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Theory and Applications of Abstract Semilinear Cauchy Problems

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To Enzo and Clara
and
To Yilin, Marion and Lucia

Foreword

Prediction is very difficult, especially about the future. – Niels Bohr

I know that in the study of material things number, order, and position are the threefold clue to exact knowledge: and that these three, in the mathematician's hands, furnish the 'first outlines for a sketch of the Universe'. – D'Arcy Thompson, On Growth and Form (1917)

The subject of differential equations has a long and storied history. At its foundation is the fundamental nature of physical change. More than two centuries ago, differential equations describing physical change were studied and applied with monumental success. The subject has grown ever since with extraordinary productivity in mathematical theory and scientific applications. The development of recent models of dynamical processes offer ever-increasing mathematical challenge. At the core of this mathematical challenge, there are fundamental ideas.

One of the most important fundamental ideas in models of physical change is the assumption of determinism. The basic idea is that the present determines the future. This idea is encompassed into differential equations of dynamical processes as a known initial condition at a specified time 0. Newton's second law states that the rate of change of momentum of a body is directly proportional to the force applied: $F = m\,dv/dt$, where m is the mass and v is the velocity. If the initial velocity $v(0)$ of the body is known, then the future velocity is known for all time. This conceptualization of deterministic behavior is a foundational mathematical description of scientific phenomena.

An alternative view of determinism is that the past determines the future. This idea encompasses into the differential equations of dynamical processes a requirement that the future forward from an initial time 0 is dependent on the history of the process up to time 0. The initial condition of such a process must incorporate more than the current state, but in addition, a past history of the current state.

The mathematical theory of the differential equations of history-determined processes has a much more recent development. The subject is known as functional differential equations. A key role in the development of functional differential equations was played by Jack Hale. In his 1977 monograph (*Theory of Functional Differential Equations*, Springer-Verlag), the theory of ordinary differential equations in finite dimensional spaces was extended to functional differential equations in a comprehensive treatment. Theoretical results about existence, uniqueness, initial conditions, stability, periodicity, asymptotic behavior, and other basic ideas were developed. One of the key ideas was the formulation of functional differential equations as abstract ordinary differential equations in infinite dimension spaces. This idea is accomplished by utilizing the theory of semigroups of linear operators in infinite dimensional spaces.

The theory of linear semigroups of operators has been developed extensively and key roles were played by the monographs of E. Hille and R.S. Phillips (*Functional Analysis and Semi-Groups*, Amer. Math. Soc.,1948; 1957), K. Yosida (*Functional Analysis*, Springer-Verlag, 1965), T. Kato (*Perturbation Theory of Linear Operators*, Springer-Verlag, 1966), and A. Pazy (*Semigroups of Linear Operator and Applications to Partial Differential Equations*, Springer-Verlag, 1983). The basic idea of a semigroup of operators is the idea of an exponential process. The solution of the abstract differential equation $dx(t)/dt = Ax(t)$, in an infinite dimensional space X, with initial condition $x(0) = x_0$, is $x(t) = e^{tA}x_0$, where $x_0 \in X$ and e^{tA} is the exponential of tA. If A is a bounded operator (matrix), then e^{tA} is $\sum_{n=0}^{\infty} t^n A^n/n!$. If A is an unbounded linear operator, then $e^{tA} = \lim_{n\to\infty}(I - t/nA)^{-n}$, where $(I - \lambda A)^{-1}$ is the resolvent of A. The operator A is called the infinitesimal generator of the semigroup of linear operators $T(t) = e^{tA}, t \geq 0$. In classical linear semigroup theory, A is densely defined in the state space X.

A linear semigroup of operators can be viewed as a generalized version of the exponential of the infinitesimal generator. Linear operator semigroup theory is called abstract Cauchy theory. The theory of first-order nonlinear perturbations of underlying linear abstract Cauchy problems is called abstract semi-linear Cauchy theory. A history dependent deterministic dynamical process can be viewed, in an appropriate setting and an appropriate formulation, as an exponential process or a nonlinear version of an exponential process. There are many applications of abstract Cauchy problems, both linear and nonlinear.

In this monograph Pierre Magal and Shigui Ruan develop an extension of linear operator semigroup theory to the case that the semigroup has an integrated form. This case arises when the infinitesimal generator is not densely defined in the state space of the operators. In this case the theoretical results for the classical case of densely defined infinitesimal generators must be extended, and sometimes with very elaborate theoretical extensions. The theory of integrated semigroups of operators with non-densely defined infinitesimal generators reveals the power of the fundamental concept of exponential processes. Pierre Magal and Shigui Ruan have been at the forefront of this development, in both its theoretical aspects and it applications to scientific problems.

One of these applications is to functional differential equations with partial derivative terms. These models have applications to problems involving spatial behavior, for example in models in which spatial diffusion plays a role. Another application is to structured population models. These models track the evolution of a population in time, but also in the organization of their structure with respect to age, size, or other individual variation. Age structure is very useful in describing many biological species, such as humans in demographic contexts. The continuum version of such models leads to an abstract Cauchy problem in a space of possible age densities of the given population. Size structured populations are another version of a useful way to organize population investigations. Size structure is sometimes more appropriate for analyzing population behavior, for example in micro-species such as cell populations. The evolution of a size structured population can be modeled as an abstract Cauchy problem in an appropriate infinite dimensional space of possible size densities. All the issues of population behavior, such as existence, uniqueness, asymptotic behavior, stability, and periodicity, can be investigated using abstract Cauchy theory and semi-linear abstract Cauchy theory.

There is a connection between structured population equations and functional differential equations. For example, the evolution of age structure in a population can be viewed as determined by the initial age structure of the population in an infinite dimensional space of age densities at initial time 0. It can also be viewed as determined by a history-dependent age structure of the population before the initial time 0. If the age of all individuals in a population is known, then their birth dates are known. Conversely, if the birth dates and the history of all individuals are known, then their age is known at the present time. Structured population models and functional differential equations models have great utility in scientific applications, and their theoretical analysis is grounded in the development found in this monograph.

The subject of abstract Cauchy theory has developed rapidly in recent years, with an expanding community of researchers. There is an important need for a comprehensive treatment of this expanding subject. This monograph provides such a comprehensive treatment and has great value to researchers in this field, both theoreticians and applied scientists.

Nashville, USA *Glenn Webb*
July 2017

Preface

Although mathematics ranks last in the Six Arts (rites, music, archery, chariot racing, calligraphy and mathematics), it is used in the most practical issues and affairs. Maximally, it enables understanding of the underlying myths of things and comprehension of their nature and developmental regularities. Minimally, it can be used in dealing with small affairs and solving multiple trivial issues. – QIN Jiushao, Preface to "Mathematical Treatise in Nine Sections" (1247)

Mathematics has a threefold purpose. It must provide an instrument for the study of nature. But this is not all: it has a philosophical purpose, and, I daresay, an aesthetic purpose. – Henri Poincaré

We first met in Nashville, Tennessee in the fall of 2001, when one of us (SR) was on sabbatical at Vanderbilt University while the other one (PM) was visiting the school. Both of us were working with Glenn Webb on various problems in mathematical biology, in particular age-structured biological models described by first order hyperbolic partial differential equations.

There are different approaches to study age-structured population models. One approach, using the theory of semigroups of operators since the late 1970s, became very powerful and important, mainly due to the work by Glenn Webb. His monograph, *"Theory of Nonlinear Age-Dependent Dynamics"* (Marcel Dekker, 1985), remains the classical reference in treating age-structured models using functional analytic techniques of nonlinear semigroups and evolution operators. The principle of linearized stability, established in Webb's monograph, says that a steady state is exponentially stable if the spectrum of the infinitesimal generator of the linearized semigroup lies entirely in the open left half-plane, whereas it is unstable if there is at least one spectral value lying in the open right half-plane (i.e. with positive real part). This not only provides a fundamental tool to study stability of age-structured models, but also indicates that periodic solutions may exist in age-structured models via Hopf bifurcation when spectral values leave the left half-plane, cross the purely imaginary axis, and enter the right half-plane as some parameter varies. The existence of non-trivial periodic solutions in age-structured models was observed in some studies by Cushing [77] (1980), Levine [227] (1983), Prüss [294] (1983), Diekmann et al. [103] (1986), Hastings [182] (1987), Swart [324] (1988) and so on in the 1980s. Our original goal was to establish a Hopf bifurcation theorem for general age-structured models. The project turned out to be much bigger than we expected.

Consider a general age-structured model

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -D(a)v(t,a) + M(\mu, v(t,.))(a), \ a \geq 0, t \geq 0, \\ v(t,0) = B(\mu, v(t,.)) \\ v(0,.) = v_0 \in L^p((0,+\infty), \mathbb{R}^n), \end{cases} \tag{1}$$

where $p \in [1,+\infty)$, $\mu \in \mathbb{R}$ is a parameter, $D(.) = \mathrm{diag}(d_1(.),...,d_n(.)) \in L^\infty((0,+\infty), M_n(\mathbb{R}^+))$, $M : \mathbb{R}\times L^1((0,+\infty),\mathbb{R}^n) \to L^1((0,+\infty),\mathbb{R}^n)$ is the mortality function, and $B : \mathbb{R}\times L^1((0,+\infty),\mathbb{R}^n) \to \mathbb{R}^n$ is the birth function. Consider the Banach space

$$X = \mathbb{R}^n \times L^p((0,+\infty), \mathbb{R}^n),$$

the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - D\varphi \end{pmatrix} \quad \text{with} \quad D(A) = \{0\} \times W^{1,p}((0,+\infty)),$$

and the function $F : \mathbb{R}\times\overline{D(A)} \to X$ defined by

$$F\left(\mu, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} B(\mu,\varphi) \\ M(\mu,\varphi) \end{pmatrix}.$$

Setting $u(t) = \begin{pmatrix} 0 \\ v(t,.) \end{pmatrix}$, we can rewrite the age-structured model as the following abstract Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(u(t),\mu), \ t \geq 0; \ \ u(0) = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \in \overline{D(A)}. \tag{2}$$

Observe that A is non-densely defined since

$$\overline{D(A)} = \{0\} \times L^p((0,+\infty), \mathbb{R}^n) \neq X$$

and A is a Hille-Yosida operator if and only if $p = 1$. Thus, problem (2) is a non-densely defined Cauchy problem in which the operator A might not be a Hille-Yosida operator. In fact, several other types of differential equations, such as functional differential equations, transport equations, parabolic partial differential equations, and partial differential equations with delay, can be formulated as non-densely defined Cauchy problems in the form of (2). Some fundamental theories for such problems have been very well studied. For example, Da Prato and Sinestrari [85] investigated the existence of different types of solutions for partial differential equations of hyperbolic and ultraparabolic type as well as equations arising from stochastic control theory that can be formulated as non-densely defined Cauchy problems.

When A is densely defined and is a Hille-Yosida operator, abstract Cauchy problems have been extensively studied (we refer to, among others, the monographs of Cazenave and Haraux [58], Engel and Nagel [126], Henry [183], Pazy [281], Sell and You [314], Temam [327], van Neerven [346], Yagi [376], and especially to the books of Haragus and Iooss [179], Hassard et al. [181], Kielhőfer [213], and Wu [374] regarding the nonlinear dynamics such as the local bifurcation, center manifold theory and normal forms). When A is non-densely defined, the constant of variation formula may not be well-defined and one must integrate the equation twice to recover the well-posedness (this is how integrated semigroups are introduced). Using integrated semigroup theory to investigate non-densely defined Cauchy problems started by Arendt in the 1980s and has been followed by many researchers (we refer to the monograph of Arendt et al. [22] for a systematic treatment of such problems).

The purpose of this monograph is to provide a self-contained presentation of the fundamental theory of nonlinear dynamics for non-densely defined semilinear Cauchy problems (in which the operator A may or may not be a Hille-Yosida operator), including the existence of integrated solutions, positivity of solutions, Lipschitz perturbation, differentiability of solutions with respect to the state variable, time differentiability of solutions, stability of equilibria, center manifold theory, normal form theory, Hopf bifurcation, and applications to age-structured models, functional differential equations and parabolic equations. It assumes a basic knowledge of real, complex and functional analyses, ordinary and partial differential equations at the senior undergraduate level and the graduate level.

In Chapter 1 we start by introducing some fundamental properties of matrices, such as the spectrum, spectral bound, spectral radius, growth bound (rate), resolvent, resolvent set, Laurent's expansion of the resolvent, and the integral resolvent formula, which can be served as a preview of the corresponding concepts for operators that will be introduced in the following chapters. Then we review some fundamental results on nonlinear dynamics, in particular the center manifold theory, Hopf bifurcation theorem, and normal form theory for Ordinary Differential Equations (ODEs) and Retarded Function Differential Equations (RFDEs). Finally we demonstrate that several classes of equations, including RFDEs, age structured models, parabolic equations, and reaction-diffusion equations with delay, can be formulated as abstract semilinear Cauchy problems.

Chapters 2-4 provide fundamentals in semigroup theory, spectral theory and Cauchy problems. Chapter 2 provides a review of the basic concepts and results on semigroups, resolvents, infinitesimal generators for linear operators and presents the Hille-Yosida theorem for strongly continuous semigroups. We also introduce Arendt's theorem which gives a Laplace transform characterization for the infinitesimal generator of a strongly continuous semigroup of bounded linear operators. Basic results on nonhomogeneous Cauchy problems with dense domain are given.

In Chapter 3 the integrated semigroup theory developed by Arendt, Hieber, Kellermann, Neubrander, Thieme and others is introduced, and the Arendt-Thieme theorem on the necessary and sufficient conditions for the existence of a non-degenerate integrated semigroup and its generator is stated. Then integrated semi-group theory is used to investigate the existence and uniqueness of integrated solutions of nonhomogeneous Cauchy problems; namely the Kellermann-Hieber theorem when A is a Hille-Yosida operator and our own results when A is not a Hille-Yosida operator are presented. Next we apply the results in this chapter to a vector valued age-structured model in L^p.

Chapter 4 covers the spectral theory for linear operators. After listing some basic properties for analytic mappings, fundamental results on the spectral theory, including Fredholm alternative theorem and Nussbaum's theorem on the radius of essential spectrum, of bounded linear operators are presented. Then the growth and essential growth bounds of linear operators are introduced and the main results are included in Webb's theorem on the relationship between the spectrum of semigroups and the spectrum of their infinitesimal generators. Finally spectral decomposition of the state space, the estimate of growth and essential growth bounds of linear operators are given which will be used in the proof of the center manifold theorem.

Chapters 5-6 present the main theory in abstract semilinear equations. In Chapter 5 we develop the fundamental theory for non-densely defined semilinear Cauchy problems, including the existence of integrated solutions, positivity of solutions, Lipschitz perturbation, differentiability of solutions with respect to the state variable, time differentiability of solutions, and stability of equilibria.

In Chapter 6 we establish the center manifold theory, Hopf bifurcation theorem, and normal form theory for abstract semilinear Cauchy problems with nondense domain.

Chapters 7-9 deal with applications of the results developed in Chapters 5-6. The goal of Chapter 7 is to apply the theories developed in Chapter 6 to functional differential equations, including retarded functional differential equations, neutral functional differential equations, and partial functional differential equations.

In Chapter 8 we treat age-structured models. Firstly we establish a Hopf bifurcation theorem for the general age-structured systems. Then we consider a susceptible-infectious epidemic model with age of infection, uniform persistence of the model is established, local and global stability of the disease-free equilibrium is studied by spectral analysis, and global stability of the unique endemic equilibrium is discussed by constructing a Liapunov functional. Finally we focus on a scalar age-structured model, detailed results on the existence of integrated solutions, local stability of equilibria, Hopf bifurcation, and normal forms are presented.

In Chapter 9, we first consider linear abstract Cauchy problems with non-densely defined and almost sectorial operators. Such problems naturally arise for parabolic equations with nonhomogeneous boundary conditions. By using the integrated semigroup theory, we then prove an existence and uniqueness result for integrated solutions. We also study the linear perturbation problem. Finally we provide detailed stability and bifurcation analyses for a scalar reaction-diffusion equation, namely, a size-structured model.

All assumptions, corollaries, definitions, examples, lemmas, propositions, remarks, and theorems are enumerated consistently by three numbers, with the first representing the chapter, the second representing the section, and the third representing the number. For instance, Proposition 3.4.3 means in Chapter 3, Section 4, property (Proposition) 4. All equations are enumerated in the same style. For example, equation (3.4.5) represents equation 5 in Chapter 3, Section 3.

We would like to express our gratitude to our Ph.D. thesis supervisors, Ovide Arino (PM) and Herbert I. Freedman[1] (SR), for their influence and inspiration which are lifetime. We are very grateful to Glenn Webb for his continuous guidance and support, not only as our mentor but also as our collaborator and friend, the writing of this monograph is indeed encouraged by his classical monograph. We are indebted to Wolfgang Arendt, Horst R. Thieme and Andre Vanderbauwhede for their mathematical work that inspired our studies on this subject. Special thanks are due to our collaborators, Jixun Chu, Arnaud Ducrot, Zhihua Liu, and Kevin Prevost, as it would have been impossible to complete this monograph without their contributions. Some parts of the book have been taught by us at Beijing Normal University, Harbin Institute of Technology and the University of Miami, and we thank the students for their feedbacks and comments. We thank the six anonymous reviewers of the earlier versions of the manuscript for their helpful comments and suggestions. Thanks are also due to our Springer editors, Donna Chernyk and Achi Dosanjh, for their patience and professional assistance.

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[1] Professor Herbert I. Freedman unfortunately passed away on November 21, 2017, when we were finalizing this monograph.

