

A system of state-dependent delay differential equation modelling forest growth I: semiflow properties

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Abstract: In this article we investigate the semiflow properties of a class of state-dependent delay differential equations which is motivated by some models describing the dynamics of the number of adult trees in forests. We investigate the existence and uniqueness of a semiflow in the space of Lipschitz and C^1 weighted functions. We obtain a blow-up result when the time approaches the maximal time of existence. We conclude the paper with an application of a spatially structured forest model.

Keywords: State-dependent delay differential equations, forest population dynamics, semiflow, time of blow-up.

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1 Introduction

Let Ω be a compact subset of \mathbb{R}^n (with $n \geq 1$). Denote for simplicity that $C(\Omega) := C(\Omega, \mathbb{R})$, $C(\Omega^2) := C(\Omega^2, \mathbb{R})$ and $C_+(\Omega) := C(\Omega, [0, +\infty))$. In this article we consider the following class of state-dependent delay differential equation: $\forall t \geq 0$ and $\forall x \in \Omega$,

$$\begin{cases} \partial_t A(t, x) = F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(t))(\cdot, \cdot))(x), \\ \int_{-\tau(t, x)}^0 f(A(t + s, \cdot))(x) ds = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \end{cases} \quad (1.1)$$

where $F : C(\Omega)^2 \times C(\Omega^2) \rightarrow C(\Omega)$ and $f : C(\Omega) \rightarrow C(\Omega)$ and $A(t - \tau(t)) \in C(\Omega^2)$ is the map defined by

$$A(t - \tau(t))(x, y) := A(t - \tau(t, x), y) \quad (1.2)$$

with the initial condition

$$A(t, x) = \varphi(t, x), \forall t \leq 0 \text{ and } \tau(0) = \tau_0 \in C_+(\Omega),$$

and the initial distribution φ belongs to

$$\text{Lip}_\alpha := \{\phi \in C((-\infty, 0], C(\Omega)) : t \rightarrow e^{-\alpha|t|}\phi(t, \cdot) \text{ is bounded and Lipschitz continuous from } (-\infty, 0] \text{ to } C(\Omega)\}, \alpha \geq 0.$$

Recall that the space Lip_α is a Banach space endowed with the norm

$$\|\phi\|_{\text{Lip}_\alpha} := \|\phi_\alpha\|_\infty + \|\phi_\alpha\|_{\text{Lip}}$$

where $\phi_\alpha : (-\infty, 0] \rightarrow C(\Omega)$ is defined by

$$\phi_\alpha(t, x) := e^{-\alpha|t|}\phi(t, x), \forall t \in (-\infty, 0], \forall x \in \Omega. \quad (1.3)$$

and

$$\|\phi_\alpha\|_\infty := \sup_{t \leq 0} \|\phi_\alpha(t, \cdot)\|_\infty$$

and for fixed t ,

$$\|\phi_\alpha(t, \cdot)\|_\infty := \sup_{x \in \Omega} |\phi_\alpha(t, x)|$$

and

$$\|\phi_\alpha\|_{\text{Lip}} := \sup_{t, s \leq 0; t \neq s} \frac{\|\phi_\alpha(t, \cdot) - \phi_\alpha(s, \cdot)\|_\infty}{|t - s|}.$$

In the rest of the paper the product space $\text{Lip}_\alpha \times C(\Omega)$ will be endowed with the usual product norm

$$\|(\phi, r)\|_{\text{Lip}_\alpha \times C(\Omega)} := \|\phi\|_{\text{Lip}_\alpha} + \|r\|_\infty, \forall \phi \in \text{Lip}_\alpha, \forall r \in C(\Omega).$$

We will make the following assumptions throughout this paper.

Assumption 1.1 *We assume that the map $F : C(\Omega)^2 \times C(\Omega^2) \rightarrow C(\Omega)$ is Lipschitz continuous on bounded sets, that is to say that for each constant $M > 0$, there exists a constant $L(M) > 0$ satisfying*

$$\|F(u, v, w) - F(\hat{u}, \hat{v}, \hat{w})\|_\infty \leq L(M) [\|u - \hat{u}\|_\infty + \|v - \hat{v}\|_\infty + \|w - \hat{w}\|_\infty]$$

whenever $\|u\|_\infty, \|\hat{u}\|_\infty, \|v\|_\infty, \|\hat{v}\|_\infty, \|w\|_\infty, \|\hat{w}\|_\infty \leq M$.

We also assume that the map $f : C(\Omega) \rightarrow C(\Omega)$ is Lipschitz continuous and there exists a real number $M > 0$ such that

$$0 < f(\varphi)(x) \leq M, \forall x \in \Omega \text{ and } \forall \varphi \in C(\Omega),$$

and f is monotone non-increasing, that is to say that

$$\varphi(x) \leq \hat{\varphi}(x), \forall x \in \Omega \Rightarrow f(\varphi)(x) \geq f(\hat{\varphi})(x), \forall x \in \Omega.$$

Examples of state-dependent delay differential equations of this form have been considered first by Smith [12, 13, 14, 15]. This idea has been successfully used in [4, 7] (see also the references therein). Our motivation to consider such a class of state-dependent delay differential equations comes from forest modelling. In [9, 10] such state-dependent delay differential equations have been used to model the competition for light between trees.

Example 1.2 (Finite number of species) *The m -species case corresponds to the case $n = 1$ and the domain Ω contains exactly m elements. We can choose for example*

$$\Omega = \{1, 2, \dots, m\}$$

and for $x = 1, \dots, m$,

$$F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(t))(\cdot, \cdot))(x) = G(x, A(t, \cdot), \tau(t, x), A(t - \tau(t, x))(\cdot))$$

where $G : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a map (see [9] for more details).

Example 1.3 (Spatially structured case) *For the spatially structured case, we can choose*

$$\Omega = [0, x_{\max}] \times [0, y_{\max}].$$

Moreover assume (for simplicity) that we have a single species, then we can choose

$$F(A(t, \cdot), \tau(t, \cdot), A_1(t, \cdot))(x, y) := e^{-\mu_J \tau(t, x, y)} \frac{f(A(t, x, y))}{f(A_1(t, x, y))} (I - \varepsilon \Delta)^{-1} [\beta A_1(t, \cdot)](x, y) - \mu_A A(t, x, y),$$

where Δ is the Laplacian operator on the domain Ω with periodic boundary conditions. This model corresponds to the spatially structured model in [9].

Let $A \in C((-\infty, r], C(\Omega))$ (for some $r \geq 0$) be given. Then for each $t \leq r$, we will use the standard notation $A_t \in C((-\infty, 0], C(\Omega))$, which is the map defined by

$$A_t(\theta, \cdot) = A(t + \theta, \cdot), \forall \theta \leq 0.$$

For clarity we will specify the notion of a solution.

Definition 1.4 *Let $r \in (0, +\infty]$. A solution of the system (1.1) on $[0, r]$ is a pair of continuous maps $A : (-\infty, r) \rightarrow C(\Omega)$ and $\tau : [0, r) \rightarrow C_+(\Omega)$ satisfying*

$$A(t, x) = \begin{cases} \varphi(0, x) + \int_0^t F(A(l, \cdot), \tau(l, \cdot), A(l - \tau(l))(\cdot, \cdot))(x) dl, \forall t \in [0, r), \forall x \in \Omega, \\ \varphi(t, x), \forall t \leq 0, \forall x \in \Omega, \end{cases}$$

and

$$\int_{t-\tau(t, x)}^t f(A(s, \cdot))(x) ds = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \forall t \in [0, r), \forall x \in \Omega.$$

In this problem the initial distribution is (φ, τ_0) . The semiflow generated by (1.1) is

$$\mathcal{U}(t)(\varphi(\cdot, x), \tau_0(x)) := (A_t(\cdot, x), \tau(t, x)),$$

where (A, τ) is the solution of (1.1) with the initial distribution (φ, τ_0) .

In order to clarify the notion of semiflow in this context, we introduce the following definition.

Definition 1.5 Let (M, d) be a metric space. Let $\mathcal{U} : D_{\mathcal{U}} \subset [0, +\infty) \times M \rightarrow M$ be a map defined on the domain

$$D_{\mathcal{U}} := \{(t, x) \in [0, +\infty) \times M : 0 \leq t < T_{BU}(x)\},$$

where $T_{BU} : M \rightarrow (0, +\infty]$ is a lower semi-continuous map (the blow-up time). We will use the notation

$$\mathcal{U}(t)x := \mathcal{U}(t, x), \forall (t, x) \in D_{\mathcal{U}}.$$

We say that \mathcal{U} is a **maximal semiflow on M** if the following properties are satisfied:

- (i) $T_{BU}(\mathcal{U}(t)x) + t = T_{BU}(x)$, $\forall x \in M$, $\forall t \in [0, T_{BU}(x))$;
- (ii) $\mathcal{U}(0)x = x$, $\forall x \in M$;
- (iii) $\mathcal{U}(t)\mathcal{U}(s)x = \mathcal{U}(t+s)x$, $\forall t, s \in [0, T_{BU}(x))$ with $t+s < T_{BU}(x)$;
- (iv) If $T_{BU}(x) < +\infty$, then

$$\lim_{t \nearrow T_{BU}(x)} d(\mathcal{U}(t)x, y) = +\infty$$

for some $y \in M$.

We will say that the semiflow \mathcal{U} is **state variable continuous** if for each $t \geq 0$ the map $x \mapsto \mathcal{U}(t)x$ is continuous around each point where $\mathcal{U}(t)$ is defined. We will say that the semiflow \mathcal{U} is **locally uniformly state variable continuous** if for each $r \in [0, T_{BU}(x_0))$,

$$\lim_{x \rightarrow x_0} \sup_{t \in [0, r]} d(\mathcal{U}(t)x, \mathcal{U}(t)x_0) = 0 \tag{1.4}$$

whenever the map $\mathcal{U}(t)$ is defined at x and x_0 and for each $t \in [0, r]$.

We will say that the semiflow \mathcal{U} is **continuous** if the map $(t, x) \mapsto \mathcal{U}(t)x$ is continuous from $D_{\mathcal{U}}$ into M .

Actually the semiflow of the state-dependent delay differential equation (1.1) is not always continuous in time. Assume for example that $\alpha = 0, \Omega = \{1\}$ and $\forall u, v \in C(\Omega), \forall w \in C(\Omega^2), \forall x \in \Omega$,

$$F(u, v, w)(x) \equiv 1 \text{ and } f(u)(x) \equiv 1.$$

Consider $(A(t), \tau(t))$ (we omit the x variable since there is only one element in Ω) the solution of (3.1) with the initial distribution

$$(\varphi, \tau_0) = (0_{\text{Lip}_\alpha}, \tau_0).$$

This solution can be solved explicitly:

$$A(t) = \begin{cases} t, \forall t > 0, \\ 0, \forall t \leq 0, \end{cases} \quad \tau(t) = \tau_0, \forall t \geq 0.$$

And the semiflow will be defined by $\mathcal{U}(t)(0_{\text{Lip}_\alpha}, \tau_0) = (A_t, \tau)$. Notice that the map $t \mapsto A(t)$ is differentiable almost everywhere and

$$A'(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Therefore for each $\widehat{t} \geq 0$,

$$\lim_{t \rightarrow \widehat{t}} \|A_t - A_{\widehat{t}}\|_{\text{Lip}} = \lim_{t \rightarrow \widehat{t}} \|A'(t + \cdot) - A'(\widehat{t} + \cdot)\|_{L^\infty(-\infty, 0)} = 1.$$

Therefore due to the discontinuity of $A'(t)$ at time $t = 0$, the semiflow is not continuous in time.

The following theorem is the main result of this section.

