susceptible hosts while $R_2(x)$ measures the impact of one infected host on the susceptible vectors. Since R_0 is difficult to visualize, our main purpose of this research is to study the relation between R_0 and $R(x)$, the latter being a function of $x \in \bar{\Omega}$.

In sections 3 and 4, we study the relation of R_0 and $R(x)$, where our approach is based on the formula

$$R_0 = r(L_1R_1L_2R_2),\ L_1 := (\lambda - \triangledown \cdot \delta_1 \triangledown)^{-1}\lambda, \ \text{ and } \ L_2 := (\mu\hat{V} - \triangledown \cdot \delta_2 \triangledown)^{-1}\mu\hat{V}, \tag{1.6}$$

where R_1 and R_2 are viewed as multiplication operators on $C(\bar{\Omega})$, and L_1 and L_2 are strongly positive compact linear operators on $C(\bar{\Omega})$. This formula establishes an interesting connection between R_0 and R as $r(L_1L_2) = r(L_1) = r(L_2) = 1$ (see Lemma 3.4). Consequences of this formula are

- if R_1 and R_2 are constant, then $R_0 = R$ (see Corollary 3.5);
- $R_0 \geq 1$ if $R_i(x) \geq 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$ and $R_0 \leq 1$ if $R_i(x) \leq 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$ (see Theorem 3.6).

When the diffusion coefficients δ_1 and δ_2 are constant, we establish a quantitative connection of R_0 and R. To this end, we prove a result (Theorem 4.1) about the convergence of spectral radii for a sequence of strongly positive compact linear operators in an ordered Banach space. Based on Theorem 4.1, we show

- $\lim_{\delta_1\to\infty} R_0 = \frac{\int_\Omega \lambda R_1(L_2R_2)dx}{\int_\Omega \lambda dx}$ for $\delta_2 > 0$ and $\lim_{\delta_2\to\infty} R_0 = \frac{\int_\Omega \mu R_2(L_1R_1)dx}{\int_\Omega \mu dx}$ for $\delta_1 > 0$ (see Theorem 4.5);
- $\lim_{(\delta_1,\delta_2)\to(\infty,\infty)} R_0 = \frac{\int_\Omega \lambda R_1 dx}{\int_\Omega \lambda dx}\frac{\int_\Omega \mu R_2 dx}{\int_\Omega \mu dx}$ (see Remark 4.8).
- $\lim_{\delta_1\to 0}\lim_{\delta_2\to 0} R_0 = \lim_{\delta_2\to 0}\lim_{\delta_1\to 0} R_0 = \lim_{(\delta_1,\delta_2)\to(0,0)} R_0 = \max\{R(x) : x \in \bar{\Omega}\}$ (see Theorems 4.9–4.11).

In section 5, we conduct numerical simulations to illustrate our results. In section 6, we give concluding remarks and provide two examples about adopting our approach to analyze R_0 for reaction-diffusion epidemic models.

2. Preliminaries. The global dynamics of (1.1) have been analyzed in [24], and we first summarize here the results that will be used. Let $V = V_u + V_i$. Then $V(x, t)$ satisfies

$$\begin{cases} V_t - \triangledown \cdot \delta_2(x)\triangledown V = \beta(x)V - \mu(x)V^2, & x \in \Omega, t > 0, \\ \partial V/\partial n = 0, & x \in \partial\Omega, t > 0, \\ V(.,0) = V_0 \in C_+(\bar{\Omega}). \end{cases} \tag{2.1}$$

The following result about (2.1) is well known (see [4, Proposition 3.17] [15, Lemma A.1], and [18, Proposition 2.5]).

LEMMA 2.1. *For any nonnegative nontrivial initial data* $V_0 \in C(\bar{\Omega})$, (2.1) *has a unique global classic solution* $V(x,t)$. *Moreover,* $V(x,t) > 0$ *for all* $(x,t) \in \bar{\Omega} \times (0,\infty)$ *and*

$$\lim_{t\to+\infty} \|V(\cdot,t) - \hat{V}\|_\infty = 0, \tag{2.2}$$

where $\hat{V}$ *is the unique positive solution of the elliptic problem* (1.3). *Moreover, if* δ_2 *is a constant parameter, then*

$$\lim_{\delta_2\to 0} \hat{V} \to \frac{\beta}{\mu} \quad \text{and} \quad \lim_{\delta_2\to\infty} \hat{V} \to \frac{\int_\Omega \beta dx}{\int_\Omega \mu dx} \quad \text{in } C(\bar{\Omega}).$$

The definition of R_0 for (1.1) is closely related to the stability of the semitrivial equilibrium $E_1 = (0, \hat{V}, 0)$ of (1.1). Linearizing the model at E_1, one can see that the stability of E_1 is determined by the sign of the principal eigenvalue of the problem:

$$\begin{cases} \kappa\varphi = \nabla\cdot\delta_1\nabla\varphi - \lambda\varphi + \sigma_1 H_u\psi, & x\in\Omega, \\ \kappa\psi = \nabla\cdot\delta_2\nabla\psi + \sigma_2\hat{V}\varphi - \mu\hat{V}\psi, & x\in\Omega, \\ \partial\varphi/\partial n = \partial\psi/\partial n = 0, & x\in\partial\Omega. \end{cases} \tag{2.3}$$

Problem (2.3) is cooperative, so it has a principal eigenvalue κ_0 associated with a positive eigenvector (φ_0, ψ_0) [17].

Let $A = B + C$, where B and C are defined in section 1. Then A and B are resolvent positive [28], and A is a positive perturbation of B. By [28, Theorem 3.5], $\kappa_0 = s(A)$ and $r(-CB^{-1}) - 1$ have the same sign, where $s(A)$ is the spectral bound of A. We then have the following result.

THEOREM 2.2. $R_0 - 1$ *and* κ_0 *have the same sign. Moreover,* E_1 *is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

The main results proved in [24] about the global dynamics of the model (1.1) are as follows.

THEOREM 2.3. *The following hold:*

- *If* $R_0 \le 1$, *then for any nonnegative initial data* $(H_{i0}, V_{u0}, V_{i0}) \in C(\bar{\Omega}; \mathbb{R}^3_+)$ *with* $V_{u0} + V_{i0} \not\equiv 0$, *the solution* (H_i, V_u, V_i) *of* (1.1) *satisfies*

$$\lim_{t\to\infty} \|(H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - E_1\|_\infty = 0, \tag{2.4}$$

  where $E_1 = (0, \hat{V}, 0)$.
- *If* $R_0 > 1$, *then for any initial data* (H_{i0}, V_{u0}, V_{i0}) *with* $V_{u0} + V_{i0} \not\equiv 0$ *and* $H_{i0} + V_{i0} \not\equiv 0$, *the solution* (H_i, V_u, V_i) *of* (1.1) *satisfies*

$$\lim_{t\to\infty} \|H_i(\cdot,t), V_u(\cdot,t), V_i(\cdot,t)) - (\hat{H}_i, \hat{V}_u, \hat{V}_i)\|_\infty = 0,$$

  where $E_2 = (\hat{H}_i, \hat{V}_u, \hat{V}_i)$ *is the unique epidemic equilibrium of* (1.1).

Let X be an ordered Banach space with positive cone X_+, and let $L_1, L_2 : X \to X$ be two bounded linear operators. Then it is well known that

$$r(L_1L_2) = r(L_2L_1) \le \|L_1\|\|L_2\|, \tag{2.5}$$

where $r(L_i)$ denotes the spectral radius of L_i, $i = 1, 2$. Indeed, this can be derived easily from Gelfand's formula

$$r(L_1) = \lim_{n\to\infty} \|L_1^n\|^{1/n}. \tag{2.6}$$

Remark 2.4. It is very important to note that (2.6) does not imply $r(L_1L_2L_3) = r(L_3L_2L_1)$.

Suppose that X_+ has a nonempty interior $int(X_+)$. Then L_1 is strongly positive if $L_1(X_+ \setminus 0) \subseteq int(X_+)$. The operator L_1 is compact if the image of the unit ball is relatively compact in X. We will need the following generalization of the Krein–Rutman theorem [2].

