A system of state-dependent delay differential equation modelling forest growth II: boundedness of solutions

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Abstract: In this article we consider a class of state-dependent delay differential equations which models the dynamics of the number of adult trees in forests. We prove the boundedness and the dissipativity of the solutions for a single species model and an n-species model.

Keywords: State-dependent delay differential equations, forest population dynamics, boundeness of solutions, dissipativity.

AMS Subject Classication : 34K05, 37L99, 37N25.

1 Introduction

In this article we are interested in a state-dependent delay differential equation modelling the growth of forest. Following Magal and Zhang [7], when the forest is composed of a single species of trees, we have the following system

$$\begin{cases} A'(t) = -\mu_A A(t) + \beta e^{-\mu_J \tau(t)} \dfrac{f(A(t))}{f(A(t-\tau(t)))} A(t-\tau(t)), \forall t \geqslant 0, \\ \displaystyle\int_{t-\tau(t)}^{t} f(A(\sigma))d\sigma = \int_{-\tau_0}^{0} f(\varphi(\sigma))d\sigma, \forall t \geqslant 0, \end{cases} \tag{1.1}$$

with the initial conditions

$$A(t) = \varphi(t) \geqslant 0, \forall t \leqslant 0 \text{ and } \tau(0) = \tau_0 \geqslant 0,$$

where φ belongs to

$$\text{Lip}_\alpha := \left\{ \phi \in C(-\infty, 0] : e^{-\alpha|.|}\phi(.) \in BUC(-\infty, 0] \cap \text{Lip}(-\infty, 0] \right\},$$

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which is a Banach space endowed with the norm

$$\|\phi\|_{\mathrm{Lip}_\alpha} := \|e^{-\alpha|.|}\phi(.)\|_{\infty,(-\infty,0]} + \|e^{-\alpha|.|}\phi(.)\|_{\mathrm{Lip}(-\infty,0]},$$

where $\alpha \geqslant 0$, $BUC(-\infty, 0]$ denotes the space of bounded uniformly continuous functions from $(-\infty, 0]$ to $\mathbb{R}$, and $\mathrm{Lip}(-\infty, 0]$ denotes the space of Lipschitz functions from $(-\infty, 0]$ to $\mathbb{R}$.

Equation (1.1) models the dynamics of the adult population of trees. Here $A(t)$ is the number of adult trees at time t, $\tau(t)$ is the time needed by newborns to become adult at time t, $\mu_A > 0$ is the mortality rate of the adult trees, $\mu_J > 0$ is the mortality rate of the juvenile trees, $\beta > 0$ is the birth rate. In the context of forest modelling (see [7]), $f(A(t))$ describes the growth rate of juveniles, and the function f is capturing the effect of the competition for light between adults and juveniles. For mathematical convenience, we will make the following assumption.

Assumption 1.1 *We assume that*

(i) The coefficients $\mu_A > 0$, $\mu_J > 0$, $\beta > 0$*;*

(ii) The function $f : \mathbb{R} \to (0, +\infty)$ *is Lipschitz continuous and continuously differentiable with*

$$f(x) > 0, \lim_{x\to+\infty} f(x) = 0 \text{ and } f'(x) \leqslant 0, \forall x \in \mathbb{R}.$$

Actually system (1.1) has been first derived by Smith [9] from a size-structured model of the form

$$\begin{cases} A'(t) = -\mu_A A(t) + f(A(t))j(t, s^*), \forall t \geqslant 0, \\ \partial_t j(t, s) + f(A(t))\partial_s j(t, s) = -\mu_J j(t, s), \forall s \in [s_-, s^*], \\ f(A(t))j(t, s_-) = \beta A(t), \\ A(0) = A_0 \geqslant 0, \\ j(0, s) = j_0(s) \geqslant 0, \forall s \in [s_-, s^*), \end{cases}$$

where $0 \leqslant s_- < s^*$ are the minimal and maximal size of juveniles, and $j(t, s)$ is the density of juveniles with size s at time t. System (1.1) has also been extensively studied by Smith in [9, 10, 11, 12], where the author introduced a change of variable to transform this kind of state-dependent delay differential equation into a constant delay differential equation. The change of variable is given by

$$x = \int_0^t f(A(\sigma))d\sigma =: \Phi(t).$$

Set

$$\delta := \int_{-\tau_0}^0 f(\varphi(\sigma))d\sigma \geqslant 0,$$

then for $x \geqslant \delta$,

$$x-\delta=\int_0^t f(A(\sigma))d\sigma-\int_{t-\tau(t)}^t f(A(\sigma))d\sigma=\int_0^{t-\tau(t)} f(A(\sigma))d\sigma=\Phi(t-\tau(t)),$$

This means that $x-\delta$ corresponds to $t-\tau(t)$ under this change of variable. Moreover by setting $W(x)=A(t)$ and using the same arguments as in Smith [9], one also has

$$\tau(t)=\int_{-\delta}^0 f(W(x+r))^{-1}dr.$$

Therefore Smith [9] obtained the following constant delay differential equation

$$W'(x)=-\mu_A\frac{W(x)}{f(W(x))}+\beta e^{-\mu_J\int_{-\delta}^0 f(W(x+r))^{-1}dr}\frac{W(x-\delta)}{f(W(x-\delta))},\forall x\geqslant 0. \quad (1.2)$$

Based on the analysis of this equation (1.2), Smith [9, 10, 11, 12] was able to prove the boundedness of solutions whenever $\delta>0$. Along the same line, he was also able to analyze the uniform persistence and Hopf bifurcation around the positive equilibrium.

Let $A\in C((-\infty,r],\mathbb{R})$ (for some $r\geqslant 0$) be given. Then for each $t\leqslant r$, we will use the standard notation $A_t\in C((-\infty,0],\mathbb{R})$, which is the map defined by

$$A_t(\theta)=A(t+\theta),\forall\theta\leqslant 0.$$

For clarity we will specify the notion of a solution.

Definition 1.2 *Let $r\in(0,+\infty]$. A solution of the system (1.1) on $[0,r)$ is a pair of continuous maps $A:(-\infty,r)\to\mathbb{R}$ and $\tau:[0,r)\to\mathbb{R}_+$ satisfying*

$$A(t)=\begin{cases}\varphi(0)+\displaystyle\int_0^t F(A(\sigma),\tau(\sigma),A(\sigma-\tau(\sigma)))d\sigma,\forall t\in[0,r),\\ \varphi(t),\forall t\leqslant 0,\end{cases}$$

and

$$\int_{t-\tau(t)}^t f(A(\sigma))d\sigma=\int_{-\tau_0}^0 f(\varphi(\sigma))d\sigma,\forall t\in[0,r),$$

where

$$F(A,\tau,A_1):=-\mu_A A+\beta e^{-\mu_J\tau}\frac{f(A)}{f(A_1)}A_1.$$

In this problem the initial distribution is (φ,τ_0). The semiflow generated by (1.1) is

$$\mathcal{U}(t)(\varphi(.),\tau_0):=(A_t(.),\tau(t)),$$

where $A(t)$ and $\tau(t)$ is the solution of (1.1) with the initial distribution (φ,τ_0). The existence and uniqueness of a maximal semiflow on $\mathrm{Lip}_\alpha\times[0,+\infty)$ (with blowup property when the time gets close to the maximal time of existence $T_{BU}=T_{BU}(\varphi,\tau_0)$) have been studied in [8].