Theorem 1.6 *There exists a maximal semiflow $\mathcal{U} : D_{\mathcal{U}} \subset [0, +\infty) \times \text{Lip}_\alpha \times C_+(\Omega) \rightarrow \text{Lip}_\alpha \times C_+(\Omega)$ and its corresponding blow-up time $T_{BU} : \text{Lip}_\alpha \times C_+(\Omega) \rightarrow (0, +\infty]$ such that for each initial distribution $(\varphi, \tau_0) \in \text{Lip}_\alpha \times C_+(\Omega)$, there exists a unique solution $A : (-\infty, T_{BU}(\varphi, \tau_0)) \rightarrow C_+(\Omega)$ and $\tau : [0, T_{BU}(\varphi, \tau_0)) \rightarrow C_+(\Omega)$ of (1.1) satisfying*

$$\mathcal{U}(t)(\varphi, \tau_0)(x) = (A_t(\cdot, x), \tau(t, x)), \forall t \in [0, T_{BU}(\varphi, \tau_0)), \forall x \in \Omega.$$

Moreover if $T_{BU}(\varphi, \tau_0) < +\infty$, then

$$\limsup_{t \nearrow T_{BU}(W_0)} \|A(t, \cdot)\|_\infty = +\infty.$$

Furthermore the semiflow \mathcal{U} has the following properties:

- (i) *The map T_{BU} is lower semi-continuous and $D_{\mathcal{U}}$ is relatively open in $[0, +\infty) \times \text{Lip}_\alpha \times C_+(\Omega)$.*
- (ii) *The semiflow \mathcal{U} is locally uniformly state variable continuous in $\text{Lip}_\alpha \times C_+(\Omega)$.*

In the sequel we will use the notation

$$BUC_\alpha := \{\phi \in C((-\infty, 0], C(\Omega)) : \phi_\alpha \in BUC((-\infty, 0], C(\Omega))\}, \alpha \geq 0$$

where

$$\phi_\alpha(t, x) := e^{-\alpha|t|} \phi(t, x)$$

and $BUC((-\infty, 0], C(\Omega))$ denotes the space of bounded uniformly continuous maps from $(-\infty, 0]$ to $C(\Omega)$. The space BUC_α is again a Banach space endowed with the norm

$$\|\phi\|_{BUC_\alpha} = \sup_{t \leq 0} \|\phi_\alpha(t, \cdot)\|_\infty.$$

We will also use the notation

$$BUC_\alpha^1 := \{\phi \in C^1((-\infty, 0], C(\Omega)) : \phi_\alpha \in BUC((-\infty, 0], C(\Omega)) \text{ and } \partial_t(\phi_\alpha) \in BUC((-\infty, 0], C(\Omega))\}$$

and the space BUC_α^1 is again a Banach space endowed with the norm

$$\|\phi\|_{BUC_\alpha^1} := \|\phi_\alpha\|_\infty + \|\partial_t(\phi_\alpha)\|_\infty = \|\phi_\alpha\|_\infty + \|\phi_\alpha\|_{\text{Lip}}.$$

Now we consider the following set D_α containing the couple (ϕ, τ_0) satisfying a compatibility condition:

$$D_\alpha := \{(\phi, \tau_0) \in BUC_\alpha^1 \times C_+(\Omega) : \partial_t \phi(0, x) = F(\phi(0, \cdot), \tau_0(\cdot), \phi(-\tau_0(\cdot), \cdot))(x), \forall x \in \Omega\}.$$

One can note that D_α is a closed subset of $BUC_\alpha^1 \times C_+(\Omega)$. Therefore D_α is a complete metric space endowed with the distance

$$d_{D_\alpha} \left((\phi, \tau_0), (\widehat{\phi}, \widehat{\tau}_0) \right) := \|\phi - \widehat{\phi}\|_{BUC_\alpha^1} + \|\tau_0 - \widehat{\tau}_0\|_\infty.$$

We also have

$$D_\alpha \subset BUC_\alpha^1 \times C_+(\Omega) \subset \text{Lip}_\alpha \times C_+(\Omega),$$

and the topology of $BUC_\alpha^1 \times C_+(\Omega)$ and $\text{Lip}_\alpha \times C_+(\Omega)$ coincide on D_α . We have the following results.

Theorem 1.7 *The sub-domain D_α is dense in $BUC_\alpha \times C_+(\Omega)$, namely*

$$\overline{D_\alpha}^{BUC_\alpha \times C(\Omega)} = BUC_\alpha \times C_+(\Omega).$$

Moreover we have following properties:

- (i) *The subdomain D_α is positively invariant by the semiflow \mathcal{U} , that is to say that for each $(\varphi, \tau_0) \in D_\alpha$,*

$$\mathcal{U}(t)(\varphi, \tau_0) \in D_\alpha, \forall t \in [0, T_{BU}(\varphi, \tau_0)).$$

- (ii) *The semiflow \mathcal{U} restricted to D_α is a continuous semiflow when D_α is endowed with the metric d_{D_α} .*

Particularly, from (ii), we know that we can choose two different state space for A_t (Lip_α or BUC_α^1), but only in the case of BUC_α^1 can we get a continuous (in time) semiflow.

In system (1.1), we can see from the second equation that the delay $\tau(t, x)$ is a solution of an integral equation. In the following (Lemma 3.2) we will see that the delay $\tau(t, x)$ can be seen as the solution of a partial differential equation, too. In Lemma 3.6, we will see that the delay $\tau(t, x)$ can be also regarded as a functional of A_t and (φ, τ_0) , which shows that it is actually a state-dependent delay. Specifically speaking, let $\delta_0 \in C_+(\Omega)$ be fixed, then we can define the map $\tau : D(\tau) \subset \text{Lip}_\alpha \rightarrow [0, +\infty)$ as the solution of

$$\int_{-\tau(\phi, x)}^0 f(\phi(s, \cdot))(x) ds = \delta_0(x) \quad (1.5)$$

with

$$D(\tau) = \left\{ \phi \in \text{Lip}_\alpha : \delta_0(x) < \int_{-\infty}^0 f(\phi(s, \cdot))(x) ds, \forall x \in \Omega \right\}.$$

Then we will see that

$$\tau(A_t, x) = \tau(t, x), \forall t \geq 0,$$

and the first equation in (1.1) can be rewritten as

$$\partial_t A(t, x) = F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(A_t, \cdot), \cdot))(x), \forall t \geq 0.$$

State-dependent delay differential equations have been used in the study of population dynamics of species [1, 2, 6, 7]. We refer in addition to [3, 5] and the references therein for a nice survey on this topic. Moreover, the semiflow properties of a general class of state-dependent delay differential equations have been recently studied by Walther [16] in D_α .

As an illustration, let us consider for example the following system

$$\begin{cases} \partial_t A(t, x) = F(A(t, \cdot))(x), \forall t \geq 0, \forall x \in \Omega, \\ \int_{-\tau(t, x)}^0 f(A(t + s, \cdot))(x) ds = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \forall t \geq 0, \forall x \in \Omega, \end{cases}$$

and the map $F : BUC_\alpha^1 \rightarrow \mathbb{R}$ is defined by

$$F(\varphi) := \varphi(-\tau(\varphi))$$

where $\tau(\varphi)$ is defined as above in (1.5). Assume in addition that f is continuously differentiable, then by Lemma 3.4, the state-dependent delay $\tau : BUC_\alpha \rightarrow C(\Omega)$ is C^1 . Then for $\varphi_0 \in BUC_\alpha^1$, we have

$$\begin{aligned} F(\psi + \varphi_0) - F(\varphi_0) &= (\psi + \varphi_0)(-\tau(\psi + \varphi_0)) - \varphi_0(-\tau(\varphi_0)) \\ &= \psi(-\tau(\psi + \varphi_0)) + \varphi_0(-\tau(\psi + \varphi_0)) - \varphi_0(-\tau(\varphi_0)), \end{aligned}$$

from which we deduce the derivative

$$DF(\varphi_0)\psi = \psi(-\tau(\varphi_0)) + \varphi_0'(-\tau(\varphi_0)) \cdot \partial_\varphi \tau(\varphi_0)\psi,$$

which satisfies the assumption (E) in Walther [16].

In this article, we consider the pair $(A_t, \tau(t, \cdot))$ as the state variable, and in this case we can also apply the result by Walther in [16] to the delay differential equation

$$\begin{cases} \partial_t A(t, x) = F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(t, \cdot), \cdot))(x), \\ \partial_t \tau(t, x) = 1 - \frac{f(A(t, \cdot))(x)}{f(A(t - \tau(t, \cdot), \cdot))(x)}. \end{cases} \quad (1.6)$$

Nevertheless the existence of a maximal semiflow as well as the blow-up time has been considered by Walther [16].

The article is organized as follows. In section 2 we prove that D_α is dense in $BUC_\alpha \times C_+(\Omega)$. In section 3 we prove some results regarding the delay $\tau(t, x)$. In sections 4 and 5 we will investigate the uniqueness and local existence of solutions, and the properties of semiflows. In the last section of the article, we will illustrate our results by proving the global existence of solutions for a spatially structured forest model.

2 Density of the domain

In this preliminary section we will prove the first result of Theorem 1.7, namely the density of D_α in the space $BUC_\alpha \times C_+(\Omega)$.

Proof. Fix $\tau_0 \in C_+(\Omega)$. Consider the space

$$\mathcal{X} := C(\Omega) \times BUC_\alpha$$

which is a Banach space endowed with the usual product norm. Define the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$ by

$$\mathcal{A} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} := \begin{pmatrix} -\partial_t \varphi(0, \cdot) \\ \partial_t \varphi \end{pmatrix}, \forall \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} \in D(\mathcal{A}),$$

with

$$D(\mathcal{A}) := \{0_{C(\Omega)}\} \times BUC_\alpha^1.$$

Then it is not difficult to prove that

$$\overline{D(\mathcal{A})} = \{0_{C(\Omega)}\} \times BUC_\alpha. \quad (2.1)$$

Moreover, the linear operator \mathcal{A} is a Hille-Yosida operator (see [8]). More precisely, we have $(0, \infty) \subset \rho(\mathcal{A})$ and for each $\lambda \in (0, \infty)$,

$$\begin{aligned} (\lambda I - \mathcal{A})^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} &= \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \\ \Leftrightarrow \psi(\theta, x) &= \frac{1}{\lambda} e^{\lambda \theta} [\alpha + \varphi(0, x)] + \int_\theta^0 e^{\lambda(\theta-l)} \varphi(l, x) dl. \end{aligned}$$

The linear operator \mathcal{A} is Hille-Yosida since we have the following estimation from [8]

$$\|(\lambda I - \mathcal{A})^{-n}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{1}{\lambda^n}, \forall n \geq 1, \forall \lambda > 0. \quad (2.2)$$

By using (2.1) and (2.2) and by the fact that

$$\lambda(\lambda I - \mathcal{A})^{-1} - I = \mathcal{A}(\lambda I - \mathcal{A})^{-1},$$

it follows that for each $\psi \in BUC_\alpha$,

$$\lim_{\lambda \rightarrow +\infty} \left\| \lambda(\lambda I - \mathcal{A})^{-1} \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} - \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \right\|_{\mathcal{X}} = 0. \quad (2.3)$$

We define the nonlinear map $\mathcal{F} : \overline{D(\mathcal{A})} \rightarrow \mathcal{X}$,

$$\mathcal{F} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} := \begin{pmatrix} F(\varphi(0, \cdot), \tau_0(\cdot), \varphi(-\tau_0(\cdot), \cdot)) \\ 0_{BUC_\alpha} \end{pmatrix}, \forall \varphi \in BUC_\alpha.$$

We observe that

$$\begin{aligned} & (\varphi, \tau_0) \in D_\alpha \\ \Leftrightarrow & (\mathcal{A} + \mathcal{F}) \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} \in \overline{D(\mathcal{A})} \text{ with } \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} \in D(\mathcal{A}) \\ \Leftrightarrow & (I - \lambda\mathcal{A} - \lambda\mathcal{F}) \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} \in \overline{D(\mathcal{A})} \text{ with } \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} \in D(\mathcal{A}), \forall \lambda > 0. \end{aligned}$$

Let $\begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \in \{0_{C(\Omega)}\} \times BUC_\alpha$ be fixed. Then for each $\lambda > 0$, consider

$$(I - \lambda\mathcal{A} - \lambda\mathcal{F}) \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} = \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \text{ with } \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} \in D(\mathcal{A}),$$

which is equivalent to the fixed point problem

$$\begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} = \lambda^{-1} (\lambda^{-1}I - \mathcal{A})^{-1} \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} + (\lambda^{-1}I - \mathcal{A})^{-1} \mathcal{F} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix}.$$

Define the map

$$\Phi_\lambda \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix} := \lambda^{-1} (\lambda^{-1}I - \mathcal{A})^{-1} \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} + (\lambda^{-1}I - \mathcal{A})^{-1} \mathcal{F} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi \end{pmatrix}.$$

Then $r > 0$ being fixed, by using the fact that \mathcal{F} is Lipschitz on bounded sets and \mathcal{A} is a Hille-Yosida operator, one can prove that there exists $\eta = \eta(r) > 0$ such that

$$\Phi_\lambda(B_{\psi,r}) \subset B_{\psi,r}, \forall \lambda \in (0, \eta]$$

and Φ_λ is a strict contraction on $B_{\psi,r}$, where

$$B_{\psi,r} := B \left(\begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix}, r \right)$$

is the ball with center $\begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix}$ and radius r in $\overline{D(\mathcal{A})} = \{0_{C(\Omega)}\} \times BUC_\alpha$. Thus

by the Banach fixed point theorem, $\forall \lambda \in (0, \eta]$, there exists $\begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} \in B_{\psi,r}$ satisfying

$$\Phi_\lambda \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} = \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix}.$$

Finally, since $\begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \in \overline{D(\mathcal{A})}$ and by using (2.2) and (2.3), we have

$$\lim_{\lambda \rightarrow 0^+} \left\| \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} - \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} \right\|_{\mathcal{X}}$$

$$\begin{aligned}
&= \lim_{\lambda \rightarrow 0^+} \left\| (\lambda^{-1} (\lambda^{-1} I - \mathcal{A})^{-1} - I) \begin{pmatrix} 0_{C(\Omega)} \\ \psi \end{pmatrix} + (\lambda^{-1} I - \mathcal{A})^{-1} \mathcal{F} \begin{pmatrix} 0_{C(\Omega)} \\ \varphi_\lambda \end{pmatrix} \right\|_{\mathcal{X}} \\
&= 0,
\end{aligned}$$

which completes the proof. \blacksquare

3 Properties of the integral equation for $\tau(t, x)$

In this section we will make the following assumption.