THEOREM 2.5. *Let X be an ordered Banach space with positive cone X_+ such that X_+ has nonempty interior. Suppose that $T : X \to X$ is a strongly positive compact linear operator. Then the spectral radius $r(T)$ is positive and a simple eigenvalue of T associated with a positive eigenvector, and there is no other eigenvalue with a positive eigenvector. Moreover if $S : X \to X$ is a linear operator such that $S \geq T$, i.e., $S(v) \geq T(v)$ for all $v \in X_+$, then $r(S) \geq r(T)$. If, in addition, $S - T$ is strongly positive, then $r(S) > r(T)$.*

3. General diffusion rates. Our basic result about the basic reproduction number R_0 of (1.1) is the following.

THEOREM 3.1. *Let $R_0 = r(-CB^{-1})$, where B and C are defined in* (1.2). *Then,*

$$R_0 = r(L_1R_1L_2R_2), \tag{3.1}$$

where R_1 and R_2 defined in (1.5) *are multiplication operators on $C(\bar{\Omega})$, and L_1 and L_2 defined in* (1.6) *are strongly positive compact linear operators on $C(\bar{\Omega})$.*

Proof. It is not hard to compute

$$B^{-1} = \begin{pmatrix} (\nabla \cdot \delta_1 \nabla - \lambda)^{-1} & -(\nabla \cdot \delta_1 \nabla - \lambda)^{-1} \sigma_1 H_u (\nabla \cdot \delta_2 \nabla - \mu \hat{V})^{-1} \\ 0 & (\nabla \cdot \delta_2 \nabla - \mu \hat{V})^{-1} \end{pmatrix}.$$

Therefore,

$$-CB^{-1} = \begin{pmatrix} 0 & 0 \\ \sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} & \sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u (\mu \hat{V} - \nabla \cdot \delta_2 \nabla)^{-1} \end{pmatrix}.$$

It then follows that

$$\begin{aligned} R_0 = r(-CB^{-1}) &= r\left(\sigma_2 \hat{V} (\lambda - \nabla \cdot \delta_1 \nabla)^{-1} \sigma_1 H_u (\mu \hat{V} - \nabla \cdot \delta_2 \nabla)^{-1}\right) \\ &= r\left(\sigma_2 \hat{V} L_1 R_1 L_2 \frac{1}{\mu \hat{V}}\right). \end{aligned}$$

From (2.5), we have

$$R_0 = r\left(L_1 R_1 L_2 \frac{1}{\mu \hat{V}} \sigma_2 \hat{V}\right) = r(L_1R_1L_2R_2).$$

It is well known that the elliptic estimates and maximum principles imply that L_1 and L_2 are strongly positive compact linear operators on $C(\bar{\Omega})$. ∎

LEMMA 3.2. $\|L_1\| = 1$ *and* $\|L_2\| = 1$.

Proof. Notice that $L_i(\pm 1) = \pm 1$ for $i = 1, 2$. For any $u \in C(\bar{\Omega})$ with $\|u\|_\infty \leq 1$, we have $-1 \leq u \leq 1$. By the comparison principle, we have

$$-1 = L_i(-1) \leq L_i u \leq L_i 1 = 1 \text{ for } i = 1, 2.$$

Therefore, $\|L_i u\|_\infty \leq 1 = \|u\|_\infty$, which implies $\|L_i\| \leq 1$ for $i = 1, 2$. Moreover, since $L_1 1 = 1$ and $L_2 1 = 1$, we must have $\|L_1\| = \|L_2\| = 1$. ∎

We immediately have the following result from (2.5).

THEOREM 3.3. *If $R_i(x) < 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$, then $R_0 < 1$.*

Proof. $R_0 = r(L_1R_1L_2R_2) \le \|L_1\|\|R_1\|\|L_2\|\|R_2\| = \|R_1\|\|R_2\| < 1$. ∎

We apply the Krein–Rutman theorem to study the spectral radii of L_1, L_2, and L_1L_2.

LEMMA 3.4. *The spectral radii of L_1, L_2, and L_1L_2 are all 1, i.e., $r(L_1) = r(L_2) = r(L_1L_2) = 1$.*

Proof. Since L_1 and L_2 are strongly positive compact operators on $C(\bar{\Omega})$, so is L_1L_2. By Theorem 2.5, $r(L_1)$, $r(L_2)$, and $r(L_1L_2)$ are simple positive eigenvalues of L_1, L_2, and L_1L_2, associated with positive eigenvectors, respectively. Moreover, there is no other eigenvalue of L_1, L_2, or L_1L_2 associated with a positive eigenvector. Since $L_11 = L_21 = L_1L_21 = 1$, we must have $r(L_1) = r(L_2) = r(L_1L_2) = 1$. ∎

Noticing that $R_0 = r(L_1R_1L_2R_2)$, Lemma 3.4 implies that there is a significant connection between the basic reproduction number R_0 and the local basic reproduction number $R(x)$. A consequence of Lemma 3.4 is the following result.

COROLLARY 3.5. *If R_1 and R_2 are constant, then $R_0 = R$.*

Our next result, based on the Krein–Rutman theorem, is stronger than Theorem 3.3.

THEOREM 3.6. *The following hold:*

1. *If $R_i(x) \ge 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$, then $R_0 \ge 1$. If, in addition, $R_1(x) \not\equiv 1$ or $R_2(x) \not\equiv 1$, then $R_0 > 1$.*
2. *If $R_i(x) \le 1$, $i = 1, 2$, for all $x \in \bar{\Omega}$, then $R_0 \le 1$. If, in addition, $R_1(x) \not\equiv 1$ or $R_2(x) \not\equiv 1$, then $R_0 < 1$.*
3. *$R_{1m}R_{2m} \le R_0 \le R_{1M}R_{2M}$, where $R_{im} = \min\{R_i(x) : x \in \bar{\Omega}\}$ and $R_{iM} = \max\{R_i(x) : x \in \bar{\Omega}\}$, $i = 1, 2$.*

Proof. We only prove part 1 as the proof of the rest is similar. If $R_i(x) \ge 1$ for all $x \in \bar{\Omega}$, then $L_1R_1L_2R_2 \ge L_1L_2$. By Theorem 2.5 and Lemma 3.4, we have $R_0 = r(L_1R_1L_2R_2) \ge r(L_1L_2) = 1$.

Let ϕ be a positive eigenfunction corresponding to principal eigenvalue R_0 of $L_1R_1L_2R_2$. If, in addition, $R_1(x) \not\equiv 1$ or $R_2(x) \not\equiv 1$, by the strong positivity of L_1 and L_2, we have

$$R_0\phi = L_1R_1L_2R_2\phi >> L_1L_2\phi.$$

Therefore, there exists $\epsilon > 0$ such that $R_0\phi \ge (1+\epsilon)L_1L_2\phi$. Let $\phi_m = \min_{x\in\bar{\Omega}} \phi(x) > 0$. Then, by the positivity of L_1L_2 and $L_1L_21 = 1$, we have

$$R_0\phi \ge (1+\epsilon)L_1L_2\phi \ge (1+\epsilon)L_1L_2\phi_m = (1+\epsilon)\phi_m.$$

Therefore, $R_0\phi \ge (1+\epsilon)\phi_m$, which implies $R_0 \ge 1+\epsilon > 1$. ∎

We next study the monotonicity of R_0. Here, we need the assumption

(H1) $\sigma_1H_u = \sigma_2\hat{V}$ or both σ_1H_u and $\sigma_2\hat{V}$ are constants.

THEOREM 3.7. *Suppose that* (H1) *holds. If δ_1 is constant, then R_0 is decreasing in δ_1.*

Proof. Let $\kappa = 1/R_0$. By the Krein–Rutman theory, κ is an eigenvalue associated with a positive eigenvector ϕ (we normalize ϕ such that $\|\phi\|_2 = 1$) of the following problem:

$$\kappa L_1R_1L_2R_2\phi = \phi.$$

Therefore, we have

$$\kappa\lambda R_1 L_2 R_2 \phi = (\lambda - \delta_1 \Delta)\phi. \tag{3.2}$$

Differentiating both sides with respect to δ_1, we have

$$\kappa_{\delta_1}\lambda R_1 L_2 R_2 \phi + \kappa\lambda R_1 L_2 R_2 \phi_{\delta_1} = -\Delta\phi + (\lambda - \delta_1\Delta)\phi_{\delta_1}. \tag{3.3}$$

Multiplying (3.3) by ϕ and (3.2) by ϕ_{δ_1}, and integrating their difference over Ω, we obtain

$$\kappa_{\delta_1}\int_\Omega \phi\lambda R_1 L_2 R_2 \phi dx = \int_\Omega |\nabla\phi|^2 dx,$$

where we used the assumption (H1) to derive

$$\int_\Omega \phi_{\delta_1}\lambda R_1 L_2 R_2 \phi dx = \int_\Omega \phi\lambda R_1 L_2 R_2 \phi_{\delta_1} dx.$$

Since $\lambda R_1 L_2 R_2$ is strongly positive, $\lambda R_1 L_2 R_2 \phi > 0$. Therefore, $\kappa_{\delta_1} \geq 0$ and κ is increasing in δ_1. Hence, R_0 is decreasing in δ_1. ∎

Remark 3.8. If β/μ is constant, $\hat{V}$ is independent of δ_2. Then, similarly to Theorem 3.7, $R_0 = r(L_2 R_2 L_1 R_1)$ is decreasing in δ_2 if (H1) holds. Moreover, from the proof of Theorem 3.7, R_0 is strictly decreasing if the eigenvector is nonconstant.