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Acronyms

$\mathbb{R}^n$	n-dimensional Euclidean space
$\lvert . \rvert_{\mathbb{R}^n}$	Norm in $\mathbb{R}^n$
$\mathbb{C}$	Set of all complex numbers
$\mathrm{Re}(\lambda)$	Real part of λ
$\mathrm{Im}(\lambda)$	Imaginary part of λ
$M_n(\mathbb{R})$	Space of all $n \times n$ real matrices
X	Banach space
$\lVert . \rVert$	Supremum norm in X
$\langle .,. \rangle$	Scalar product for the duality X^*, X
X^*	Dual space of X consisted of $x^*(x) = \langle x^*, x \rangle$ for $x \in X$
$\mathscr{L}(X,Y)$	Space of all linear operators from space X to space Y
$\mathscr{L}(X)$	$\mathscr{L}(X,X)$
$C(X,Y)$	Space of continuous maps from space X to space Y
$BC(X,Y)$	Space of bounded continuous maps from space X to space Y
$UBC(X,Y)$	Space of bounded and uniformly continuous maps from X to Y
$\mathrm{Lip}(X,Y)$	Space of Lipschitz continuous maps from X to Y
$BC^{\eta}(J,Y)$	$\{u \in C(J,Y) : \sup_{t \in J} e^{-\eta \lvert t \rvert} \lVert u(t) \rVert_Y < +\infty\}$
$L^p(J,X)$	Space of L^p-integrable functions from interval J to space X
$W^{k,p}(\Omega)$	$\{u \in L^p(\Omega) : D^{\alpha} u \in L^p(\Omega)\ \forall \lvert\alpha\rvert \leq k\}$, Sobolev space of locally summable functions $u : \Omega \to \mathbb{R}$ such that for every multi-index α with $\lvert\alpha\rvert \leq k$ the weak derivative $D^{\alpha} u$ exists and belongs to $L^p(\Omega)$
$C_c((a,b),\mathbb{R})$	Space of continuous functions with compact support in (a,b)
$X^{\mathbb{C}}$	$X + iX$, complexified Banach space of X
Π	Projector
$B_X(y,r)$	Ball centered at y with radius r in X
$S_{\mathbb{C}}(\widehat{\lambda}, \varepsilon)$	$\{\lambda \in \mathbb{C} : \lvert \lambda - \widehat{\lambda} \rvert = \varepsilon\}$
A	Linear operator
$D(A)$	Domain of the operator A
$G(A)$	$\{(x, Ax) : x \in D(A)\}$, graph of the operator A
A_Y	The part of A in a subspace $Y \subset X$, $A_Y(x) = A(x), \forall x \in D(A_Y)$
$A\vert_Y$	The restriction of A in a subspace $Y \subset X$

$\sigma(A)$	Spectrum of the operator A
$\sigma_p(A)$	Point spectrum of the operator A
$\sigma_c(A)$	Continuous spectrum of the operator A
$\sigma_r(A)$	Residual spectrum of the operator A
$\sigma_{\mathrm{ess}}(A)$	Essentail spectrum of the operator A
$\sigma_d(A)$	Discrete spectrum of the operator A
$s(A)$	Spectral bound of the operator A
$r(A)$	Spectral radius of the operator A
$r_{\mathrm{ess}}(A)$	Essential spectral radius of the operator A
$\rho(A)$	Resolvent set of the operator A
$(\lambda I - A)^{-1}$	Resolvent of the operator A
$\omega(A)$	Growth bound of the operator A
$\omega_{0,\mathrm{ess}}(A)$	Essential growth bound of the operator A
$\mathscr{C}$	$C([-r,0],\mathbb{R}^n)$, space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$
$\mathscr{N}(L)$	$\{x \in D(L) : Lx = 0\}$, null space of the operator $L : X \to Y$
$\mathscr{R}(L)$	$\{y \in Y : \exists x \in D(L) \text{ s.t. } y = Lx\}$, range of the operator $L : X \to Y$
$\{T_A(t)\}_{t\geq 0}$	C^0-semigroup generated by A on a Banach space X
$\{S_A(t)\}_{t\geq 0}$	Integrated semigroup generated by A on a Banach space X
$(S_A * f)(t)$	$\int_0^t S_A(s) f(t-s) ds$
$(S_A \diamond f)(t)$	$\frac{d}{dt}(S_A * f)(t) = S_A(t) f(0) + \int_0^t S_A(s) f'(t-s) ds$
$V^\infty(S_A, 0, \tau)$	semi-variation of $\{S_A(t)\}_{t\geq 0}$ on $[0,\tau]$
$VL^p(J,H)$	Bounded L^p-variation of function H on J
$\{U(a,s)\}_{a\geq s\geq 0}$	Exponential bounded evolution family
$\kappa(B)$	Kuratovsky measure of non-compactness of $B \subset X$
X/E	$\{\{x + \nu : \nu \in E\} : x \in X\}$, quotient space
$[A,G](x)$	$DG(x)(Ax) - AG(x), \forall x \in X$, Lie bracket
$\widehat{f}(\lambda)$	$\mathscr{L}(f)(\lambda) = \int_0^\infty e^{-\lambda s}(s) ds$, Laplace transform of f
$1_{[a,b]}(x)$	$\begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$

Chapter 1
Introduction

The goal of this chapter is to introduce some fundamental theories for Ordinary Differential Equations (ODEs), Retarded Functional Differential Equations (RFDEs), and Age-structured Models and to derive abstract semilinear Cauchy problems from these equations. It serves two purposes: to present a brief review of the basic results on the nonlinear dynamics of these three types of equations and to give a quick preview about the types of results we will develop for the abstract semilinear Cauchy problems in this monograph.

1.1 Ordinary Differential Equations

1.1.1 Spectral Properties of Matrices

Let $M_n(\mathbb{R})$ be the space of all $n \times n$ real matrices with the usual matrix norm. Consider a matrix $A \in M_n(\mathbb{R})$. Define a family of matrices $\left\{e^{At}\right\}_{t \in \mathbb{R}}$ by

$$e^A := I + A + \frac{A^2}{2!} + ... = \sum_{k=0}^{+\infty} \frac{A^k}{k!}, \tag{1.1.1}$$

where I is the $n \times n$ identity matrix. Then

$$e^{A+B} = e^A e^B$$

whenever A and B commute (i.e., $AB = BA$). Then we know that $\{e^{At}\}_{t \in \mathbb{R}}$ forms a *group (flow)* under composition:

$$(i)\ e^{A0} = I;\ (ii)\ e^{At} e^{As} = e^{A(t+s)};\ (iii)\ e^{At} e^{-At} = I,\ \forall t, s \in \mathbb{R}.$$

One may also observe that the map $t \to e^{At}$ is continuously differentiable from $\mathbb{R}$ into $M_n(\mathbb{R})$ and

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A, \ \forall t \in \mathbb{R}.$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{C}$ $(m \leq n)$ be the eigenvalues of A with algebraic multiplicity $n_1, n_2, ..., n_m$, respectively, $n_1 + n_2 + \cdots + n_m = n$. Then Jordan's decomposition says that there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that

$$A = P^{-1}JP,$$

where $J \in M_n(\mathbb{R})$ is a block diagonal matrix

$$J = \begin{bmatrix} J_{n_1}^{\lambda_1} & 0 & \cdots & \cdots & 0 \\ 0 & J_{n_2}^{\lambda_2} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J_{n_m}^{\lambda_m} \end{bmatrix},$$

in which the elementary Jordan blocks are defined by

$$J_{n_k}^{\lambda_k} = [\lambda_k] \text{ if } n_k = 1$$

and

$$J_{n_k}^{\lambda_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_k \end{bmatrix} \in M_{n_k}(\mathbb{C}) \text{ if } n_k > 1.$$

Since J is a triangular matrix, its diagonal elements are the eigenvalues of A. The *spectrum* of A is the set of all eigenvalues of A given by

$$\sigma(A) = \{\lambda_1, ..., \lambda_m\}. \tag{1.1.2}$$

Observe that for each $k = 1, ..., m$, one has

$$J_{n_k}^{\lambda_k} = \lambda_k I + N_{n_k},$$

where N_{n_k} is nilpotent of order n_k; that is,

$$N_{n_k} \neq 0, \ (N_{n_k})^2 \neq 0, \ldots, (N_{n_k})^{n_k - 1} \neq 0, \text{ and } (N_{n_k})^{n_k} = 0.$$

Now we have

$$e^{J_{n_k}^{\lambda_k} t} = e^{\left(\lambda_k I + J_{n_k}^0\right) t} = e^{\lambda_k t} \ e^{J_{n_k}^0 t}$$

and

$$e^{J_{n_k}^0 t} = \begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{(n_k-1)}/(n_k-1)! \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t^2/2! \\ \vdots & & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Therefore, by using the spectral theory of matrices, the asymptotic behavior of e^{At} is entirely determined by

$$e^{At} = Pe^{Jt}P^{-1}, \forall t \geq 0.$$

The *growth bound (rate)* of A is defined as

$$\omega(A) := \lim_{t \to +\infty} \frac{\ln\left(\left\|e^{At}\right\|_{\mathscr{L}(\mathbb{R}^n)}\right)}{t} \in (-\infty, +\infty), \tag{1.1.3}$$

where $\mathscr{L}(\mathbb{R}^n)$ is the space of all linear operators on $\mathbb{R}^n$ with the operator norm $\|.\|$, namely,

$$\|A\|_{\mathscr{L}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n : 0 < \|x\| \leq 1} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathbb{R}^n : \|x\| = 1} \|Ax\|. \tag{1.1.4}$$

Remark 1.1.1. In general

$$e^{-\omega(A)t} \left\|e^{At}\right\| \to +\infty \text{ as } t \to +\infty.$$

Indeed, for example, take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then we have

$$\omega(A) = 0.$$

By using the explicit formula for the elementary Jordan blocks we have

$$\left\|e^{At}\right\|_{\mathscr{L}(\mathbb{R}^n)} \to +\infty \text{ as } t \to +\infty.$$

We have the following result.

Theorem 1.1.2. *For the growth bound of a matrix A, one has*

$$\omega(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}. \tag{1.1.5}$$

The right hand side of the above equality (1.1.5) is called the *spectral bound* of A, denoted by $s(A)$; namely,

$$s(A) = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\}. \tag{1.1.6}$$

The *spectral radius* of A is defined as

$$r(A) := \lim_{k\to+\infty} \|A^k\|^{1/k}. \tag{1.1.7}$$

For any given matrix it is usually more convenient to use the following characterization of the spectral radius

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}. \tag{1.1.8}$$

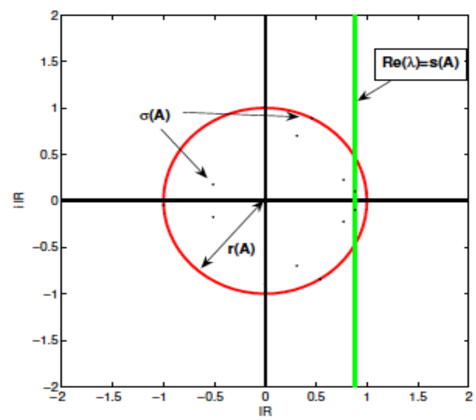

Fig. 1.1: *The spectrum* $\sigma(A)$, *spectral radius* $r(A)$, *and spectral bound* $s(A)$ *of a matrix* A.

If $\lambda \notin \sigma(A)$ (that is, λ is not an eigenvalue of the matrix A), then the matrix $\lambda I - A$ is invertible, so we can define a function

$$(\lambda I - A)^{-1} : \mathbb{C} \setminus \sigma(A) \to \mathscr{L}(\mathbb{R}^n),$$

which is called the *resolvent* of A. The set $\mathbb{C} \setminus \sigma(A)$ is called the *resolvent set* of A, denoted by $\rho(A)$; that is,

$$\rho(A) = \mathbb{C} \setminus \sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}. \tag{1.1.9}$$

For each $\lambda \in \rho(A)$, J is invertible and

$$(\lambda I - J)^{-1} = \begin{bmatrix} \left(\lambda I - J_{n_1}^{\lambda_1}\right)^{-1} & 0 & \cdots & 0 \\ 0 & \left(\lambda I - J_{n_2}^{\lambda_2}\right)^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \left(\lambda I - J_{n_m}^{\lambda_m}\right)^{-1} \end{bmatrix},$$

where (note that $J_{n_k}^{\lambda_k} = \lambda_k I + N_{n_k}$)

$$\begin{aligned}\left(\lambda I - J_{n_k}^{\lambda_k}\right)^{-1} &= \left((\lambda - \lambda_k) I - N_{n_k}\right)^{-1} \\ &= (\lambda - \lambda_k)^{-1}\left(I - (\lambda - \lambda_k)^{-1} N_{n_k}\right)^{-1} \\ &= (\lambda - \lambda_k)^{-1} \sum_{j=0}^{n_k - 1} \frac{1}{(\lambda - \lambda_k)^j} N_{n_k}^j.\end{aligned}$$

Hence

$$\left(\lambda I - J_{n_k}^{\lambda_k}\right)^{-1} = \sum_{j=1}^{n_k} (\lambda - \lambda_k)^{-j} N_{n_k}^{j-1}.$$

It follows that $\lambda \to (\lambda I - A)^{-1}$ is analytic from $\rho(A)$ into $\mathscr{L}(\mathbb{R}^n)$. Since a given eigenvalue $\widehat{\lambda} \in \sigma(A)$ may appear in several Jordan's blocks, we deduce that the resolvent of A has the following *Laurent's expansion of resolvent* (for matrices) around $\widehat{\lambda}$:

$$(\lambda I - A)^{-1} = \sum_{n=-\widehat{m}}^{+\infty} \left(\lambda - \widehat{\lambda}\right)^n B_n, \tag{1.1.10}$$

where $\widehat{m} := \max\{n_k : k = 1, ..., m \text{ and } \lambda_k = \widehat{\lambda}\}$ and B_n is given by

$$B_n = \frac{1}{2\pi i} \int_{S_{\mathbb{C}}\left(\widehat{\lambda}, \varepsilon\right)^+} \left(\lambda - \widehat{\lambda}\right)^{-(n+1)} (\lambda I - A)^{-1} d\lambda$$

for each $\varepsilon > 0$, where $S_{\mathbb{C}}\left(\widehat{\lambda}, \varepsilon\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \widehat{\lambda}\right| = \varepsilon\right\}$, and $S_{\mathbb{C}}\left(\widehat{\lambda}, \varepsilon\right)^+$ is the counter-clockwise oriented circumference $\left|\lambda - \widehat{\lambda}\right| = \varepsilon$ for sufficiently small $\varepsilon > 0$ such that $\left|\lambda - \widehat{\lambda}\right| \leq \varepsilon$ does not contain other point of the spectrum than $\widehat{\lambda}$. A point of the spectrum that is isolated and around which the resolvent has the above expansion (i.e. (1.1.10)) is called *a pole of the resolvent* $(\lambda I - A)^{-1}$.

Remark 1.1.3. The expansion formula (1.1.10) is also interesting because the projector on the generalized eigenspace associated to $\widehat{\lambda}$ is B_{-1}.

We can also establish a relationship between the resolvent $(\lambda I - A)^{-1}$ and e^{At}.

Theorem 1.1.4 (Integral Resolvent Formula (for Matrices)). *Consider a matrix $A \in M_n(\mathbb{R})$. For each $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > \omega(A)$, $\lambda I - A$ is invertible and*

$$(\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} e^{At} dt. \tag{1.1.11}$$

Proof. We have

$$(\lambda I - A) \int_0^{+\infty} e^{-\lambda t} e^{At} dt = \int_0^{+\infty} e^{-\lambda t} e^{At} dt \, (\lambda I - A)$$

$$= \lambda \int_0^{+\infty} e^{-\lambda t} e^{At} dt - \int_0^{+\infty} e^{-\lambda t} A e^{At} dt$$
$$= \lambda \int_0^{+\infty} e^{-\lambda t} e^{At} dt - \int_0^{+\infty} e^{-\lambda t} \frac{d}{dt} e^{At} dt.$$

By integrating by parts the last integral we obtain

$$(\lambda I - A) \int_0^{+\infty} e^{-\lambda t} e^{At} dt = \int_0^{+\infty} e^{-\lambda t} e^{At} dt \, (\lambda I - A) = I.$$

The result follows. □

1.1.2 State Space Decomposition

Consider the linear Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) \text{ for } t \geq 0, \ x(0) = x_0 \in \mathbb{R}^n, \tag{1.1.12}$$

where $A \in M_n(\mathbb{R})$. Problem (1.1.12) has a unique solution given by

$$x(t) = e^{At} x_0 \text{ for each } t \geq 0.$$

In order to be more precise about the asymptotic behavior of the linear system, we introduce some notation. Define

$$\sigma_s(A) = \{\lambda \in \sigma(A) : \ \mathrm{Re}\,(\lambda) < 0\} \text{ (stable spectrum)},$$
$$\sigma_c(A) = \{\lambda \in \sigma(A) : \ \mathrm{Re}\,(\lambda) = 0\} \text{ (central spectrum)},$$
$$\sigma_u(A) = \{\lambda \in \sigma(A) : \ \mathrm{Re}\,(\lambda) > 0\} \text{ (unstable spectrum)}.$$

By using Jordan's theorem again, we have a state space decomposition

$$\mathbb{R}^n = X_s \oplus X_c \oplus X_u,$$

where X_s, X_c and X_u are three linear subspaces of $\mathbb{R}^n$ (with possibly $X_k = \{0\}$ for some $k = s, c, u$) satisfying the following properties:

$$AX_k \subset X_k, \ \forall k = s, c, u,$$

and the spectrum of the linear map $A_k : X_k \to X_k$ is defined by

$$A_k x = Ax,$$

and

$$\sigma(A_k) = \sigma_k(A), \ \forall k = s, c, u.$$

Remark 1.1.5. In this book, we will often use the notion of the part of a linear operator in a subspace. Actually A_k defined above is the part of A in X_k. One may observe that $A_k : X_k \to X_k$ is a linear map on X_k such that

$$A_k x = Ax, \ \forall x \in X_k,$$

The linear map A_k is not equal to $A\mid_{X_k}$, the restriction of A to X_k, since $A\mid_{X_k}$ goes from X_k into $\mathbb{R}^n$ and

$$A\mid_{X_k} x = Ax, \ \forall x \in X_k.$$

Definition 1.1.6. The spaces X_s, X_c, and X_u are called the *linear stable, center, and unstable subspaces*, respectively.