In order to obtain a global existence result for the solution, we now focus on the positive solution. From the form of the equation we can prove that

$$\varphi \geqslant 0 \Rightarrow A(t) \geqslant 0, \forall t \in [0, T_{BU}(\varphi, \tau_0)).$$

The number of juvenile individuals at time $t \in [0, T_{BU}(\varphi, \tau_0))$ is given by

$$J(t) := \int_{t-\tau(t)}^{t} e^{-\mu_J (t-\sigma)} \beta A(\sigma) d\sigma, \forall t \in [0, T_{BU}(\varphi, \tau_0)),$$

and $A \geqslant 0$ implies that

$$J(t) \geqslant 0, \forall t \in [0, T_{BU}(\varphi, \tau_0)).$$

Moreover we have

$$J'(t) = \beta A(t) - e^{-\mu_J \tau(t)} \frac{f(A(t))}{f(A(t-\tau(t)))} \beta A(t-\tau(t)) - \mu_J J(t).$$

By summing the A and J equations we obtain

$$[A(t) + J(t)]' = \beta A(t) - \mu_A A(t) - \mu_J J(t). \tag{1.3}$$

Set

$$U(t) := A(t) + J(t),$$

then since $A \geqslant 0$ we have

$$U'(t) \leqslant (\beta - \mu) U(t),$$

where $\mu := \min\{\mu_A, \mu_J\}$. By using a comparison argument we deduce that

$$U(t) \leqslant e^{(\beta-\mu)t} U(0), \forall t \in [0, T_{BU}(\varphi, \tau_0)),$$

and since $J \geqslant 0$ we deduce that

$$A(t) \leqslant e^{(\beta-\mu)t} U(0), \forall t \in [0, T_{BU}(\varphi, \tau_0)),$$

and by using Theorem 1.6 in [8], the maximal time of existence $T_{BU}(\varphi, \tau_0)$ is equal to $+\infty$. Therefore the well-posedness and the global existence of solutions of system (1.1) is guaranteed on $M := (\mathrm{Lip}_\alpha \times [0, +\infty)) \cap (C_+ \times [0, +\infty))$.

The result on boundedness of solutions for this case is as follows.

Theorem 1.3 *Let Assumption 1.1 be satisfied. Assume that $\tau_0 > 0$. Then for each $\varphi \geqslant 0$ and $\varphi \in \mathrm{Lip}_\alpha$, the corresponding solution of system* (1.1) *is bounded.*

Remark 1.4 *One may observe that the boundedness of solutions might not be true when $\tau_0 = 0$. Indeed, by the second equation of (1.1),*

$$\tau_0 = 0 \Rightarrow \tau(t) = 0, \forall t \geqslant 0,$$

and in this special case the first equation of (1.1) becomes linear:

$$A'(t) = (\beta - \mu_A) A(t), \forall t \geqslant 0. \tag{1.4}$$

The solution of (1.4) exists but when $\beta - \mu_A > 0$, every strictly positive solution is unbounded.

In [7], we also constructed a mathematical model for a forest composed of two species of trees. And by comparing it with the forest model SORTIE, we find that it is capable of describing the dynamics of the two-species forest. Inspired by this, we now take a step forward and consider the following n-species model

$$\begin{cases} A_i'(t) = -\mu_{A_i} A_i(t) + \beta_i e^{-\mu_{J_i}\tau_i(t)} \dfrac{f_i(Z_i(t))}{f_i(Z_i(t-\tau_i(t)))} A_i(t-\tau_i(t)), \forall t \geqslant 0, \\ \displaystyle\int_{t-\tau_i(t)}^{t} f_i(Z_i(\sigma))d\sigma = \int_{-\tau_{i0}}^{0} f_i(Z_{i\varphi}(\sigma))d\sigma, \forall t \geqslant 0, \end{cases} \tag{1.5}$$

with the initial conditions

$$A_i(t) = \varphi_i(t) \in \mathrm{Lip}_\alpha, \varphi_i(t) \geqslant 0, \forall t \leqslant 0 \text{ and } \tau_i(0) = \tau_{i0} \geqslant 0,$$

where

$$Z_i(t) = \sum_{j=1}^{n} \zeta_{ij} A_j(t), Z_{i\varphi}(t) := \sum_{j=1}^{n} \zeta_{ij}\varphi_j(t)$$

with $\zeta_{ij} \geqslant 0$, $i = 1, \ldots, n$. We will use the following assumptions.

Assumption 1.5 *We assume that* $\forall i = 1, \ldots, n$,

(i) The coefficients $\mu_{A_i} > 0$, $\mu_{J_i} > 0$, $\beta_i > 0$ *and* $\zeta_{ii} > 0$*;*

(ii) The function f_i *satisfies Assumption 1.1-(ii) and*

$$\sup_{x \geqslant 0} \frac{f_i(x)}{f_i(cx)} < +\infty, \forall c \geqslant 1. \tag{1.6}$$

By using the same kind of notion of solutions as in the single species case (Definition 1.2) and by using the result in [8] the well-posedness of (1.5) and the global existence of positive solutions follow.

In this article, we will prove the following result for n-species model (1.5).

Theorem 1.6 *Let Assumption 1.5 be satisfied. Then for each nonnegative initial values* $\varphi_i \geqslant 0$ *and* $\varphi_i \in \mathrm{Lip}_\alpha$ *and each* $\tau_{i0} > 0$*, the corresponding solution of equation (1.5) is bounded.*

Remark 1.7 *The proof of Theorem 1.3 (single species case) uses a similar argument as the proof of Theorem 1.6 (n-species case), which will be presented in Section 3. But for the single species case, the condition (1.6) in Assumption 1.5 is no longer needed.*

Remark 1.8 *For the n-species case we can no longer use the change of variable employed by Smith in [9, 10] since the delays* $\tau_i(t)$ *are different in general. Nevertheless, in this article we show that the arguments employed to prove the boundedness of solutions and the dissipativity in [9, 10] can be adapted to the n-species case. The notion of dissipativity will be described in details in Theorem 3.4 and Theorem 5.2.*

Remark 1.9 *It is necessary to assume that $\tau_{i0} > 0$ because we possibly have*

$$\zeta_{ij} = 0, \forall i \neq j.$$

Hence it is necessary to assume that in the case of species without coupling, the solution is bounded.

State-dependent delay differential equations have been used by several authors to describe the stage-structured population dynamics. We refer to [1, 2, 3, 4, 5, 6] for more results on this topic. We also refer to Walther [13] for a very general analysis of the semiflow generated by state-dependent delay differential equations.

The paper is organized as follows. In section 2 we will present some results about the delay $\tau(t)$. Section 3 is devoted to the single species model (1.1). The goal is to clarify the arguments of proof that we will extend later in Sections 4 and 5 to the n-species case. Section 4 is devoted to the proof the boundedness of solutions for the n-species model (1.5). In section 5, we prove a dissipativity result for such a system.

2 Properties of the integral equation for $\tau(t)$

For simplicity, we focus on the single species model (1.1) in this section. The same result can be similarly deduced for the n-species model (1.5). We have the following lemma of the equivalence of the integral equation for $\tau(t)$ and an ordinary differential equation.

Lemma 2.1 *Let $A : (-\infty, r) \to \mathbb{R}$ be a given continuous function with $r > 0$. Then there exists a uniquely determined function $\tau : [0, r) \to [0, +\infty)$ satisfying*

$$\int_{t-\tau(t)}^{t} f(A(\sigma))d\sigma = \int_{-\tau_0}^{0} f(\varphi(\sigma))d\sigma, \forall t \in [0, r). \tag{2.1}$$

Moreover this uniquely determined function $t \mapsto \tau(t)$ is continuously differentiable and satisfies the ordinary differential equation

$$\tau'(t) = 1 - \frac{f(A(t))}{f(A(t-\tau(t)))}, \forall t \in [0, r), \text{ and } \tau(0) = \tau_0. \tag{2.2}$$

Conversely if $t \mapsto \tau(t)$ is a C^1 function satisfying the above ordinary differential equation (2.2), then it also satisfies the above integral equation (2.1).