Assumption 3.1 *Let $(\varphi, \tau_0) \in C((-\infty, 0], C(\Omega)) \times C_+(\Omega)$. Let $A \in C((-\infty, r), C(\Omega))$ (with $r \in (0, +\infty]$) be given and satisfy*

$$A(t, \cdot) = \varphi(t, \cdot), \forall t \leq 0.$$

Lemma 3.2 *There exists a uniquely determined map $\tau : [0, r) \rightarrow C(\Omega)$ satisfying*

$$\int_{t-\tau(t,x)}^t f(A(s, \cdot))(x) ds = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \forall t \in [0, r), \forall x \in \Omega. \quad (3.1)$$

Moreover this uniquely determined map $t \mapsto \tau(t, x)$ is continuously differentiable and satisfies the following equation

$$\begin{cases} \partial_t \tau(t, x) = 1 - \frac{f(A(t, \cdot))(x)}{f(A(t - \tau(t, x), \cdot))(x)}, \forall t \in [0, r), \forall x \in \Omega, \\ \tau(0, x) = \tau_0(x). \end{cases} \quad (3.2)$$

Conversely if $t \mapsto \tau(t, x)$ is a C^1 map satisfying the above ordinary differential equation (3.2), then it also satisfies the above integral equation (3.1).

Remark 3.3 *By using equation (3.2), it is easy to check that*

$$\tau_0(x) > 0 \Rightarrow \tau(t, x) > 0, \forall t \in [0, r), \forall x \in \Omega,$$

and

$$\tau_0(x) = 0 \Rightarrow \tau(t, x) = 0, \forall t \in [0, r), \forall x \in \Omega.$$

Proof. Step 1 (Existence of $\tau(t, x)$): By Assumption 1.1, f is strictly positive, so fix $t \in [0, r)$ and $x \in \Omega$, and by considering the function $\tau \mapsto \int_{t-\tau}^t f(A(s, \cdot))(x) ds$ and observing that

$$\begin{aligned}
\int_{t-0}^t f(A(s, \cdot))(x) ds &= 0 \leq \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \\
\int_{t-(t+\tau_0(x))}^t f(A(s, \cdot))(x) ds &\geq \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds,
\end{aligned}$$

it follows by the intermediate value theorem that there exists a unique $\tau(t, x) \in [0, t + \tau_0(x)]$ satisfying (3.1).

Step 2 (The map $t \mapsto t - \tau(t, x)$ is increasing): First we prove that the function $t \mapsto t - \tau(t, x)$ is increasing. Indeed, assume by contradiction that $t_1 \leq t_2$ while $t_1 - \tau(t_1, x) \geq t_2 - \tau(t_2, x)$, $\forall x \in \Omega$, namely we have

$$t_2 - \tau(t_2, x) \leq t_1 - \tau(t_1, x) \leq t_1 \leq t_2.$$

Then by (3.1) we have

$$\begin{aligned} & \int_{t_1 - \tau(t_1, x)}^{t_1} f(A(s, \cdot))(x) ds = \int_{t_2 - \tau(t_2, x)}^{t_2} f(A(s, \cdot))(x) ds \\ = & \int_{t_2 - \tau(t_2, x)}^{t_1 - \tau(t_1, x)} f(A(s, \cdot))(x) ds + \int_{t_1 - \tau(t_1, x)}^{t_1} f(A(s, \cdot))(x) ds + \int_{t_1}^{t_2} f(A(s, \cdot))(x) ds, \end{aligned}$$

thus

$$\int_{t_2 - \tau(t_2, x)}^{t_1 - \tau(t_1, x)} f(A(s, \cdot))(x) ds + \int_{t_1}^{t_2} f(A(s, \cdot))(x) ds = 0,$$

which is impossible since the function f is strictly positive.

Step 3 (The continuity of the map $x \mapsto \tau(t, x)$): Next we will prove the continuity of the map $x \mapsto \tau(t, x)$. By step 2 we have for each $t \in [0, r)$,

$$t - \tau(t, x) \leq -\tau_0(x) \Leftrightarrow 0 \leq \tau(t, x) \leq t - \tau_0(x), \forall x \in \Omega.$$

Now the boundness of the function $x \mapsto \tau(t, x)$ follows from the boundedness of $\tau_0(x)$. Then for any $t \in [0, r)$, we know that

$$\tau^\infty(t) := \sup_{x \in \Omega} \tau(x, t) < +\infty.$$

Let $\xi(x) := \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds$, $\forall x \in \Omega$. Then fix $x_0 \in \Omega$, for any $x \in \Omega$, we have

$$\begin{aligned} |\xi(x_0) - \xi(x)| &= \left| \int_{-\tau_0(x_0)}^0 f(\varphi(s, \cdot))(x_0) ds - \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds \right| \\ &\leq \left| \int_{-\tau_0(x_0)}^{-\tau_0(x)} f(\varphi(s, \cdot))(x_0) ds \right| + \int_{-\tau_0(x)}^0 |f(\varphi(s, \cdot))(x_0) - f(\varphi(s, \cdot))(x)| ds \\ &\leq |\tau_0(x_0) - \tau_0(x)| f(m_1)(x_0) + \tau_0^\infty \sup_{s \in [-\tau_0^\infty, 0]} |f(\varphi(s, \cdot))(x_0) - f(\varphi(s, \cdot))(x)|, \end{aligned}$$

where m_1 is the constant function defined with the constant value

$$m_1(x) = \inf_{s \in [-\tau_0^\infty, 0]} \|\varphi(s, \cdot)\|_\infty, \forall x \in \Omega$$

and $\tau_0^\infty := \sup_{x \in \Omega} \tau_0(x)$. Then the continuity of $\xi(x)$ in x follows from

$$\lim_{x \rightarrow x_0} |\xi(x_0) - \xi(x)| = 0$$

by the continuity of $\tau_0(x)$ in x and $f(\varphi)$ in φ . Now for each $t \in [0, r)$ fixed, by (3.1) we have

$$\begin{aligned} \xi(x_0) - \xi(x) &= \int_{t-\tau(t, x_0)}^t f(A(s, \cdot))(x_0) ds - \int_{t-\tau(t, x)}^t f(A(s, \cdot))(x) ds \\ &= \int_{t-\tau(t, x_0)}^{t-\tau(t, x)} f(A(s, \cdot))(x_0) ds + \int_{t-\tau(t, x)}^t (f(A(s, \cdot))(x_0) - f(A(s, \cdot))(x)) ds, \end{aligned}$$

thus

$$\begin{aligned} \int_{t-\tau(t, x_0)}^{t-\tau(t, x)} f(A(s, \cdot))(x_0) ds &\leq |\xi(x_0) - \xi(x)| + \tau^\infty(t) \sup_{s \in [t-\tau^\infty(t), t]} |f(A(s, \cdot))(x_0) \\ &\quad - f(A(s, \cdot))(x)|. \end{aligned}$$

On the other hand, we have

$$\int_{t-\tau(t, x_0)}^{t-\tau(t, x)} f(A(s, \cdot))(x_0) ds \geq |\tau(t, x_0) - \tau(t, x)| f(M_1)(x_0),$$

where M_1 is the constant function defined by

$$M_1(x) := \sup_{s \in [-\tau_0^\infty, t]} \|A(s, \cdot)\|_\infty, \forall x \in \Omega.$$

Then there exists a constant $\eta := \eta(t, x_0)$ such that

$$|\tau(t, x_0) - \tau(t, x)| \leq \eta \left(|\xi(x_0) - \xi(x)| + \sup_{s \in [t-\tau^\infty(t), t]} |f(A(s, \cdot))(x_0) - f(A(s, \cdot))(x)| \right).$$

Then the continuity of $\tau(t, x)$ in x follows from

$$\lim_{x \rightarrow x_0} |\tau(t, x_0) - \tau(t, x)| = 0$$

by the continuity of $\xi(x)$ in x and $f(A)$ in A .

Step 4 (Differentiability of the map $t \mapsto \tau(t, x)$): By applying the implicit function theorem to the map $\psi : (0, r) \times C(\Omega) \rightarrow C(\Omega)$ defined by

$$\psi(t, \gamma)(x) = \int_{\gamma(x)}^t f(A(s, \cdot))(x) ds - \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds$$

(which is possible since $\frac{\partial \psi(t, \gamma)}{\partial \gamma}(\hat{\gamma})(x) = -f(A(\gamma(x), \cdot))(x)\hat{\gamma}(x)$ and by Assumption 1.1, f is strictly positive), we deduce that $t \mapsto t - \tau(t, x)$ is continuously

differentiable on $(0, r)$. Since the above formula of the derivative is also valid at $t = 0$ and $t = r$, the map $t \mapsto t - \tau(t, x)$ is continuously differentiable on $[0, r]$. By calculating the time derivative on both sides of (3.1), we get that $\tau(t, x)$ is a solution of (3.2).

Step 5 ((3.2) \Rightarrow (3.1)): Conversely, assume that $\tau(t, x)$ is a solution of (3.2). Then

$$f(A(t, \cdot))(x) = \left(1 - \frac{\partial \tau(t, x)}{\partial t}\right) f(A(t - \tau(t, x), \cdot))(x), \forall t \in [0, r], \forall x \in \Omega.$$

Integrating both sides with respect to t , we have

$$\int_0^t f(A(s, \cdot))(x) ds = \int_0^t f(A(s - \tau(s, x), \cdot))(x) \left(1 - \frac{\partial \tau(s, x)}{\partial s}\right) ds.$$

Make the change of variable $l = s - \tau(s, x)$, we have $\forall t \in [0, r], \forall x \in \Omega$,

$$\begin{aligned} \int_0^t f(A(s, \cdot))(x) ds &= \int_{-\tau_0(x)}^{t - \tau(t, x)} f(A(l, \cdot))(x) dl \\ \Leftrightarrow \int_{t - \tau(t, x)}^t f(A(s, \cdot))(x) ds + \int_0^{t - \tau(t, x)} f(A(s, \cdot))(x) ds &= \int_{-\tau_0(x)}^{t - \tau(t, x)} f(A(s, \cdot))(x) ds \\ \Leftrightarrow \int_{t - \tau(t, x)}^t f(A(s, \cdot))(x) ds &= \int_{-\tau_0(x)}^{t - \tau(t, x)} f(A(s, \cdot))(x) ds + \int_{t - \tau(t, x)}^0 f(A(s, \cdot))(x) ds \\ \Leftrightarrow \int_{t - \tau(t, x)}^t f(A(s, \cdot))(x) ds &= \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \end{aligned}$$

so $\tau(t, x)$ also satisfies the equation (3.1). \blacksquare

In order to see that the delay τ is also a functional of A_t and (φ, τ_0) , we define the following functional. We define the map $\hat{\tau} : D(\hat{\tau}) \subset BUC_\alpha \times C(\Omega) \rightarrow C(\Omega)$ as the solution of

$$\int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\bar{\phi}(s, \cdot))(x) ds = \delta(x) \quad (3.3)$$

where

$$\bar{\phi}(s, x) := \begin{cases} \phi(s, x), & \text{if } s \leq 0 \\ \phi(0, x), & \text{if } s \geq 0. \end{cases}$$

Since by Assumption 1.1 the map $f(\phi(0, \cdot))(x) > 0$, then if $\delta(x) \leq 0$ we have $\hat{\tau}(\phi, \delta)(x) \leq 0$ and

$$\int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\phi(0, \cdot))(x) ds = \delta(x) \Leftrightarrow \hat{\tau}(\phi, \delta)(x) = \frac{\delta(x)}{f(\phi(0, \cdot))(x)}.$$

We define the domain $D(\hat{\tau})$ by

$$D(\hat{\tau}) = \left\{ (\phi, \delta) \in BUC_\alpha \times C(\Omega) : \delta(x) < \int_{-\infty}^0 f(\phi(s, \cdot))(x) ds \text{ if } \delta(x) > 0 \right\}.$$

For clarity we prove the following lemma.

Lemma 3.4 For each $(\phi, \delta) \in D(\hat{\tau})$ there exists $\hat{\tau}(\phi, \delta) \in C(\Omega)$.

Proof. Let $(\phi, \delta) \in D(\hat{\tau})$ be fixed.