4. Small or large diffusion rates. We prove the following result on the convergence of spectral radii for strongly positive compact linear operators, which is essential for our investigation of the role of diffusion rates for the basic reproduction number R_0.

THEOREM 4.1. *Let X be an ordered Banach space with positive cone X_+ such that X_+ has nonempty interior. Let $T_n, n \geq 1$, and T be strongly positive compact linear operators on X. Suppose $T_n \xrightarrow{SOT} T$ (strong operator topology) which means $T_n(u) \to T(u)$ for any $u \in X$. If $\cup_{n\geq 1} T_n(B)$ is precompact, where B is the closed unit ball of X, and $r(T_n) \geq r_0$ for some $r_0 > 0$, then $r(T_n) \to r(T)$.*

Proof. Since T and T_n are compact and strongly positive, by Theorem 2.5, $r(T)$ and $r(T_n)$ are positive simple eigenvalues of T and T_n, respectively. So there exists $e_n \in int(X_+)$ with $\|e_n\| = 1$ such that $T_n e_n = r(T_n) e_n$ for all $n \geq 1$. Since $\cup_{n\geq 1} T_n(B)$ is precompact and $r(T_n) \geq r_0 > 0$, $\{e_n\}$ is precompact. So there exists a subsequence $\{e_{n_k}\}$ of $\{e_n\}$ such that $e_{n_k} \to e$ for some $e \in X$.

We claim $T_{n_k} e_{n_k} \to Te$. Note that $\sup_{n\geq 1} \|T_n(u)\| < \infty$ for any $u \in X$ by the convergence assumption $T_n \xrightarrow{\text{SOT}} T$. Then by the uniform boundedness principle, there exists $M > 0$ such that $\sup_{n\geq 1} \|T_n\| < M$. Let $\epsilon > 0$ be arbitray. Since $e_{n_k} \to e$ and $T_{n_k} e \to Te$, there eixsts $N > 0$ such that $\|e_{n_k} - e\| < \epsilon$ and $\|T_{n_k} e - Te\| < \epsilon$ for all $k > N$. Hence for all $k > N$, we have

$$\|T_{n_k} e_{n_k} - Te\| \leq \|T_{n_k}(e_{n_k} - e)\| + \|T_{n_k} e - Te\| \leq M\epsilon + \epsilon.$$

Since $\epsilon > 0$ was abitrary, $T_{n_k} e_{n_k} \to Te$.

Since $T_{n_k} e_{n_k} = r(T_{n_k}) e_{n_k}$, $T_{n_k} e_{n_k} \to Te$, and $e_{n_k} \to e$, we have $r(T_{n_k}) = \|T_{n_k} e_{n_k}\| \to \|Te\|$ and $Te = \|Te\| e$. Since $e_n \in X_+$ and $\|e_n\| = 1$, $e \in X_+$ and $\|e\| = 1$. Thus e is a positive eigenvector of T corresponding to eigenvalue $\|Te\|$. Again by Theorem 2.5, we have $r(T) = \|Te\|$. Hence $r(T_{n_k}) \to r(T)$ and $r(T_n) \to r(T)$ (here we use a well-known result: if every subsequence of the sequence $\{a_n\}$ has a convergent subsequence with limit a, then $a_n \to a$). ∎

The convergence of a sequence of compact operators in the SOT is not sufficient to guarantee the convergence of their spectral radii. We use the following simple example to illustrate this fact.

Example 4.2. Let H be a Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. For $n \geq 1$, define $T_n : H \to H$ by

$$T_n(a) = a_n e_n \text{ for any } a = \sum_{i=1}^{\infty} a_i e_i \in H.$$

Then $\{T_n\}$ is a sequence of compact operators with $r(T_n) = 1$, and $T_n \xrightarrow{\text{SOT}} 0$. Since $r(T_n) = 1$ and $r(T) = 0$, $r(T_n) \not\to r(T)$.

It is interesting to see whether some of the hypotheses in Theorem 4.1 can be dropped. We leave this as an open problem.

4.1. Large diffusion rates. In the following two subsections, we investigate R_0 quantitatively when the diffusion rates are large or small. To this end, we assume that δ_1 and δ_2 are constants. Define two integral operators $L_{1,\infty}, L_{2,\infty} : C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$L_{1,\infty}(\phi) = \frac{\int_\Omega \lambda(x)\phi(x)dx}{\int_\Omega \lambda(x)dx} \quad \text{and} \quad L_{2,\infty}(\phi) = \frac{\int_\Omega \mu(x)\phi(x)dx}{\int_\Omega \mu(x)dx} \quad \text{for any } \phi \in C(\bar{\Omega}).$$

LEMMA 4.3. *$L_1 \xrightarrow{SOT} L_{1,\infty}$ in $C(\bar{\Omega})$ as $\delta_1 \to \infty$.*

Proof. Let $u \in C(\bar{\Omega})$ be given. We need to prove that $L_1(u) \to L_{1,\infty}(u)$ in $C(\bar{\Omega})$ as $\delta_1 \to \infty$. For any $\delta_1 > 0$, let $v_{\delta_1} = L_1(u)$. Then v_{δ_1} is the solution of the problem

$$\begin{cases} \lambda v_{\delta_1} - \delta_1 \Delta v_{\delta_1} = \lambda u, & x \in \Omega, \\ \frac{\partial}{\partial n} v_{\delta_1} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

By the comparison principle, we have $-\|u\|_\infty \leq v_{\delta_1} \leq \|u\|_\infty$ for all $\delta_1 > 1$. Hence by the L^p estimate, $\{v_{\delta_1}\}_{\delta_1 > 1}$ is uniformly bounded in $W^{2,p}(\Omega)$ for any $p > 1$. Since the embedding $W^{2,p}(\Omega) \subseteq C(\bar{\Omega})$ is compact for $p > n$, up to a subsequence, $v_{\delta_1} \to v$ weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ for some $v \in W^{2,p}(\Omega)$ as $\delta_1 \to \infty$. Moreover, v satisfies

$$\begin{cases} -\Delta v = 0, & x \in \Omega, \\ \frac{\partial}{\partial n} v = 0, & x \in \partial\Omega. \end{cases}$$

By the maximum principle, v is a constant. Integrating both sides of the first equation of (4.1) and taking $\delta_1 \to \infty$, we find $v = \frac{\int_\Omega \lambda u dx}{\int_\Omega \lambda dx}$. ∎

LEMMA 4.4. *$L_2 \xrightarrow{SOT} L_{2,\infty}$ in $C(\bar{\Omega})$ as $\delta_2 \to \infty$.*

Proof. Let $u \in C(\bar{\Omega})$ be given. We need to prove that $L_2(u) \to L_{2,\infty}(u)$ in $C(\bar{\Omega})$ as $\delta_2 \to \infty$. For any $\delta_2 > 0$, let $v_{\delta_2} = L_2(u)$. Then v_{δ_2} is the solution of the problem

$$\begin{cases} \mu \hat{V} v_{\delta_2} - \delta_2 \Delta v_{\delta_2} = \mu \hat{V} u, & x \in \Omega, \\ \frac{\partial}{\partial n} v_{\delta_2} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Noticing that $\hat{V}$ is the positive solution of

$$\begin{cases} -\delta_2 \Delta V = \beta V - \mu V^2, & x \in \Omega, \\ \frac{\partial}{\partial n} V = 0, & x \in \partial\Omega, \end{cases}$$

it satisfies

$$\hat{V} \to \frac{\int_\Omega \beta dx}{\int_\Omega \mu dx} \quad \text{as } \delta_2 \to \infty \tag{4.3}$$

(see [4, Proposition 3.17] and [18, Proposition 2.5]). The rest of the proof is essentially the same as the proof of Lemma 4.3. ∎

We now investigate R_0 for large diffusion rates by Theorem 4.1.