Define the projections $\Pi_s, \Pi_c, \Pi_u \in M_n(\mathbb{R})$ such that

$$\Pi_s(\mathbb{R}^n) = X_s \text{ and } (I - \Pi_s)(\mathbb{R}^n) = X_c \oplus X_u,$$

$$\Pi_c(\mathbb{R}^n) = X_c \text{ and } (I - \Pi_c)(\mathbb{R}^n) = X_s \oplus X_u,$$

$$\Pi_u(\mathbb{R}^n) = X_u \text{ and } (I - \Pi_u)(\mathbb{R}^n) = X_s \oplus X_c.$$

By using the properties of the elementary Jordan blocks, one may observe that $\eta > 0$ can be chosen such that

$$\omega(A_s) := \sup_{\lambda \in \sigma_s(A)} \operatorname{Re}(\lambda) < -\eta < 0 < \eta < \inf_{\lambda \in \sigma_u(A)} \operatorname{Re}(\lambda) =: \omega(-A_u).$$

Since the inequalities are strict and $\eta < \min(-\omega(A_s), \omega(-A_u))$, we have

$$\begin{aligned} M_s &:= \sup_{t \geq 0} e^{\eta t} \left\| e^{At} \Pi_s \right\|_{\mathscr{L}(\mathbb{R}^n)} = \sup_{t \geq 0} e^{\eta t} \left\| e^{A_s t} \right\|_{\mathscr{L}(X_s)} < +\infty, \\ M_u &:= \sup_{t \geq 0} e^{\eta t} \left\| e^{-At} \Pi_u \right\|_{\mathscr{L}(\mathbb{R}^n)} = \sup_{t \geq 0} e^{\eta t} \left\| e^{-A_u t} \right\|_{\mathscr{L}(X_u)} < +\infty. \end{aligned} \tag{1.1.13}$$

Remark 1.1.7. In general we have

$$\|A \Pi_k\|_{\mathscr{L}(\mathbb{R}^n)} \neq \|A\|_{\mathscr{L}(X_k)}.$$

But this property becomes true if we use the equivalent norm

$$|x| = \|\Pi_s x\| + \|\Pi_c x\| + \|\Pi_u x\|.$$

By fixing a constant $M \geq \max(M_s, M_u) \geq 1$, we obtain

$$\left\| e^{At} \Pi_s \right\| \leq M e^{-\eta t}, \ \left\| e^{-At} \Pi_u \right\| \leq M e^{-\eta t}, \ \forall t \geq 0.$$

Actually, the non-exponentially growing part is contained in the center part. It is described by

$$e^{A_c t} = e^{At} \Pi_c,$$

which grows like a polynomial when t goes to $\pm\infty$.

1.1.3 Semilinear Systems

Consider the nonhomogeneous Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) + f(t),\ t \in [0, \tau];\ \ x(0) = x_0 \in \mathbb{R}^n, \tag{1.1.14}$$

where $f \in L^1\left((0, \tau), \mathbb{R}^n\right)$.

Lemma 1.1.8. *The solution of (1.1.14) is given by the so-called variation of constants formula*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s)ds,\ \forall t \in [0, \tau]. \tag{1.1.15}$$

We should emphasize here that the variation of constants formula plays a crucial role in analyzing the qualitative behavior of nonlinear differential equations locally around an equilibrium.

Consider a semilinear ordinary differential system of the form

$$\frac{dx(t)}{dt} = Ax(t) + F\left(x(t)\right),\ t \geq 0;\ \ x(0) = x_0 \in \mathbb{R}^n, \tag{1.1.16}$$

where $A \in M_n(\mathbb{R})$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a k-time ($k \geq 1$) continuously differentiable function. The notion of a *solution* of system (1.1.16) must be understood as a continuous function $x \in C\left([0, \tau], \mathbb{R}^n\right)$ satisfying

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} F(x(s))ds \ \text{ for each } t \in [0, \tau]. \tag{1.1.17}$$

In other words, x is a solution of the fixed point problem

$$x(t) = \Psi(x)(t), \forall t \in [0, \tau],$$

where $\Psi : C\left([0, \tau], \mathbb{R}^n\right) \to C\left([0, \tau], \mathbb{R}^n\right)$ is a nonlinear operator defined by

$$\Psi(x)(t) := e^{At}x_0 + \int_0^t e^{A(t-s)} F(x(s))ds.$$

Definition 1.1.9. The map F is said to be *Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|F(x) - F(y)\| \leq k\|x - y\|,\ \forall x, y \in \mathbb{R}^n,$$

and the *Lipschitz norm of* F is defined by

$$\|F\|_{\mathrm{Lip}} := \sup_{x,y \in \mathbb{R}^n : x \neq y} \frac{\|F(x) - F(y)\|}{\|x - y\|}.$$

The map F is said to be *Lipschitz continuous on bounded sets of* $\mathbb{R}^n$ if for each constant $M > 0$ there exists $k = k(M) > 0$ such that

$$\|F(x) - F(y)\| \leq k \|x - y\|, \; \forall x, y \in B_{\mathbb{R}^n}(0, M),$$

where $B_{\mathbb{R}^n}(0, M)$ is the closed ball of radius M centered at 0; namely,

$$B_{\mathbb{R}^n}(0, M) := \{x \in \mathbb{R}^n : \|x\| \leq M\}.$$

Remark 1.1.10. Assume that F is C^1. Set

$$G(s) = F(sx + (1-s)y).$$

Then

$$G(1) - G(0) = \int_0^1 G'(s) ds.$$

Therefore, we obtain the fundamental formula of differential calculus (Lang [224])

$$F(x) - F(y) = \int_0^1 DF(sx + (1-s)y)(x-y) ds.$$

From this formula, one deduces that every C^1 map on $\mathbb{R}^n$ is Lipschitz continuous on bounded sets. This property is only true in spaces with finite dimensions.

(a) Flows and semiflows. A very important concept in the context of dynamical systems is the notion of a semiflow or a flow whenever the semiflow can be extended in a unique manner for negative time.

Definition 1.1.11. Let (M, d) be a metric space. Let $\{U(t)\}_{t \geq 0}$ (respectively $\{U(t)\}_{t \in \mathbb{R}}$) be a famiIly of continuous maps from M into itself. $\{U(t)\}_{t \geq 0}$ is called a *continuous semiflow on M* (respectively $\{U(t)\}_{t \in \mathbb{R}}$ is a *continuous flow on M*) if the following properties are satisfied:

(i) $U(0) = I$;
(ii) $U(t)U(s) = U(t+s), \forall t, s \geq 0$ (respectively $\forall t, s \in \mathbb{R}$);
(iii) The map $(t, x) \to U(t)x$ is continuous from $[0, +\infty) \times M$ into M (respectively continuous from $\mathbb{R} \times M$ into M).

Now we recall the classical Picard's existence theorem for flows.

Theorem 1.1.12 (Picard's Theorem). *Assume that F is Lipschitz continuous. Then equation (1.1.16) generates a unique flow* $\{U(t)\}_{t \in \mathbb{R}}$ *on* $\mathbb{R}^n$*; that is, for each* $x_0 \in \mathbb{R}^n$ *there exists a unique solution* $t \to x(t)$ *of system (1.1.16) on* $\mathbb{R}$*. Moreover,*

$$U(t)x_0 := x(t)$$

defines a flow on $\mathbb{R}^n$.

If F is only Lipschitz continuous on bounded sets, then blowup may occur. Thus, we need to define the time of (eventual) blowup, $\tau(x_0) \in (0, +\infty]$, as follows

$$\tau(x_0) := \sup\{\widehat{\tau} \geq 0 : \text{ equation (1.1.16) has a solution } x \in C([0, \widehat{\tau}], \mathbb{R}^n)\}.$$

For simplicity, we only introduce the notion of a maximal semiflow. One can define a maximal flow similarly, but would need to introduce two times of blowup for both positive and negative times.

Definition 1.1.13. Let (M, d) be a metric space. Let $\tau : M \to (0, +\infty]$ be a map. Define

$$D_\tau := \{(t, x) : 0 \leq t < \tau(x)\}.$$

Let $U : D_\tau \to \mathbb{R}^n$ be a map. For convenience we denote

$$U(t)x := U(t, x), \forall (t, x) \in D_\tau.$$

We say that U is a *maximal semiflow* if the following properties are satisfied:

(i) $\tau(U(t)x) = \tau(x) - t, \forall (t, x) \in D_\tau$;
(ii) $U(0) = I$;
(iii) $U(t)U(s)x = U(t+s)x, \forall (t, x) \in D_\tau, \forall (s, x) \in D_\tau$ such that $(t+s, x) \in D_\tau$;
(iv) If $\tau(x) < +\infty$, then

$$\lim_{t(<\tau(x)) \to \tau(x)} \|U(t)x\| = +\infty.$$

Moreover, we say that U is a *maximal continuous semiflow* if it satisfies in addition the following property:

(v) The set D_τ is relatively open in $[0, +\infty) \times M$ and the map $(t, x) \to U(t)x$ is continuous from D_τ into M.

When F is only Lipschitz continuous on bounded sets we have the following theorem.

Theorem 1.1.14 (Existence and Uniqueness). *Assume that F is Lipschitz continuous on bounded sets. Then system (1.1.16) generates a unique maximal continuous semiflow U on $\mathbb{R}^n$. More precisely, there exists $\tau : \mathbb{R}^n \to (0, +\infty]$, which is lower semi-continuous, such that for each $x_0 \in \mathbb{R}^n$ there exists a unique solution $t \to x(t)$ of system (1.1.16) on $[0, \tau(x_0))$, and*

$$U(t)x_0 := x(t)$$

defines a maximal semiflow on $\mathbb{R}^n$.

In order to understand the notion of linearized equations around a given solution, we introduce the following result.

Theorem 1.1.15 (Linearized Semiflow). *Assume that F is one-time continuously differentiable. Then for each $x_0 \in \mathbb{R}^n$ and each $t \in [0, \tau(x_0))$, the map $x \to U(t)x$ is well defined locally around x_0 (in other words, there is an $\varepsilon > 0$ such that $t^* < \tau(x)$ for each $x \in B(x_0, \varepsilon)$). Moreover, the map $x \to U(t)x$ is differentiable, and if we set $V(t)y := \partial_x U(t)(x_0)y$, then the map $t \to V(t)y$ is defined on $[0, \tau(x_0))$ and satisfies the following (nonautonomous and linear) ordinary differential equation*

$$\frac{dV(t)y}{dt} = AV(t)y + \partial_x F(U(t)(x_0))(V(t)y), \; \forall t \in [0, \tau(x_0)); \; V(0)y = y.$$

Definition 1.1.16. We say that $\bar{x} \in \mathbb{R}^n$ is an *equilibrium* (or *equilibrium solution*) of system (1.1.16) if

$$x(t) = \bar{x} \text{ for all } t \geq 0$$

is a constant solution of system (1.1.16), or equivalently if

$$A\bar{x} + F(\bar{x}) = 0_{\mathbb{R}^n}.$$

(b) Linearized equation around an equilibrium. Assume that $\bar{x} \in \mathbb{R}^n$ is an equilibrium of system (1.1.16). By applying Theorem 1.1.15 around $U(t)\bar{x} = \bar{x}$, $\forall t \geq 0$, we deduce that

$$\partial_x U(t)(\bar{x})y = e^{Bt}y,$$

where

$$B = A + \partial_x F(\bar{x}).$$

The linear system

$$\frac{dy(t)}{dt} = (A + \partial_x F(\bar{x}))y(t), \; t \geq 0; \;\; y(0) = y$$

is called the *linearized system* of (1.1.16) at $\bar{x}$.

Assumption 1.1.17. Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Assume in addition that there exists an equilibrium $\bar{x} \in \mathbb{R}^n$ of system (1.1.16) such that

$$F(\bar{x}) = 0 \text{ and } DF(\bar{x}) = 0_{M_n(\mathbb{R})}.$$

Assumption 1.1.17 is equivalent to assuming that Ax is the only linearized part of system (1.1.16).

Definition 1.1.18. The equilibrium $\bar{x}$ is said to be *hyperbolic* if and only if

$$\mathrm{Re}(\lambda) \neq 0, \; \forall \lambda \in \sigma(A).$$

Otherwise, it is *nonhyperbolic*.

For convenience, we assume that

$$\bar{x} = 0.$$

Indeed, we can use the change of variables

$$V(t)x = U(t)(x+\overline{x}) - \overline{x}$$

and obtain that

$$\frac{dV(t)x}{dt} = \frac{dU(t)(x+\overline{x})}{dt} = AU(t)(x+\overline{x}) + F(U(t)(x+\overline{x})).$$

Therefore, $V(t)$ is a semiflow generated by

$$\frac{dV(t)x}{dt} = AV(t)x + G(V(t)x)$$

and

$$G(x) = F(x+\overline{x}) + A\overline{x}.$$

The problem is unchanged since

$$DG(0) = DF(\overline{x}).$$

Theorem 1.1.19 (Exponential Stability). *Let Assumption 1.1.17 be satisfied. Assume that the spectrum $\sigma(A)$ of the matrix A contains only complex numbers with strictly negative real part. Then the equilibrium $\overline{x}$ of system (1.1.16) is exponentially asymptotically stable; that is, there exist $\eta > 0$ and $M > 0$ such that*

$$\|x-\overline{x}\| \leq \eta \Rightarrow \|U(t)x - \overline{x}\| \leq Me^{-\alpha t}\|x-\overline{x}\|, \ \forall t \geq 0.$$

Theorem 1.1.20 (Instability). *Let Assumption 1.1.17 be satisfied. Assume that there exists $\lambda \in \sigma(A)$ such that*

$$\mathrm{Re}(\lambda) > 0,$$

then the equilibrium $\overline{x}$ of system (1.1.16) is unstable. This means that there exist a constant $\varepsilon > 0$, a sequence $\{x_n\} \to \overline{x}$, and a sequence $\{t_n\} \to +\infty$, such that

$$\|U(t_n)x_n - \overline{x}\| \geq \varepsilon.$$

(c) Center Manifold Theorem. We return to the state space decomposition

$$\mathbb{R}^n = X_s \oplus X_c \oplus X_u.$$

Set

$$X_h := X_s \oplus X_u \text{ (the hyperbolic subspace)}$$
$$X_{cu} := X_c \oplus X_u \text{ (the linear center-unstable subspace).}$$

Before stating the main result about the local center manifold theorem, we will first explain the idea about the global center manifold theorem. Actually this class of problems can be regarded as persistent results for manifolds. Consider first the linear Cauchy problem (1.1.12). Then the linear center subspace X_c is invariant under e^{At}; that is,

$$e^{At}X_c = X_c, \ \forall t \in \mathbb{R}.$$

Moreover, X_c is a linear manifold. More precisely, we can find a map $L_c : X_c \to X_h$ such that

$$X_c = \{x_c + L_c(x_c) : x_c \in X_c\},$$

and L_c is defined by

$$L_c x_c = 0, \ \forall x_c \in X_c.$$

So it is natural to ask if such an invariant set X_c persists if one considers a "reasonable" perturbation of the linear Cauchy problem (1.1.12). Consider the perturbed system (1.1.16). Let $\eta \in (0, \min(-\omega(A_s), \omega(-A_u)))$. For the linear problem we have

$$X_c := \left\{ x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \left\| e^{At}x \right\| < +\infty \right\}.$$

Based on this observation, it becomes "natural" to define

$$M_c^\eta := \left\{ x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \| U(t)x \| < +\infty \right\},$$

and the global center manifold theorem says that if $\|F\|_{\mathrm{Lip}}$ is small enough, then there exists a map $\Psi_c : X_c \to X_h$, which is Lipschitz continuous, such that

$$M_c := \{x_c + \Psi_c(x_c) : x_c \in X_c\}.$$

By the definition of M_c, one may realize that

$$U(t)M_c = M_c, \ \forall t \in \mathbb{R}.$$

Let $x \in M_c$ be given. Consider a solution $u(t) = U(t)x$. Since

$$u(t) \in M_c, \ \forall t \in \mathbb{R},$$

we have

$$u(t) = \Pi_c u(t) + \Psi_c(\Pi_c u(t))$$

and $u_c(t) = \Pi_c u(t) = \Pi_c U(t)x$ satisfies the equation on X_c :

$$\frac{du_c(t)}{dt} = A_c u_c(t) + \Pi_c F(u_c(t) + \Psi_c(u_c(t))).$$

The last equation is called the *reduced system* since the dimension of X_c is smaller than the dimension of the original phase space.