Step 1 (Existence of $\hat{\tau}(\phi, \delta)(x)$): Let $x \in \Omega$ be fixed. If $\delta(x) \leq 0$, we have

$$\hat{\tau}(\phi, \delta)(x) = \frac{\delta(x)}{f(\phi(0, \cdot))(x)}.$$

If $\delta(x) > 0$, by the definition of the domain $D(\hat{\tau})$ we have

$$\delta(x) < \int_{-\infty}^0 f(\phi(s, \cdot))(x) ds,$$

therefore by the intermediate value theorem, we can find $\hat{\tau}(\phi, \delta)(x) \in \mathbb{R}$ such that

$$\delta(x) = \int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\phi(s, \cdot))(x) ds.$$

Step 2 (Boundedness of $\hat{\tau}(\phi, \delta)(x)$): Assume by contradiction that $x \mapsto \hat{\tau}(\phi, \delta)(x)$ is unbounded. Since Ω is compact, we can find a converging sequence $x_n \rightarrow \bar{x} \in \Omega$ as $n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow +\infty} \hat{\tau}(\phi, \delta)(x_n) = +\infty.$$

It is sufficient to consider the case $\delta(x_n) > 0$, since the case $\delta(x_n) \leq 0$ is explicit. By the continuity of the function δ we can assume that $\delta(\bar{x}) \geq 0$. By the definition of the domain $D(\hat{\tau})$ we have

$$\delta(\bar{x}) < \int_{-\infty}^0 f(\phi(s, \cdot))(\bar{x}) ds.$$

So we can find a constant $M > 0$ such that

$$\delta(\bar{x}) < \int_{-M}^0 f(\phi(s, \cdot))(\bar{x}) ds,$$

and by continuity we can find a neighborhood U of \bar{x} such that

$$\delta(x) < \int_{-M}^0 f(\phi(s, \cdot))(x) ds, \forall x \in U.$$

It follows that for all integer n large enough,

$$\hat{\tau}(\phi, \delta)(x_n) \leq M$$

a contradiction.

Step 3 (Continuity of the map $x \mapsto \hat{\tau}(\phi, \delta)(x)$): From the previous part, we know that

$$\hat{\tau}^\infty := \sup_{x \in \Omega} |\hat{\tau}(\phi, \delta)(x)| < +\infty.$$

Fix $x_0 \in \Omega$. If $\delta(x_0) < 0$ there is nothing to prove. Let us assume that

$$\delta(x_0) \geq 0.$$

Let $x_n \rightarrow x_0$ be a converging sequence. If $\delta(x_0) = 0$ and there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ satisfying $\delta(x_{n_k}) < 0, \forall k \in \mathbb{N}$, then it is clear that $\hat{\tau}(\phi, \delta)(x_{n_k}) \rightarrow \hat{\tau}(\phi, \delta)(x_0) = 0$. Assume that $\delta(x_n) \geq 0$ for each integer $n \geq 0$. By (3.3) we have

$$\begin{aligned} \delta(x_0) - \delta(x) &= \int_{-\hat{\tau}(\phi, \delta)(x_0)}^0 f(\bar{\phi}(s, \cdot))(x_0) ds - \int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\bar{\phi}(s, \cdot))(x) ds \\ &= \int_{-\hat{\tau}(\phi, \delta)(x_0)}^{-\hat{\tau}(\phi, \delta)(x)} f(\bar{\phi}(s, \cdot))(x_0) ds + \int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\bar{\phi}(s, \cdot))(x_0) ds \\ &\quad - \int_{-\hat{\tau}(\phi, \delta)(x)}^0 f(\bar{\phi}(s, \cdot))(x) ds. \end{aligned}$$

Assume without loss of generality that $\hat{\tau}(\phi, \delta)(x_0) > \hat{\tau}(\phi, \delta)(x)$, then we have

$$\begin{aligned} &\int_{-\hat{\tau}(\phi, \delta)(x_0)}^{-\hat{\tau}(\phi, \delta)(x)} f(\bar{\phi}(s, \cdot))(x_0) ds \\ &= (\delta(x_0) - \delta(x)) + \int_{-\hat{\tau}(\phi, \delta)(x)}^0 (f(\bar{\phi}(s, \cdot))(x) - f(\bar{\phi}(s, \cdot))(x_0)) ds \\ &\leq |\delta(x_0) - \delta(x)| + \hat{\tau}^\infty \sup_{s \in [-\hat{\tau}^\infty, 0]} |f(\bar{\phi}(s, \cdot))(x) - f(\bar{\phi}(s, \cdot))(x_0)|, \end{aligned}$$

and

$$\int_{-\hat{\tau}(\phi, \delta)(x_0)}^{-\hat{\tau}(\phi, \delta)(x)} f(\bar{\phi}(s, \cdot))(x_0) ds \geq |\hat{\tau}(\phi, \delta)(x_0) - \hat{\tau}(\phi, \delta)(x)| f(M_1)(x_0),$$

where M_1 is the constant function assigned with the single value $\sup_{s \in [-\hat{\tau}^\infty, 0]} \|\phi(s, \cdot)\|_\infty$.

Thus there exists a constant $\eta := \eta(x_0)$ such that

$$|\hat{\tau}(\phi, \delta)(x_0) - \hat{\tau}(\phi, \delta)(x)| \leq \eta \left(|\delta(x_0) - \delta(x)| + \sup_{s \in [-\hat{\tau}^\infty, 0]} |f(\bar{\phi}(s, \cdot))(x) - f(\bar{\phi}(s, \cdot))(x_0)| \right).$$

Then the result follows by the continuity of the functions δ and $f(\phi)$ in x . ■

Lemma 3.5 *Assume in addition that f is continuously differentiable. Then the domain $D(\hat{\tau})$ is an open subset of $BUC_\alpha \times C(\Omega)$ and the map $\hat{\tau} : D(\hat{\tau}) \subset BUC_\alpha \times C(\Omega) \rightarrow C(\Omega)$ is continuously differentiable.*

Proof. Define the map $\Gamma : BUC_\alpha \times C(\Omega) \times C(\Omega) \rightarrow C(\Omega)$ by

$$\Gamma(\phi, \delta, \gamma)(x) := \int_{-\gamma(x)}^0 f(\bar{\phi}(s, \cdot))(x) ds - \delta(x).$$

Since by Assumption 1.1 the map f is C^1 , so is the map Γ . By (3.3), we have $\Gamma(\phi, \delta, \widehat{\tau}(\phi, \delta))(x) = 0$ and

$$\partial_\gamma \Gamma(\phi, \delta, \gamma)(\widehat{\gamma})(x) = f(\overline{\phi}(-\gamma(x), \cdot))\widehat{\gamma}(x).$$

Since f is strictly positive, it follows that $\partial_\gamma \Gamma(\phi, \delta, \gamma)$ is invertible. The result follows by applying the implicit function theorem. \blacksquare

Lemma 3.6 *Set*

$$\delta_0(x) := \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds, \forall x \in \Omega.$$

Then we have the following equality

$$\widehat{\tau}(A_t, \delta_0)(x) = \tau(t, x), \forall t \in (0, r),$$

where A is given by Assumption 3.1 and $\tau(t, x)$ is the solution of (3.1).

Proof. It is sufficient to observe that $\forall t \in [0, r)$ and $x \in \Omega$,

$$\begin{aligned} & \int_{-\widehat{\tau}(A_t, \delta_0)(x)}^0 f(A_t(s, \cdot))(x) ds = \delta_0(x) = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds \\ & = \int_{-\tau(t, x)}^0 f(A(t + s, \cdot))(x) ds \end{aligned}$$

\blacksquare

For simplicity, we will write $\tau(\phi, x)$ instead of $\widehat{\tau}(\phi, \delta_0)(x)$ if the function δ_0 is defined as in Lemma 3.6.

Lemma 3.7 *Let $\phi, \widehat{\phi} \in BUC_\alpha$. If $\phi \leq \widehat{\phi}$, namely $\phi(s, x) \leq \widehat{\phi}(s, x), \forall s \leq 0, x \in \Omega$, then*

$$\tau(\phi, x) \leq \tau(\widehat{\phi}, x), \forall x \in \Omega.$$

Proof. By (3.3) we have

$$\int_{-\tau(\phi, x)}^0 f(\phi(s, \cdot))(x) ds = \delta_0(x) = \int_{-\tau(\widehat{\phi}, x)}^0 f(\widehat{\phi}(s, \cdot))(x) ds.$$

Assume by contradiction that there exists $x \in \Omega$ such that $\tau(\phi, x) > \tau(\widehat{\phi}, x)$, then

$$\begin{aligned} 0 &= \int_{-\tau(\phi, x)}^0 f(\phi(s, \cdot))(x) ds - \int_{-\tau(\widehat{\phi}, x)}^0 f(\widehat{\phi}(s, \cdot))(x) ds \\ &= \int_{-\tau(\phi, x)}^{-\tau(\widehat{\phi}, x)} f(\phi(s, \cdot))(x) ds + \int_{-\tau(\widehat{\phi}, x)}^0 [f(\phi(s, \cdot)) - f(\widehat{\phi}(s, \cdot))](x) ds. \end{aligned}$$

Since by assumption $f(\phi(s, \cdot))(x) > 0$ and $\tau(\phi, x) > \tau(\widehat{\phi}, x)$, we have

$$\int_{-\tau(\phi, x)}^{-\tau(\widehat{\phi}, x)} f(\phi(s, \cdot))(x) ds > 0.$$

By assumption we also have $f(\phi(s, \cdot))(x) \geq f(\widehat{\phi}(s, \cdot))(x)$ when $\tau(\phi, x) > \tau(\widehat{\phi}, x)$, then

$$\int_{-\tau(\widehat{\phi}, x)}^0 [f(\phi(s, \cdot)) - f(\widehat{\phi}(s, \cdot))](x) ds \geq 0.$$

Then we have

$$0 = \int_{-\tau(\phi, x)}^{-\tau(\widehat{\phi}, x)} f(\phi(s, \cdot))(x) ds + \int_{-\tau(\widehat{\phi}, x)}^0 [f(\phi(s, \cdot)) - f(\widehat{\phi}(s, \cdot))](x) ds > 0 + 0 = 0,$$

which is a contradiction. \blacksquare

In the following lemma we obtain some a priori estimates for the delay.

Lemma 3.8 *Let Assumption 3.1 be satisfied. Assume that there exists a constant $M > 0$ such that*

$$\sup_{t \in [0, r]} \|A(t, \cdot)\|_\infty \leq M. \quad (3.4)$$

Then

$$\tau_{\min} \leq \tau(A_t, x) \leq \tau_{\max}, \forall t \in [0, r], \forall x \in \Omega,$$

where the constants τ_{\min} and τ_{\max} are defined as follows:

$$0 \leq \tau_{\min} := \frac{\inf_{x \in \Omega} [\tau_0(x) f(\varphi_{\max})(x)]}{\sup_{x \in \Omega} f(-M_1)(x)} \leq \tau_{\max} := \frac{\sup_{x \in \Omega} [\tau_0(x) f(-\varphi_{\max})(x)]}{\inf_{x \in \Omega} f(M_1)(x)}$$

with M_1 and φ_{\max} being defined as constant functions as follows

$$M_1(x) := \max\{M, \varphi_{\max}\} \text{ and } \varphi_{\max}(x) := \sup_{t \in [-\tau_0^\infty, 0]} \|\varphi(t, \cdot)\|_\infty, \forall x \in \Omega \quad (3.5)$$

and

$$\tau_0^\infty := \sup_{x \in \Omega} \tau_0(x).$$

Proof. For any $x \in \Omega$ and $t \in [0, r]$, since by Assumption 1.1, the map f is decreasing, it follows that

$$\begin{aligned} \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds &= \int_{-\tau(A_t, x)}^0 f(A(t+s, \cdot))(x) ds \\ &= \int_{t-\tau(A_t, x)}^t f(A(l, \cdot))(x) dl \\ &\leq \int_{t-\tau(A_t, x)}^t f(-M_1)(x) ds = \tau(A_t, x) \sup_{x \in \Omega} f(-M_1)(x). \end{aligned}$$

Then $\forall x \in \Omega, \forall t \in [0, r)$,

$$\tau(A_t, x) \geq \frac{\int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds}{\sup_{x \in \Omega} f(-M_1)(x)} \geq \frac{\inf_{x \in \Omega} [\tau_0(x) f(\varphi_{\max})(x)]}{\sup_{x \in \Omega} f(-M_1)(x)}.$$

The derivation of the estimation from above for $\tau(t, x)$ is similar. \blacksquare

Lemma 3.9 *Let $(\varphi, \tau_0), (\tilde{\varphi}, \tilde{\tau}_0), A, \tilde{A}$ satisfy Assumption 3.1. Assume that*

$$M := \max \left\{ \sup_{t \in [0, r)} \|A(t, \cdot)\|_{\infty}, \sup_{t \in [0, r)} \|\tilde{A}(t, \cdot)\|_{\infty} \right\} < +\infty.$$

Then there exists a constant $L_{\tau} > 0$ such that $\forall t \in [0, r), \forall x \in \Omega$,

$$|\hat{\tau}(A_t, \delta_0)(x) - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)| \leq L_{\tau} \left[\sup_{s \in [-\tilde{\tau}_0^{\infty}, r)} \|A(s, \cdot) - \tilde{A}(s, \cdot)\|_{\infty} + \|\delta_0 - \tilde{\delta}_0\|_{\infty} \right],$$

where δ_0 and $\tilde{\delta}_0$ are defined as in Lemma 3.6 respectively with (φ, τ_0) and $(\tilde{\varphi}, \tilde{\tau}_0)$ and

$$\tilde{\tau}_0^{\infty} := \max \left\{ \sup_{x \in \Omega} \tau_0(x), \sup_{x \in \Omega} \tilde{\tau}_0(x) \right\}.$$