Remark 2.2 *By using equation (2.2), it is easy to check that*

$$\tau_0 > 0 \Rightarrow \tau(t) > 0, \forall t \in [0, r)$$

and

$$\tau_0 = 0 \Rightarrow \tau(t) = 0, \forall t \in [0, r).$$

Proof. Let $t \in [0, r]$. Since by Assumption 1.1, f is strictly positive, then by considering the function $\tau \mapsto \int_{t-\tau}^{t} f(A(\sigma))d\sigma$, and observing that

$$\int_{t-0}^{t} f(A(\sigma))d\sigma = 0 < \int_{-\tau_0}^{0} f(\varphi(\sigma))d\sigma \text{ and } \int_{t-(t+\tau_0)}^{t} f(A(\sigma))d\sigma \geqslant \int_{-\tau_0}^{0} f(\varphi(\sigma))d\sigma,$$

it follows by the intermediate value theorem that there exists a unique $\tau(t) \in [0, t + \tau_0]$.

By applying the implicit function theorem to the map $\psi : (t, \gamma) \mapsto \int_{\gamma}^{t} f(A(\sigma))d\sigma$ (which is possible since $\dfrac{\partial \psi}{\partial \gamma} = -f(A(\gamma))$ and by Assumption 1.1, f is strictly positive), we deduce that $t \mapsto t - \tau(t)$ is continuously differentiable, and by computing the derivative with respect to t on both sides of (2.1), we deduce that $\tau(t)$ is a solution of (2.2).

Conversely, assume that $\tau(t)$ is a solution of (2.2). Then

$$f(A(t)) = (1 - \tau'(t))f(A(t - \tau(t))), \forall t \in [0, r).$$

Integrating both sides with respect to t, we have

$$\int_0^t f(A(\sigma))d\sigma = \int_0^t f(A(\sigma - \tau(\sigma)))\,(1 - \tau'(\sigma))\,d\sigma.$$

Make the change of variable $l = \sigma - \tau(\sigma)$, we have $\forall t \in [0, r)$,

$$\begin{aligned} &\int_0^t f(A(\sigma))d\sigma = \int_{-\tau_0}^{t-\tau(t)} f(A(l))dl \\ \Leftrightarrow &\int_{t-\tau(t)}^{t} f(A(\sigma))d\sigma + \int_0^{t-\tau(t)} f(A(\sigma))d\sigma = \int_{-\tau_0}^{t-\tau(t)} f(A(l))dl \\ \Leftrightarrow &\int_{t-\tau(t)}^{t} f(A(\sigma))d\sigma = \int_{-\tau_0}^{t-\tau(t)} f(A(l))dl - \int_0^{t-\tau(t)} f(A(\sigma))d\sigma, \end{aligned}$$

this implies that $\tau(t)$ also satisfies the equation (2.1). ■

The delay $\tau(t)$ can be regarded as a functional of $A_t \in \mathrm{Lip}_\alpha$. Indeed, given a constant $C > 0$, we can define the map $\widehat{\tau} : D(\widehat{\tau}) \subset C(-\infty, 0] \times [0, +\infty) \to [0, +\infty)$ as the solution of the integral equation

$$\int_{-\widehat{\tau}(\phi, C)}^{0} f(\phi(\sigma))d\sigma = C \tag{2.3}$$

and the map $\widehat{\tau}$ is defined on the domain

$$D(\widehat{\tau}) = \left\{ (\phi, C) \in C((-\infty, 0]) \times [0, +\infty) : C < \int_{-\infty}^{0} f(\phi(\sigma))d\sigma \right\},$$

where the last integral is defined as the limit $\lim_{x\to-\infty}\int_x^0 f(\phi(\sigma))d\sigma$ (which always exists since $f \geqslant 0$).

Lemma 2.3 *Set* $C_0 := \int_{-\tau_0}^0 f(\varphi(\sigma))d\sigma$*, then we have the following relation*

$$\widehat{\tau}(A_t, C_0) = \tau(t), \forall t \in (0, r),$$

where $\tau(t)$ *is the solution of (2.1).*

Proof. It is sufficient to observe that

$$\int_{-\widehat{\tau}(A_t,C_0)}^0 f(A_t(\sigma))d\sigma = \int_{t-\widehat{\tau}(A_t,C_0)}^t f(A(\sigma))d\sigma = C_0.$$

∎

3 Boundedness and dissipativity of solutions for single species case

The following property is fundamental in this problem (see [8] for a proof).

Lemma 3.1 *Let Assumption 1.1 be satisfied. Then the function* $t - \tau(t)$ *is strictly increasing with respect to* t*.*

The first step to prove the boundedness is to prove that the map $t \mapsto t - \tau(t)$ crosses 0.

Lemma 3.2 *Let Assumption 1.1 be satisfied. Then there exists* $t^* > 0$ *such that* $t^* - \tau(t^*) = 0$*. Moreover, if* $A(t)$ *is a solution of system (1.1),* $\forall t \geqslant 0$*, then* $A(t)$ *is bounded on* $[0, t^*]$*.*

Proof. Rewrite the first equation of (1.1) as follows:

$$A'(t) = f(A(t))[-\mu_A \mathscr{B}(A(t)) + \beta e^{-\mu_J \tau(t)} \mathscr{B}(A(t - \tau(t)))], \ \forall t \geqslant 0,$$

where

$$\mathscr{B}(x) = \frac{x}{f(x)}$$

is an increasing function and $\mathscr{B}(x) \nearrow +\infty$ as $x \to +\infty$, since we have

$$\mathscr{B}(x) \geqslant \frac{x}{f(0)} \text{ when } x \geqslant 0.$$

We define $\hat{t}$ as

$$\hat{t} := \sup\{t \geqslant 0 : l - \tau(l) \leqslant 0, \ \forall l \in [0, t]\}.$$

This is well defined because the set on the right side contains at least one element 0. By Lemma 3.1, we know that the function $t-\tau(t)$ is strictly increasing, then we can assume by contradiction that $\hat{t}=+\infty$, which means that

$$t-\tau(t)<0, \forall t \geqslant 0,$$

or more precisely,

$$t-\tau(t) \in [-\tau_0, 0),\ \forall t \geqslant 0.$$

Then the equation can be written as

$$A'(t)=f(A(t))[-\mu_A \mathscr{B}(A(t))+\beta e^{-\mu_J \tau(t)} \mathscr{B}(\varphi(t-\tau(t)))],\ \forall t \geqslant 0, \tag{3.1}$$

We define

$$\Gamma := \beta \mathscr{B}\left(\sup_{t\in[-\tau_0,0]} \varphi(t)\right) > 0,$$

Then $A(t) \leqslant \hat{A}(t), \forall t \geqslant 0$, where $\hat{A}(t)$ is the solution of

$$\begin{cases} \hat{A}'(t)=-\mu_A \hat{A}(t)+\Gamma f(\hat{A}(t)) =: g_\Gamma(\hat{A}(t)), \\ \hat{A}(0)=\varphi(0) \geqslant 0. \end{cases} \tag{3.2}$$

Apparently $g_\Gamma(\hat{A})$ is monotone decreasing with respect to $\hat{A}$ and we have

$$g_\Gamma(0)=\Gamma f(0)>0,\ \lim_{\hat{A}\to+\infty} g_\Gamma(\hat{A})=-\infty.$$