Theorem 1.1.21 (Global Center Manifold). *Let Assumption 1.1.17 be satisfied. Assume that*

$$\bar{x} = 0$$

and

$$\sigma_c(A) \neq \varnothing.$$

Let $\eta \in (0, \min(-\omega(A_s), \omega(-A_u)))$. *Then there exists a constant* $\kappa = \kappa(\eta) > 0$ *so that if*

$$\|F\|_{\mathrm{Lip}} \leq \kappa,$$

then there exists a map $\Psi_c : X_c \to X_h$, *which is Lipschitz continuous and satisfies*

$$\Psi_c(0) = 0,$$

such that

$$M_c^\eta := \{x_c + \Psi_c(x_c) : x_c \in X_c\}.$$

Proof. Recall that

$$BC^\eta(\mathbb{R}, \mathbb{R}^n) := \left\{ u \in C(\mathbb{R}, \mathbb{R}^n) : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|u(t)\| < +\infty \right\}$$

is a Banach space endowed with the norm

$$\|u\|_\eta = \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|u(t)\|.$$

Assume that $x \in M_c^\eta$. Then the map $t \to u(t) := U(t)x$ belongs to $BC^\eta(\mathbb{R}, \mathbb{R}^n)$, and by using the variation of constants formula we have

$$u(t) = e^{A(t-s)}u(s) + \int_s^t e^{A(t-l)}F(u(l))\,dl, \ \forall t \geq s. \tag{1.1.18}$$

By projecting on X_s we obtain

$$\Pi_s u(t) = e^{A_s(t-s)}\Pi_s u(s) + \int_s^t e^{A_s(t-l)}\Pi_s F(u(l))\,dl$$

and by using the fact that $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$, we deduce when s goes to $-\infty$ that

$$\Pi_s u(t) = \int_{-\infty}^t e^{A_s(t-l)}\Pi_s F(u(l))\,dl.$$

Similarly, by projecting on X_u we obtain

$$\Pi_u u(t) = e^{A_u(t-s)}\Pi_u u(s) + \int_s^t e^{A_u(t-l)}\Pi_u F(u(l))\,dl.$$

Therefore,

$$\Pi_u u(s) = e^{-A_u(t-s)}\Pi_u u(t) - \int_s^t e^{-A_u(l-s)}\Pi_u F(u(l))\,dl$$

and when t goes to $+\infty$, we obtain

$$\Pi_u u(t) = -\int_t^{+\infty} e^{-A_u(l-t)} \Pi_u F(u(l))\, dl.$$

Thus, u must satisfy the following equality for each $t \in \mathbb{R}$:

$$\begin{aligned} u(t) = {} & e^{A_c t} \Pi_u x_c + \int_0^t e^{A_c(t-l)} \Pi_c F(u(l))\, dl \\ & + \int_{-\infty}^t e^{A_s(t-l)} \Pi_s F(u(l))\, dl - \int_t^{+\infty} e^{-A_u(l-t)} \Pi_u F(u(l))\, dl. \end{aligned}$$

We leave as an exercise on the converse implication; namely, if $u \in BC^\eta(\mathbb{R}, \mathbb{R}^n)$ satisfies the above equality then u satisfies (1.1.18). One then observes that this problem can be reformulated as a fixed point problem:

$$u = K_1 x_c + K_2 F(u), \tag{1.1.19}$$

where $K_1 : X_c \to BC^\eta(\mathbb{R}, \mathbb{R}^n)$ is a bounded linear operator defined by

$$K_1(x_c) := e^{A_c t} \Pi_u x_c$$

and (by using (1.1.13)) $K_2 : BC^\eta(\mathbb{R}, \mathbb{R}^n) \to BC^\eta(\mathbb{R}, \mathbb{R}^n)$ is a bounded linear operator defined by

$$\begin{aligned} K_2(f) : = {} & \int_0^t e^{A_c(t-l)} \Pi_c f(l) dl + \int_{-\infty}^t e^{A_s(t-l)} \Pi_s f(l) dl \\ & - \int_t^{+\infty} e^{-A_u(l-t)} \Pi_u f(l) dl. \end{aligned}$$

Assume that

$$\|F\|_{\mathrm{Lip}} \|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R}, \mathbb{R}^n))} < 1,$$

it follows that (1.1.19) has a unique fixed point

$$u_{x_c} = (I - K_2 F)^{-1} K_1 x_c \in BC^\eta(\mathbb{R}, \mathbb{R}^n).$$

Therefore, the first part of the theorem is proved by defining

$$\Psi_c(x_c) := \Pi_h u_{x_c}(0).$$

To prove that Ψ_c is Lipschitz continuous it is sufficient to observe that

$$u_{x_c} - u_{\widehat{x}_c} = K_1(x_c - \widehat{x}_c) + K_2 F(u_{x_c}) - K_2 F(u_{\widehat{x}_c}).$$

Therefore,

$$\left\| u_{x_c} - u_{\widehat{x}_c} \right\|_\eta \leq \|K_1\|_{\mathscr{L}(X_c, BC^\eta(\mathbb{R}, \mathbb{R}^n))} \|x_c - \widehat{x}_c\| + \|F\|_{\mathrm{Lip}} \|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R}, \mathbb{R}^n))} \left\| u_{x_c} - u_{\widehat{x}_c} \right\|_\eta$$

and we obtain

$$\left\|u_{x_c}-u_{\widehat{x}_c}\right\|_\eta \leq \frac{\|K_1\|_{\mathscr{L}(X_c,BC^\eta(\mathbb{R},\mathbb{R}^n))}}{1-\|F\|_{\mathrm{Lip}}\|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R},\mathbb{R}^n))}}\left\|x_c-\widehat{x}_c\right\|.$$

The result follows since

$$\|\Psi_c(x_c)-\Psi_c(\widehat{x}_c)\| \leq \|\Pi_h\|\left\|u_{x_c}-u_{\widehat{x}_c}\right\|_\eta.$$

This completes the proof. □

(d) Truncation method. Let $\{U_\varepsilon(t)x\}_{t\geq 0}$ be the semiflow generated by the truncated problem

$$\frac{dU_\varepsilon(t)}{dt}=AU_\varepsilon(t)x+F_\varepsilon(U_\varepsilon(t)x)$$

for $\varepsilon>0$ small enough. The map F_ε is a truncation of F; namely,

$$F_\varepsilon(x)=\rho\left(\varepsilon^{-1}x\right)F(x),$$

where $\rho:\mathbb{R}^n\to[0,+\infty)$ is a C^k map satisfying

$$\rho(x)=\begin{cases}1 & \text{if } \|x\|\leq 1\\ \in[0,1] & \text{if } 1\leq\|x\|\leq 2\\ 0 & \text{if } \|x\|\geq 2,\end{cases}$$

where $\|.\|$ is the Euclidean norm.

Since $DF(0)=0$, one deduces that

$$\|F_\varepsilon\|_{\mathrm{Lip}}\to 0 \text{ as } \varepsilon\to 0.$$

Moreover, U and U_ε coincide in $B_{\mathbb{R}^n}(\bar{x},\varepsilon)$. This means that for each $x\in B_{\mathbb{R}^n}(\bar{x},\varepsilon)$ and $t>0$,

$$U_\varepsilon(s)x\in B_{\mathbb{R}^n}(\bar{x},\varepsilon) \text{ or } U(s)x\in B_{\mathbb{R}^n}(\bar{x},\varepsilon), \forall s\in[0,t]$$

implies that

$$U_\varepsilon(t)x=U(t)x.$$

Since in general F is Lipschitz continuous, a local center manifold result is in order. The main difficulty to prove the local center manifold theorem is the regularity part (i.e., it is not an application of the implicit function theorem).

Theorem 1.1.22 (Local Center Manifold). *Let Assumption 1.1.17 be satisfied. Assume that*

$$\bar{x}=0$$

and

$$\sigma_c(A)\neq\varnothing.$$

Then there exists a one-time continuously differentiable map $\Psi_c:X_c\to X_h$ such that

$$\Psi_c(0) = 0 \text{ and } D\Psi_c(0) = 0.$$

The local center manifold (which is not uniquely determined)

$$M_c := \{x_c + \Psi_c(x_c) : x_c \in X_c\}$$

is locally invariant under $U(t)$ in some neighbordhood of 0. More precisely, there exists an $\varepsilon > 0$ such that the following properties hold:

(i) If $I \subset \mathbb{R}$ is an interval and $u_c : I \to X_c$ is a solution of the ordinary differential equation on X_c

$$\textbf{(Reduced equation)} \begin{cases} \dfrac{du_c(t)}{dt} = A_c u_c(t) + \Pi_c F(u_c(t) + \Psi_c(u_c(t))) \text{ for } t \in \mathbb{R}, \\ u_c(0) = x_c \in X_c \end{cases}$$

satisfying

$$u_c(t) + \Psi_c(u_c(t)) \in B_{\mathbb{R}^n}(0, \varepsilon), \ \forall t \in I,$$

then $x(t) := u_c(t) + \Psi_c(u_c(t))$ is a solution of (1.1.16);

(ii) If $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of (1.1.16) such that

$$x(t) \in B_{\mathbb{R}^n}(0, \varepsilon), \ \forall t \in \mathbb{R},$$

then

$$x(t) \in M_c, \ \forall t \in \mathbb{R},$$

and $u_c(t) := \Pi_c x(t)$ is a solution of the reduced equation;

*(iii) **(Regularity)** Let $k \geq 1$ be an integer. If F is k-time continuously differentiable locally around 0, then Ψ_c is also k-time continuously differentiable.*

(e) Normal form theory. To determine the qualitative behavior of a nonlinear system in the neighborhood of a nonhyperbolic equilibrium point, the center manifold theorem implies that it could be reduced to the problem of determining the qualitative behavior of the nonlinear system restricted on the center manifold, which reduces the dimension of a local bifurcation problem near the nonhyperbolic equilibrium point. The normal form theory provides a way of finding a nonlinear analytic transformation of coordinates in which the nonlinear system restricted to the center manifold takes the "simplest" form, called *normal form*.

Assume that the reduced system takes the form

$$\frac{du(t)}{dt} = Au(t) + F_2(u) + F_3(u) + \cdots + F_{m-1}(u) + O(|u|^m), \tag{1.1.20}$$

where $F_k(u)$ contains the terms of precise order k. The idea is to choose a coordinate transformation to simplify or eliminate the quadratic terms. Let $u = x + h_2(x)$, where h_2 is a quadratic polynomial. Substituting into system (1.1.20) yields that

$$(I + Dh_2(x))\frac{dx}{dt} = A(x + h_2(x)) + F_2(x + h_2(x)) + \cdots + F_{m-1}(x + h_2(x)) + O(|x|^m).$$

Note that

$$F_k(x+h_2(x)) = F_k(x) + O(|x|^{k+1}), \;\; 2 \le k \le m-1.$$

Thus we have

$$(I+Dh_2(x))\frac{dx}{dt} = Ax + Ah_2(x) + F_2(x) + \tilde{F}_3(x) + \cdots + \tilde{F}_{m-1}(x) + O(|x|^m), \tag{1.1.21}$$

where $\tilde{F}_k(x)$ are the corresponding modified $O(|x|^k)$ terms.

If $|x|$ is sufficiently small, then $I+Dh_2(x)$ is invertible and

$$(I+Dh_2(x))^{-1} = I - Dh_2(x) + O(|x|^2).$$

Substituting into system (1.1.21), we have

$$\frac{dx}{dt} = Ax - [Dh_2(x)Ax - Ah_2(x)] + F_2(x) + \tilde{F}_3(x) + \cdots + \tilde{F}_{m-1}(x) + O(|x|^m). \tag{1.1.22}$$

Now introduce the notation of *Lie bracket* (Marsden and McCracken [257]) as follows:

$$[A,h](x) \triangleq L_A(h(x)) = Dh(x)(Ax) - Ah(x).$$

One can see that the second term becomes $[Ah_2(x)) - Dh_2(x)Ax] = -[A,h_2](x)$. Thus, if $h_2(x)$ can be selected so that

$$[A,h_2](x) = F_2(x), \tag{1.1.23}$$

then the quadratic terms in system (1.1.22) can be eliminated. Note that a solution to (1.1.23) is possible only when $F_2(x)$ belongs to the range of linear operator $[A,h_2]$.

Notice that restricting $[A,h_2]$ to second order polynomials transforms (1.1.23) to a linear algebraic problem. Let $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \cdots, \overrightarrow{e}_n\}$ be a basis for $\mathbb{R}^n$. A *vector monomial of degree k* takes the form

$$(x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n})\overrightarrow{e}_i, \;\; \sum_{j=1}^{n} m_j = k,$$

where $m_j \ge 0$ are integers. The vector monomials of degree k form a basis for the finite dimensional vector space H_k of all vector-valued polynomials of degree k.

Take $n=2$ and let $\overrightarrow{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\overrightarrow{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ denote the standard basis for $\mathbb{R}^2$. Then

$$H_2 = \operatorname{span}\left\{ \begin{pmatrix}x_1^2\\0\end{pmatrix}, \begin{pmatrix}x_1x_2\\0\end{pmatrix}, \begin{pmatrix}x_2^2\\0\end{pmatrix}, \begin{pmatrix}0\\x_1^2\end{pmatrix}, \begin{pmatrix}0\\x_1x_2\end{pmatrix}, \begin{pmatrix}0\\x_2^2\end{pmatrix} \right\}.$$

For the linear map $L_A = [A,\cdot] : H_2 \to H_2$, we can write

$$H_2 = L_A(H_2) \oplus G_2,$$

where G_2 is a complementary subspace of the range of L_A acting on H_2. Now rewrite

$$F_2(x) = F_2^{nr}(x) + F_2^r(x), \;\; F_2^{nr} \in L_A(H_2), \;\; F_2^r \in G_2.$$

Choosing h_2 so that $L_A(h_2(x)) = [A, h_2](x) = F_2^{nr}(x)$, we then obtain the following theorem.

Theorem 1.1.23 (Poincaré Normal Form Theorem). *Consider system (1.1.20) and define a linear transmormation $L_A : H_k \to H_k$ by*

$$L_A(h(x)) \triangleq [A, h](x) = Dh(x)(Ax) - Ah(x).$$

Then by using the decomposition $H_k = L_A(H_k) \oplus G_k$, there exists a sequence of transmormations $x \to x + h_k(x)$ (with $h_k \in H_k$) which transforms system (1.1.20) into the normal form

$$\frac{dx}{dt} = Ax + F_2^r(x) + \cdots + F_{m-1}^r(x) + O(|x|^m), \tag{1.1.24}$$

where

$$F_k^r \in G_k, \;\; \forall k = 2, 3, \cdots, m-1.$$

Remark 1.1.24. Suppose that $A = \mathrm{diag}[\lambda_1, \cdots, \lambda_n]$ is a diagonal matrix. Let $h(x) = (x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}) \overrightarrow{e}_i \in H_k$, where $\sum_{j=1}^n m_j = k$. Then

$$L_A(h(x)) = [A, h](x) = Dh(x)(Ax) - Ah(x) = \left[\sum_{j=1}^n m_j \lambda_j - \lambda_i \right] h(x).$$

Hence, L_A is also diagonal on H_k in the standard basis and is not invertible if zero is an eigenvalue; that is, if $\sum_{j=1}^n m_j \lambda_j - \lambda_i = 0$ for some i. If the eigenvalues of A satisfy a relation of this form where the m_j are non-negative integers, then the eigenvalues are in *resonance* of order $\sum_{j=1}^n m_j$. For this reason, the terms $F_k^r(x)$ in (1.1.24) are called *resonance terms.*

Remark 1.1.25. It is important to understand that the simplified system (1.1.24) is strongly depending on the specific choice of the complementary spaces G_k. In other words, changing the complementary spaces will change the form of the simplified system (1.1.24) which is obtained by making a succession of changes of variables.

(f) Hopf bifurcation theorem. In order to explain the idea of Hopf bifurcation theorem (see Hopf [191]), we first consider a system of two scalar ordinary differential equations

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \kappa \left(x(t)^2 + y(t)^2 \right) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where the bifurcation parameter α varies from negative values to positive values, and the parameters

$$\omega \neq 0 \text{ and } \kappa \neq 0$$

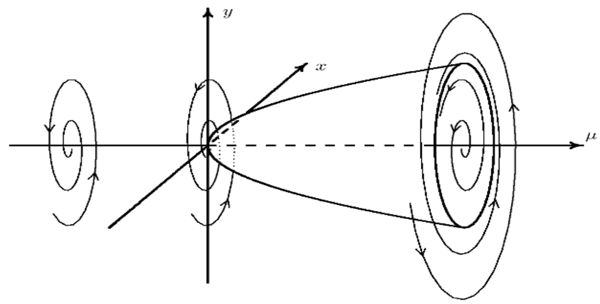

Fig. 1.2: When $\kappa < 0$ we use $\mu := \alpha$ as a bifurcation parameter, and when μ passes through 0 a stable periodic orbit is appearing. The case $\kappa > 0$ can be understood from the case $\kappa < 0$ by going backward in time; that is, by considering $\widehat{x}(t) := x(-t)$ and $\widehat{y}(t) := y(-t)$. When $\kappa > 0$ we use $\mu := -\alpha$ as a bifurcation parameter, when μ passes through 0 an unstable periodic orbit is appearing.

are fixed.