Proof. Let $t \in [0, r)$ and $x \in \Omega$. Recall from Lemma 3.6 that

$$\delta_0(x) = \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds \text{ and } \tilde{\delta}_0(x) = \int_{-\tilde{\tau}_0(x)}^0 f(\tilde{\varphi}(s, \cdot))(x) ds, \forall x \in \Omega.$$

Without loss of generality we may assume that $\hat{\tau}(A_t, \delta_0)(x) \geq \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x) > 0$. Then we have

$$\begin{aligned} \delta_0(x) - \tilde{\delta}_0(x) &= \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds - \int_{-\tilde{\tau}_0(x)}^0 f(\tilde{\varphi}(s, \cdot))(x) ds \\ &= \int_{t - \hat{\tau}(A_t, \delta_0)(x)}^t f(A(s, \cdot))(x) ds - \int_{t - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)}^t f(\tilde{A}(s, \cdot))(x) ds \\ &= \int_{t - \hat{\tau}(A_t, \delta_0)(x)}^{t - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)} f(A(s, \cdot))(x) ds + \int_{t - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)}^t \left[f(A(s, \cdot))(x) - f(\tilde{A}(s, \cdot))(x) \right] ds. \end{aligned}$$

Since by Assumption 1.1, f is Lipschitz continuous, then

$$\begin{aligned} &\int_{t - \hat{\tau}(A_t, \delta_0)(x)}^{t - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)} f(A(s, \cdot))(x) ds \\ &= \int_{t - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)}^t \left[f(\tilde{A}(s, \cdot))(x) - f(A(s, \cdot))(x) \right] ds + (\delta_0(x) - \tilde{\delta}_0(x)) \end{aligned}$$

$$\begin{aligned}
&\leq \|f\|_{\text{Lip}} \int_{t-\hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)}^t \|\tilde{A}(s, \cdot) - A(s, \cdot)\|_{\infty} ds + \|\delta_0 - \tilde{\delta}_0\|_{\infty} \\
&\leq \bar{\tau}_{\max} \|f\|_{\text{Lip}} \sup_{s \in [-\bar{\tau}_0^{\infty}, r)} \|A(s, \cdot) - \tilde{A}(s, \cdot)\|_{\infty} + \|\delta_0 - \tilde{\delta}_0\|_{\infty}
\end{aligned}$$

where $\bar{\tau}_{\max} := \max\{\tau_{\max}, \tilde{\tau}_{\max}\}$ and $\tau_{\max}, \tilde{\tau}_{\max}$ are obtained in Lemma 3.8. On the other hand, we have

$$\int_{t-\hat{\tau}(A_t, \delta_0)(x)}^{t-\hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x)} f(A(s, \cdot))(x) ds \geq \left(\hat{\tau}(A_t, \delta_0)(x) - \hat{\tau}(\tilde{A}_t, \tilde{\delta}_0)(x) \right) \inf_{x \in \Omega} f(M_1)(x)$$

where M_1 is the constant function defined with the single value $\max\{\varphi_{\max}, \tilde{\varphi}_{\max}, M\}$ and $\varphi_{\max}, \tilde{\varphi}_{\max}$ are defined in (3.5) respectively with $\varphi, \tilde{\varphi}$. The result follows. \blacksquare

4 Existence and uniqueness of solutions

We start this section with two technical lemmas.

Lemma 4.1 *Let $a < b$ be two real numbers. Let $\chi \in \text{Lip}([a, b], C(\Omega))$. Then for each $c \in (a, b)$ we have the following estimation*

$$\|\chi\|_{\text{Lip}([a, b], C(\Omega))} \leq \|\chi\|_{\text{Lip}([a, c], C(\Omega))} + \|\chi\|_{\text{Lip}([c, b], C(\Omega))}.$$

where

$$\|\chi\|_{\text{Lip}(I, C(\Omega))} := \sup_{t, s \in I: t \neq s} \frac{\|\chi(t, \cdot) - \chi(s, \cdot)\|_{\infty}}{|t - s|}.$$

Proof. Let $t, s \in [a, b]$ with $t > s$. Define the function $\rho : [0, 1] \rightarrow \mathbb{R}$ by

$$\rho(h) := \|\chi((t-s)h + s, \cdot) - \chi(s, \cdot)\|_{\infty}, \forall h \in [0, 1].$$

Then we have $\forall h, \hat{h} \in [0, 1]$,

$$\begin{aligned}
|\rho(h) - \rho(\hat{h})| &\leq \left| \|\chi((t-s)h + s, \cdot) - \chi(s, \cdot)\|_{\infty} - \|\chi((t-s)\hat{h} + s, \cdot) - \chi(s, \cdot)\|_{\infty} \right| \\
&\leq \|\chi((t-s)h + s, \cdot) - \chi((t-s)\hat{h} + s, \cdot)\|_{\infty} \\
&\leq \|\chi\|_{\text{Lip}([a, b], C(\Omega))} |t-s| |h - \hat{h}|,
\end{aligned}$$

thus ρ is Lipschitz continuous. Denote

$$\text{Lip}(\rho)(h) := \limsup_{\varepsilon \rightarrow 0^+} \frac{\rho(h + \varepsilon) - \rho(h)}{\varepsilon},$$

then

$$\text{Lip}(\rho)(h) \leq \begin{cases} \|\chi\|_{\text{Lip}([a, c], C(\Omega))} |t-s|, & \text{if } (t-s)h + s \in [a, c], \\ \|\chi\|_{\text{Lip}([c, b], C(\Omega))} |t-s|, & \text{if } (t-s)h + s \in [c, b]. \end{cases}$$

Since ρ is Lipschitz continuous, by using Theorem 8.17 in page 158 of Rudin [11], we deduce that ρ is differentiable everywhere on a subset of the form $[0, 1] \setminus N$ (where N has null Lebesgue measure) and

$$\rho(t) = \rho(0) + \int_0^t \rho'(l) dl, \forall t \in [0, 1].$$

By using the definition of $\text{Lip}(\rho)(t)$ we deduce that

$$\rho'(t) \leq \text{Lip}(\rho)(t) \leq C, \forall t \in [0, 1] \setminus N,$$

where $C := [\|\chi\|_{\text{Lip}([a,c],C(\Omega))} + \|\chi\|_{\text{Lip}([c,b],C(\Omega))}] |t - s|$. Therefore we obtain

$$\|\chi(t, \cdot) - \chi(s, \cdot)\|_\infty = \rho(1) - \rho(0) = \int_0^1 \rho'(l) dl \leq \int_0^1 C dl$$

which completes the proof. \blacksquare

Lemma 4.2 *Let $t \geq 0$. Assume that $A \in C((-\infty, t], C(\Omega))$ and $A_0 = \varphi$. Define for each $(\theta, x) \in (-\infty, 0] \times \Omega$,*

$$A_{t,\alpha}(\theta, x) := e^{-\alpha|\theta|} A_t(\theta, x) \text{ and } \varphi_\alpha(\theta, x) := e^{-\alpha|\theta|} \varphi(\theta, x).$$

Then we have the following estimations

$$\|A_{t,\alpha}\|_\infty \leq \sup_{\theta \in [0,t]} \|e^{\alpha(\theta-t)} A(\theta, \cdot)\|_\infty + e^{-\alpha t} \|\varphi_\alpha\|_\infty, \quad (4.1)$$

$$\|A_{t,\alpha}\|_{\text{Lip}((-\infty,0],C(\Omega))} \leq \|A_{t,\alpha}\|_{\text{Lip}([-t,0],C(\Omega))} + e^{-\alpha t} \|\varphi_\alpha\|_{\text{Lip}((-\infty,0],C(\Omega))} \quad (4.2)$$

and

$$\|A_t\|_{\text{Lip}_\alpha} \leq \sup_{\theta \in [0,t]} \|e^{\alpha(\theta-t)} A(\theta, \cdot)\|_\infty + \|A_{t,\alpha}\|_{\text{Lip}([-t,0],C(\Omega))} + e^{-\alpha t} \|\varphi\|_{\text{Lip}_\alpha}. \quad (4.3)$$

Proof. We have for the supremum norm

$$\begin{aligned} \|A_{t,\alpha}\|_\infty &= \sup_{\theta \leq 0} \|e^{\alpha\theta} A(t + \theta, \cdot)\|_\infty = e^{-\alpha t} \sup_{\theta \leq 0} \|e^{\alpha(t+\theta)} A(t + \theta, \cdot)\|_\infty \\ &\leq \sup_{s \in [0,t]} \|e^{\alpha(s-t)} A(s, \cdot)\|_\infty + e^{-\alpha t} \|\varphi_\alpha\|_\infty. \end{aligned}$$

The result follows by using similar arguments combined with Lemma 4.1 for the Lipschitz semi-norm. \blacksquare

Lemma 4.3 (Uniqueness of solutions) *Let $\varphi \in \text{Lip}_\alpha$ and $\tau_0 \in C_+(\Omega)$ satisfy*

$$\|\varphi\|_{\text{Lip}_\alpha} + \|\tau_0\|_\infty \leq M_0.$$

where $M_0 > 0$ is a given real number. Let $r \in (0, +\infty)$ be given. Then the equation (1.1) admits at most one solution $(A, \tau) \in C((-\infty, r], C(\Omega)) \times C([0, r], C(\Omega))$.

Proof. Suppose that there exist $(A^1, \tau^1), (A^2, \tau^2) \in C((-\infty, r], C(\Omega)) \times C([0, r], C(\Omega))$ two solutions of (1.1) on $(-\infty, r]$ with

$$(A_0^1, \tau^1(0, \cdot)) = (A_0^2, \tau^2(0, \cdot)) = (\varphi, \tau_0).$$

Define

$$t_0 = \sup \{t \in [0, r] : A^1(s, x) = A^2(s, x), \tau^1(s, x) = \tau^2(s, x), \forall s \in [0, t], \forall x \in \Omega\}.$$

Assume that $t_0 < r$. We first observe that since r is finite, we have

$$\tilde{K}_0 := \sup_{s \in [0, r]} \|A^1(s, \cdot)\|_\infty + \sup_{s \in [0, r]} \|A^2(s, \cdot)\|_\infty < +\infty.$$

By Lemma 3.8, $\tau^1(t, x)$ and $\tau^2(t, x)$ are also bounded from above (by τ_{\max}^1 and τ_{\max}^2 respectively) on $t \in [0, r]$. Since by Assumption 1.1, $F : C(\Omega)^2 \times C(\Omega^2) \rightarrow C(\Omega)$ is Lipschitz on bounded sets and for each $i = 1, 2$, each $t \in [0, r]$ and each $x \in \Omega$,

$$A^i(t, x) = \varphi(0, x) + \int_0^t F(A^i(l, \cdot), \tau^i(l, \cdot), A^i(l - \tau^i(l))(\cdot, \cdot))(x) dl,$$

it follows by using Lemma 3.8 that for each $i = 1, 2$,

$$\sup_{s \in [0, r]} \|F(A^i(s, \cdot), \tau^i(s, \cdot), A^i(s - \tau^i(s))(\cdot, \cdot))\|_\infty < +\infty,$$

and

$$K_L := \|A^1\|_{\text{Lip}([0, r], C(\Omega))} + \|A^2\|_{\text{Lip}([0, r], C(\Omega))} < +\infty.$$

Set

$$K_0 := 2(\tilde{K}_0 + \varphi_{\max}) + \tau_{\max}^1 + \tau_{\max}^2,$$

where φ_{\max} is defined in (3.5). For each $t \in [t_0, r]$ and each $x \in \Omega$ we have

$$\begin{aligned} A^1(t, x) - A^2(t, x) &= \int_0^t [F(A^1(l, \cdot), \tau^1(l, \cdot), A^1(l - \tau^1(l))(\cdot, \cdot))(x) \\ &\quad - F(A^2(l, \cdot), \tau^2(l, \cdot), A^2(l - \tau^2(l))(\cdot, \cdot))(x)] dl, \end{aligned}$$

thus by using the fact that F is Lipschitz on bounded sets, we obtain

$$\begin{aligned} \|A^1(t, \cdot) - A^2(t, \cdot)\|_\infty &\leq (t - t_0)L(K_0) \sup_{s \in [t_0, t]} [\|A^1(s, \cdot) - A^2(s, \cdot)\|_\infty \\ &\quad + \|A^1(s - \tau^1(s))(\cdot, \cdot) - A^2(s - \tau^2(s))(\cdot, \cdot)\|_\infty \\ &\quad + \|\tau^1(s, \cdot) - \tau^2(s, \cdot)\|_\infty]. \end{aligned}$$

Define

$$\|A_t^1 - A_t^2\|_\infty := \sup_{s \leq 0} \|A^1(t + s, \cdot) - A^2(t + s, \cdot)\|_\infty.$$

By Lemma 3.9, for each $s \in [t_0, t]$ we have

$$\|\tau^1(s, \cdot) - \tau^2(s, \cdot)\|_\infty = \|\tau(A_s^1, \cdot) - \tau(A_s^2, \cdot)\|_\infty \leq L_\tau \|A_s^1 - A_s^2\|_\infty \leq L_\tau \|A_t^1 - A_t^2\|_\infty.$$