Fixing $\hat{A}^* \in [\varphi(0), +\infty)$ such that $g_\Gamma(\hat{A}^*) \leqslant 0$, we have

$$A(t) \leqslant \hat{A}(t) \leqslant \hat{A}^*, \forall t \geqslant 0.$$

Now

$$1=\int_{t-\tau(t)}^t f(A(\sigma))d\sigma \geqslant \int_0^t f(A(\sigma))d\sigma \geqslant \int_0^t f(\hat{A}^*)d\sigma = tf(\hat{A}^*)$$

which is not possible for all $t \geqslant 0$ (since $f(\hat{A}^*)>0$). ■

Proof of Theorem 1.3. We have that

$$A'(t)=-\mu_A A(t)+\beta e^{-\mu_J \tau(t)} f(A(t)) \mathscr{B}(A(t-\tau(t))) \geqslant -\mu_A A(t),$$

and that the solution of

$$z'(t)=-\mu_A z(t),\ z(0)=m. \tag{3.3}$$

is

$$z(t)=z(t;m)=me^{-\mu_A t},\ t \geqslant 0.$$

Step 1: For each $m \geqslant 0$, we define $\tau_m>0$ as the unique solution of the integral equation

$$\int_0^{\tau_m} f(z(\sigma))d\sigma=1,$$

which is equivalent to the integral equation

$$\int_0^{\tau_m} f(me^{-\mu_A \sigma}) d\sigma = 1.$$

Then one can prove $\tau_m \to +\infty$ as $m \to +\infty$ (see section 4 for a similar detailed proof).

Step 2: Let $m > 0$ large enough such that

$$\beta e^{-\mu_J \tau_m} < \mu_A. \tag{3.4}$$

Step 3: Due to the fact that the function $\mathscr{B}$ is increasing and unbounded and $\mathscr{B}(0) = 0$, we can find $N > 0$, such that

$$\mathscr{B}(x) \geqslant \frac{\mu_A}{\beta} \mathscr{B}(N) \Rightarrow x \geqslant m.$$

Step 4: By Lemma 3.2, we can find $K \geqslant N$, such that

$$A(t) \leqslant K, \forall t \in [0, t^*].$$

Step 5: Next we will show that $\forall t > t^*$, $A(t) \leqslant K$. Define

$$t_K := \sup\{t \geqslant 0 : A(l) \leqslant K, \forall l \in [0, t]\},$$

and assume by contradiction that t_K is finite. Then $t_K \geqslant t^*$ and satisfies the following properties

$$A(t) \leqslant K, \forall t \in [0, t_K); A(t_K) = K, A'(t_K) \geqslant 0. \tag{3.5}$$

Now by using (3.1) and the fact that $A'(t_K) \geqslant 0$, we obtain

$$\beta \mathscr{B}(A(t_K - \tau(t_K))) \geqslant \beta e^{-\mu_J \tau(t_K)} \mathscr{B}(A(t_K - \tau(t_K))) \geqslant \mu_A \mathscr{B}(A(t_K)) = \mu_A \mathscr{B}(K) \geqslant \mu_A \mathscr{B}(N),$$

and by using step 3 we deduce that

$$A(t_K - \tau(t_K)) \geqslant m.$$

By using a comparison principle on

$$A'(t) \geqslant -\mu_A A(t), \forall t \geqslant t_K - \tau(t_K), A(t_K - \tau(t_K)) \geqslant m$$

and the equation (3.3), we have

$$A(t) \geqslant z(t - t_K + \tau(t_K)), \forall t \geqslant t_K - \tau(t_K).$$

Now since $x \mapsto f(x)$ is decreasing we deduce that

$$1 = \int_{t_K - \tau(t_K)}^{t_K} f(A(\sigma)) d\sigma \leqslant \int_{t_K - \tau(t_K)}^{t_K} f(z(\sigma - t_K + \tau(t_K))) d\sigma = \int_0^{\tau(t_K)} f(z(\sigma)) d\sigma.$$

By the definition of τ_m, we must have

$$\tau(t_K) \geqslant \tau_m. \tag{3.6}$$

By using (3.4)-(3.6), we obtain

$$\begin{aligned} 0 &\leqslant A'(t_K) = f(A(t_K)) \left[-\mu_A \mathscr{B}(A(t_K)) + \beta e^{-\mu_J \tau(t_K)} \mathscr{B}(A(t_K - \tau(t_K))) \right] \\ &= f(K) \left[-\mu_A \mathscr{B}(K) + \beta e^{-\mu_J \tau(t_K)} \mathscr{B}(A(t_K - \tau(t_K))) \right] \\ &\leqslant f(K) \left[-\mu_A \mathscr{B}(K) + \beta e^{-\mu_J \tau_m} \mathscr{B}(A(t_K - \tau(t_K))) \right] \\ &\leqslant f(K) \left(-\mu_A \mathscr{B}(K) + \beta e^{-\mu_J \tau_m} \mathscr{B}(K) \right) \leqslant K \left(-\mu_A + \beta e^{-\mu_J \tau_m} \right) < 0. \end{aligned} \tag{3.7}$$

This contradiction shows that t_K can not be finite. By using the definition of t_K we deduce that $A(t) \leqslant K, \forall t > t^*$. ■

In the rest of the section we study the dissipativity of the system, namely we look for an asymptotic uniform bound for solutions starting in some bounded sets. In order to study this property we need the following lemma.

Lemma 3.3 *Let Assumptions 1.1 be satisfied. Suppose that $(A(t), \tau(t))$ is the solution of system (1.1), then*

$$\lim_{t \to +\infty} [t - \tau(t)] = +\infty.$$

Proof. If $\tau_0 = 0$, then $\tau(t) = 0, \forall t \geqslant 0$ and there is nothing to prove. If $\tau_0 > 0$, then by Theorem 1.3 we know that $t \mapsto A(t)$ is bounded from above by a certain constant $K > 0$. Since $\tau(t)$ is the unique solution of the integral equation

$$\int_{-\tau(t)}^{0} f(A(t + \sigma)) d\sigma = \delta, \forall t \geqslant 0$$

where $\delta := \displaystyle\int_{-\tau_0}^{0} f(\varphi(\sigma)) d\sigma > 0$, by using the fact that $x \mapsto f(x)$ is decreasing we deduce that $\tau(t) f(K) \leqslant \delta, \forall t \geqslant 0$, and it follows that $t \mapsto \tau(t)$ is bounded by $f(K)^{-1}\delta$. This completes the proof. ■

Theorem 3.4 (Dissipativity) *Let $\alpha > 0$. Let Assumption 1.1 be satisfied. Let $B \subset \mathrm{Lip}_\alpha$ be a bounded subset and $[\tau_{\min}, \tau_{\max}] \subset (0, +\infty)$ be a fixed interval. Denote*

$$\delta_{\min} := \inf_{(\varphi, \tau_0) \in B \times [\tau_{\min}, \tau_{\max}]} \int_{-\tau_0}^{0} f(\varphi(\sigma)) d\sigma.$$

Then for each initial condition $(\varphi, \tau_0) \in B \times [\tau_{\min}, \tau_{\max}]$, there exists a constant $M^* = M^*(\delta_{\min}) > 0$ (independent of the initial condition) such that

$$\limsup_{t \to +\infty} A(t) \leqslant M^*.$$

Proof. Similarly as in step 1 of the proof of Theorem 1.3, we consider τ_m the unique solution of the integral equation

$$\int_0^{\tau_m} f(me^{-\mu_A \sigma})d\sigma = \delta_{\min}.$$

Then we can find $M^* > 0$ (large enough) such that for each $M \geqslant M^*$, the two following inequalities

$$-\mu_A + \beta e^{-\mu_J \tau_m} < 0 \text{ with } m := \frac{\mu_A M}{\beta} \tag{3.8}$$

and

$$-\mu_A + \beta e^{-\mu_J \frac{\delta_{\min}}{f(M)}} < 0 \tag{3.9}$$

are satisfied.