Embedding the system into the complex plan, namely, setting

$$\lambda(t) = x(t) + iy(t) \Leftrightarrow x(t) := \mathrm{Re}(\lambda(t)) \text{ and } y(t) = \mathrm{Im}(\lambda(t)),$$

we obtain the Poincaré normal form [291, 290]

$$\lambda'(t) = (\alpha + i\omega)\,\lambda(t) + \kappa\,|\lambda(t)|^2\,\lambda(t). \tag{1.1.25}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\,|\lambda(t)|^2 &= \overline{\lambda(t)}\lambda'(t) + \overline{\lambda(t)}'\lambda(t)\\ &= (\alpha + i\omega)\,|\lambda(t)|^2 + \kappa\,|\lambda(t)|^2\,|\lambda(t)|^2\\ &\quad + (\alpha - i\omega)\,|\lambda(t)|^2 + \kappa\,|\lambda(t)|^2\,|\lambda(t)|^2.\end{aligned}$$

So by setting $r(t) := |\lambda(t)|^2$ we deduce that $r(t)$ satisfies the logistic equation

$$\frac{dr(t)}{dt} = 2r(t)(\alpha + \kappa r(t)). \tag{1.1.26}$$

By using this equation, we deduce that the curve

$$x(t)^2 + y(t)^2 = \bar{r}^2 := -\frac{\alpha}{\kappa}$$

is invariant by the flow as long as

$$-\frac{\alpha}{\kappa} > 0.$$

Moreover, on this curve (i.e. when $x(t)^2 + y(t)^2 = \bar{r}^2$), the Poincaré normal form (1.1.25) becomes

$$\lambda'(t) = i\omega\lambda(t),$$

which gives

$$\lambda(t) = \sqrt{r}e^{i\omega t}.$$

Thus, this curve is a periodic orbit of period $\frac{2\pi}{\omega}$. By using the logistic equation (1.1.26) one may also analyze the stability of this periodic solution.

The Hopf bifurcation theorem is an extension of the above idea. Consider a parametrized system of ordinary differential equations

$$\frac{dx(t)}{dt} = Ax(t) + F(\mu, x(t)) \text{ for } t \geq 0 \text{ with } x(0) = x_0 \in \mathbb{R}^n, \tag{1.1.27}$$

where $\mu \in \mathbb{R}$ is a parameter. In order to clarify the statement under the assumptions for the Hopf bifurcation theorem, we first recall a definition.

Definition 1.1.26. An eigenvalue $\lambda_0 \in \sigma(A)$ is said to be *simple* if one of the following equivalent conditions are satisfied:

(i) λ_0 is a root of order 1 of the characteristic polynomial of A; namely, a root of order 1 of the polynomial

$$\lambda \to \det(\lambda I - A);$$

(ii) $\dim(\ker(\lambda_0 I - A)) = 1$ and $\dim(\ker(\lambda_0 I - A)^2) = 1$.

We make the following assumption.

Assumption 1.1.27. Let $\varepsilon > 0$ and $F \in C^k((-\varepsilon, \varepsilon) \times B_{\mathbb{R}^n}(0, \varepsilon); \mathbb{R}^n)$ for some $k \geq 4$. Assume that the following conditions are satisfied:

(i) $F(\mu, 0) = 0$, $\forall \mu \in (-\varepsilon, \varepsilon)$, and $\partial_x F(0,0) = 0$.

(ii) **(Transversality condition)** For each $\mu \in (-\varepsilon, \varepsilon)$, there exists a pair of conjugated simple eigenvalues of $(A + \partial_x F(\mu, 0))_0$, denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, such that

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu),$$

the map $\mu \to \lambda(\mu)$ is continuously differentiable,

$$\omega(0) > 0,\ \alpha(0) = 0,\ \frac{d\alpha(0)}{d\mu} \neq 0,$$

and

$$\sigma(A) \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\}. \tag{1.1.28}$$

To prove the Hopf bifurcation theorem one may apply the center manifold theorem to obtain a 3-dimensional reduced system for the system

$$\begin{cases} \dfrac{d\mu(t)}{dt} = 0 \\ \dfrac{dx(t)}{dt} = Ax(t) + F(\mu(t), x(t)). \end{cases}$$

Then by using the Hopf bifurcation theorem for the 2-dimensional parametrized system, one may prove the following theorem (Hopf [191] and Hassard et al. [181]).

Theorem 1.1.28 (Hopf Bifurcation). *Let Assumption 1.1.27 be satisfied. Then there exist a constant $\varepsilon^* > 0$ and three C^{k-1} maps, $\varepsilon \to \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to x_\varepsilon$ from $(0, \varepsilon^*)$ into $\mathbb{R}^n$, and $\varepsilon \to T(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $T(\varepsilon)$-periodic function $x_\varepsilon \in C^k\left(\mathbb{R}^{n+1}\right)$, which is a solution of (1.1.27) with the parameter value $\mu = \mu(\varepsilon)$ and the initial value $x_\varepsilon(0) = x_0$. So for each $t \geq 0$, $x_\varepsilon(t)$ satisfies*

$$\frac{dx_\varepsilon(t)}{dt} = Ax_\varepsilon(t) + F(\mu(\varepsilon), x_\varepsilon(t)) \text{ for } t \geq 0 \text{ and } x_\varepsilon(0) = x_0.$$

Moreover, we have the following properties:

(i) There exist a neighborhood N of 0 in $\mathbb{R}^n$ and an open interval I in $\mathbb{R}$ containing 0, such that for $\widehat{\mu} \in I$ and any periodic solution $\widehat{x}(t)$ in N with minimal period $\widehat{T}$ close to $\frac{2\pi}{\omega(0)}$ of (1.1.27) for the parameter value $\widehat{\mu}$, there exists $\varepsilon \in (0, \varepsilon^*)$ such that $\widehat{x}(t) = x_\varepsilon(t + \theta)$ (for some $\theta \in [0, \gamma(\varepsilon))$), $\mu(\varepsilon) = \widehat{\mu}$, and $T(\varepsilon) = \widehat{T}$;

(ii) The map $\varepsilon \to \mu(\varepsilon)$ is a C^{k-1} function and we have the Taylor expansion

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0, \varepsilon^*),$$

where $[\frac{k-2}{2}]$ is the integer part of $\frac{k-2}{2}$;

(iii) The period $T(\varepsilon)$ of $t \to x_\varepsilon(t)$ is a C^{k-1} function and

$$T(\varepsilon) = \frac{2\pi}{\omega(0)}[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \tau_{2n} \varepsilon^{2n}] + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0, \varepsilon^*),$$

where $\omega(0)$ is the imaginary part of $\lambda(0)$ defined in Assumption 1.1.27;

(iv) The nonzero Floquet exponent $\beta(\varepsilon)$ is a C^{k-1} function satisfying $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and having the Taylor expansion

$$\beta(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \beta_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0, \varepsilon^*).$$

The periodic solution $x_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\varepsilon) < 0$ and unstable if $\beta(\varepsilon) > 0$.

Remark 1.1.29. In applications, we usually have the following approximations

$$\mu(\varepsilon) = \mu_2 \varepsilon^2 + O(\varepsilon^4),\ T(\varepsilon) = \frac{2\pi}{\omega(0)}[1 + \tau_2 \varepsilon^2] + O(\varepsilon^4),\ \beta(\varepsilon) = \beta_2 \varepsilon^2 + O(\varepsilon^4)$$

for all$\varepsilon \in (0, \varepsilon^*)$. Therefore, the direction of the Hopf bifurcation, the stability and period of the bifurcation periodic solutions are determined as follows: if $\mu_2 > 0 (< 0)$, then the bifurcating periodic solutions exist for $\mu > 0 (< 0)$ and the bifurcation is called *supercritical* (*subcritical*); if $\beta_2 < 0 (> 0)$, then the bifurcating periodic solutions are stable (unstable); if $\tau_2 > 0(< 0)$, then the period of the bifurcating periodic solutions increases (decreases).

1.2 Retarded Functional Differential Equations

In this section we introduce some concepts in Retarded Functional Differential Equations (RFDEs), also called Delay Differential Equations (DDEs), and state some very basic results on the subject. This part will be especially useful to readers who are not familiar with delay differential equations. Our goal is to use delay differential equations as a motivating example for the applications of the semigroup theory. We refer to the monographs of Hale [170], Hale and Verduyn Lunel [175], Diekmann et al. [106], Wu [374], and Arino et al. [27] for fundamental theories and results on RFDEs. See also the books of Kuang [221] and the surveys of Ruan [299, 300] for more examples of RFDEs in the context of population dynamics.

Let $r > 0$ be a fixed constant. The first prototype equation is the following delay differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t-r)) \text{ for } t \geq 0, \\ x(\theta) = \phi(\theta), \ \forall \theta \in [-r, 0], \end{cases} \tag{1.2.1}$$

where $\phi \in C([-r,0],\mathbb{R})$, the space of all continuous functions from $[-r,0]$ to $\mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

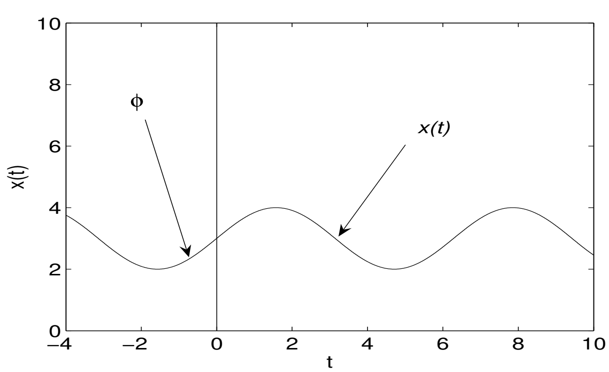

Fig. 1.3: The solution of a RFDE depending on the initial value $\phi(\theta), \theta \in [-r,0]$.

In such a problem the function ϕ is called the *initial value* of system (1.2.1). Moreover, a *solution* of system (1.2.1) is understood as a continuous function $x \in C([-r,\tau),\mathbb{R})$ satisfying

$$x(t) = \begin{cases} \phi(0) + \int_0^t f(x(s-r))ds & \text{if } t \geq 0, \\ \phi(t) & \text{if } -r \leq t \leq 0. \end{cases} \tag{1.2.2}$$

We observe that in this case the solution can be constructed inductively. Indeed, for each $t \in [0, r]$, we have

$$x(t) = \phi(0) + \int_0^t f(\phi(s-r))ds \text{ if } t \geq 0.$$

Since ϕ is given, we know that $x(t)$ exists and is uniquely determined on $[0, r]$ by ϕ. Similarly, we deduce that for each $n \geq 0$, the solution $x(t)$ restricted to $[n, n+r]$ is entirely and uniquely determined by $x(t)$ on $[n-r, n]$. By using this inductive procedure, we deduce that there exists a solution $x \in C([-r, +\infty), \mathbb{R})$ of system (1.2.1) which is uniquely determined by the initial value ϕ.

Now we reformulate this example in a general form. Let $\tau > 0$ be a given constant and let $x \in C([-r, \tau], \mathbb{R})$. For each $t \in [0, \tau]$, define $x_t \in C([-r, 0], \mathbb{R})$ by

$$x_t(\theta) = x(t+\theta), \ \forall \theta \in [-r, 0].$$

Consider a map $G : C([-r, 0], \mathbb{R}) \to \mathbb{R}$ defined by

$$G(\phi) = f(\phi(-r)).$$

Then the delay differential equation (1.2.1) can be rewritten as

$$\begin{cases} \dfrac{dx(t)}{dt} = G(x_t) \text{ for } t \geq 0, \\ x(\theta) = \phi(\theta), \ \forall \theta \in [-r, 0], \end{cases} \tag{1.2.3}$$

and a solution of system (1.2.3) is understood as

$$x(t) = \begin{cases} \phi(0) + \int_0^t G(x_s)ds & \text{if } t \geq 0, \\ \phi(t) & \text{if } -r \leq t \leq 0. \end{cases} \tag{1.2.4}$$

The second prototype delay differential equation is the following

$$\begin{cases} \dfrac{dx(t)}{dt} = bx(t) + f(x(t-r)) \text{ for } t \geq 0, \\ x(\theta) = \phi(\theta), \ \forall \theta \in [-r, 0], \end{cases} \tag{1.2.5}$$

Again we define $G : C([-r, 0], \mathbb{R}) \to \mathbb{R}$ as

$$G(\phi) = b\phi(0) + f(\phi(-r)),$$

and as before we can rewrite the problem in the form (1.2.3).

Remark 1.2.1. In equation (1.2.5) the solution can also be computed step by step by writing it as a continuous function satisfying

$$x(t) = \begin{cases} e^{bt}\phi(0) + \int_0^t e^{b(t-s)} f(x(s-r))ds & \text{if } t \geq 0, \\ \phi(t) & \text{if } -r \leq t \leq 0. \end{cases}$$

Example 1.2.2 (Nicholson's Blowflies Model). Let $N(t)$ denote the population of sexually mature adult blowflies. Assume that the average per capita fecundity drops exponentially with increasing population, then the following delay differential equation describes the total number of mature individuals (Gurney et al. [160])

$$\frac{dN}{dt} = \underbrace{PN(t-\tau)e^{-\frac{N(t-\tau)}{N_0}}}_{\text{birth}} - \underbrace{\delta N(t)}_{\text{mortality}}, \tag{1.2.6}$$

where P is the maximum per capita daily egg production rate, N_0 is the size at which the blowflies population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the time units that all eggs take to develop into sexually mature adults.

1.2.1 Existence and Uniqueness of Solutions

Let $n \geq 1$ be an integer. Consider $\mathscr{C} := C([-r,0],\mathbb{R}^n)$, the space of all continuous functions from $[-r,0]$ to $\mathbb{R}^n$, endowed with the usual supremum norm

$$\|\phi\| = \sup_{\theta\in[-r,0]} |\phi(\theta)|.$$

In this section we consider the delay differential equation of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + G(x_t), \\ x_0 = \phi \in \mathscr{C}, \end{cases} \tag{1.2.7}$$

where $G : \mathscr{C} \to \mathbb{R}^n$ is a continuous map and $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix.

In the following definition we introduce some terminology commonly used for delay differential equations.

Definition 1.2.3. The equation (1.2.7) is called a *scalar delay differential equation* if $n = 1$. The delay differential equation (1.2.7) is called a *discrete delay differential equation* if it can be written as the following special form

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + H(x(t-r_1),\ldots,x(t-r_p)), \\ x_0 = \phi \in C([-r,0],\mathbb{R}^n), \end{cases}$$

where $r_1, ..., r_p \in [0,r]$, and $H : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n}_{p \text{ times}} \to \mathbb{R}^n$ is a continuous map. Otherwise, the delay differential equation (1.2.7) is called a *distributed delay differential equation*.

Definition 1.2.4. For each $\tau \in (0,+\infty]$, we say that $x \in C([-r,\tau), \mathbb{R}^n)$ is a *solution* of (1.2.7) if it satisfies

$$x(t) = \begin{cases} e^{Bt}\phi(0) + \int_0^t e^{B(t-s)} G(x_s) ds & \text{if } 0 \le t < \tau, \\ \phi(t) & \text{if } -r \le t \le 0. \end{cases}$$

The first main result of this section is the following theorem in which we summarize some basic results on delay differential equations (Hale and Verduyn Lunel [175]).

Theorem 1.2.5. *Assume that $G : \mathscr{C} \to \mathbb{R}^n$ is Lipschitz continuous; that is, there exists some $K > 0$ such that*

$$|G(\phi) - G(\psi)| \le K \|\phi - \psi\|, \ \forall \phi, \psi \in \mathscr{C}.$$

Then for each $\phi \in \mathscr{C}$, there exists a unique solution $x_\phi \in C([-r,+\infty), \mathbb{R}^n)$. Moreover, there exist two constants $C > 0$ and $M \ge 1$ such that

$$|x_\phi(t) - x_\psi(t)| \le M e^{Ct} \|\phi - \psi\|, \ \forall t \ge 0, \forall \phi, \psi \in \mathscr{C}.$$

Furthermore, if we consider the family of operators $\{U(t)\}_{t \ge 0}$ from $\mathscr{C}$ into itself defined by

$$U(t)(\phi) = x_{\phi,t} \Leftrightarrow U(t)(\phi)(\theta) = x_\phi(t+\theta), \ \forall \theta \in [-r,0],$$

*then $\{U(t)\}_{t \ge 0}$ defines a **continuous semiflow**; that is,*

(i) $U(t) \circ U(s) = U(t+s), \forall t, s \ge 0$, *and* $U(0) = I$;
(ii) $(t,\phi) \to U(t)\phi$ *is continuous from* $[0,+\infty) \times \mathscr{C}$ *into* $\mathscr{C}$.

Definition 1.2.6. An *equilibrium solution* of (1.2.7) is a solution which is constant in time; that is

$$x(t) = \bar{x}, \ \forall t \ge -r.$$

So

$$0 = B\bar{x} + G\left(\bar{x} 1_{[-r,0]}(.)\right),$$

where

$$1_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.2.7. One may also observe that if $\bar{x}$ is an equilibrium of the RFDE, then $\bar{x} 1_{[-r,0]}(.)$ satisfies

$$U(t)\left(\bar{x} 1_{[-r,0]}(.)\right) = \bar{x} 1_{[-r,0]}(.), \ \forall t \ge 0.$$

So $\bar{x}1_{[-r,0]}(.)$ an equilibrium of the semiflow $\{U(t)\}_{t\geq 0}$. Thus, we also have an interpretation in terms of semiflows.

Example 1.2.8 (Hutchinson's Equation). Consider the equation (Hutchinson [194])

$$\frac{dx(t)}{dt} = \alpha x(t)\left(1 - \frac{x(t-r)}{\kappa}\right) \tag{1.2.8}$$

with $\alpha \in \mathbb{R}$, $r > 0$, and $\kappa > 0$. Then the equilibria are

$$\bar{x} = 0 \text{ and } \bar{x} = \kappa.$$

The second main result of this section is the following theorem on linearized delay differential equations (Hale and Verduyn Lunel [175]).