THEOREM 4.5. *The following statements hold:*

1. *For fixed* $\delta_2 > 0$,

$$R_0 \to r(L_{1,\infty} R_1 L_2 R_2) = \frac{\int_\Omega \lambda R_1 (L_2 R_2) dx}{\int_\Omega \lambda dx} \ \textit{as } \delta_1 \to \infty.$$

2. *For fixed* $\delta_1 > 0$,

$$R_0 \to r(L_{2,\infty} R_2 L_1 R_1) = \frac{\int_\Omega \mu R_2 (L_1 R_1) dx}{\int_\Omega \mu dx} \ \textit{as } \delta_2 \to \infty.$$

Proof. For $i = 1, 2$, define two bounded linear operators $H_{i,\infty} : C(\bar{\Omega}) \to C(\bar{\Omega})$ by

$$H_{1,\infty}(\phi) = \frac{\int_\Omega \lambda R_1 L_2 R_2 \phi dx}{\int_\Omega \lambda dx} \quad \text{and} \quad H_{2,\infty}(\phi) = \frac{\int_\Omega \mu R_2 L_1 R_1 \phi dx}{\int_\Omega \mu dx} \quad \text{for any } \phi \in C(\bar{\Omega}).$$

Then $H_{1,\infty} = L_{1,\infty} R_1 L_2 R_2$ and $H_{2,\infty} = L_{2,\infty} R_2 L_1 R_1$. By Lemmas 4.3–4.4, we have

$$L_1 R_1 L_2 R_2 \xrightarrow{\text{SOT}} H_{1,\infty} \ \text{ as } \ \delta_1 \to \infty \ \text{ and } \ L_2 R_2 L_1 R_1 \xrightarrow{\text{SOT}} H_{2,\infty} \ \text{ as } \ \delta_2 \to \infty.$$

Clearly, $L_1 R_1 L_2 R_2, L_2 R_2 L_1 R_1, H_{1,\infty}$, and $H_{2,\infty}$ are strongly positive compact operators on $C(\bar{\Omega})$. In the proof of Lemma 3.2, we have shown that $L_i(B) \subset B$, $i = 1, 2$. This implies that $\cup_{\delta_1 > 1} L_1 R_1 L_2 R_2 (B) \subset L_1 R_1 (R_{2M} B)$ and $\cup_{\delta_2 > 1} L_2 R_2 L_1 R_1 (B) \subset L_2 R_2 (R_{1M} B)$ are precompact in $C(\bar{\Omega})$, where R_{1M} and R_{2M} are defined in Theorem 3.6. By Theorem 3.6, we have $r(L_1 R_1 L_2 R_2) = r(L_2 R_2 L_1 R_1) \geq R_{1m} R_{2m} > 0$. Then by Theorem 4.1, we have $R_0 = r(L_1 R_1 L_2 R_2) \to r(H_{1,\infty})$ as $\delta_1 \to \infty$ and $R_0 = r(L_2 R_2 L_1 R_1) \to r(H_{2,\infty})$ as $\delta_2 \to \infty$. Finally, we observe that the eigenfunctions of $H_{1\infty}$ and $H_{2\infty}$ must be constants, and

$$r(H_{1,\infty}) = \frac{\int_\Omega \lambda R_1 (L_2 R_2) dx}{\int_\Omega \lambda dx} \quad \text{and} \quad r(H_{2,\infty}) = \frac{\int_\Omega \mu R_2 (L_1 R_1) dx}{\int_\Omega \mu dx}.$$

∎

Remark 4.6. If R_2 is constant, then $L_2 R_2 = R_2$ and

$$R_0 \to \frac{\int_\Omega \lambda R_1 (L_2 R_2) dx}{\int_\Omega \lambda dx} = \frac{\int_\Omega \lambda R dx}{\int_\Omega \lambda dx} \quad \text{as } \delta_1 \to \infty,$$

which is independent of δ_2. Similarly, if R_1 is constant, then

$$R_0 \to \frac{\int_\Omega \mu R_2 (L_1 R_1) dx}{\int_\Omega \lambda dx} = \frac{\int_\Omega \mu R dx}{\int_\Omega \mu dx} \quad \text{as } \delta_2 \to \infty,$$

which is independent of δ_1.

Define

$$\hat{R}_1 := \frac{\int_\Omega \lambda R_1 dx}{\int_\Omega \lambda dx} = \frac{\int_\Omega \sigma_1 H_u dx}{\int_\Omega \lambda dx} \text{ and } \hat{R}_2 := \frac{\int_\Omega \mu R_2 dx}{\int_\Omega \mu dx} = \frac{\int_\Omega \sigma_2 dx}{\int_\Omega \mu dx}.$$

THEOREM 4.7. *The following statements hold:*
1. $r(L_{1,\infty}R_1L_2R_2) \to \hat{R}_1\hat{R}_2$. *as* $\delta_2 \to \infty$.
2. $r(L_{2,\infty}R_2L_1R_1) \to \hat{R}_1\hat{R}_2$ *as* $\delta_1 \to \infty$.

Proof. By Lemmas 4.3–4.4, we have

$$L_2R_2 \to \frac{\int_\Omega \mu R_2 dx}{\int_\Omega \mu dx} \quad \text{and} \quad L_1R_1 \to \frac{\int_\Omega \lambda R_1 dx}{\int_\Omega \lambda dx} \text{ in } C(\bar{\Omega}).$$

Our claim now just follows from Theorem 4.5. ∎

Remark 4.8. By Theorems 4.5–4.7, we have

$$\lim_{\delta_1\to\infty}\lim_{\delta_2\to\infty} R_0 = \lim_{\delta_2\to\infty}\lim_{\delta_1\to\infty} R_0 = \hat{R}_1\hat{R}_2.$$

We can actually prove

$$\lim_{(\delta_1,\delta_2)\to(\infty,\infty)} R_0 = \hat{R}_1\hat{R}_2, \tag{4.4}$$

by making use of $L_1R_1L_2R_2 \xrightarrow{\text{SOT}} L_{1,\infty}R_1L_{2,\infty}R_2$ and Theorem 4.1.

4.2. Small diffusion rates. We next study R_0 when the diffusion rates are small.

THEOREM 4.9. *The following statements hold:*
1. *For fixed* $\delta_2 > 0$, $R_0 \to r(RL_2)$ *as* $\delta_1 \to 0$.
2. *For fixed* $\delta_1 > 0$, $R_0 \to r(RL_1)$ *as* $\delta_2 \to 0$.

Proof. 1. It is well known that, for each $\phi \in C(\bar{\Omega})$, $L_1\phi \to \phi$ in $C(\bar{\Omega})$ as $\delta_1 \to 0$. So we have $R_1L_2R_2L_1 \xrightarrow{\text{SOT}} R_1L_2R_2$ as $\delta_1 \to 0$. Let B be the closed unit ball in $C(\bar{\Omega})$. Since $L_1(B) \subseteq B$, we have $\cup_{\delta_1<1}R_1L_2R_2L_1(B) \subseteq R_1L_2R_2(B)$. By the compactness of L_2, $\cup_{\delta_1<1}R_1L_2R_2L_1(B)$ is precompact in $C(\bar{\Omega})$. By Theorem 3.6, we have $r(R_1L_2R_2L_1) \geq R_{1m}R_{2m} > 0$. Noticing that $R_1L_2R_2L_1$ and $R_1L_2R_2$ are strongly positive compactor operators on $C(\bar{\Omega})$, by Theorem 4.1, we have

$$R_0 = r(R_1L_2R_2L_1) \to r(R_1L_2R_2) = r(R_2R_1L_2) = r(RL_2) \quad \text{as} \quad \delta_1 \to 0.$$

2. By [15, Lemma A.1], $\hat{V} \to \beta/\mu$ in $C(\bar{\Omega})$ and $L_2\phi \to \phi$ for any $\phi \in C(\bar{\Omega})$ as $\delta_2 \to 0$. Hence $R_2L_1R_1L_2 \xrightarrow{\text{SOT}} R_2L_1R_1$ as $\delta_2 \to 0$. The rest of the proof is similar to part 1. ∎

Let $R_M = \max\{R(x) : x \in \bar{\Omega}\}$. The proof of the following result is similar to [21, Lemma 3.1], and we attach it in the appendix for readers' convenience. Unfortunately, we cannot apply Theorem 4.1, since R is not compact. Can we generalize Theorem 4.1 so that it can be used to prove the following result? We leave this as an open question.

THEOREM 4.10. *The following statements hold:*
1. $r(RL_2) \to R_M$ *as* $\delta_2 \to 0$.
2. $r(RL_1) \to R_M$ *as* $\delta_1 \to 0$.

Combining Theorems 4.9–4.10, we actually have

$$\lim_{\delta_1\to 0}\lim_{\delta_2\to 0} R_0 = \lim_{\delta_2\to 0}\lim_{\delta_1\to 0} R_0 = \max\{R(x) : x \in \bar\Omega\}. \tag{4.5}$$

We can apply [17] to prove the following result.

Theorem 4.11. *The following statement holds:*

$$\lim_{(\delta_1,\delta_2)\to(0,0)} R_0 = \max\{R(x) : x \in \bar\Omega\}. \tag{4.6}$$

Proof. Let $R_M = \max\{R(x) : x \in \bar\Omega\}$. First, suppose $R_M = 1$ and $\hat V$ is independent of δ_2. We need to show that $R_0 \to 1$ as $(\delta_1, \delta_2) \to (0, 0)$. Let $\kappa = 1/R_0$ and view it as a function of (δ_1, δ_2). Since R_0 is the principal eigenvalue of $L_1R_1L_2R_2$, there exists a positive $\Phi_0 = (\varphi_0, \psi_0)^T$ (satisfying homogeneous Neumann boundary conditions) such that κ satisfies

$$A\Phi_0 + \kappa B\Phi_0 = 0, \tag{4.7}$$

where

$$A = \begin{pmatrix} \delta_1\Delta - \lambda & 0 \\ \mu\hat V R_2 & \delta_2\Delta - \mu\hat V \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & \lambda R_1 \\ 0 & 0 \end{pmatrix}.$$

For any positive a, δ_1, and δ_2, let $e = e(a, \delta_1, \delta_2)$ be the principal eigenvalue of the following eigenvalue problem (with homogeneous Neumann boundary conditions):

$$A\Phi + aB\Phi = e\Phi. \tag{4.8}$$

Then, we have $e(\kappa, \delta_1, \delta_2) = 0$.