Thus

$$\begin{aligned} & \|A^1(s - \tau^1(s)) - A^2(s - \tau^2(s))\|_\infty \\ & \leq \|A^1(s - \tau^1(s)) - A^1(s - \tau^2(s))\|_\infty + \|A^1(s - \tau^2(s)) - A^2(s - \tau^2(s))\|_\infty \\ & \leq K_L \|\tau^1(s, \cdot) - \tau^2(s, \cdot)\|_\infty + \|A_t^1 - A_t^2\|_\infty, \end{aligned}$$

hence

$$\|A^1(s - \tau^1(s)) - A^2(s - \tau^2(s))\|_\infty \leq (K_L L_\tau + 1) \|A_t^1 - A_t^2\|_\infty.$$

So we obtain for each $t \in [t_0, r]$,

$$\|A_t^1 - A_t^2\|_\infty \leq (t - t_0) L(K_0) ((K_L + 1) L_\tau + 2) \|A_t^1 - A_t^2\|_\infty.$$

It follows that we can find $\varepsilon \in (0, r - t_0)$ such that

$$\|A_t^1 - A_t^2\|_\infty = 0, \forall t \in [t_0, t_0 + \varepsilon],$$

which contradicts with the definition of t_0 . Thus $t_0 = r$. \blacksquare

Theorem 4.4 (Local existence of solutions) *Let $M_0 > 0$ be fixed. Then for $M > M_0$, there exists a time $r = r(M_0, M) > 0$ such that for each $(\varphi, \tau_0) \in \text{Lip}_\alpha \times C_+(\Omega)$ satisfying*

$$\|\varphi\|_{\text{Lip}_\alpha} + \|\tau_0\|_\infty \leq M_0,$$

system (1.1) admits a unique solution $(A, \tau) \in C((-\infty, r], C(\Omega)) \times C([0, r], C(\Omega))$. Moreover,

$$\|A(t, \cdot)\|_\infty \leq M, \forall t \in [0, r].$$

Proof. Step 1 (Fixed point problem): We start by defining the fixed point problem. Let $M_0 > 0$ be fixed. Let $\varphi \in \text{Lip}_\alpha$ and $\tau_0 \in C_+(\Omega)$ satisfy

$$\|\varphi\|_{\text{Lip}_\alpha} + \|\tau_0\|_\infty \leq M_0.$$

Let $M > M_0$ be fixed. Define

$$E_\varphi := \{A \in C((-\infty, r], C(\Omega)) : A_0 = \varphi \text{ and } \sup_{t \in [0, r]} \|A(t, \cdot)\|_\infty \leq M\}$$

where r will be determined later on.

Let $\Phi : E_\varphi \rightarrow C((-\infty, r], C(\Omega))$ be the map defined as follows: for each $t \in [0, r]$ and $x \in \Omega$,

$$\Phi(A)(t)(x) := \varphi(0, x) + \int_0^t F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot))(x) dl \quad (4.4)$$

where $\tau(A_t, x)$ is the unique solution of the integral equation (3.1), and for $t \leq 0$ and $x \in \Omega$,

$$\Phi(A)(t)(x) := \varphi(t, x).$$

Set

$$\widetilde{M} := \max\{M, \varphi_{\max}\} \quad (4.5)$$

where φ_{\max} is defined in (3.5). For any $A \in E_\varphi$ and $t \in [0, r]$, we have

$$\begin{aligned} \|\Phi(A)(t)\|_\infty &\leq \|\varphi(0, \cdot)\|_\infty + \int_0^t \|F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot))\|_\infty dl \\ &\leq M_0 + \int_0^t \|F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot)) - F(0, 0, 0)\|_\infty dl \\ &\quad + \int_0^t \|F(0, 0, 0)\|_\infty dl \\ &\leq M_0 + r[(2\widetilde{M} + \tau_{\max})L(2\widetilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_\infty]. \end{aligned}$$

Since $M > M_0$ we can find $r_1 > 0$ such that for each $r \in (0, r_1]$,

$$M_0 + r[(2\widetilde{M} + \tau_{\max})L(2\widetilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_\infty] \leq M,$$

and it follows that

$$\Phi(E_\varphi) \subset E_\varphi.$$

Step 2 (Lipschitz estimation): Set

$$M_L := \frac{M - M_0}{r} \geq (2\widetilde{M} + \tau_{\max})L(2\widetilde{M} + \tau_{\max}) + \|F(0, 0, 0)\|_\infty. \quad (4.6)$$

For each $t, s \in [0, r]$ with $t \geq s$ and $A \in E_\varphi$ we have

$$\begin{aligned} \frac{\|\Phi(A)(t) - \Phi(A)(s)\|_\infty}{|t - s|} &\leq \frac{\int_s^t \|F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot))\|_\infty dl}{|t - s|} \\ &\leq \frac{\int_s^t \|F(A(l, \cdot), \tau(A_l, \cdot), A(l - \tau(A_l, \cdot), \cdot)) - F(0, 0, 0)\|_\infty dl}{|t - s|} + \frac{\int_s^t \|F(0, 0, 0)\|_\infty dl}{|t - s|}, \end{aligned}$$

thus

$$\|\Phi(A)\|_{\text{Lip}([0, r], C(\Omega))} \leq M_L, \forall A \in E_\varphi.$$

Step 3 (Iteration procedure): Consider the sequence $\{A^n\}_{n \in \mathbb{N}} \subset E_\varphi$ defined by iteration as follows: for each $(t, x) \in (-\infty, r] \times \Omega$,

$$A^0(t, x) = \begin{cases} \varphi(0, x), & \text{if } t \in [0, r], \\ \varphi(t, x), & \text{if } t \leq 0, \end{cases}$$

and for each integer $n \geq 0$,

$$A^{n+1}(t, x) = \begin{cases} \Phi(A^n)(t)(x), & \text{if } t \in [0, r], \\ \varphi(t, x), & \text{if } t \leq 0. \end{cases}$$

From the step 2 and the definition of A^0 , we know that for each integer $n \geq 0$,

$$A^n \in \text{Lip}([- \tau_0^\infty, r], C(\Omega)).$$

and

$$\|A^n\|_{\text{Lip}([- \tau_0^\infty, r], C(\Omega))} \leq \max\{M_L, \|\varphi\|_{\text{Lip}([- \tau_0^\infty, 0], C(\Omega))}\} =: \widehat{M}_L.$$

For each integer $n, p \geq 0$, the maps A^n and A^p coincide for negative time t , therefore we can define

$$\|A^n - A^p\|_\infty := \sup_{t \leq r} \|A^n(t, \cdot) - A^p(t, \cdot)\|_\infty.$$

Next, we have $\forall n \in \mathbb{N}$,

$$\begin{aligned} \|A^{n+1} - A^n\|_\infty &= \sup_{t \in [0, r]} \|\Phi(A^n)(t)(\cdot) - \Phi(A^{n-1})(t)(\cdot)\|_\infty \\ &\leq \int_0^r \|F(A^n(l, \cdot), \tau(A_l^n, \cdot), A^n(l - \tau(A_l^n, \cdot), \cdot)) \\ &\quad - F(A^{n-1}(l, \cdot), \tau(A_l^{n-1}, \cdot), A^{n-1}(l - \tau(A_l^{n-1}, \cdot), \cdot))\|_\infty dl \\ &\leq rL(\widetilde{M} + \tau_{\max}) \left[\|A^n - A^{n-1}\|_\infty + \sup_{l \in [0, r]} \|\tau(A_l^n, \cdot) - \tau(A_l^{n-1}, \cdot)\|_\infty \right. \\ &\quad \left. + \sup_{l \in [0, r]} \|A^n(l - \tau(A_l^n, \cdot), \cdot) - A^{n-1}(l - \tau(A_l^{n-1}, \cdot), \cdot)\|_\infty \right], \\ &\leq rL(\widetilde{M} + \tau_{\max}) \left[\|A^n - A^{n-1}\|_\infty + (1 + \|A^n\|_{\text{Lip}([- \tau_0^\infty, r], C(\Omega))}) \cdot \right. \\ &\quad \left. \sup_{l \in [0, r]} \|\tau(A_l^n, \cdot) - \tau(A_l^{n-1}, \cdot)\|_\infty \right]. \end{aligned}$$

By Lemma 3.9, for each integer $n \geq 1$ we obtain

$$\|A^{n+1} - A^n\|_\infty \leq rC \|A^n - A^{n-1}\|_\infty$$

with $C := (2 + L_\tau(1 + \widehat{M}_L))L(\widetilde{M} + \tau_{\max})$.

Now we can find $r_2 \in (0, r_1]$ such that $r_2C < 1/2$, and for each $r \in (0, r_2]$ we have

$$\|A^{n+1} - A^n\|_\infty \leq \frac{1}{2^n} \|A^1 - A^0\|_\infty, \forall n \geq 1.$$

It follows that $\{A^n|_{[0, r]}\}$ is a Cauchy sequence in the space $C([0, r], C(\Omega))$ and $\{A^n\}$ coincides with φ for negative t . Define

$$A(t, x) := \begin{cases} \lim_{n \rightarrow +\infty} A^n(t, x), & \text{if } t \in [0, r], x \in \Omega, \\ \varphi(t, x), & \text{if } t \leq 0, x \in \Omega. \end{cases}$$

Then we have

$$\lim_{n \rightarrow +\infty} \|A^n - A\|_\infty = 0.$$

By using again Lemma 3.9, we have

$$\sup_{l \in [0, r]} \|\tau(A_l^n, \cdot) - \tau(A_l, \cdot)\|_\infty \leq L_\tau \|A^n - A\|_\infty, \forall n \geq 1.$$

Finally by taking the limit on both sides of the equation

$$A^{n+1}(t, x) := \varphi(0, x) + \int_0^t F(A^n(l, \cdot), \tau(A_l^n, \cdot), A^n(l - \tau(A_l^n, \cdot), \cdot))(x) dl,$$

we deduce that the couple $(A, \tau(A_t, \cdot))$ is a solution of equation (1.1).

Step 4: (Estimation of the solution) We observe that

$$\begin{aligned} \|A^{n+1} - A^0\|_\infty &\leq \|A^{n+1} - A^n\|_\infty + \|A^n - A^{n-1}\|_\infty + \dots + \|A^1 - A^0\|_\infty \\ &\leq \sum_{k=0}^n \left(\frac{1}{2}\right)^k \|A^1 - A^0\|_\infty, \end{aligned}$$

therefore

$$\|A^{n+1} - A^0\|_\infty \leq 2\|A^1 - A^0\|_\infty,$$

and by taking the limit when n goes to infinity we obtain

$$\|A - A^0\|_\infty \leq 2\|A^1 - A^0\|_\infty.$$

From the definition of A^1 and A^0 we have

$$\|A^1 - A^0\|_\infty \leq \int_0^r \|F(A^0(l, \cdot), \tau(A_l^0, \cdot), A^0(l - \tau(A_l^0, \cdot), \cdot))\|_\infty dl,$$

so

$$\|A^1 - A^0\|_\infty \leq rC_1,$$

where $C_1 = \sup_{l \in [0, r]} \|F(A^0(l, \cdot), \tau(A_l^0, \cdot), A^0(l - \tau(A_l^0, \cdot), \cdot))\|_\infty$. It follows that for each $t \in [0, r]$,

$$\|A(t, \cdot)\|_\infty \leq \|A(t, \cdot) - \varphi(0, \cdot)\|_\infty + \|\varphi(0, \cdot)\|_\infty \leq \|A - A^0\|_\infty + M_0,$$

thus

$$\|A(t, \cdot)\|_\infty \leq M_0 + 2rC_1,$$

and by choosing r small enough we obtain $M_0 + 2rC_1 \leq M$. The proof is completed. \blacksquare

From step 2 in the above proof combined with Lemma 4.2, we have the following corollary.

Corollary 4.5 *With the same notation as in Theorem 4.4, we have*

$$A_t \in \text{Lip}_\alpha, \forall t \in [0, r],$$

and there exists $\widehat{M} := \widehat{M}(M, \tau_{\max}, \alpha) > M$ such that

$$\|A_t\|_{\text{Lip}_\alpha} \leq \widehat{M}, \forall t \in [0, r].$$

5 Properties of the semiflow

For each initial distribution $W_0 = \begin{pmatrix} \varphi \\ \tau_0 \end{pmatrix} \in \text{Lip}_\alpha \times C_+(\Omega)$, define

$$T_{BU}(W_0) = \sup\{t > 0 : \text{there exists a solution of (3.1) on the interval } [0, t] \text{ with the initial distribution } W_0\},$$

Observe by Theorem 4.4 that we must have $T_{BU}(W_0) > 0$.