Now suppose that we can find $t \mapsto (A(t), \tau(t))$ the solution of system (1.1) with the initial condition $(\varphi, \tau_0) \in B \times [\tau_{\min}, \tau_{\max}]$ satisfying

$$M := \limsup_{t \to +\infty} A(t) \geqslant M^*.$$

Then we have the following alternatives:

Case 1: There exists a time sequence $\{t_n\}_{n \in \mathbb{N}}$ which satisfies $\lim\limits_{n \to +\infty} t_n = +\infty$ and for any t_n,

$$A'(t_n) = 0,$$

and

$$A(t_n) \to M \text{ as } n \to +\infty.$$

Then we have

$$0 = A'(t_n) = -\mu_A A(t_n) + \beta e^{-\mu_J \tau(t_n)} \frac{f(A(t_n))}{f(A(t_n - \tau(t_n)))} A(t_n - \tau(t_n)).$$

By taking the limit on both sides when $n \to +\infty$, we have

$$0 \leqslant -\mu_A M + \beta f(M) \limsup_{n \to +\infty} \frac{A(t_n - \tau(t_n))}{f(A(t_n - \tau(t_n)))},$$

and since the map $x \mapsto \dfrac{1}{f(x)}$ is increasing, we deduce that

$$0 \leqslant -\mu_A M + \beta \frac{f(M)}{f(M)} \limsup_{n \to +\infty} A(t_n - \tau(t_n)).$$

Hence

$$\limsup_{n \to +\infty} A(t_n - \tau(t_n)) \geqslant \frac{\mu_A M}{\beta} = m. \tag{3.10}$$

Now by using the same method as in step 5 of the proof of Theorem 1.3 and noticing that

$$\delta_{\min} \leqslant \int_{t_n-\tau(t_n)}^{t_n} f(A(\sigma))d\sigma \leqslant \int_{t_n-\tau(t_n)}^{t_n} f(z(\sigma - t_n + \tau(t_n)))d\sigma = \int_0^{\tau(t_n)} f(z(\sigma))d\sigma,$$

we get $\tau(t_n) \geqslant \tau_m$. Thus we can repeat the procedure in (3.7) and get a contradiction

$$0 = \lim_{n\to+\infty} A'(t_n) \leqslant M\left(-\mu_A + \beta e^{-\mu_J \tau_m}\right) < 0.$$

Case 2: The solution $A(t)$ is eventually monotone. So we can assume that there exists a time $\bar{t} > 0$ such that

$$A'(t) \geqslant 0, \forall t \geqslant \bar{t}$$

(the case $A'(t) \leqslant 0, \forall t \geqslant \bar{t}$ being similar). Since $A(t)$ is eventually increasing, we deduce that

$$\lim_{t\to+\infty} A_t = M \text{ in } C_\alpha := \left\{\phi \in C(-\infty, 0] : e^{-\alpha|.|}\phi(.) \text{ is bounded}\right\}$$

where C_α is the Banach space endowed with the norm $\|\phi\|_{C_\alpha} := \|e^{-\alpha|.|}\phi(.)\|_\infty$.

As $A(t)$ is bounded, A'_t is relatively compact in Lip_α (since $\alpha > 0$, $A(t)$ satisfies system (1.1) and by applying Arzelà-Ascoli theorem locally on the bounded interval $[-\theta^*, 0]$ for each $\theta^* > 0$ and by using the step method to extend to $(-\infty, 0]$), we get

$$\lim_{t\to+\infty} A'_t = 0 \text{ in } L^\infty_\alpha := \left\{\phi \in C(-\infty, 0] : e^{-\alpha|.|}\phi(.) \in L^\infty(-\infty, 0]\right\}$$

where L^∞_α is the Banach space endowed with the norm $\|\phi\|_{L^\infty_\alpha} := \|e^{-\alpha|.|}\phi(.)\|_{L^\infty}$.

Moreover, we have

$$\delta_{\min} \leqslant \int_{t-\tau(t)}^{t} f(A(\sigma))d\sigma =: \delta,$$

and by taking the limit when $t \to +\infty$ (and since by Lemma 3.3 $t - \tau(t) \to +\infty$) we obtain

$$\lim_{t\to+\infty} \tau(t) = \frac{\delta}{f(M)} \geqslant \frac{\delta_{\min}}{f(M)}.$$

By taking the limit when $t \to +\infty$ in the first equation of system (1.1) we obtain the following contradiction

$$0 = \lim_{t\to+\infty} A'_t \leqslant -\mu_A M + \beta e^{-\mu_J \frac{\delta_{\min}}{f(M)}} \frac{f(M)}{f(M)} M < 0.$$

Both cases lead to a contradiction, which implies that

$$\limsup_{t\to+\infty} A(t) \leqslant M^*.$$

■

4 Boundedness of solutions for n-species case

In this section we will investigate the boundedness of a trajectory of system (1.5) with the initial conditions satisfying

$$\int_{-\tau_{i0}}^{0} f_i(Z_{i\varphi}(\sigma))d\sigma > 0, \forall i = 1, ..., n.$$

Multiplying each of the above integrals by a positive constant, we can assume without loss of generality that

Assumption 4.1

$$\int_{-\tau_{i0}}^{0} f_i(Z_{i\varphi}(\sigma))d\sigma = 1, \forall i = 1, ..., n.$$

We have the following lemma from [8].

Lemma 4.2 *Let Assumptions 1.5 and 4.1 be satisfied. Then the functions $t - \tau_i(t)$ are strictly increasing with respect to t, $\forall i = 1, \ldots, n$.*

Next we will prove the following result.

Lemma 4.3 *Let Assumptions 1.5 and 4.1 be satisfied. Then for each $i = 1, \ldots, n$ there exists $t_i^* > 0$ such that $t_i^* - \tau_i(t_i^*) = 0$.*

Proof. For each $i = 1, \ldots, n$ we define

$$t_i^* := \sup\{t \geqslant 0 : s - \tau_i(s) \leqslant 0, \forall s \in [0, t]\}.$$

Case 1: We assume that all the elements of $\{t_i^*\}_{i=1}^n$ are infinite, and we will prove that this is not possible. By the above definition of t_i^*, we have $\forall t \geqslant 0$, $t - \tau_i(t) \leqslant 0$, or precisely,

$$t - \tau_i(t) \in [-\tau_{i0}, 0].$$

Then the equation for $A_i(t)$ becomes

$$A_i'(t) = -\mu_{A_i} A_i(t) + \beta_i e^{-\mu_{J_i}\tau_i(t)} \frac{f_i(Z_i(t))}{f_i(Z_{i\varphi}(t - \tau_i(t)))} \varphi_i(t - \tau_i(t)), \forall t \geqslant 0.$$

We set

$$\Gamma_i := \beta_i \sup_{t\in[-\tau_{i0},0]} \frac{\varphi_i(t)}{f_i(Z_{i\varphi}(t))} > 0.$$

Since $f_i(Z_i(t)) \leqslant f_i(\zeta_{ii} A_i(t)), \forall t \geqslant 0$, then by the comparison principle, we have $A_i(t) \leqslant \hat{A}_i(t)$, $\forall t \geqslant 0$, where $\hat{A}_i(t)$ is the solution of

$$\begin{cases} \hat{A}_i'(t) = -\mu_{A_i} \hat{A}_i(t) + \Gamma_i f_i(\zeta_{ii} \hat{A}_i(t)) =: g_{\Gamma_i}(\hat{A}_i(t)), \forall t \geqslant 0, \\ \hat{A}_i(0) = \varphi_i(0) \geqslant 0. \end{cases}$$