Theorem 1.2.9 (Linearized Equation). *Assume that G is Lipschitz continuous and continuously differentiable. Then for each $t \geq 0$, the semiflow $\phi \to U(t)(\phi)$ is continuously differentiable. Moreover, if for each $\psi \in \mathscr{C}$ we set*

$$v(t) = V(t)(\psi) = \partial_\phi U(t)(\phi)(\psi),$$

then

$$v(t)(\theta) = y_\psi(t+\theta), \forall \theta \in [-r,0],$$

where $y_\psi(t)$ is the unique solution of

$$\begin{cases} \dfrac{dy_\psi(t)}{dt} = By_\psi(t) + D_\phi G\left(x_{\phi,t}\right)\left(y_{\psi,t}\right), \ \forall t \geq 0 \\ y_{\psi,0} = \psi, \end{cases}$$

in which $x_{\phi,t} = U(t)(\phi)$ is the solution of (1.2.7) with the initial value ϕ.

1.2.2 Linearized Equation at an Equilibrium

If we consider the special case of an equilibrium solution $x(t) = \bar{x}$, $\forall t \geq -r$, then the linearized equation of (1.2.7) is given by

$$\begin{cases} \dfrac{dy_\psi(t)}{dt} = By_\psi(t) + D_\phi G\left(\bar{x}1_{[-r,0]}(.)\right)\left(y_{\psi,t}\right), \ \forall t \geq 0, \\ y_{\psi,0} = \psi \in C([-r,0],\mathbb{R}^n). \end{cases} \tag{1.2.9}$$

By applying Theorem 1.2.5 to system (1.2.9) and by using the fact that $\psi \to B\psi(0) + D_\phi G\left(\bar{x}1_{[-r,0]}(.)\right)(\psi)$ is a bounded linear operator from $\mathscr{C}$ into $\mathbb{R}^n$, we deduce that if we set

$$T(t)(\psi) := \partial_\phi U(t)\left(\bar{x}1_{[-r,0]}(.)\right)(\psi) = y_{\psi,t},$$

then $\{T(t)\}_{t\geq 0}$ satisfies the following properties:

(i) For each $t \geq 0$, $T(t)$ is a bounded linear operator from $\mathscr{C}$ into itself;
(ii) $T(t)T(s) = T(t+s), \forall t,s \geq 0$, and $T(0) = I$;
(iii) $(t,\phi) \to T(t)\phi$ is continuous from $[0,+\infty) \times \mathscr{C}$ into $\mathscr{C}$.

Such a family of linear operators $\{T(t)\}_{t\geq 0}$ is called a *strongly continuous semigroup* of bounded linear operators on $\mathscr{C}$.

Next we explain how to compute the linearized equation for a discrete delay differential equation. Consider a delay differential equation

$$\frac{dx(t)}{dt} = f\left(x\left(t-r_1\right),...,x\left(t-r_p\right)\right), \tag{1.2.10}$$

where $0 \leq r_1 < r_2 < ... < r_{p-1} < r_p =: r$ and $f : \mathbb{R}^p \to \mathbb{R}$ is a C^1-map satisfying

$$f(\bar{x},...,\bar{x}) = 0$$

for some $\bar{x} \in \mathbb{R}$. Then $\bar{x}$ is an equilibrium solution and we can rewrite (1.2.10) as

$$\frac{dx(t)}{dt} = G\left(x_t\right)$$

with

$$G(\phi) = \left(f \circ L_1\right)\phi,$$

in which $L_1 : C\left([-r,0],\mathbb{R}\right) \to \mathbb{R}^p$ is the bounded linear operator defined by

$$L_1\phi = \begin{pmatrix} \phi\left(-r_1\right) \\ \vdots \\ \phi\left(-r_p\right) \end{pmatrix}.$$

By using the differentiation of composed maps we obtain

$$D_\phi G\left(\bar{x}1_{[-r,0]}(.)\right)(\psi) = Df\left(L_1\left(\bar{x}1_{[-r,0]}(.)\right)\right)L_1(\psi).$$

Thus,

$$\begin{aligned} D_\phi G\left(\bar{x}1_{[-r,0]}(.)\right)(\psi) &= \left(\frac{\partial f(\bar{x},...,\bar{x})}{\partial x_1}, \frac{\partial f(\bar{x},...,\bar{x})}{\partial x_2}, ..., \frac{\partial f(\bar{x},...,\bar{x})}{\partial x_p}\right)\begin{pmatrix} \psi\left(-r_1\right) \\ \vdots \\ \psi\left(-r_p\right) \end{pmatrix} \\ &= \sum_{i=1}^{p} \frac{\partial f(\bar{x},...,\bar{x})}{\partial x_i}\psi\left(-r_i\right). \end{aligned}$$

So the linearized equation around $\bar{x}$ is given by

$$\frac{dy(t)}{dt} = \sum_{i=1}^{p} \frac{\partial f(\bar{x},...,\bar{x})}{\partial x_i} y\left(t-r_i\right).$$

Example 1.2.10. For the Hutchinson equation (1.2.8), the linearized equation at $\bar{x} = 0$ is

$$\begin{cases} \dfrac{dy(t)}{dt} = \alpha y(t), \\ y_{\psi,0} = \psi \in C\left([-r,0], \mathbb{R}\right), \end{cases}$$

which is an ordinary differential equation, and the linearized equation at $\bar{x} = \kappa$

$$\begin{cases} \dfrac{dy(t)}{dt} = -\alpha y(t-r), \\ y_{\psi,0} = \psi \in C\left([-r,0], \mathbb{R}\right) \end{cases}$$

is a linear delay differential equation.

1.2.3 Characteristic Equations

Consider a linear delay differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}(x_t) \\ x_0 = \phi \in \mathscr{C}, \end{cases} \tag{1.2.11}$$

where $B \in M_n(\mathbb{R})$ and $\widehat{L} \in \mathscr{L}(\mathscr{C}, \mathbb{R}^n)$, the space of all bounded linear operators from $\mathscr{C}$ to $\mathbb{R}^n$.

One may observe that we can apply Riesz's representation theorem for the dual of the space of continuous functions and deduce that

$$\widehat{L}(\varphi) = \int_{-r}^{0} d\eta(\theta)\, \varphi(\theta)$$

is a Stieltjes integral, where $\eta : [-r,0] \to M_n(\mathbb{R})$ is a function with bounded variation. We recall that η has a bounded variation on $[-r,0]$ if

$$V(\eta, [-r,0]) = \sup \sum_{i=1}^{n} \|\eta(\theta_{i+1}) - \eta(\theta_i)\| < +\infty,$$

where the supremum is taken over all subdivisions $-r = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} = 0$. Then the Stieltjes integral has the following limit

$$\int_{-r}^{0} d\eta(\theta)\, \varphi(\theta) = \lim_{\Delta \to 0} \sum_{i=1}^{n} [\eta(\theta_{i+1}) - \eta(\theta_i)]\, \varphi(\widehat{\theta}_i),$$

where

$$\Delta := \max_{i=1,\ldots,n} (\theta_{i+1} - \theta_i)$$

and

$$\widehat{\theta}_i \in [\theta_i, \theta_{i+1}], \forall i = 1, ..., n.$$

Example 1.2.11. Consider

$$\widehat{L}(\varphi) = M\varphi(-r_1)$$

for some matrix $M \in M_n(\mathbb{R})$ and some $r_1 \in [-r, 0]$. If $r_1 = 0$, take

$$\eta(\theta) = \begin{cases} M & \text{if } \theta = 0, \\ 0 & \text{if } \theta < 0, \end{cases}$$

and if $r_1 < 0$, take

$$\eta(\theta) = \begin{cases} M & \text{if } \theta > r_1, \\ 0 & \text{if } \theta \leq r_1. \end{cases}$$

Then we obtain the desired property.

In order to describe the behavior of such a linear system one needs to study the spectral properties of (1.2.11). An elementary approach to do that is to look for solutions of (1.2.11) of the following form

$$x(t) = e^{\lambda t} z, \;\; \forall t \geq -r$$

with $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n \setminus \{0\}$. Substituting $x(t) = e^{\lambda t} z$ into (1.2.11), we obtain

$$\lambda e^{\lambda t} z = \frac{dx(t)}{dt} = Be^{\lambda t} z - \widehat{L}\left(e^{-\lambda(t+.)} z\right).$$

Cancelling $e^{\lambda t}$, we have

$$\lambda z - Bz - \widehat{L}\left(e^{-\lambda \cdot} z\right) = 0.$$

The *characteristic function* of (1.2.11) is defined by

$$\Delta(\lambda) = \lambda I_{\mathbb{C}^n} - B - \widehat{L}\left(e^{-\lambda \cdot} I_{\mathbb{C}^n}\right) \tag{1.2.12}$$

and the *characteristic equation* of (1.2.11) is defined by

$$\det(\Delta(\lambda)) = 0. \tag{1.2.13}$$

The *spectrum* of the linear RFDE (1.2.11) is defined by

$$\sigma = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\}.$$

Definition 1.2.12. The equilibrium $\bar{x}$ of the RFDE (1.2.7) is *exponentially asymptotically stable* if there exist three constants $\varepsilon > 0$ (sufficiently small), $M \geq 1$, and $\alpha > 0$, such that for each $\phi \in \mathscr{C}$,

$$\left\|\phi - \bar{x}1_{[-r,0]}(.)\right\| \leq \varepsilon \Rightarrow \left\|U(t)\phi - \bar{x}1_{[-r,0]}(.)\right\| \leq Me^{-\alpha t}\left\|\phi - \bar{x}1_{[-r,0]}(.)\right\|, \; \forall t \geq 0,$$

or equivalently there exist three constants $\varepsilon > 0$ (small enough), $\widehat{M} \geq 1$, and $\alpha > 0$, such that for each $\phi \in \mathscr{C}$,

$$\left\| \phi - \overline{x} 1_{[-r,0]}(.) \right\| \leq \varepsilon \Rightarrow |x_\phi(t) - \overline{x}| \leq \widehat{M} e^{-\alpha t} \left\| \phi - \overline{x} 1_{[-r,0]}(.) \right\|, \forall t \geq 0.$$

Theorem 1.2.13 (Exponential Asymptotic Stability). *Let $G : \mathscr{C} \to \mathbb{R}^n$ be a continuous map. Assume that $\overline{x} \in \mathbb{R}^n$ is an equilibrium of the RFDE (1.2.7) and G is C^1 locally around $\overline{x} 1_{[-r,0]}(.)$. Then the equilibrium is (locally) exponentially asymptotically stable if the spectrum of the linearized equation RFDE (1.2.9) at $\overline{x}$ contains only complex numbers with strictly negative real part.*

Example 1.2.14. As an example consider the linearized equation of the Hutchinson equation (1.2.8) at the positive equilibrium $\overline{x} = \kappa$,

$$\begin{cases} \dfrac{dy(t)}{dt} = -\alpha y(t-r) \\ y_{\psi,0} = \psi \in C([-r,0], \mathbb{R}) \end{cases} \tag{1.2.14}$$

Looking for solutions of (1.2.14) of the form

$$y(t) = e^{\lambda t},$$

we obtain the characteristic equation

$$\lambda = -\alpha e^{-\lambda r}, \ \lambda \in \mathbb{C}. \tag{1.2.15}$$

Assume that $\lambda = a + ib$ is a solution of the characteristic equation (1.2.15). Then by taking the modulus on both sides we obtain

$$a^2 + b^2 = \alpha^2 e^{-2ar}.$$

So

$$b^2 = \alpha^2 e^{-2ar} - a^2.$$

We must have

$$a^2 \leq \alpha^2 e^{-2ar}$$

and

$$b = \pm \sqrt{\alpha^2 e^{-2ar} - a^2}. \tag{1.2.16}$$

Moreover, by taking the real and the imaginary parts, we have

$$a = -\alpha e^{-ar} \cos(rb), \quad b = \alpha e^{-ar} \sin(rb). \tag{1.2.17}$$

Proposition 1.2.15. *If $r\alpha < \dfrac{\pi}{2}$, then all roots of characteristic equation (1.2.15) have strictly negative real parts. Hence, the positive equilibrium $\overline{x} = \kappa$ of the Hutchinson equation (1.2.8) is exponentially asymptotically stable.*

Proof. Indeed, assume by contradiction that there exists a solution of the characteristic equation $\lambda = a + ib$ with $a \geq 0$. From (1.2.16) and the characteristic equation, we can assume that $b > 0$ and must have

$$b \leq \alpha.$$

Now by using (1.2.17) we deduce that

$$0 < \cos(r\alpha) < \cos(rb) < 1,$$

a contradiction. This proves the claim. $\square$

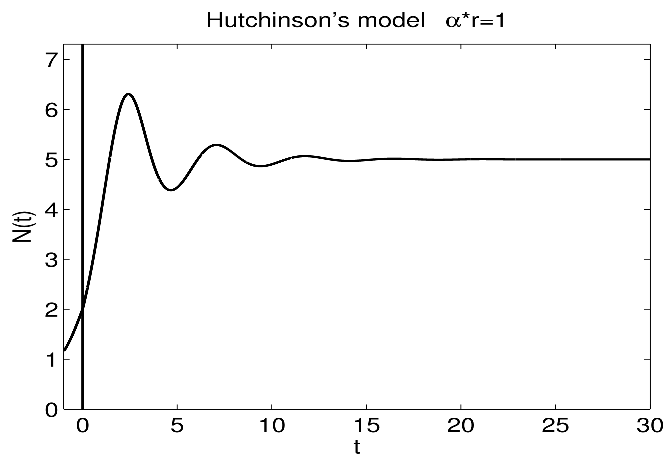

Fig. 1.4: The exponential asymptotic stability of the positive equilibrium of the Hutchinson equation (1.2.8) with $r = 1$ and $\alpha = 1$.

1.2.4 Center Manifolds

Consider the linear RFDE

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + L(x_t) \\ x_0 = \phi \in \mathscr{C}, \end{cases} \tag{1.2.18}$$

where $B \in M_n(\mathbb{R})$ and $L \in \mathscr{L}(\mathscr{C}, \mathbb{R}^n)$ is a bounded linear operator.

The infinitesimal generator of the strongly continuous semigroup generated by the linear RFDE (1.2.18) is defined by $A_0 : D(A_0) \subset \mathscr{C} \to \mathscr{C}$ as

$$A_0\varphi := \varphi'$$

with domain

$$D(A_0) := \{\varphi \in C^1([-r,0],\mathbb{R}^n) : \varphi'(0) = B\varphi(0) + L(\varphi)\}.$$

The linear RFDE can be rewritten into the following abstract form

$$u'(t) = A_0 u(t) \text{ for } t \geq 0 \text{ with } u(0) = \varphi \in \mathscr{C}. \tag{1.2.19}$$

Since the strongly continuous semigroup generated by A_0 is eventually compact, the space $\mathscr{C}$ can be decomposed accordingly to the spectral decomposition $\sigma = \sigma_u \cup \sigma_c \cup \sigma_s$ in which σ_u, σ_c, and σ_s are the sets of eigenvalues with positive, zero, and negative real parts, respectively. We can find three closed subspace of $\mathscr{C}$

$$\mathscr{C} = U \oplus N \oplus S,$$

which define three bounded linear projectors

$$\begin{aligned} &\pi_U \mathscr{C} = U \quad \text{and } (I - \pi_U)\mathscr{C} = N \oplus S, \\ &\pi_N \mathscr{C} = N \quad \text{and } (I - \pi_N)\mathscr{C} = U \oplus S, \\ &\pi_S \mathscr{C} = S \quad \text{and } (I - \pi_S)\mathscr{C} = U \oplus N. \end{aligned}$$

It is well known that the dual space of $\mathscr{C}$ is the space of random measures which is a space much bigger than $C([0,r],(\mathbb{R}^n)^*)$. In order to compute the projectors on the eigenspaces, one can define a formal adjoint relationship between $\mathscr{C} = C([-r,0],\mathbb{R}^n)$ and $\mathscr{C}^* := C([0,r],(\mathbb{R}^n)^*)$ by using the following bilinear form

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-r}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi$$

for $\phi \in \mathscr{C}$ and $\psi \in \mathscr{C}^*$.