In this section we investigate the semiflow properties of the map $W : D(W) \subset [0, +\infty) \times \text{Lip}_\alpha \times C(\Omega) \rightarrow \text{Lip}_\alpha \times C(\Omega)$ defined for each initial distribution W_0 as

$$W(t, W_0)(x) = \begin{pmatrix} A_t(\cdot, x) \\ \tau(t, x) \end{pmatrix}$$

where $\{(A_t, \tau(t, \cdot))\}_{t \in [0, T_{BU}(W_0))}$ is the solution of (1.1) with initial distribution W_0 which can be defined up to the maximal time of existence $T_{BU}(W_0)$. The domain is

$$D(W) = \bigcup_{W_0 \in \text{Lip}_\alpha \times C_+(\Omega)} [0, T_{BU}(W_0)) \times \{W_0\}.$$

Proof. (First part of Theorem 1.6) Suppose that $W_0 \in \text{Lip}_\alpha \times C_+(\Omega)$ satisfies $\|W_0\|_{\text{Lip}_\alpha \times C(\Omega)} \leq M_0$, where M_0 is a positive constant, and define \mathcal{U} by

$$\mathcal{U}(t)W_0(x) := \begin{pmatrix} A_t(\cdot, x) \\ \tau(t, x) \end{pmatrix}$$

where the map $t \mapsto (A_t, \tau(t, \cdot))$ belongs to $C([0, T_{BU}(W_0)), \text{Lip}_\alpha \times C(\Omega))$ is the maximal solution of (1.1) with the initial value W_0 . In this proof, we will verify that \mathcal{U} satisfies the properties (i)-(iv) of Definition 1.5.

The property (ii) of Definition 1.5 is trivially satisfied, since by construction $\mathcal{U}(0)W_0 = W_0$.

Step 1 ((i) and (iii) of Definition 1.5): By Lemma 3.2, the map $t \mapsto (A(t, \cdot), \tau(t, \cdot))$ is the solution of (1.1) if and only if for each $t \in [0, T_{BU}(W_0))$ and each $x \in \Omega$ the equations

$$\begin{cases} A(t, x) = \varphi(0, x) + \int_0^t F(A(l, \cdot), \tau(l, \cdot), A(l - \tau(l))(\cdot, \cdot))(x) dl, \\ \tau(t, x) = \tau_0(x) + \int_0^t \left[1 - \frac{f(A(l, \cdot))(x)}{f(A(l - \tau(l))(\cdot, \cdot))(x)} \right] dl, \end{cases} \quad (5.1)$$

are satisfied together with the initial condition

$$(A_0, \tau(0, \cdot)) = W_0.$$

Let $s \in [0, T_{BU}(W_0))$. Let us prove that

$$s + T_{BU}(\mathcal{U}(s)W_0) \leq T_{BU}(W_0)$$

and for each $t \in [0, T_{BU}(\mathcal{U}(s)W_0))$,

$$\mathcal{U}(t+s)W_0 = \mathcal{U}(t)\mathcal{U}(s)W_0. \quad (5.2)$$

For each $t \in [0, T_{BU}(\mathcal{U}(s)W_0))$,

$$\mathcal{U}(t)\mathcal{U}(s)(W_0) = \mathcal{U}(t) \begin{pmatrix} A_s \\ \tau(s, \cdot) \end{pmatrix} = \begin{pmatrix} \tilde{A}_t \\ \tilde{\tau}(t) \end{pmatrix}$$

where $t \mapsto (A(t, \cdot), \tau(t, \cdot))$ is the solution of (1.1) with initial condition W_0 and $t \mapsto (\tilde{A}(t, \cdot), \tilde{\tau}(t, \cdot))$ is the solution of (1.1) with initial condition $\mathcal{U}(s)(W_0)$. Then we have for each $t \in [0, T_{BU}(\mathcal{U}(s)W_0))$,

$$\begin{cases} \tilde{A}(t, x) = A(s, x) + \int_0^t F(\tilde{A}(l, \cdot), \tilde{\tau}(l, \cdot), \tilde{A}(l - \tilde{\tau}(l))(\cdot, \cdot))(x) dl \\ \tilde{\tau}(t, x) = \tau(s, x) + \int_0^t \left[1 - \frac{f(\tilde{A}(l, \cdot))(x)}{f(\tilde{A}(l - \tilde{\tau}(l))(\cdot, \cdot))(x)} \right] dl \end{cases} \quad (5.3)$$

with initial condition

$$\tilde{A}_0 = A_s, \tilde{\tau}(0, x) = \tau(s, x).$$

Now by setting

$$(\bar{A}_t, \bar{\tau}(t, \cdot)) := \begin{cases} (\tilde{A}_{t-s}, \tilde{\tau}(t-s, \cdot)), & \text{if } t \in [s, s + T_{BU}(\mathcal{U}(s)W_0)), \\ (A_t, \tau(t, \cdot)), & \text{if } t \in [0, s], \end{cases}$$

then by using (5.1) and (5.3) we deduce that $t \mapsto (\bar{A}(t, \cdot), \bar{\tau}(t, \cdot))$ is a solution of (1.1) on the time interval $[0, s + T_{BU}(\mathcal{U}(s)W_0))$ with initial condition W_0 . It follows that

$$s + T_{BU}(\mathcal{U}(s)W_0) \leq T_{BU}(W_0).$$

Now assume that $t \mapsto (A(t, \cdot), \tau(t, \cdot))$ is a solution of (1.1) on the time interval $[0, T_{BU}(W_0))$ with initial condition W_0 . Let $s \in [0, T_{BU}(W_0))$. Then by using (5.1) it follows that $t \mapsto (A(t+s, \cdot), \tau(t+s, \cdot))$ defined on $[0, T_{BU}(W_0) - s)$ is a solution of (1.1) with initial condition $\mathcal{U}(s)W_0$. It follows that

$$T_{BU}(W_0) - s \leq T_{BU}(\mathcal{U}(s)W_0)$$

and the properties (i) and (iii) of Definition 1.5 follow.

Step 2 ((iv) of Definition 1.5): Assume that $T_{BU}(W_0) < +\infty$. Suppose that $\|\mathcal{U}(t)W_0\|_{\text{Lip}_\alpha \times C(\Omega)}$ does not go to $+\infty$ when $t \nearrow T_{BU}(W_0)$. Then there exists a constant $M_0 > 0$ and a sequence $\{t_n\} \subset [0, T_{BU}(W_0))$ such that $\lim_{n \rightarrow +\infty} t_n = T_{BU}(W_0)$, and for any $n \in \mathbb{N}$,

$$\|\mathcal{U}(t_n)W_0\|_{\text{Lip}_\alpha \times C(\Omega)} = \left\| \begin{pmatrix} A_{t_n} \\ \tau(t_n, \cdot) \end{pmatrix} \right\|_{\text{Lip}_\alpha \times C(\Omega)} \leq M_0.$$

Let $W_{0,n} := \begin{pmatrix} A_{t_n} \\ \tau(t_n, \cdot) \end{pmatrix}$. Then by the local existence Theorem 4.4 and Corollary 4.5 we can find a constant $\widehat{M} > M_0$ and a time $r = r(M_0, \widehat{M}) > 0$ such that for any $n \in \mathbb{N}$,

$$T_{BU}(W_{0,n}) \geq r.$$

Moreover, from the previous part of the proof we have

$$T_{BU}(W_0) = t_n + T_{BU}(W_{0,n}) \geq t_n + r,$$

and when $n \rightarrow +\infty$ we obtain

$$T_{BU}(W_0) \geq T_{BU}(W_0) + r$$

a contradiction since $r > 0$. Thus we have

$$\lim_{t \nearrow T_{BU}(W_0)} \|\mathcal{U}(t)W_0\|_{\text{Lip}_\alpha \times C(\Omega)} = +\infty. \quad (5.4)$$

Proof. (Second part of Theorem 1.6) Let us prove that if $T_{BU}(W_0) < +\infty$ then ■

$$\limsup_{t \nearrow T_{BU}(W_0)} \|A(t, \cdot)\|_\infty = +\infty.$$

Assume that $T_{BU}(W_0) < +\infty$ and assume by contradiction that

$$\limsup_{t \nearrow T_{BU}(W_0)} \|A(t, \cdot)\|_\infty < +\infty.$$

Since the map $t \mapsto t - \tau(t, x)$ is increasing, we have

$$-\tau_0(x) \leq t - \tau(t, x) \leq t < T_{BU}(W_0), \forall x \in \Omega,$$

therefore

$$0 \leq \tau(t, x) \leq T_{BU}(W_0) + \tau_0(x).$$

And by assumption $T_{BU}(W_0) < +\infty$, then

$$\limsup_{t \nearrow T_{BU}(W_0)} \|\tau(t, \cdot)\|_\infty < +\infty.$$

Moreover, for each $t \in [0, T_{BU}(W_0))$,

$$\partial_t A(t, x) = F(A(t, \cdot), \tau(t, \cdot), A(t - \tau(t))(\cdot, \cdot))(x),$$

and since F is Lipschitz on bounded sets and by Lemma 4.2 we deduce that

$$\limsup_{t \nearrow T_{BU}(W_0)} \|e^{\alpha \cdot} A_t(\cdot)\|_{\text{Lip}} < +\infty,$$

which contradicts (5.4). ■

Lemma 5.1 *We have the following results:*

- (i) *The map $W_0 \mapsto T_{BU}(W_0)$ is lower semi-continuous on $\text{Lip}_\alpha \times C_+(\Omega)$.*
- (ii) *For every $x \in \Omega$, for every $W_0 \in \text{Lip}_\alpha \times C_+(\Omega)$, $\widehat{T} \in (0, T_{BU}(W_0))$, and every sequence $\{W_0^{(n)}\}_{n \in \mathbb{N}} \subset \text{Lip}_\alpha \times C_+(\Omega)$ satisfying*

$$\lim_{n \rightarrow +\infty} W_0^{(n)} = W_0 \text{ in } \text{Lip}_\alpha \times C_+(\Omega),$$

we have

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, \widehat{T}]} \left\| \mathcal{U}(t)W_0^{(n)} - \mathcal{U}(t)W_0 \right\|_{\text{Lip}_\alpha \times C(\Omega)} = 0. \quad (5.5)$$

Proof. Step 1 (Fixed point problem): Let $t \mapsto (\bar{A}(t, \cdot), \bar{\tau}(t, \cdot))$ be a solution of system (1.1) which exists up to the maximal time of existence $T_{BU}(\bar{W}_0)$ with the initial distribution $\bar{W}_0 = \begin{pmatrix} \bar{\varphi} \\ \bar{\tau}_0 \end{pmatrix} \in \text{Lip}_\alpha \times C_+(\Omega)$.

Let $t^* \in (0, T_{BU}(\bar{W}_0))$ be fixed. By construction the map $t \mapsto (\bar{A}_t, \bar{\tau}(t, \cdot))$ is continuous from $[0, T_{BU}(\bar{W}_0)]$ to $BUC_\alpha \times C_+(\Omega)$. Therefore

$$\sup_{t \in [0, t^*]} [\|\bar{A}_t\|_{BUC_\alpha} + \|\bar{\tau}(t, \cdot)\|_\infty] < +\infty,$$

and since $\bar{A}(t, \cdot)$ satisfies the equation (1.1) for positive time t , it follows that

$$\widehat{M} := \sup_{t \in [0, t^*]} \|\bar{A}_t\|_{\text{Lip}_\alpha} < +\infty.$$

Let $t_0 \in [0, t^*]$ and $r > 0$ with $t_0 + r < t^*$ where r will be determined later on.

Let $\varepsilon > 0$ be fixed. Let $W_0 = \begin{pmatrix} \varphi \\ \tau_0 \end{pmatrix} \in \text{Lip}_\alpha \times C_+(\Omega)$ satisfy

$$\|\varphi - \bar{A}_{t_0}\|_{\text{Lip}_\alpha} \leq \varepsilon \text{ and } \|\tau_0 - \bar{\tau}(t_0, \cdot)\|_\infty \leq \varepsilon.$$

Let $M > \varepsilon$ be fixed. Define the space

$$E_{\varphi, t_0} := \{A \in BUC_\alpha((-\infty, r], C(\Omega)) : A_0 = \varphi \text{ and } \sup_{t \in [0, r]} \|A(t, \cdot) - \bar{A}_{t_0}(t, \cdot)\|_\infty \leq M\}.$$

Let $\Phi : E_{\varphi, t_0} \rightarrow C((-\infty, r], C(\Omega))$ be the map defined by

$$\Phi(A)(t)(x) := \varphi(0, x) + \int_0^t F(A(l, \cdot), \widehat{\tau}(A_l, \delta_0), A(l - \widehat{\tau}(A_l, \delta_0), \cdot))(x) dl \quad (5.6)$$

whenever $t \in [0, r]$ and $x \in \Omega$, and

$$\Phi(A)(t)(x) := \bar{A}_{t_0}(t, x)$$

whenever $t \leq 0$ and $x \in \Omega$.