It has been shown in [17, Theorem 1.4] that

$$\lim_{(\delta_1,\delta_2)\to(0,0)} e = \max_{x\in\bar\Omega} \hat e(C_a(x)),$$

where $\hat e(C_a(x))$ denotes the eigenvalue of the matrix $C_a(x)$ with a greater real part for each $x \in \bar\Omega$ (by the Perron–Frobenius theorem, the eigenvalues of $C_a(x)$ are real), and

$$C_a = \begin{pmatrix} -\lambda & a\lambda R_1 \\ \mu\hat V R_2 & -\mu\hat V \end{pmatrix}.$$

Therefore, for each a, $e = e(a, \delta_1, \delta_2)$ can be extended to be a continuous function of (δ_1, δ_2) on $(0,\infty) \times (0,\infty) \cup \{(0, 0)\}$ by $e(a, 0, 0) := \max_{x\in\bar\Omega} \hat e(C_a(x))$.

We claim that e is increasing in a for each $(\delta_1, \delta_2) \in (0,\infty) \times (0,\infty)$. To see this, we can choose $\Phi = (\varphi, \psi)$ to be a positive eigenvector with $\|\varphi\|_2 + \|\psi\|_2 = 1$ of (4.8). Then differentiate both sides of (4.8) with respect to a, we obtain

$$A\Phi_a + aB\Phi_a + B\Phi = e_a\Phi + e\Phi_a. \tag{4.9}$$

Multiplying (4.9) by Φ^T to the left and (4.8) by Φ_a^T to the left, and integrating their difference over Ω, we obtain $\Phi^T B\Phi = e_a\Phi^T\Phi$. Therefore, $e_a = \int_\Omega \lambda R_1\varphi\psi dx > 0$ and e is strictly increasing in a.

Noticing $\max\{R(x) : x \in \bar\Omega\} = 1$, it is not hard to check that $e(a, 0, 0) = \max_{x\in\bar\Omega} \hat e(C_a(x)) = 0$ if and only if $a = 1$. Moreover, $e(a, 0, 0)$ is strictly increasing in a. Assume to the contrary that $\kappa(\delta_1, \delta_2) \not\to 1$ as $(\delta_1, \delta_2) \to (0, 0)$. Then there exists a sequence $\{(\delta_{1n}, \delta_{2n})\}_{n=1}^{\infty}$ and $a_0 \neq 1$ such that $\kappa_n := \kappa(\delta_{1n}, \delta_{2n}) \to a_0$ as $n \to \infty$. Without loss of generality, we may assume $a_0 > 1$. Choose $\epsilon_0 > 0$ such that $a_0 - \epsilon_0 > 1$, which implies $\kappa(a_0 - \epsilon_0, 0, 0) > \kappa(1, 0, 0) = 0$. Then there exists $N > 0$ such that $\kappa_n > a_0 - \epsilon_0$ for all $n \geq N$. By the monotonicity of e, we have

$$0 = e(\kappa_n, \delta_{1n}, \delta_{2n}) > e(a_0 - \epsilon_0, \delta_{1n}, \delta_{2n}) \text{ for all } n \geq N.$$

Taking $n \to \infty$ and by the continuity of $e(a_0 - \epsilon_0, \cdot, \cdot)$, we have

$$0 \geq \lim_{n\to\infty} e(a_0 - \epsilon_0, \delta_{1n}, \delta_{2n}) = e(a_0 - \epsilon_0, 0, 0) > 0,$$

which is a contradiction. Therefore, $\kappa(\delta_1, \delta_2) \to 1$ as $(\delta_1, \delta_2) \to (0, 0)$. This proves the case $\max\{R(x) : x \in \bar{\Omega}\} = 1$.

Then, we drop the assumption $R_M = 1$ but still suppose that $\hat{V}$ is independent of δ_2. We have

$$\frac{R_0}{R_M} = r\left(L_1 R_1 L_2 \frac{R_2}{R_M}\right) \to \max\left\{R_1(x)\frac{R_2(x)}{R_M} : x \in \bar{\Omega}\right\} = 1 \text{ as } (\delta_1, \delta_2) \to (0, 0).$$

This means $R_0 \to R_M$ as $(\delta_1, \delta_2) \to (0, 0)$.

Finally, we drop the assumption that $\hat{V}$ is independent of δ_2. Let $\epsilon > 0$ be given. By Lemma 2.1, there exists $\delta > 0$ such that $\|\hat{V} - \beta/\mu\|_\infty < \epsilon$ for all $\delta_2 < \delta$. By the comparison principle, for $\delta_2 < \delta$, we have

$$\begin{aligned}\left(\mu\left(\frac{\beta}{\mu} + \epsilon\right) - \delta_2\Delta\right)^{-1} \mu\left(\frac{\beta}{\mu} - \epsilon\right) &\leq L_2 \\ = (\mu\hat{V} - \delta_2\Delta)^{-1}\mu\hat{V} &\leq \left(\mu\left(\frac{\beta}{\mu} - \epsilon\right) - \delta_2\Delta\right)^{-1} \mu\left(\frac{\beta}{\mu} + \epsilon\right).\end{aligned}$$

Define

$$\hat{L}_{2\epsilon} = \left(\mu\left(\frac{\beta}{\mu} - \epsilon\right) - \delta_2\Delta\right)^{-1} \mu\left(\frac{\beta}{\mu} - \epsilon\right) \tag{4.10}$$

and

$$\hat{R}_{2\epsilon} = \frac{\frac{\beta}{\mu} + \epsilon}{\frac{\beta}{\mu} - \epsilon} R_2. \tag{4.11}$$

Similarly, we define $\check{L}_{2\epsilon}$ and $\check{R}_{2\epsilon}$ only with ϵ replaced by $-\epsilon$ in (4.10)–(4.11). Then, we have

$$L_1 R_1 \check{L}_{2\epsilon} \check{R}_{2\epsilon} \leq L_1 R_1 L_2 R_2 \leq L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon} \text{ for } \delta_2 < \delta.$$

It follows from Theorem 2.5 that

$$r(L_1 R_1 \check{L}_{2\epsilon} \check{R}_{2\epsilon}) \leq R_0 \leq r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}) \text{ for } \delta_2 < \delta. \tag{4.12}$$

By the previous step,

$$\lim_{(\delta_1,\delta_2)\to(0,0)} r(L_1 R_1 \check{L}_{2\epsilon} \check{R}_{2\epsilon}) = \max\{R_1(x)\check{R}_{2\epsilon}(x) : x \in \bar{\Omega}\} := \check{R}_{M\epsilon}$$

and

$$\lim_{(\delta_1,\delta_2)\to(0,0)} r(L_1 R_1 \hat{L}_{2\epsilon} \hat{R}_{2\epsilon}) = \max\{R_1(x)\hat{R}_{2\epsilon}(x) : x \in \bar{\Omega}\} := \hat{R}_{M\epsilon}.$$

Taking $(\delta_1, \delta_2) \to (0,0)$ in (4.12), we obtain

$$\check{R}_{M\epsilon} \le \liminf_{(\delta_1,\delta_2)\to(0,0)} R_0 \le \limsup_{(\delta_1,\delta_2)\to(0,0)} R_0 \le \hat{R}_{M\epsilon}.$$

Taking $\epsilon \to 0$, we have

$$\liminf_{(\delta_1,\delta_2)\to(0,0)} R_0 = \limsup_{(\delta_1,\delta_2)\to(0,0)} R_0 = R_M.$$

∎

By Theorem 4.11, we have the following result.