As $g_{\Gamma_i}(\hat{A}_i)$ is decreasing with $\hat{A}_i$ and we have

$$g_{\Gamma_i}(0) = \Gamma_i f_i(0) > 0, \lim_{\hat{A}_i \to +\infty} g_{\Gamma_i}(\hat{A}_i) = -\infty,$$

so fixing $\hat{A}_i^* \in [\varphi_i(0), +\infty)$ such that $g_{\Gamma_i}(\hat{A}_i^*) \leqslant 0$, we have

$$A_i(t) \leqslant \hat{A}_i(t) \leqslant \hat{A}_i^*, \forall t \geqslant 0.$$

Now since by assumption $t - \tau_i(t) \leqslant 0, \forall t \geqslant 0$, we obtain for each $t \geqslant 0$

$$1 = \int_{t-\tau_i(t)}^{t} f_i(Z_i(\sigma))d\sigma \geqslant \int_0^t f_i(Z_i(\sigma))d\sigma \geqslant t f_i\left(\sum_{j=1}^{n} \zeta_{ij}\hat{A}_j^*\right) \tag{4.1}$$

which is impossible.

Case 2: We assume that exactly j elements of $\{t_i^*\}_{i=1}^n$ are finite, where $1 \leqslant j < n$, and we will prove that this is not possible, either. Without loss of generality we might assume that $t_1^*, \dots, t_j^*$ are finite and $t_{j+1}^*, \dots, t_n^*$ are infinite. Firstly we prove that $A_1(t), \dots, A_n(t)$ are bounded on $[0, +\infty)$.

Following a similar argument as in case 1, for each $i = j+1, \dots, n$, as t_i^* is infinite, we can find $\hat{A}_i^* \in [\varphi_i(0), +\infty)$ such that

$$A_i(t) \leqslant \hat{A}_i^*, \forall t \geqslant 0.$$

For each $k = 1, \dots, j$, consider

$$z_k(t) = z_k(t; m_k) = m_k e^{-\mu_{A_k} t}, t \geqslant 0$$

where $m_k > 0$ will be fixed later on and as before $z_k(t)$ is a solution of the following ordinary differential equation

$$z_k'(t) = -\mu_{A_k} z_k(t), z_k(0) = m_k. \tag{4.2}$$

We define $\tau_{k,m_k} > 0$ as the unique solution of the integral equation

$$\int_0^{\tau_{k,m_k}} f_k(\zeta_{kk} z_k(\sigma))d\sigma = 1. \tag{4.3}$$

By Assumption 1.5-(i), we have $\zeta_{kk} > 0$ and

$$\int_0^{\tau} f_k(\zeta_{kk} z_k(\sigma))d\sigma \geqslant \int_0^{\tau} f_k(\zeta_{kk} m_k)d\sigma = \tau f_k(\zeta_{kk} m_k) > 0 \text{ when } \tau > 0,$$

therefore $\tau_{k,m_k} > 0$ exists and is finite. Next we observe that we have

$$\tau_{k,m_k} \to +\infty \text{ as } m_k \to +\infty. \tag{4.4}$$

Indeed, assume by contradiction that there exists a subsequence $\{m_{k,l}\}_{l \geqslant 0} \to +\infty$ and a sequence $\{\tau_{k,m_{k,l}}\}_{l \geqslant 0}$ bounded by $\tau^* > 0$. Then we have

$$1 = \int_0^{\tau_{k,m_{k,l}}} f_k(\zeta_{kk} z_k(\sigma))d\sigma \leqslant \int_0^{\tau^*} f_k(\zeta_{kk} z_k(\sigma))d\sigma \to 0 \text{ as } l \to +\infty$$

which is impossible.
By Assumption 1.5-(ii), for each $c \geqslant 1$,

$$M_{f_k}(c) := \sup_{x \geqslant 0} \frac{f_k(x)}{f_k(cx)} < +\infty.$$

By using (4.4) we can fix m_k (large enough) such that

$$-\mu_{A_k} + \beta_k e^{-\mu_{J_k}\tau_{k,m_k}} M_{f_k}\left(\frac{\zeta_{k1} + \cdots + \zeta_{kn}}{\zeta_{kk}}\right) < 0. \tag{4.5}$$

For a constant $K > 0$, define

$$t_K := \sup\{t \geqslant 0 : \max\{A_1(s), \dots, A_j(s)\} \leqslant K, \forall s \in [0, t]\}.$$

Let us now prove that $A_1(t), \dots, A_j(t)$ are bounded on $[0, +\infty)$. Assume by contradiction that t_K is finite for each $K > 0$ large enough. Then at least one of $A_k(t), k = 1, \dots, j$ reaches K at t_K. Assume for example that $A_1(t_K) = K$. Firstly we prove that

$$t_K - \tau_1(t_K) \geqslant 0 \tag{4.6}$$

for each $K > 0$ large enough. Otherwise $t_K - \tau_1(t_K) < 0$ and thus t_K must be smaller than t_1^*, then we can use the same comparison principle arguments as in case 1 on the interval of time $[0, t_1^*]$, and we can find $\hat{A}_1^* > 0$ (independent of K) such that

$$K = A_1(t_K) \leqslant \hat{A}_1^*,$$

which becomes impossible whenever K becomes large enough. We deduce that (4.6) holds true.

Now we will prove $A_1(t_K - \tau_1(t_K)) \to +\infty$ when $K \to +\infty$. By assumption t_K is finite, and by definition of t_K we have

$$A_1(t) \leqslant K, \forall t \in [0, t_K]$$

and we must have

$$A_1'(t_K) \geqslant 0.$$

Then

$$\begin{aligned} 0 \quad &\leqslant \quad A_1'(t_K) = -\mu_{A_1} A_1(t_K) + \beta_1 e^{-\mu_{J_1}\tau_1(t_K)} \frac{f_1(Z_1(t_K))}{f_1(Z_1(t_K - \tau_1(t_K)))} A_1(t_K - \tau_1(t_K)) \\ &\leqslant \quad -\mu_{A_1} K + \beta_1 \frac{f_1(\zeta_{11} K)}{f_1((\zeta_{11} + \cdots + \zeta_{1n})\hat{K})} A_1(t_K - \tau_1(t_K)), \end{aligned}$$

where

$$\hat{K} := \max\left\{K, \hat{A}_{j+1}^*, \dots, \hat{A}_n^*, \max_{t \in [-\tau_{10}, 0]} \varphi_1(t), \dots, \max_{t \in [-\tau_{n0}, 0]} \varphi_n(t)\right\}.$$

Notice that $\dfrac{(\zeta_{11}+\cdots+\zeta_{1n})\hat{K}}{\zeta_{11}K}>1$, we have

$$A_1(t_K-\tau_1(t_K)) \geqslant \frac{\mu_{A_1}K}{\beta_1}\cdot\frac{f_1((\zeta_{11}+\cdots+\zeta_{1n})\hat{K})}{f_1(\zeta_{11}K)} \geqslant \frac{\mu_{A_1}K}{\beta_1}\cdot\frac{1}{M_{f_1}\left(\frac{(\zeta_{11}+\cdots+\zeta_{1n})\hat{K}}{\zeta_{11}K}\right)}.$$