In the formula (5.6) the delay $\hat{\tau}(A_t, \delta_0)(x)$ is the unique solution of

$$\int_{-\hat{\tau}(A_t, \delta_0)(x)}^0 f(A(t+s, \cdot))(x) ds = \delta_0(x)$$

where

$$\delta_0(x) := \int_{-\tau_0(x)}^0 f(\varphi(s, \cdot))(x) ds.$$

For any $A \in E_{\varphi, t_0}$, $t \in [0, r]$ and $x \in \Omega$, we have

$$\begin{aligned} & |\Phi(A)(t)(x) - \bar{A}_{t_0}(t, x)| \\ &= \left| \varphi(0, x) + \int_0^t F(A(l, \cdot), \hat{\tau}(A_l, \delta_0), A(l - \hat{\tau}(A_l, \delta_0), \cdot))(x) dl \right. \\ &\quad \left. - \bar{A}_{t_0}(0, x) - \int_0^t F(\bar{A}_{t_0}(l, \cdot), \hat{\tau}(\bar{A}_{t_0+l}, \bar{\delta}_0), \bar{A}_{t_0}(l - \hat{\tau}(\bar{A}_{t_0+l}, \bar{\delta}_0), \cdot))(x) dl \right| \\ &\leq \varepsilon + \int_0^t |F(A(l, \cdot), \hat{\tau}(A_l, \delta_0), A(l - \hat{\tau}(A_l, \delta_0), \cdot))(x) \\ &\quad - F(\bar{A}_{t_0}(l, \cdot), \hat{\tau}(\bar{A}_{t_0+l}, \bar{\delta}_0), \bar{A}_{t_0}(l - \hat{\tau}(\bar{A}_{t_0+l}, \bar{\delta}_0), \cdot))(x)| dl, \end{aligned}$$

where

$$\bar{\delta}_0(x) = \int_{-\bar{\tau}(t_0, x)}^0 f(\bar{A}(t_0 + s, \cdot))(x) ds = \int_{-\bar{\tau}_0(x)}^0 f(\bar{\varphi}(s, \cdot))(x) ds.$$

By Lemma 3.9, we can find a constant $L_\tau > 0$ such that

$$\|\hat{\tau}(A_l, \delta_0) - \hat{\tau}(\bar{A}_{t_0+l}, \bar{\delta}_0)\|_\infty \leq L_\tau \left[\sup_{s \in [-\max\{\tau_0^\infty, \bar{\tau}_0^\infty\}, r]} \|A(s, \cdot) - \bar{A}_{t_0}(s, \cdot)\|_\infty + \|\delta_0 - \bar{\delta}_0\|_\infty \right]$$

where τ_0^∞ and $\bar{\tau}_0^\infty$ are defined as in Lemma 3.8.

By using the definition of δ_0 and $\bar{\delta}_0$ we have

$$\begin{aligned} \|\delta_0 - \bar{\delta}_0\|_\infty &\leq \sup_{x \in \Omega} \left| \int_{-\tau_0(x)}^{-\bar{\tau}(t_0, x)} f(\varphi(s, \cdot))(x) ds \right| \\ &\quad + \sup_{x \in \Omega} \left| \int_{-\bar{\tau}(t_0, x)}^0 [f(\varphi(s, \cdot))(x) - f(\bar{A}(t_0 + s, \cdot))(x)] ds \right| \\ &\leq \|\tau_0(\cdot) - \bar{\tau}(t_0, \cdot)\|_\infty \sup_{x \in \Omega} f(-\varphi_{\max})(x) \\ &\quad + \bar{\tau}_0^\infty \|f\|_{\text{Lip}} \sup_{s \in [-\bar{\tau}_0^\infty, 0]} \|\varphi(s, \cdot) - \bar{A}(t_0 + s, \cdot)\|_\infty \\ &\leq \varepsilon \left(\sup_{x \in \Omega} f(-\varphi_{\max})(x) + \bar{\tau}_0^\infty e^{\alpha \bar{\tau}_0^\infty} \|f\|_{\text{Lip}} \right) \end{aligned}$$

where φ_{\max} is defined as in Lemma 3.8.

From the above estimations, it follows that there exists a constant $M_1 > 0$ such that

$$\|\widehat{\tau}(A_l, \delta_0) - \widehat{\tau}(\bar{A}_{t_0+t}, \bar{\delta}_0)\|_\infty \leq M_1.$$

Now, similarly as in step 3 of the proof of Theorem 4.4 to evaluate $\|A^{n+1} - A^n\|_\infty$, we deduce that there exists a constant $r_1 > 0$ such that for each $r \in (0, r_1]$, we have $\Phi(E_{\varphi, t_0}) \subset E_{\varphi, t_0}$.

Step 2 (Lipschitz estimation): Similarly as in step 2 of the proof of Theorem 4.4, we can deduce that there exists a constant $M_L > 0$ such that

$$\|\Phi(A)\|_{\text{Lip}([0, r], C(\Omega))} \leq M_L, \forall A \in E_{\varphi, t_0}.$$

Step 3 (Iteration procedure): Consider the sequence $\{A^n\}_{n \in \mathbb{N}} \subset E_{\varphi, t_0}$ defined by iteration as follows: for each $(t, x) \in (-\infty, r] \times \Omega$,

$$A^0(t, x) = \bar{A}_{t_0}(t, x),$$

and for each integer $n \geq 0$,

$$A^{n+1}(t, x) := \begin{cases} \Phi(A^n)(t)(x), & \text{if } t \in [0, r], \\ \varphi(t, x), & \text{if } t \leq 0. \end{cases}$$

From step 2, we deduce that there exists a constant $\widehat{M}_L > 0$ such that for each integer $n \geq 0$,

$$\|A^n\|_{\text{Lip}([-\tau_0^\infty, r], C(\Omega))} \leq \widehat{M}_L.$$

By using the same argument as in step 3 of the proof of Theorem 4.4, we can find a constant $r_2 \in (0, r_1]$ such that $\forall r \in (0, r_2]$,

$$\|A^{n+1} - A^n\|_\infty \leq \frac{1}{2^n} \|A^1 - A^0\|_\infty, \forall n \geq 1.$$

It follows that $\{A^n|_{[0, r]}\}$ is a Cauchy sequence in the space $C([0, r], C(\Omega))$. Define

$$A(t, x) := \begin{cases} \lim_{n \rightarrow +\infty} A^n(t, x), & \text{if } t \in [0, r], x \in \Omega, \\ \varphi(t, x), & \text{if } t \leq 0, x \in \Omega. \end{cases}$$

Then we have

$$\lim_{n \rightarrow +\infty} \|A^n - A\|_\infty = 0,$$

and we deduce that $(A, \widehat{\tau}(A_l, \delta_0))$ is a solution of (1.1) with the initial distribution (φ, τ_0) .

Step 4 (Estimation of the solution): As in step 4 of the proof of Theorem 4.4, we also have

$$\|A - A^0\|_\infty \leq 2\|A^1 - A^0\|_\infty.$$

Since we have

$$\|A^1 - A^0\|_\infty$$

$$\begin{aligned}
&= \sup_{t \in [0, r]} \sup_{x \in \Omega} \left| \varphi(0, x) + \int_0^t F(A^0(l, \cdot), \hat{\tau}(A_l^0, \delta_0), A^0(l - \hat{\tau}(A_l^0, \delta_0), \cdot))(x) dl - \bar{A}_{t_0}(t, x) \right| \\
&\leq \sup_{t \in [0, r]} [\|\varphi(0, \cdot) - \bar{A}_{t_0}(0, \cdot)\|_\infty \\
&\quad + \sup_{x \in \Omega} \left| \bar{A}_{t_0}(0, \cdot) + \int_0^t F(A^0(l, \cdot), \hat{\tau}(A_l^0, \delta_0), A^0(l - \hat{\tau}(A_l^0, \delta_0), \cdot))(x) dl - \bar{A}_{t_0}(t, x) \right|]
\end{aligned}$$

and since $\bar{A}(t, x)$ is a solution, the last term in the above inequality is null, it follows that

$$\|A^1 - A^0\|_\infty \leq \varepsilon.$$

Then we obtain

$$\|A(t, \cdot) - \bar{A}_{t_0}(t, \cdot)\|_\infty \leq 2\varepsilon, \forall t \in [t_0, t_0 + r].$$

Step 5 (Convergence result): Fix $r = \frac{t^*}{n} \leq r_2$ for some integer $n \geq 1$. Choose an initial value satisfying

$$\|\varphi - \bar{A}_{t_0}\|_{\text{Lip}_\alpha} \leq \frac{\varepsilon}{2^{n+1}} \text{ and } \|\tau_0 - \bar{\tau}(t_0, \cdot)\|_\infty \leq \frac{\varepsilon}{2^{n+1}}.$$

By using the above prove result when $t_0 = 0$, we deduce that

$$\|A(t, \cdot) - \bar{A}(t, \cdot)\|_\infty \leq \frac{\varepsilon}{2^{n+1}}, \forall t \in [0, r].$$

and by induction $t_0 = kr$ for $k = 0, \dots, n$ we obtain

$$\|A(t, \cdot) - \bar{A}(t, \cdot)\|_\infty \leq \varepsilon, \forall t \in [0, t^*],$$

the result follows. \blacksquare

The part of of Theorem 1.7: time continuity of the semiflow in BUC_α^1 is left to the reader.

6 Application to the forest model with space

For $x \in \Omega := [0, 1]$ and $t \geq 0$ we consider the forest spatial model with one species (see [9] for more details)

$$\partial_t A(t, x) = e^{-\mu_J \tau(t, x)} \frac{f(A(t, x))}{f(A(t - \tau(t, x), x))} B(t - \tau(t, x), x) - \mu_A A(t, x), \quad (6.1)$$

where the birth is defined by

$$B(t, x) := (I - \varepsilon \Delta)^{-1} [\beta A(t, \cdot)](x),$$

where Δ is the Laplacian operator on the domain Ω with periodic boundary conditions, and the state-dependent delay satisfies

$$\int_{t-\tau(t, x)}^t f(A(\sigma, x)) d\sigma = \int_{-\tau_0(x)}^0 f(\varphi(\sigma, x)) d\sigma,$$

and $A(t, x)$ satisfies the initial condition

$$A(t, x) = \varphi(t, x), \forall t \leq 0, x \in \Omega$$

with

$$\varphi \in \text{Lip}_\alpha \text{ and } \varphi \geq 0.$$

Then it is well known that $(I - \varepsilon\Delta)^{-1}$ is a positive operator, i.e.

$$(I - \varepsilon\Delta)^{-1}C_+(\Omega) \subset C_+(\Omega)$$

and

$$\|(I - \varepsilon\Delta)^{-1}\|_{\mathcal{L}(C(\Omega))} = 1.$$

6.1 Positivity

Assume that

$$\varphi(t, x) \geq 0, \forall (t, x) \in (-\infty, 0] \times \Omega.$$

Assume that the solution starting from this initial distribution exists up to the time $T_{BU}(\varphi, \tau_0) > 0$. Then we have for each $t \in [0, T_{BU}(\varphi, \tau_0))$,

$$A(t, x) = e^{-\mu_A t} \varphi(0, x) + \int_0^t e^{-\mu_A(t-s)} e^{-\mu_J \tau(s, x)} \frac{f(A(s, x))}{f(A(s - \tau(s, x), x))} (I - \varepsilon\Delta)^{-1}[\beta A(s - \tau(s, x), \cdot)](x) ds.$$

Since the operator $(I - \varepsilon\Delta)^{-1}$ preserves the positivity of the distribution and by Assumption 1.1, f is strictly positive, then the positivity of the solution follows by using fixed point arguments on $A(t, x)$ in the above integral equation.

6.2 Global existence

Consider the spatial density of juveniles

$$J(t, x) := \int_{t-\tau(t, x)}^t e^{-\mu_J(t-s)} \beta (I - \varepsilon\Delta)^{-1}(A(s, \cdot))(x) ds$$

for each $t \in [0, T_{BU}(\varphi, \tau_0))$. It is clear that

$$J(t, x) \geq 0, \forall t \in [0, T_{BU}(\varphi, \tau_0)). \quad (6.2)$$

Moreover we have

$$\begin{aligned} \partial_t J(t, x) = & \beta (I - \varepsilon\Delta)^{-1}(A(t, \cdot))(x) - e^{-\mu_J \tau(t, x)} \frac{f(A(t, x))}{f(A(t - \tau(t, x), x))} \\ & \beta (I - \varepsilon\Delta)^{-1}(A(t - \tau(t, x), \cdot))(x) - \mu_J J(t, x). \end{aligned}$$

By summing equation (6.1) and the above equation we obtain

$$\partial_t [A(t, x) + J(t, x)] = \beta (I - \varepsilon\Delta)^{-1}(A(t, \cdot))(x) - \mu_A A(t, x) - \mu_J J(t, x). \quad (6.3)$$

Set

$$U(t, x) := A(t, x) + J(t, x),$$

then we have

$$\partial_t U(t, x) \leq \beta(I - \varepsilon \Delta)^{-1}(U(t, \cdot))(x) - \mu U(t, x),$$

where $\mu := \min\{\mu_A, \mu_J\}$. By using a comparison argument we deduce that

$$U(t, x) \leq e^{[\beta(I - \varepsilon \Delta)^{-1} - \mu]t}(U(0, \cdot))(x), \forall t \in [0, T_{BU}(\varphi, \tau_0)).$$

Therefore by using (6.2), we deduce that

$$A(t, x) \leq e^{[\beta(I - \varepsilon \Delta)^{-1} - \mu]t}(U(0, \cdot))(x), \forall t \in [0, T_{BU}(\varphi, \tau_0)),$$

and by using Theorem 1.6, we must have $T_{BU}(\varphi, \tau_0) = +\infty$.

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