The subspace $N \subset C([-r,0],\mathbb{R}^n)$ is a direct sum of the generalized eigenspace associated with eigenvalues with zero real part for the infinitesimal generator of the linear RFDE (1.2.18) which can be rewritten as

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \int_{-r}^{0} d\eta(\theta) x(t+\theta), & t \geq 0, \\ x_0 = \varphi \in C([-r,0],\mathbb{R}^n). \end{cases}$$

The subspace $N^* \subset C([0,r],(\mathbb{R}^n)^*)$ is a direct sum of the generalized eigenspace associated with eigenvalues with zero real part for the infinitesimal generator of the linear RFDE

$$\begin{cases} \dfrac{dy(s)}{ds} = -y(s)B - \int_{-r}^{0} y(s-\theta) d\eta(\theta), & s \leq 0, \\ y^0 = \psi \in C([0,r],(\mathbb{R}^n)^*). \end{cases}$$

Then N and N^* have the same finite dimension. Moreover, let Φ be a basis for N and Ψ be a basis for N^* with

$$\langle \Phi, \Psi \rangle = I.$$

Assume that $\dim N = m \geq 1$. One may observe that $\Phi \in D(A_0)$. Then one can rewrite A_0 in the basis Φ which gives an $m \times m$ matrix B_m. Furthermore, by projecting (1.2.19) on N (i.e. by applying π_N on both sides of (1.2.19) and expressing this into the basis Φ), it follows that

$$\dot{\Phi} = B_m \Phi.$$

Let BC be the set of all functions from $[-r,0]$ to $\mathbb{R}^n$ that are uniformly continuous on $[-r,0)$ and may have a possible jump discontinuity at 0. Define $X_0 : [-r,0] \to M_n(\mathbb{R}^n)$ by

$$X_0(\theta) = \begin{cases} I & \text{if } \theta = 0, \\ 0 & \text{if } \theta \in [-r,0). \end{cases}$$

Then

$$BC = \{\phi + X_0\xi : \phi \in \mathscr{C}, \xi \in \mathbb{R}^n\}.$$

Clearly BC is a Banach space equipped with norm

$$\|\phi + X_0\xi\|_{BC} = \|\phi\| + |\xi|.$$

Consider an extension $A : D(A) \subset \mathscr{C} \to BC$ of A_0 to BC

$$A\psi = \dot{\psi} + X_0[B\psi(0) + L\psi - \dot{\psi}(0)]$$

with domain

$$D(A) := C^1\left([-r,0], \mathbb{R}^n\right).$$

Remark 1.2.16. One may observe that A_0 is the part of A in $\mathscr{C}$; that is,

$$A_0\varphi := A\varphi \text{ for } \varphi \in D(A_0)$$

and

$$D(A_0) := \{\phi \in D(A) : A\varphi \in \mathscr{C}\}.$$

Now consider the functional differential equation

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + L(x_t) + F(x_t) \\ x_0 = \phi \in \mathscr{C}, \end{cases} \tag{1.2.20}$$

where $F : \mathscr{C} \to \mathbb{R}^n$ is a k-time continuously differentiable map satisfying

$$F(0_{\mathscr{C}}) = 0_{\mathbb{R}^n} \text{ and } DF(0_{\mathscr{C}}) = 0_{\mathscr{L}(\mathscr{C},\mathbb{R}^n)}.$$

By setting

$$u(t,\theta) := x_t(\theta) = x(t+\theta),$$

the RFDE (1.2.20) can be rewritten as an abstract Cauchy problem in the Banach space $\mathscr{C}$:

$$\frac{du(t)}{dt} = Au(t) + X_0F(u(t)) \text{ for } t \geq 0 \text{ with } u(0) = \varphi \in \mathscr{C}. \tag{1.2.21}$$

Remark 1.2.17. Of course this problem is not classical since the ranges of A and X_0F do not belong to $\mathscr{C}$.

Then consider a map $\widehat{\pi}_N : BC \to N$ defined as follows

$$\widehat{\pi}_N(\phi + X_0\xi) = \Phi[\langle \Psi, \phi \rangle + \Psi(0)\xi]$$

for $\phi \in \mathscr{C}$ and $\xi \in \mathbb{R}^n$. One can observe that $\widehat{\pi}_N$ extends the projector π_N and we have

$$BC = N \oplus \ker \widehat{\pi}_N$$

with

$$S \oplus U \subset \ker \widehat{\pi}_N.$$

Next we can decompose the solution x_t of (1.2.20) as

$$x_t = \Phi z(t) + y(t)$$

with

$$z(t) \in \mathbb{R}^n \text{ and } y(t) := (I - \pi_N)x_t \in (I - \pi_N)\mathscr{C}.$$

Then (1.2.20) is equivalent to

$$\begin{aligned} \dot{z}(t) &= B_m z(t) + \Psi(0)F(\Phi z + y), \\ \frac{dy}{dt} &= B_h y + (I - \widehat{\pi}_N)X_0F(\Phi z + y), \end{aligned} \tag{1.2.22}$$

where B_h is the part of $B : D(B) \subset \mathscr{C} \to \mathscr{C}$

$$B\phi = \phi' \text{ and } D(B) := C^1\left([-r, 0], \mathbb{R}^n\right)$$

in the hyperbolic space

$$X_h := (I - \widehat{\pi}_N)BC;$$

that is,

$$B_h\varphi = B\varphi = \phi' \text{ for } \varphi \in D(B_h)$$

and

$$D(B_h) := \{\varphi \in D(B) \cap X_h : B\varphi \in X_h\}.$$

Remark 1.2.18. The variable $u(t) = \Phi z(t) + y(t)$ satisfies the original equation (1.2.21).

We have the following result.

Theorem 1.2.19 (Center Manifold). *Assume that $N \neq 0$ and F is a k-time continuously differentiable map satisfying*

$$F(0_{\mathscr{C}}) = 0_{\mathbb{R}^n} \text{ and } DF(0_{\mathscr{C}}) = 0_{\mathscr{L}(\mathscr{C}, \mathbb{R}^n)}.$$

Then there exist a map $W \in C^k(\mathbb{R}^m, \ker \pi_N)$ *with*

$$W(0) = 0 \text{ and } D_z W(0) = 0,$$

and a neighborhood V *of* 0 *in* $\mathscr{C}$ *such that the center manifold*

$$W^c_{\text{loc}}(0) = \{\Phi z + W(z) : z \in \mathbb{R}^m\}$$

has the following properties

(i) $W^c_{\text{loc}}(0)$ *is locally invariant with respect to (1.2.21); that is, if* $\phi \in W^c_{\text{loc}}(0) \cap V$ *and*

$$u(t,\phi) = x_t(\phi) \in V, \ \ \forall t \in I(\phi),$$

then

$$u(t,\phi) = x_t(\phi) \in W^C_{\text{loc}}(0)$$

for all $t \in I(\phi)$ *(the interval of existence);*

(ii) $W^c_{\text{loc}}(0)$ *contains all solutions of (1.2.21) remaining in* V *for all* $t \in \mathbb{R}$.

Note that $W^c_{\text{loc}}(0)$ is a C^k-manifold of (1.2.21) parameterized by $z \in \mathbb{R}^m$. Thus, $W^c_{\text{loc}}(0)$ has the same dimension m, passes through 0, and is tangent to N at 0.

1.3 Age-structured Models

Let $u(t,a)$ be the density of a population with age a at time $t \geq 0$, so that for each $0 \leq a_1 \leq a_2$,

$$\int_{a_1}^{a_2} u(t,a)da$$

is the number of individuals with age a between a_1 and a_2 and the total number of individuals at time t is

$$\int_0^{+\infty} u(t,a)da.$$

Consider the following age-structured model

$$\begin{cases} \underbrace{\dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a)}_{\text{mortality}} \text{ for } t \geq 0 \text{ and } a \geq 0 \\ \underbrace{u(t,0) = \alpha h\left(\int_0^{+\infty} \beta(a)u(t,a)da\right)}_{\text{birth}} \text{ for } t \geq 0 \end{cases} \tag{1.3.1}$$

with the initial distribution

$$u(0,.) = \phi \in L^1_+((0,+\infty),\mathbb{R}).$$

In (1.3.1) the function

$$\mu \in L^1_{+,loc}((0,+\infty),\mathbb{R})$$

is the mortality rate, $\alpha > 0$ is the birth rate of mature individuals, and the function

$$\beta(.) \in L^\infty_+((0,+\infty),\mathbb{R})$$

is the probability for an individual with age a to be mature. Therefore,

$$\int_0^{+\infty} \beta(a)u(t,a)da$$

is the total number of mature individuals at time t.

The function $h(x)$ describes the birth limitation whenever the size of the population increases. A classical example for such a function is the Ricker's function of the form

$$h(x) = xe^{-\delta x},$$

where $\delta \geq 0$, which was introduced by Ricker [297, 298] to describe cannibalism in fish population. Ricker's function can be derived by using a singular limit procedure. We refer to Ducrot et al. [118] for more results about this topic.

1.3.1 Volterra formulation

We observe that if the map $(t,a) \to u(t,a)$ is C^1 and

$$\lim_{h \to 0} \frac{u(t+h,a+h) - u(t,a)}{h} = \frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a},$$

then the first equation of (1.3.1) means that $U(h) := u(t+h,a+h)$ satisfies an ordinary differential equation along the characteristic curve $a = t + c$. More precisely, we have

$$U'(h) = \mu(a+h)U(h), \tag{1.3.2}$$

or equivalently

$$U(0) = \exp\left(-\int_{a-h}^{a} \mu(s))ds\right) U(-h). \tag{1.3.3}$$

Some characteristic curves $a = t + c$ are represented in Fig. 1.5. This figure shows that we need to distinguish the case $a > t$ and the case $a < t$ in order to compute the solution from the value of the distribution on the boundary of the domain $(0,+\infty)^2$.

Set

$$B(t) := \int_0^{+\infty} \beta(a)u(t,a)da.$$

Then the first equation of (1.3.1) means that

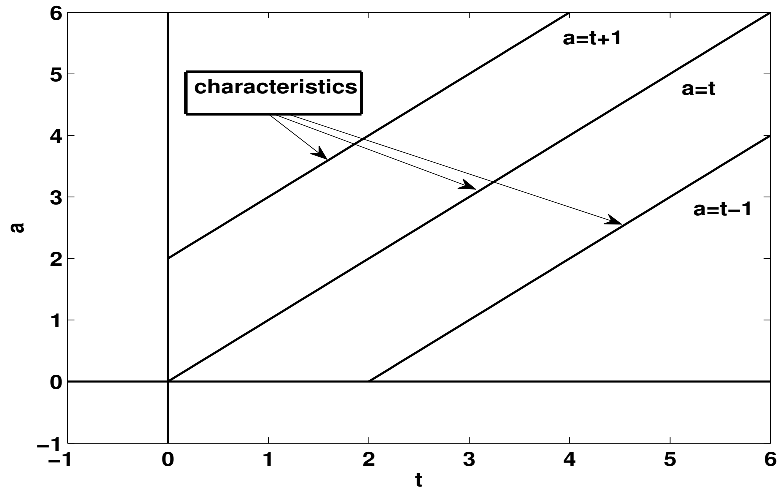

Fig. 1.5: Some characteristics curves $a = t + c$.

$$u(t,a) = \begin{cases} \exp(-\int_{a-t}^{a} \mu(s)ds)\phi(a-t) & \text{if } a > t, \\ \exp(-\int_{0}^{a} \mu(s)ds)\alpha h(B(t-a)) & \text{if } a < t, \end{cases}$$

or equivalently in a more condensed form

$$u(t,a) = \begin{cases} \dfrac{\Pi(a)}{\Pi(a-t)}\phi(a-t) & \text{if } a > t, \\ \Pi(a)\alpha h(B(t-a)) & \text{if } a < t, \end{cases} \tag{1.3.4}$$

where

$$\Pi(a) := \exp(-\int_0^a \mu(s)ds) \tag{1.3.5}$$

is the probability for a newborn to survive to the age a. Now we have

$$B(t) = \int_0^{+\infty} \beta(a)u(t,a)da = \int_0^t \beta(a)u(t,a)da + \int_t^{+\infty} \beta(a)u(t,a)da.$$

So by using (1.3.4) we deduce that $B(t)$ must satisfy a nonlinear *Volterra integral equation*

$$B(t) = \int_0^t \beta(a)\Pi(a)\alpha h(B(t-a))\,da + F(t), \tag{1.3.6}$$

where $F(t)$ corresponds to the contribution of the initial distribution $\phi(.)$ to the number of mature individuals at time t; namely,

$$F(t) := \int_t^{+\infty} \beta(a)\frac{\Pi(a)}{\Pi(a-t)}\phi(a-t)da. \tag{1.3.7}$$

Set

$$K(a) := \beta(a)\Pi(a). \tag{1.3.8}$$

Then the nonlinear Volterra integral equation (1.3.6) can be rewritten as

$$B(t) := (K * G(B))(t) + F(t), \tag{1.3.9}$$

which is also called the *Lotka integral equation*, where

$$G(x) := \alpha h(x)$$

and the operator of convolution is defined by

$$(K * B)(t) := \int_0^t K(a)B(t-a)da = \int_0^t K(t-a)B(a)da. \tag{1.3.10}$$

1.3.2 Age-structured Models without Birth

Assume first that the birth rate

$$\alpha = 0.$$

Then the solution becomes

$$u(t,a) = \begin{cases} e^{\left(-\int_{a-t}^{a} \mu(s)ds\right)}\phi(a-t) = \dfrac{\Pi(a)}{\Pi(a-t)}\phi(a-t) & \text{if } a > t, \\ 0 & \text{if } a < t. \end{cases} \tag{1.3.11}$$

The following theorem summarizes some known results about this special case.

Theorem 1.3.1. *Under the above assumptions, the family of bounded linear operators* $\{T_{\widehat{A}_0}(t)\}_{t\geq 0}$ *on* $L^1((0,+\infty),\mathbb{R})$ *defined by*

$$T_{\widehat{A}_0}(t)(\phi)(a) = \begin{cases} e^{\left(-\int_{a-t}^{a} \mu(s)ds\right)}\phi(a-t) & \text{if } a > t, \\ 0 & \text{if } a < t. \end{cases} \tag{1.3.12}$$

is a strongly continuous semigroup of bounded linear operators; that is,

(i) $T_{\widehat{A}_0}(t)T_{\widehat{A}_0}(s) = T_{\widehat{A}_0}(t+s), \forall t,s \geq 0$, *and* $T_{\widehat{A}_0}(0) = I$;
(ii) $(t,\phi) \to T_{\widehat{A}_0}(t)\phi$ *is continuous from* $[0,+\infty) \times L^1((0,+\infty),\mathbb{R})$ *into* $L^1((0,+\infty),\mathbb{R})$.

Moreover, the linear operator $\widehat{A}_0 : D(\widehat{A}_0) \subset L^1((0,+\infty),\mathbb{R}) \to L^1((0,+\infty),\mathbb{R})$ *defined by*

$$\widehat{A}_0\phi = \phi'$$

with domain

$$D(\widehat{A}_0) = \{\phi \in W^{1,1}((0,+\infty),\mathbb{R}) : \phi(0) = 0\}$$

is the infinitesimal generator of $\{T_{\widehat{A}_0}(t)\}_{t\geq 0}$; *that is,*

$$\lim_{h\to 0}\frac{\left(T_{\widehat{A}_0}(h)\phi-\phi\right)}{h}$$

exists and is equal to χ *if and only if*

$$\phi\in D(\widehat{A}_0) \text{ and } \widehat{A}_0(\phi)=\chi.$$

We also have the following result.

Lemma 1.3.2. *Assume that there exist a constant* $\mu_0>0$ *and an age* $a_0>0$ *such that*

$$\mu(a)>\mu_0 \text{ for almost every } a>a_0. \tag{1.3.13}$$

Then there exists a constant $M>0$ *such that*

$$\|T_{\widehat{A}_0}(t)\|_{\mathscr{L}\left(L^1((0,+\infty),\mathbb{R})\right)}\leq Me^{-\mu_0 t},\ \ \forall t\geq 0.$$

Since the map $\phi\to\int_0^{+\infty}\beta(a)\phi(a)da$ is a bounded linear functional on $L^1((0,+\infty),\mathbb{R})$ and since

$$F(t)=\int_0^{+\infty}\beta(a)T_{\widehat{A}_0}(t)(\phi)(a)da,$$

by applying Theorem 1.3.1, it follows that $F(t)$ is a continuous map and by Lemma 1.3.2

$$|F(t)|\leq\|\beta\|_{L^\infty}Me^{-\mu_0 t},\ \ \forall t\geq 0.$$

1.3.3 Age-structured Models with Birth

Observe that with our assumptions on $\mu(a)$ and $\beta(a)$ the map

$$K:=\beta\Pi\in L^\infty_+((0,+\infty),\mathbb{R}).$$

Moreover, the map $h(x)$ is Lipschitz continuous on $[0,+\infty)$. In order to prove the existence of solutions we can apply the following fixed point procedure

$$B^{n+1}(t)=(K*G(B^n))(t)+F(t),\ \forall t\geq 0 \tag{1.3.14}$$

in some convenient space of continuous functions. Namely, we consider

$$C_\eta([0,+\infty),\mathbb{R})=\{\chi\in C([0,+\infty),\mathbb{R}):\sup_{t\geq 0}e^{-\eta t}|\chi(t)|<+\infty\},$$

which is a Banach space endowed with the norm

$$\|\chi\|_\eta := \sup_{t\geq 0} e^{-\eta t}|\chi(t)|.$$

Theorem 1.3.3. *Under the above assumptions, for each $\eta > 0$ such that*

$$\int_0^{+\infty} e^{-\eta l} K(l)dl < 1,$$

there exists a unique function $B \in C_\eta([0,+\infty),\mathbb{R})$ (for some $\eta > 0$ large enough) satisfying the Volterra integral equation (1.3.6). Moreover,

$$B(t) \geq 0, \ \ \forall t \geq 0.$$

If, in addition, μ satisfies the condition (1.3.13), then $B(t)$ is bounded and

$$\limsup_{t\to+\infty} B(t) \leq \int_0^{+\infty} K(a)da \sup_{x\geq 0} G(x).$$

Proof. Let B_1 and B_2 be two functions in $C_\eta([0,+\infty),\mathbb{R})$. Then

$$\begin{aligned}
&e^{-\eta t}|(K * G(B_1))(t) - (K * G(B_2))(t)| \\
&= e^{-\eta t}|\int_0^t K(t-a)[G(B_1)(a) - G(B_2)(a)]da| \\
&= \left|\int_0^t e^{-\eta(t-a)}K(t-a)e^{-\eta a}[G(B_1)(a) - G(B_2)(a)]da\right| \\
&\leq \int_0^t e^{-\eta(t-a)}K(t-a)da\|G\|_{\mathrm{Lip}}\|B_1 - B_2\|_\eta .
\end{aligned}$$

Thus,

$$\|(K * G(B_1)) - (K * G(B_2))\|_\eta \leq k_\eta \|B_1 - B_2\|_\eta ,$$

where

$$k_\eta := \|G\|_{\mathrm{Lip}} \int_0^{+\infty} e^{-\eta l} K(l)dl \to 0 \text{ as } \eta \to +\infty.$$

By applying the Banach fixed point procedure to (1.3.14), the first part of Theorem 1.3.3 follows.