Now since for all $K>0$ large enough $\hat{K}=K$, we deduce that

$$A_1(t_K-\tau_1(t_K)) \to +\infty \text{ as } K \to +\infty.$$

By using (4.6), we can fix K large enough such that

$$A_1(t_K-\tau_1(t_K)) \geqslant m_1 \text{ and } t_K-\tau_1(t_K) \geqslant 0.$$

By using the comparison principle on equation (4.2) and

$$A_1'(t) \geqslant -\mu_{A_1}A_1(t), \forall t \geqslant t_K-\tau_1(t_K)$$

with

$$A_1(t_K-\tau_1(t_K)) \geqslant m_1,$$

we have

$$A_1(t) \geqslant z_1(t-t_K+\tau_1(t_K)), \forall t \geqslant t_K-\tau_1(t_K).$$

An integration shows that

$$\begin{aligned} 1 &= \int_{t_K-\tau_1(t_K)}^{t_K} f_1(Z_1(\sigma))d\sigma \leqslant \int_{t_K-\tau_1(t_K)}^{t_K} f_1(\zeta_{11}A_1(\sigma))d\sigma \\ &\leqslant \int_{t_K-\tau_1(t_K)}^{t_K} f_1(\zeta_{11}z_1(\sigma-t_K+\tau_1(t_K)))d\sigma = \int_0^{\tau_1(t_K)} f_1(\zeta_{11}z_1(\sigma))d\sigma. \end{aligned}$$

By the definition of $\tau_{1,m_1}>0$ (defined as the solution of (4.3)), we have

$$\tau_1(t_K) \geqslant \tau_{1,m_1}.$$

Now by using (4.5), we have

$$\begin{aligned} 0 &\leqslant A_1'(t_K) = -\mu_{A_1}A_1(t_K) + \beta_1 e^{-\mu_{J_1}\tau_1(t_K)}\frac{f_1(Z_1(t_K))}{f_1(Z_1(t_K-\tau_1(t_K)))}A_1(t_K-\tau_1(t_K)) \\ &= f_1(Z_1(t_K))\left[-\mu_{A_1}\frac{A_1(t_K)}{f_1(Z_1(t_K))} + \beta_1 e^{-\mu_{J_1}\tau_1(t_K)}\frac{A_1(t_K-\tau_1(t_K))}{f_1(Z_1(t_K-\tau_1(t_K)))}\right] \\ &\leqslant f_1(Z_1(t_K))\left[-\mu_{A_1}\frac{K}{f_1(\zeta_{11}K)} + \beta_1 e^{-\mu_{J_1}\tau_{1,m_1}}\frac{K}{f_1((\zeta_{11}+\cdots+\zeta_{1n})K)}\right] \\ &= \frac{f_1(Z_1(t_K))K}{f_1(\zeta_{11}K)}\left[-\mu_{A_1} + \beta_1 e^{-\mu_{J_1}\tau_{1,m_1}}\frac{f_1(\zeta_{11}K)}{f_1((\zeta_{11}+\cdots+\zeta_{1n})K)}\right] \\ &\leqslant \frac{f_1(Z_1(t_K))K}{f_1(\zeta_{11}K)}\left[-\mu_{A_1} + \beta_1 e^{-\mu_{J_1}\tau_{1,m_1}}M_{f_1}\left(\frac{\zeta_{11}+\cdots+\zeta_{1n}}{\zeta_{11}}\right)\right] < 0, \end{aligned} \tag{4.7}$$

which leads to a contradiction. Thus for $K > 0$ large enough t_K is infinite, namely

$$A_k(t) \leqslant K, \forall t \geqslant 0, \forall k = 1, \dots, j.$$

Observe that by assumption $t^*_{j+1}, \dots, t^*_n$ are infinite, which means that $t - \tau_i(t) \leqslant 0, \forall t \geqslant 0, \forall i = j+1, \dots, n$, therefore we deduce that for all $t \geqslant 0$,

$$\begin{aligned} 1 &= \int_{t-\tau_i(t)}^{t} f_i(Z_i(\sigma))d\sigma \geqslant \int_0^t f_i(Z_i(\sigma))d\sigma \\ &\geqslant \int_0^t f_i((\zeta_{i1} + \cdots + \zeta_{ij})K + \zeta_{i,j+1}\hat{A}^*_{j+1} + \cdots + \zeta_{in}\hat{A}^*_n)d\sigma \\ &= t f_i((\zeta_{i1} + \cdots + \zeta_{ij})K + \zeta_{i,j+1}\hat{A}^*_{j+1} + \cdots + \zeta_{in}\hat{A}^*_n) \end{aligned}$$

which is impossible when t is large enough. The proof is completed. ■

Proof of Theorem 1.6. For each $i = 1, \dots, n$, we define τ_{i,m_i} satisfying

$$\int_0^{\tau_{i,m_i}} f_i(\zeta_{ii} z_i(\sigma))d\sigma = 1,$$

where $z_i(t) = m_i e^{-\mu_{A_i} t}$, $t \geqslant 0$. Same as (4.5) in case 2 of Lemma 4.3, we can fix m_i large enough such that

$$\beta_i e^{-\mu_{J_i}\tau_{i,m_i}} M_{f_i}\left(\frac{\zeta_{i1} + \cdots + \zeta_{in}}{\zeta_{ii}}\right) < \mu_{A_i}.$$

For a constant $K > 0$, we define

$$t_K := \sup\{t > 0 : \max\{A_1(s), \dots, A_n(s)\} \leqslant K, \forall s \in [0, t]\}.$$

Then similar to the procedure of case 2 in the proof of Lemma 4.3, we can get a fixed K large enough and we can deduce that $t_K = +\infty$. Thus $A_i(t)$ is bounded for all $t \geqslant 0$. ■

5 Dissipativity of the system

In this section we will investigate the dissipativity of system (1.5). First, we have the following lemma similar as Lemma 3.3.

Lemma 5.1 *Let Assumptions 1.5 be satisfied. Suppose that $(A_i(t), \tau_i(t))$ is the solution of system (1.5), then*

$$\lim_{t \to +\infty} [t - \tau_i(t)] = +\infty.$$

Proof. If $\tau_{i0} = 0$, then again there is nothing to prove. When $\tau_{i0} > 0$, by Theorem 1.6 we know that $t \mapsto A_i(t)$ is bounded from above by a certain constant $K > 0$. Since $\tau_i(t)$ is the unique solution of the integral equation

$$\int_{t-\tau_i(t)}^{t} f_i(Z_i(\sigma))d\sigma = \hat{\delta}, \forall t \geqslant 0$$

where $\hat{\delta} := \int_{-\tau_{i0}}^{0} f_i(Z_{i\varphi}(\sigma))d\sigma > 0$, then similar as the proof of Lemma 3.3, we deduce that $t \mapsto \tau_i(t)$ is bounded by $f_i((\zeta_{i1} + \ldots + \zeta_{in})K)^{-1}\hat{\delta}$. This completes the proof. ■

Theorem 5.2 (Dissipativity) *Let Assumption 1.5 be satisfied. Let $B_i \subset \mathrm{Lip}_\alpha$ be a bounded subset and $[\tau_{i,\min}, \tau_{i,\max}] \subset (0, +\infty)$ be a fixed interval, $i = 1, \ldots, n$. Let*

$$B := \prod_{i=1}^{n} B_i \text{ and } I_\tau := \prod_{i=1}^{n} [\tau_{i,\min}, \tau_{i,\max}].$$

Denote

$$\hat{\delta}_{\min} := \inf_{(\varphi,\tau_0)\in B\times I_\tau} \int_{-\tau_{i0}}^{0} f_i(Z_{i\varphi}(\sigma))d\sigma$$

where $\varphi := (\varphi_1, \ldots, \varphi_n)$ and $\tau_0 = (\tau_{10}, \ldots, \tau_{n0})$. Then for each initial condition $(\varphi, \tau_0) \in B \times I_\tau$, there exists a constant $M^* = M^*(\hat{\delta}_{\min}) > 0$ (independent of the initial condition) such that

$$\limsup_{t\to+\infty} \max_{i=1,\ldots,n} \{A_i(t)\} \leqslant M^*.$$

Proof. Similarly as in case 2 of the proof of Lemma 4.3, we consider τ_{i,m_i} the unique solution of the integral equation

$$\int_{0}^{\tau_{k,m_k}} f_k(\zeta_{kk} z_k(\sigma))d\sigma = \hat{\delta}_{\min}.$$