PROPOSITION 4.12. *The following statements hold:*

1. *If $R(x) < 1$ for all $x \in \bar{\Omega}$, then there exists $\hat{\delta} > 0$ such that $R_0 < 1$ for all (δ_1, δ_2) with $\delta_1, \delta_2 \le \hat{\delta}$.*
2. *If $R(x) > 1$ for some $x \in \bar{\Omega}$, then there exists $\tilde{\delta} > 0$ such that $R_0 > 1$ for all (δ_1, δ_2) with $\delta_1, \delta_2 \le \tilde{\delta}$.*

5. Simulations.

5.1. Dependence on δ_1.

In this section, we investigate the dependence of R_0 on δ_1. Let $\Omega = [0,1] \times [0,1]$. We fix all the coefficients except for δ_1: $\delta_2 = 4, \sigma_1 = 5\sin(x) + 3, \sigma_2 = \mu = \beta = (x+1)^2 + 0.1, H_u = \cos(y) + 1.5, \lambda = 12$. Since $\beta/\mu = 1$, the unique positive solution of (1.3) is $\hat{V} = 1$. By Theorem 3.6, $R_0 \le \max\{R(x) : x \in \bar{\Omega}\} = 1.5015$. Noticing that $R_2 = \sigma_2/\mu = 1$ and λ are constant, by Remark 4.6,

$$R_0 \to \frac{\int_\Omega \lambda R dx}{\int_\Omega \lambda dx} = \frac{\int_\Omega R dx}{|\Omega|} = 0.5854 \quad \text{as } \delta_1 \to \infty. \tag{5.1}$$

We then find $r(RL_2)$. Using the fact that $\kappa' = 1/r(RL_2)$ is the principal eigenvalue of the following problem (with homogenous Neumann boundary conditions),

$$(\mu\hat{V} - \delta_2\Delta)\phi = \kappa\mu\hat{V}R\phi,$$

we can compute $r(RL_2) = 1.0075$ numerically. By Theorem 4.9, we expect

$$R_0 \to r(RL_2) = 1.0075 \text{ as } \delta_1 \to 0. \tag{5.2}$$

We now compute R_0. By definition, $\kappa = 1/R_0$ is the principal eigenvalue of the following problem (with homogeneous Neumann boundary conditions):

$$\begin{pmatrix} -\nabla\cdot\delta_1\nabla\varphi \\ -\nabla\cdot\delta_2\nabla\psi \end{pmatrix} + \begin{pmatrix} \lambda & -\sigma_1 H_u \\ 0 & \mu\hat{V} \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \kappa \begin{pmatrix} 0 & 0 \\ \sigma_2\hat{V} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

For different values of $\delta_1 \in [0.001, 400]$, we solve the eigenvalue problem numerically and plot R_0 in Figure 1. In particular, $R_0 = 1.0074$ when $\delta_1 = 0.001$ and $R_0 = 0.5904$ when $\delta_1 = 400$, which agrees with (5.1)–(5.2). Moreover, we observe that R_0 is decreasing in σ_1. We conjecture that this is true in general.

5.2. Simulations in a realistic situation.

In this section, we will simulate the model using geometric and population data of Puerto Rico. The domain Ω is taken as the geometric boundary of Puerto Rico, which can be obtained from Mathematica as a polygon. The population density data of the 76 districts of Puerto Rico can also be found in Mathematica, which can be used to construct the susceptible human distribution, i.e., $H_u(x)$, by interpolation. H_{i0} is assumed to be 100 people, distributed

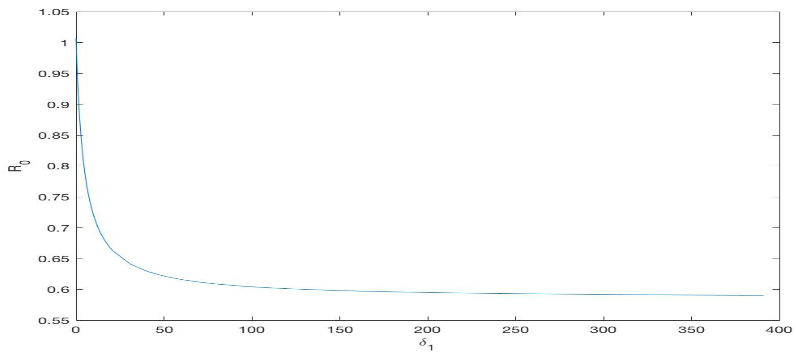

FIG. 1. *The basic reproduction number R_0 for different values of δ_1.*

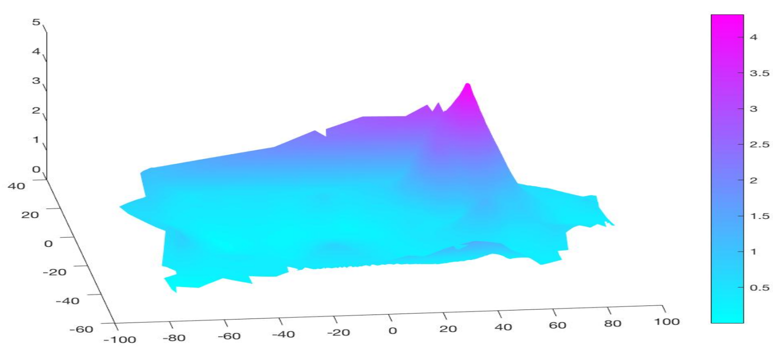

FIG. 2. *Local basic reproduction number $R(x)$.*

normally, centered at $(0, -20)$. Set $V_{i0} = 10H_{i0}$, $V_{u0} = 150$, $\sigma_1 = 0.000001$, $\sigma_2 = 0.7$, $\lambda = 1$, $\beta = 5$, and $\mu = 0.0005$. The local basic reproduction number $R(x) = \sigma_1\sigma_2 H_u/\lambda\mu$ is shown in Figure 2.

Then we compute $\max\{R(x) : x \in \bar{\Omega}\} = 4.3167$ and $\frac{\int_\Omega \lambda R_1(L_2R_2)dx}{\int_\Omega \lambda dx} = \frac{\int_\Omega Rdx}{|\Omega|} = 0.6513$. By Theorems 2.3, (4.5)–(4.7), and (4.9)–(4.10), we expect that the solution of (1.1) converges to a positive steady state when the diffusion rates are small and to the semitrivial equilibrium $(0, \hat{V}, 0)$ when δ_2 is large. For verification, we choose different diffusion rates and use the finite element method in MATLAB to solve (1.1).

Case 1. Set $\delta_1 = \delta_2 = 4$. We plot the total infected host cases in Figure 3 and the density of infected hosts for $t = 4, 8, 12, 16$ in Figure 4. In this case, the solution converges to the positive steady state and the disease persists.

Case 2. Set $\delta_1 = 4$ and $\delta_2 = 4000$. We plot the total infected host cases in Figure 5 and the density of infected hosts in Figure 6. In this case, the density of infected hosts converges to zero and the disease dies out.

6. Discussion. In this paper, we have shown that the basic reproduction number R_0 of the reaction-diffusion model (1.1) can be written as $R_0 = r(L_1R_1L_2R_2)$, where the local basic reproduction number $R(x) = R_1(x)R_2(x)$ is a multiplication operator on $C(\bar{\Omega})$, and L_1 and L_2 are strongly positive compact linear operators with spectral radii one. We are then able to study the relation of R_0 and $R(x)$. We prove that

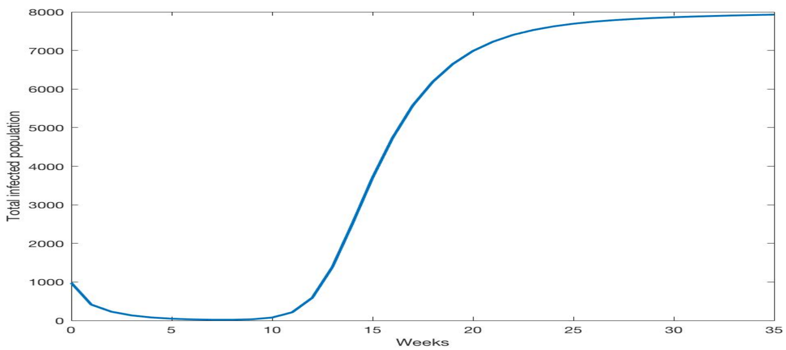

FIG. 3. *Total infected host cases, i.e.,* $\int_\Omega H_i(x,t)dx$ *with* $\delta_1 = \delta_2 = 4$.

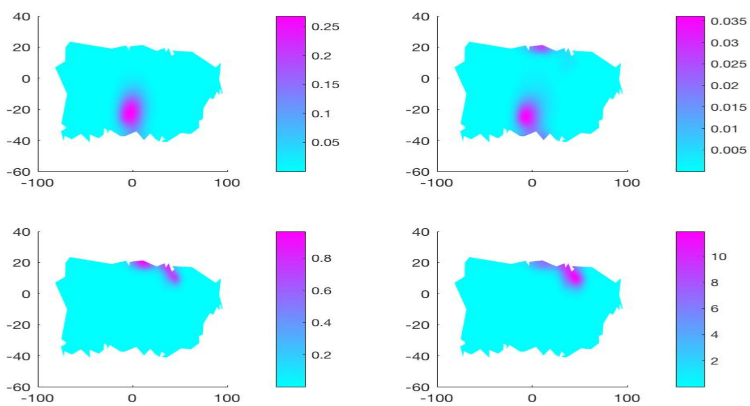

FIG. 4. *The density of infected hosts, i.e.,* $H_i(x,t)$, *at* $t = 4, 8, 12, 16$ *with* $\delta_1 = \delta_2 = 4$.

$R_0 \geq 1$ if $R_1(x) \geq 1$ and $R_2(x) \geq 1$ for all $x \in \bar{\Omega}$, and $R_0 \leq 1$ if $R_1(x) \leq 1$ and $R_2(x) \leq 1$. Actually, R_0 is bounded below and above by the products of the minimum and maximum of R_1 and R_2. When the diffusion rates are small, $R_0 > 1$ provided that $R(x) > 1$ for some $x \in \bar{\Omega}$. When the diffusion rates are large, R_0 approximates $\hat{R}_1\hat{R}_2$. Moreover, our numerical simulations suggest that R_0 is decreasing in δ_1, however, we are only able to prove it under the assumption (H1). The dependence of R_0 on δ_2 is more difficult to study since $\hat{V}$ is also dependent on δ_2. We only know that if β/μ is constant, then $\hat{V}$ is independent of δ_2 and R_0 is decreasing in δ_2 under the assumption (H1).

We remark that our approach can be applied to many other reaction-diffusion epidemic models. For example, if we adopt our approach to analyze R_0 for the diffusive SIS model in Allen et al. [1], we will compute $R_0 = r(-CB^{-1}) = r(\beta(\gamma - d_I\Delta)^{-1})$. Then we can write R_0 as $R_0 = r(RL)$, where $R(x) = \beta(x)/\gamma(x)$ is the local basic reproduction number and $L = (\gamma - d_I\Delta)^{-1}\gamma$ is a strongly positive compact linear operator in $C(\bar{\Omega})$ with spectral radius one. To further illustrate this, we briefly adopt this approach to study the basic reproduction number of some other models in the following two subsections.

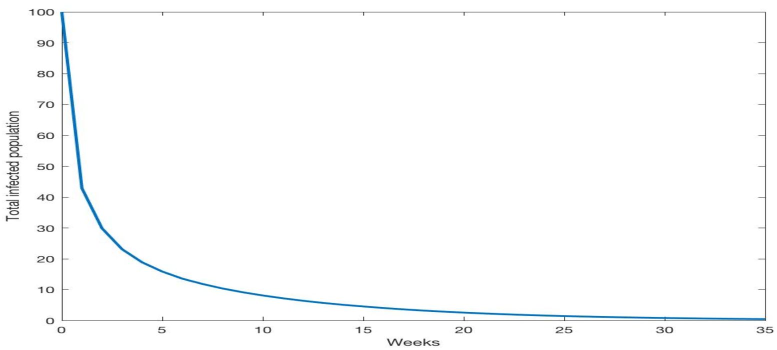

FIG. 5. *Total infected host cases, i.e.,* $\int_\Omega H_i(x,t)dx$ *with* $\delta_1 = 4, \delta_2 = 4000$.

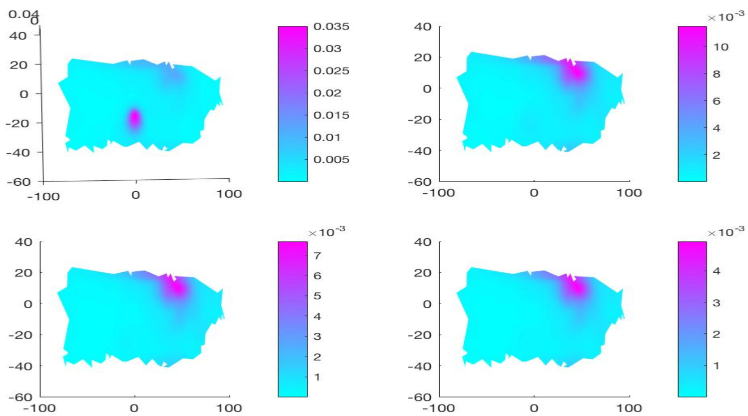

FIG. 6. *The density of infected hosts, i.e.,* $H_i(x,t)$*, at* $t = 4, 8, 12, 16$ *with* $\delta_1 = 4, \delta_2 = 4000$.

6.1. A within-host model on viral dynamics. Suppose that $T(x,t)$, $I(x,t)$, and $V(x,t)$ are the density of target cells, infected cells, and free virus particles at position x and time t, respectively. The model proposed in [19] to study the repulsion effect of superinfecting virion by infected cells is the following:

$$\begin{cases} \frac{\partial T}{\partial t} = D_T \Delta T + h(x) - d_T T - \beta(x) TV, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \beta(x) TV - d_I I, \\ \frac{\partial V}{\partial t} = \nabla \cdot (D_V(I) \nabla V) + \gamma(x) I - d_V V, \end{cases} \tag{6.1}$$

subject to homogeneous Neumann boundary conditions and nonnegative initial conditions.

Let $\hat{T}(x)$ be the unique positive solution of

$$D_T \Delta T + h(x) - d_T T = 0.$$

Linearizing (6.1) at the equilibrium $(\hat{T}, 0, 0)$, the stability of it is related to the following eigenvalue problem,

$$\begin{cases} \kappa\varphi = D_I \Delta\varphi - d_I \varphi + \beta \hat{T} \psi, \\ \kappa\psi = D_0 \Delta\psi + \gamma\varphi - d_V \psi, \end{cases}$$

where $D_0 = D_V(0)$. As before, we define

$$B = \begin{pmatrix} D_I\Delta & 0 \\ 0 & D_0\Delta \end{pmatrix} + \begin{pmatrix} -d_I & \beta\hat{T} \\ 0 & -d_V V \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

and the basic reproduction number

$$R_0 = r(-CB^{-1}).$$

Similarly to Theorem 3.1, we write R_0 as

$$R_0 = r\left(\gamma(d_I - D_I\Delta)^{-1}\beta\hat{T}(d_V - D_0)^{-1}\right).$$

We have

$$R_0 = r(L_1 R_1 L_2 R_2) \tag{6.2}$$

with

$$L_1 = (d_I - D_I\Delta)^{-1}d_I, \quad L_2 = (d_V - D_0\Delta)^{-1}d_V,$$

and

$$R_1 = \frac{\beta\hat{T}}{d_I}, \qquad R_2 = \frac{\gamma}{d_V}.$$

The local basic reproduction number is defined as

$$R = R_1 R_2 = \frac{\gamma\beta\hat{T}}{d_I d_V}.$$

Here, L_1 and L_2 are strongly positive compact linear operators on $C(\bar{\Omega})$ with spectral radius one, and $\hat{T} = (d_T - D_T\Delta)^{-1}h$ satisfies

$$\lim_{D_T\to 0}\hat{T} = R_3, \quad \lim_{D_T\to\infty}\hat{T} = \frac{\int_\Omega d_T R_3 dx}{\int_\Omega d_T dx},$$

and

$$\min\{R_3(x) : x \in \bar{\Omega}\} \le \hat{T} \le \max\{R_3(x) : x \in \bar{\Omega}\}$$

with

$$R_3 = \frac{h}{d_T}.$$

An immediate consequence of (6.2) is the following result.

THEOREM 6.1. *The following statements hold:*

- *If R_1 and R_2 are constant, then $R_0 = R$.*
- *Let $R_{im} = \min\{R_i(x) : x \in \bar{\Omega}\}$ and $R_{iM} = \max\{R_i(x) : x \in \bar{\Omega}\}$ for $i = 1, 2$, then*

$$R_{1m}R_{2m} \le R_0 \le R_{1M}R_{2M}.$$

- $$\lim_{(D_I, D_T, D_V)\to(\infty,\infty,\infty)} R_0 = \frac{\bar{\beta}\bar{\gamma}\bar{h}}{\bar{d}_I\bar{d}_V\bar{d}_T},$$

  where $\bar{f}$ denotes the average of f, i.e., $\bar{f} = \int_\Omega f dx/|\Omega|$ for $f = \beta$, γ, h, d_I, d_V, d_T.

- $$\lim_{D_I\to 0}\lim_{D_V\to 0} R_0 = \lim_{D_V\to 0}\lim_{D_I\to 0} R_0 = \lim_{(D_I,D_V)\to(0,0)} R_0 = \max\{R(x) : x\in\bar{\Omega}\}.$$

We notice that R is consistent with the basic reproduction number defined using [13] (R can be viewed as the total number of newly infected cells produced by one infected cell) for the corresponding ordinary differential equation model. We will leave the interested readers to investigate the monotonicity of R_0 with respect to the diffusion rates.

6.2. An HIV model with cell-to-cell transmission. Let $T(x,t)$, $T^*(x,t)$, and $V(x,t)$ be the density of healthy T cells, infected T cells, and virions at position x and time t, respectively. The model proposed in [26] to describe the cell-to-cell HIV transmission is the following:

$$\text{(6.3)}\qquad \begin{cases} \frac{\partial T}{\partial t} = d_1\Delta T + \lambda(x) - d(x)T - \beta_1(x)TV - \beta_2(x)TT^*, \\ \frac{\partial T^*}{\partial t} = d_2\Delta T^* + \beta_1(x)TV + \beta_2(x)TT^* - \gamma(x)T^*, \\ \frac{\partial V}{\partial t} = d_3\Delta V + N(x)T^* - e(x)V, \end{cases}$$

subject to homogeneous Neumann boundary conditions and nonnegative initial conditions.