Then we can find $M^* > 0$ (large enough) such that for each $M \geqslant M^*$, the two following inequalities

$$-\mu_{A_i} + \beta_i e^{-\mu_{J_i}\tau_{i,m_i}} M_{f_i}\left(\frac{\zeta_{i1} + \cdots + \zeta_{in}}{\zeta_{ii}}\right) < 0 \tag{5.1}$$

with $m_i := \dfrac{\mu_{A_i} M}{\beta_i} \cdot \dfrac{1}{M_{f_i}\left(\frac{\zeta_{i1}+\cdots+\zeta_{in}}{\zeta_{ii}}\right)}$ and

$$-\mu_{A_i} + \beta_i e^{-\mu_{J_i}\frac{\hat{\delta}_{\min}}{f_i(\zeta_{ii}M)}} M_{f_i}\left(\frac{\zeta_{i1} + \cdots + \zeta_{in}}{\zeta_{ii}}\right) < 0 \tag{5.2}$$

are satisfied for any $i = 1, \ldots, n$.

Now suppose that we can find $t \mapsto (A(t), \tau(t))$ the solution of system (1.5) with the initial condition $(\varphi, \tau_0) \in B \times I_\tau$ satisfying

$$M := \limsup_{t\to+\infty} \max_{i=1,\ldots,n} \{A_i(t)\} \geqslant M^*.$$

Without loss of generality we might assume that $M = \limsup_{t\to+\infty} A_1(t)$. Then we have the following alternatives:

Case 1: There exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ which satisfies $\lim_{n\to+\infty} t_n = +\infty$ and for any t_n,

$$A_1'(t_n) = 0,$$

and

$$A_1(t_n) \to M \text{ as } n \to +\infty.$$

Then we have

$$0 = A_1'(t_n) = -\mu_{A_1} A_1(t_n) + \beta_1 e^{-\mu_{J_1}\tau_1(t_n)} \frac{f_1(Z_1(t_n))}{f_1(Z_1(t_n - \tau_1(t_n)))} A_1(t_n - \tau_1(t_n)).$$

By taking the limit on both sides when $n \to +\infty$, we have

$$0 \leqslant -\mu_{A_1} M + \beta_1 f_1(\zeta_{11} M) \limsup_{n\to+\infty} \frac{A_1(t_n - \tau_1(t_n))}{f_1(Z_1(t_n - \tau_1(t_n)))},$$

and since the map $x \mapsto \dfrac{1}{f(x)}$ is increasing, we deduce that

$$0 \leqslant -\mu_{A_1} M + \beta_1 \frac{f_1(\zeta_{11} M)}{f_1((\zeta_{11} + \ldots + \zeta_{1n})M)} \limsup_{n\to+\infty} A_1(t_n - \tau_1(t_n)).$$

Hence

$$\limsup_{n\to+\infty} A_1(t_n - \tau_1(t_n)) \geqslant \frac{\mu_{A_1} M}{\beta_1} \cdot \frac{1}{M_{f_1}\left(\frac{\zeta_{11}+\cdots+\zeta_{1n}}{\zeta_{11}}\right)} = m_1. \tag{5.3}$$

Now by using the same method as in case 2 of the proof of Lemma 4.3 and noticing that

$$\begin{aligned} \hat{\delta}_{\min} &\leqslant \int_{t_n-\tau_1(t_n)}^{t_n} f_1(Z_1(\sigma))d\sigma \leqslant \int_{t_n-\tau_1(t_n)}^{t_n} f_1(\zeta_{11} A_1(\sigma))d\sigma \\ &\leqslant \int_{t_n-\tau_1(t_n)}^{t_n} f_1(\zeta_{11} z_1(\sigma - t_n + \tau_1(t_n)))d\sigma = \int_0^{\tau_1(t_n)} f_1(\zeta_{11} z_1(\sigma))d\sigma, \end{aligned}$$

we get $\tau_1(t_n) \geqslant \tau_{1,m_1}$. Thus we can repeat the procedure in (4.7) and get a contradiction

$$0 = \lim_{n\to+\infty} A_1'(t_n) \leqslant M\left(-\mu_{A_1} + \beta_1 e^{-\mu_{J_1}\tau_{1,m_1}} M_{f_1}\left(\frac{\zeta_{11} + \cdots + \zeta_{1n}}{\zeta_{11}}\right)\right) < 0.$$

Case 2: The solution $A_1(t)$ is eventually monotone. So we can assume that there exists a time $\bar{t} > 0$ such that

$$A_1'(t) \geqslant 0, \forall t \geqslant \bar{t}$$

(the case $A_1'(t) \leqslant 0, \forall t \geqslant \bar{t}$ being similar). Since $A_1(t)$ is eventually increasing, we deduce that

$$\lim_{t\to+\infty} A_{1,t} = M \text{ in } C_\alpha := \left\{\phi \in C(-\infty, 0] : e^{-\alpha|.|}\phi(.) \text{ is bounded}\right\}$$

where C_α is the Banach space endowed with the norm $\|\phi\|_{C_\alpha} := \|e^{-\alpha|.|}\phi(.)\|_\infty$.

As $A_1(t)$ is bounded, $A'_{1,t}$ is relatively compact in Lip_α (since $\alpha > 0$, $A_i(t), i = 1, \ldots, n$ satisfy the system (1.5) and by applying Arzelà-Ascoli theorem locally on the bounded interval $[-\theta^*, 0]$ for each $\theta^* > 0$ and by using the step method to extend to $(-\infty, 0]$), we get

$$\lim_{t\to+\infty} A'_{1,t} = 0 \text{ in } L^\infty_\alpha := \left\{\phi \in C(-\infty, 0] : e^{-\alpha|.|}\phi(.) \in L^\infty(-\infty, 0]\right\}$$

where L^∞_α is the Banach space endowed with the norm $\|\phi\|_{L^\infty_\alpha} := \|e^{-\alpha|.|}\phi(.)\|_{L^\infty}$.

Moreover, we have

$$\hat{\delta}_{\min} \leqslant \int_{t-\tau_1(t)}^{t} f_1(Z_1(\sigma))d\sigma =: \delta_1,$$

and by taking the limit when $t \to +\infty$ (and since by Lemma 5.1 $t-\tau_1(t) \to +\infty$) we obtain

$$\lim_{t\to+\infty} \tau_1(t) = \frac{\delta_1}{f((\zeta_{11} + \ldots + \zeta_{1n})M)} \geqslant \frac{\hat{\delta}_{\min}}{f(\zeta_{11}M)}.$$

By taking the limit when $t \to +\infty$ in the first equation of system (1.5) we obtain the following contradiction

$$0 = \lim_{t\to+\infty} A'_{1,t} \leqslant -\mu_{A_1}M + \beta_1 e^{-\mu_{J_1}\frac{\hat{\delta}_{\min}}{f_1(\zeta_{11}M)}} \frac{f_1(\zeta_{11}M)}{f_1((\zeta_{11} + \cdots + \zeta_{1n})M)} M < 0.$$

Both cases lead to a contradiction, which proves that

$$\limsup_{t\to+\infty} \max_{i=1,\ldots,n} \{A_i(t)\} \leqslant M^*.$$

■

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