Let $T_0(x)$ be the unique positive solution of

$$d_1\Delta T + \lambda(x) - d(x)T = 0.$$

Linearizing (6.1) at the equilibrium $(T_0,0,0)$, we obtain the following eigenvalue problem,

$$\text{(6.4)}\qquad \begin{cases} \kappa\varphi = d_2\Delta\varphi + (\beta_2T_0-\gamma)\varphi + \beta_1T_0\psi, \\ \kappa\psi = d_3\Delta\psi + N\varphi - e\psi. \end{cases}$$

We define

$$B = \begin{pmatrix} d_2\Delta & 0 \\ 0 & d_3\Delta \end{pmatrix} + \begin{pmatrix} -\gamma & 0 \\ N & -e \end{pmatrix} \quad\text{and}\quad C = \begin{pmatrix} \beta_2T_0 & \beta_1T_0 \\ 0 & 0 \end{pmatrix}$$

and the basic reproduction number

$$R_0 = r(-CB^{-1}).$$

Similarly to Theorem 3.1, we compute R_0 as

$$R_0 = r\left(\beta_2T_0(\gamma-d_2\Delta)^{-1} + \beta_1T_0(e-d_3\Delta)^{-1}N(\gamma-d_2\Delta)^{-1}\right).$$

So we have

$$\text{(6.5)}\qquad R_0 = r(L_2(R_2^2 + R_2^1L_3R_3))$$

with

$$L_2 = (\gamma-d_2\Delta)^{-1}\gamma, \quad L_3 = (e-d_3\Delta)^{-1}e,$$

and

$$R_2^1 = \frac{\beta_1T_0}{\gamma}, \quad R_2^2 = \frac{\beta_2T_0}{\gamma}, \quad R_3 = \frac{N}{e}.$$

Here L_1 and L_2 are strongly positive compact linear operators on $C(\bar{\Omega})$ with spectral radius one, and $L_i 1 = 1$ for $i = 1, 2$. The local basic reproduction number R is defined as

$$R = R_2^2 + R_2^1 R_3 = \frac{(\beta_1 N + \beta_2 e) T_0}{er},$$

where $T_0 = (d - d_1 \Delta)^{-1} \lambda$ satisfies

$$\lim_{d_1 \to 0} T_0 = R_1, \quad \lim_{d_1 \to \infty} T_0 = \frac{\int_\Omega d R_1}{\int_\Omega d},$$

and

$$\min\{R_1(x) : x \in \bar{\Omega}\} \le T_0 \le \max\{R_1(x) : x \in \bar{\Omega}\}$$

with

$$R_1 = \frac{\lambda}{d}.$$

We can also prove the following.

THEOREM 6.2. *The following statements hold:*

- *If R_2^1, R_2^2, and R_3 are constant, then $R_0 = R$.*
- *Let $S_m = \min\{S(x) : x \in \bar{\Omega}\}$ and $S_M = \max\{S(x) : x \in \bar{\Omega}\}$ for $S = R_2^1, R_2^2, R_3$, then*

$$R_{2m}^1 + R_{2m}^2 R_{3m} \le R_0 \le R_{2M}^1 + R_{2M}^2 R_{3M}.$$

- $$\lim_{(d_1,d_2,d_3) \to (\infty,\infty,\infty)} R_0 = \frac{(\bar{\beta}_1 \bar{N} + \bar{\beta}_2 \bar{e}) \bar{\lambda}}{\bar{e} \bar{r} \bar{d}},$$

  where $\bar{f}$ denotes the average of f over Ω, i.e., $\bar{f} = \int_\Omega f dx / |\Omega|$ for $f = \beta_1, \beta_2, e, r, d, \lambda$.
- $\lim_{d_2 \to 0} \lim_{d_3 \to 0} R_0 = \max\{R(x) : x \in \bar{\Omega}\}$.

Proof. We will only sketch the proof of the last part. Noticing that $L_3 \phi \to \phi$ in $C(\bar{\Omega})$, we have $L_2(R_2^2 + R_2^1 L_3 R_3) \xrightarrow{\text{SOT}} L_2(R_2^2 + R_2^1 R_3) = L_2 R$ as $d_3 \to 0$. Let $B \subset C(\bar{\Omega})$ be the closed unit ball, then

$$\cup_{\delta_3 > 0} L_2(R_2^2 + R_2^1 L_3 R_3)(B) \subset L_2((R_{2M}^1 + R_{2M}^2 R_{3M}) B),$$

which is compact. By Theorem 4.1, we have $R_0 = r(L_2(R_2^2 + R_2^1 L_3 R_3)) \to r(L_2 R)$ as $d_3 \to 0$. The proof of $r(L_2 R) \to \max\{R(x) : x \in \bar{\Omega}\}$ as $d_2 \to 0$ is the same with Theorem 4.10. ∎

Appendix A. Proof of Theorem 4.10.

Proof. We only prove part 1. Define $r_{\delta_2} =: r(RL_2) = r(L_2 R)$. Then $\kappa_{\delta_2} = 1/r_{\delta_2}$ is the principal eigenvalue of the problem

$$\text{(A.1)} \qquad \begin{cases} (\mu V - \delta_2 \Delta) v = \kappa \mu \hat{V} R v, & x \in \Omega, \\ \frac{\partial}{\partial n} v = 0, & x \in \partial\Omega. \end{cases}$$

By (A.1),

$$\kappa_{\delta_2} = \frac{1}{r_{\delta_2}} = \min\left\{ \frac{\delta_2 \int_\Omega |\nabla v|^2 dx + \int_\Omega \mu \hat{V} v^2 dx}{\int_\Omega R \mu \hat{V} v^2 dx} : v \in H^1(\Omega) \text{ and } v \ne 0 \right\}$$
$$\ge \frac{1}{R_M} \min\left\{ \frac{\delta_2 \int_\Omega |\nabla v|^2 dx + \int_\Omega \mu \hat{V} v^2 dx}{\int_\Omega \mu \hat{V} v^2 dx} : v \in H^1(\Omega) \text{ and } v \ne 0 \right\} = \frac{1}{R_M}.$$

It then follows that $\liminf_{\delta_2 \to 0} \kappa_{\delta_2} \ge 1/R_M$.

We only need to show $\limsup_{\delta_2\to 0}\kappa_{\delta_2}\le 1/R_M$. Assume to the contrary that the statement does not hold, i.e., $\limsup_{\delta_2\to 0}\kappa_{\delta_2}>1/R_M$. Then there exists $\epsilon_0>0$ and a sequence $\{\delta_{2,n}\}$ with $\delta_{2,n}\to 0$ such that $\kappa_{\delta_{2,n}}>1/(R_M-\epsilon_0)$. Let $x_0\in\Omega$ and $a>0$ such that $R(x)>R_M-\epsilon_0/2$ in $B(x_0,a)$. Let $v_{\delta_{2,n}}$ be a positive eigenvector of (A.1) associated with the principal eigenvalue $\kappa_{\delta_{2,n}}$. Then in $B(x_0,a)$, we have

$$(\mu\hat{V}-\delta_{2,n}\Delta)v_{\delta_{2,n}}=\kappa_{\delta_{2,n}}\mu\hat{V}Rv_{\delta_{2,n}}>\frac{(R_M-\epsilon_0/2)\mu\hat{V}v_{\delta_{2,n}}}{R_M-\epsilon_0}.$$

It follows that, in $B(x_0,a)$,

$$-\frac{\Delta v_{\delta_{2,n}}}{v_{\delta_{2,n}}}>\frac{\epsilon_0}{2\delta_{2,n}(R_M-\epsilon_0)}\mu\hat{V}.$$

Let κ' be the principal eigenvalue of $-\Delta$ in domain $B(x_0,a)$ with a Dirichlet boundary condition. By a minimax formulation of κ' [3], we have

$$\text{(A.2)}\qquad \kappa'=\sup_{u\in W^{2,p}(B(x_0,a)),u>0}\inf_{x\in B(x_0,a)}\frac{-\Delta u}{u}>\frac{\epsilon_0}{2\delta_{2,n}(R_M-\epsilon_0)}\inf_{x\in B(x_0,a)}\{\mu\hat{V}\}.$$

Noticing that $\hat{V}\ge\min\{\beta(x):x\in\bar{\Omega}\}/\max\{\mu(x):x\in\bar{\Omega}\}$, the right-hand side of (A.2) tends to ∞ as $\delta_{2,n}\to 0$. This is a contradiction. Hence, $\kappa_{\delta_2}\to 1/R_M$ and $r_{\delta_2}\to R_M$ as $\delta_2\to 0$. ∎

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