To prove the last part of Theorem 1.3.3 we use Lemma 1.3.2 and observe that under the additional condition $K \in L^1((0,+\infty),\mathbb{R})$. Therefore,

$$\limsup_{t\to+\infty} B(t) \leq \limsup_{t\to+\infty} \int_0^{+\infty} K(t-a)G(B(a))da + F(t).$$

Hence

$$\limsup_{t\to+\infty} B(t) \leq \limsup_{t\to+\infty} \int_0^{+\infty} K(t-a)da \sup_{x\geq 0} G(x)$$

and the result follows. □

As a consequence we obtain the following theorem.

Theorem 1.3.4. *Under the above assumptions, there exists a unique continuous semiflow $\{U(t)\}_{t\geq 0}$ generated by the solutions integrated along the characteristics for age-structured model (1.3.1). In other words, if for each $\phi \in L^1_+((0,+\infty),\mathbb{R})$ we define $U(t)(\phi)$ as*

$$U(t)(\phi)(a) = u_\phi(t,a),$$

where $u_\phi(t,a)$ is given by

$$u_\phi(t,a) = \begin{cases} \dfrac{\Pi(a)}{\Pi(a-t)}\phi(a-t) & \text{if } a > t, \\ \Pi(a)G\left(B_\phi(t-a)\right) & \text{if } a < t \end{cases} \tag{1.3.15}$$

with $B_\phi(t)$ being a solution of the nonlinear Volterra integral equation

$$B_\phi(t) = \int_0^t \beta(a)\Pi(a)G\left(B_\phi(t-a)\right)da + F_\phi(t) \tag{1.3.16}$$

and

$$F_\phi(t) := \int_t^{+\infty} \beta(a)\frac{\Pi(a)}{\Pi(a-t)}\phi(a-t)da, \tag{1.3.17}$$

then U is a continuous semiflow on $\phi \in L^1_+((0,+\infty),\mathbb{R})$; that is,

(i) $U(t)\circ U(s) = U(t+s), \forall t,s \geq 0$, *and* $U(0) = I$;
(ii) $(t,\phi) \to U(t)\phi$ *is continuous from* $[0,+\infty)\times L^1((0,+\infty),\mathbb{R})$ *into* $L^1_+((0,+\infty),\mathbb{R})$.

1.3.4 Equilibria and Linearized Equations

A positive equilibrium solution for the age-structured model (1.3.1) satisfies $\overline{u} \in W^{1,1}((0,+\infty),\mathbb{R})$ (i.e. $\overline{u} \in L^1((0,+\infty),\mathbb{R})$ and $\overline{u}' \in L^1((0,+\infty),\mathbb{R})$) and

$$\begin{cases} \overline{u}'(a) = -\mu(a)\overline{u}(a) \text{ for a.e. } a \geq 0, \\ \overline{u}(0) = G(\int_0^{+\infty}\beta(a)\overline{u}(a)da). \end{cases} \tag{1.3.18}$$

Thus

$$\overline{u}(a) = \Pi(a)\overline{u}(0) \text{ for all } a \geq 0$$

and

$$\overline{u}(0) = r\overline{u}(0)\exp(-\kappa\overline{u}(0)),$$

where

$$r := \alpha\int_0^{+\infty}\beta(a)\Pi(a)da \text{ and } \kappa := \delta\int_0^{+\infty}\beta(a)\Pi(a)da.$$

Lemma 1.3.5. *We have the following alternatives:*

(i) *If $r \leq 1$, then $\overline{u}_0(a) = 0, \forall a \geq 0$, is the unique equilibrium of the age-structured model (1.3.1);*

(ii) *If $r > 1$, then $\overline{u}_0(a) = 0, \forall a \geq 0$, and $\overline{u}_1(a) = \Pi(a)\dfrac{\ln(r)}{\kappa}, \forall a \geq 0$, is the only positive equilibrium of the age-structured model (1.3.1).*

The linearized equation around $\overline{u}$ is given by

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -\mu(a)v(t,a), \ t \geq 0, \ a \geq 0 \\ v(t,0) = G'(\int_0^{+\infty} \beta(a)\overline{u}(a)da) \int_0^{+\infty} \beta(a)v(t,a)da, \ t \geq 0. \end{cases} \tag{1.3.19}$$

But

$$G'(x) = \alpha(1 - x\delta)\exp(-\delta x).$$

It follows that

$$G'(0) = \alpha. \tag{1.3.20}$$

For the positive equilibrium (when it exists) we have

$$\int_0^{+\infty} \beta(a)\overline{u}_1(a)da = \int_0^{+\infty} \beta(a)\Pi(a)da\frac{\ln(r)}{\kappa} = \frac{\ln(\alpha \int_0^{+\infty} \beta(a)\Pi(a)da)}{\delta},$$

$$G'\left(\int_0^{+\infty} \beta(a)\overline{u}_1(a)da\right) = \frac{1 - \ln\left(\alpha \int_0^{+\infty} \beta(a)\Pi(a)da\right)}{\int_0^{+\infty} \beta(a)\Pi(a)da}.$$

Assume for simplicity that

$$\int_0^{+\infty} \beta(a)\Pi(a)da = 1,$$

we obtain

$$G'\left(\int_0^{+\infty} \beta(a)\overline{u}_1(a)da\right) = 1 - \ln(\alpha). \tag{1.3.21}$$

By using (1.3.19) and (1.3.20), we deduce that the linearized equation around $\overline{u}_0 = 0$ is given by

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -\mu(a)v(t,a) \text{ for } t \geq 0 \text{ and } a \geq 0 \\ v(t,0) = \alpha \int_0^{+\infty} \beta(a)v(t,a)da \text{ for } t \geq 0, \end{cases} \tag{1.3.22}$$

and by using (1.3.19) and (1.3.21), we deduce that the linearized equation around $\overline{u}_1$ is given by

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -\mu(a)v(t,a) \text{ for } t \geq 0 \text{ and } a \geq 0 \\ v(t,0) = (1 - \ln(\alpha)) \int_0^{+\infty} \beta(a)v(t,a)da \text{ for } t \geq 0. \end{cases} \tag{1.3.23}$$

In order to derive the characteristic equation we look for (nontrivial) solution of the form

$$v(t,a) = e^{\lambda t}v_0(a). \tag{1.3.24}$$

By substituting it into the linearized equation we obtain

$$\begin{cases} \lambda v_0(a) + v_0'(a) = -\mu(a)v_0(a) \text{ for all } a \geq 0 \\ v_0(0) = G'\left(\int_0^{+\infty} \beta(a)\overline{u}(a)da\right) \int_0^{+\infty} \beta(a)v_0(a)da. \end{cases} \tag{1.3.25}$$

By using the first equation of (1.3.25) we have

$$v_0(a) = v_0(0)\Pi(a)e^{-\lambda a}.$$

So by plugging this last expression into the second equation of (1.3.25), we obtain the *characteristic equation* to find $\lambda \in \mathbb{C}$:

$$1 = G'\left(\int_0^{+\infty} \beta(a)\overline{u}(a)da\right) \int_0^{+\infty} \beta(a)\Pi(a)e^{-\lambda a}da. \tag{1.3.26}$$

1.3.5 Age-structured Models Reduce to DDEs and ODEs

We consider the age-structured model (1.3.1) in the following cases.

(i) $\beta(a) = e^{-\beta a}1_{[\tau,+\infty)}(a),\ \mu(0) = \mu$, where $\beta > 0,\ \tau \geq 0$ and $\mu > 0$ are constants. Let $\hat{u}(t) = \int_0^\infty e^{-\beta a}u(t,a)da$. Then, we obtain a delay differential equation

$$\begin{cases} \frac{d\hat{u}(t)}{dt} = \alpha e^{-\mu\tau}h(\hat{u}(t-\tau)) - (\mu+\beta)\hat{u}(t), t \geq \tau, \\ \hat{u}(t) = e^{-\mu t}\int_\tau^\infty e^{-\beta a}u_0(a-t)da \\ \qquad = e^{-(\mu+\beta)t}\int_{t-\tau}^\infty e^{-\beta b}u_0(b)db,\ t \in [0,\tau]. \end{cases} \tag{1.3.27}$$

(ii) $\beta(a) = 1_{[0,\infty)}(a)$. Let $\tilde{u}(t) = \int_0^\infty u(t,a)da$. Then we obtain an ODE

$$\begin{cases} \frac{d\tilde{u}}{dt} = \alpha h(\tilde{u}(t)) - \mu\tilde{u}(t),\ \ t \geq 0, \\ \tilde{u}(0) = \tilde{u}_0 \geq 0. \end{cases} \tag{1.3.28}$$

(iii) With juveniles and adults. Let

$$A(t) = \int_\tau^\infty u(t,a)da$$

denote the number of adults in the population at time t. Consider the boundary condition

$$u(t,0) = \int_\tau^\infty \beta(A(t))u(t,a)da = \beta(A(t))A(t).$$

Then $A(t)$ satisfies a delay differential equation

$$\begin{aligned} \frac{dA}{dt} &= u(t,\tau) - \mu A(t) \\ &= \beta(A(t-\tau))A(t-\tau)e^{-\mu\tau} - \mu A(t). \end{aligned} \tag{1.3.29}$$

Let

$$\beta(A) = \beta_0 e^{-\frac{A}{N_0}}, \;\; \delta = \mu.$$

Then equation (1.3.29) becomes

$$\frac{dA}{dt} = \beta_0 e^{-\delta\tau} A(t-\tau) e^{-\dfrac{A(t-\tau)}{N_0}} - \delta A(t), \tag{1.3.30}$$

which is the Nicholson's blowflies model (1.2.6) if we denote $P = \beta_0 e^{-\delta\tau}$.

1.4 Abstract Semilinear Formulation

In this section we formulate several types of equations, including functional differential equations, age-structured models, parabolic equations, and partial functional differential equations, as abstract Cauchy problems with nondense domain.

1.4.1 Functional Differential Equations

(a) From RFDE to PDE. Consider the retarded functional differential equations of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}(x_t) + G(x_t), \; \forall t \geq 0, \\ x_0 = \phi \in C([-r,0], \mathbb{R}^n), \end{cases} \tag{1.4.1}$$

where $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, $\widehat{L} : C([-r,0], \mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator, and $G : C([-r,0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous map.

In order to study the RFDE (1.4.1) by using the integrated semigroup theory, we need to consider RFDE (1.4.1) as an abstract non-densely defined Cauchy problem. Firstly, we regard the RFDE (1.4.1) as a Partial Differential Equation (PDE). Define $v \in C([0,+\infty) \times [-r,0], \mathbb{R}^n)$ by

$$v(t,\theta) = x(t+\theta), \; \forall t \geq 0, \; \forall \theta \in [-r,0].$$

Note that if $x \in C^1([-r,+\infty), \mathbb{R}^n)$, then

$$\frac{\partial v(t,\theta)}{\partial t} = x'(t+\theta) = \frac{\partial v(t,\theta)}{\partial \theta}.$$

Hence, we must have

$$\frac{\partial v(t,\theta)}{\partial t} - \frac{\partial v(t,\theta)}{\partial \theta} = 0, \; \forall t \geq 0, \; \forall \theta \in [-r,0].$$

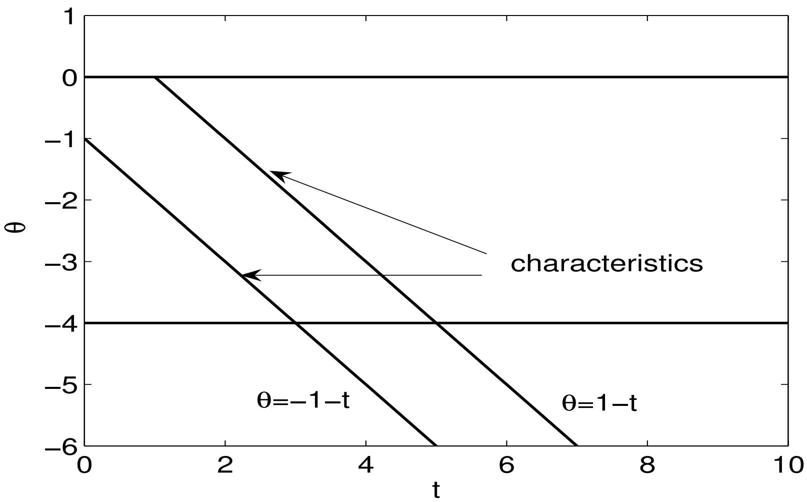

Fig. 1.6: Some characteristics curves $\theta = c - t$.

In Fig. 1.6 we take $r = -4$. The PDE $\frac{\partial v(t,\theta)}{\partial t} - \frac{\partial v(t,\theta)}{\partial \theta} = 0$ implies that the solution $v(t,\theta)$ must be constant along the characteristics $\theta = c - t$. So in order to define the solutions of the PDE, we need an additional boundary condition at $\theta = 0$ for $t > 0$. For $\theta = 0$, we obtain

$$\frac{\partial v(t,0)}{\partial \theta} = x'(t) = Bx(t) + \widehat{L}(x_t) + G(x_t) = Bv(t,0) + \widehat{L}(v(t,.)) + G(v(t,.)), \forall t \geq 0.$$

Therefore, we deduce formally that v must satisfy a PDE

$$\begin{cases} \dfrac{\partial v(t,\theta)}{\partial t} - \dfrac{\partial v(t,\theta)}{\partial \theta} = 0, \ \forall t \geq 0, \forall \theta \in [-r,0], \\ \dfrac{\partial v(t,0)}{\partial \theta} - Bv(t,0) = \widehat{L}(v(t,.)) + G(v(t,.)), \ \forall t \geq 0, \\ v(0,.) = \phi \in C([-r,0], \mathbb{R}^n). \end{cases} \tag{1.4.2}$$

The above PDE is a linear transport type equation with nonlinear Robin's type boundary condition. One may observe that the delay induces some nonlocal terms in the boundary condition. Because of that the problem becomes difficult to study from the PDE point of view, and the real question is how to define the solutions of such a PDE problem. To do so, we rewrite the PDE (1.4.2) as an abstract non-densely defined Cauchy problem.

We start by extending the state space to take into account the boundary conditions. This can be accomplished by adopting the following state space

$$X = \mathbb{R}^n \times C([-r,0], \mathbb{R}^n) = \mathbb{R}^n \times \mathscr{C}$$

taken with the usual product norm

$$\left\| \begin{pmatrix} x \\ \phi \end{pmatrix} \right\| = \|x\|_{\mathbb{R}^n} + \|\phi\|.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi'(0) + B\phi(0) \\ \phi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} \in D(A),$$

with

$$D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-r,0], \mathbb{R}^n).$$

Note that A is non-densely defined because

$$\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times \mathscr{C} \neq X.$$

We also define $L : \overline{D(A)} \to X$ by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} = \begin{pmatrix} \widehat{L}(\phi) \\ 0_{\mathscr{C}} \end{pmatrix}$$

and $F : \overline{D(A)} \to X$ by

$$F \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} = \begin{pmatrix} G(\phi) \\ 0_{\mathscr{C}} \end{pmatrix}.$$

Set

$$u(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ v(t) \end{pmatrix}.$$

Now we consider the RFDE (1.4.2) as the following non-densely defined Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + L(u(t)) + F(u(t)), \ t \geq 0, \ u(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} \in \overline{D(A)}. \tag{1.4.3}$$

The abstract Cauchy problem can be written as

$$\frac{d}{dt} \begin{pmatrix} 0 \\ v(t,.) \end{pmatrix} = \begin{pmatrix} -\dfrac{dv(t,0)}{d\theta} + Bv(t,0) + \widehat{L}(v(t,.)) + G(v(t,.)) \\ \dfrac{dv(t,.)}{d\theta} \end{pmatrix}. \tag{1.4.4}$$

Unfortunately, depending on the initial value of the problem, such a solution does not exist in general. In fact for a RFDE, if we take an initial value $\phi \in C^1([-r,0], \mathbb{R}^n)$ satisfying the so called *compatibility condition*

$$\phi'(0) = B\phi(0) + \widehat{L}(\phi) + G(\phi),$$

then the solution $x(t)$ of the RFDE belongs to $C^1([-r,+\infty),\mathbb{R}^n)$ and $u(t,.) = x_t$ satisfies (1.4.4).

But if we only assume that $\phi \in C([-r,0],\mathbb{R}^n)$, then we need to extend this notion of solution by considering the so-called *integrated solutions* (or *mild solutions*); that is,

$$\int_0^t u(l)dl \in D(A)$$

and

$$u(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} + A\int_0^t u(l)dl + \int_0^t L(u(l)) + F(u(l))dl,\ t \geq 0.$$

In fact, it is not difficult to prove that $v(t,.) = x_t$ satisfies the following properties

$$\int_0^t v(l,.)dl \in C^1([-r,0],\mathbb{R}^n),$$

$$\begin{pmatrix} 0 \\ v(t,.) \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \phi \end{pmatrix} + \begin{pmatrix} (-\dfrac{d}{d\theta} + B)\mid_{\theta=0} \int_0^t v(l,.)dl + \int_0^t \widehat{L}(v(l,.)) + G(v(l,.))\,dl \\ \dfrac{d}{d\theta}\int_0^t v(l,.)dl \end{pmatrix}.$$

(b) Linear RFDEs. Consider the linear RFDE

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}(x_t), \forall t \geq 0, \\ x_0 = \phi \in \mathscr{C}. \end{cases} \tag{1.4.5}$$

As we saw, the linear RFDE can be formulated as the following PDE

$$\begin{cases} \dfrac{\partial v(t,\theta)}{\partial t} - \dfrac{\partial v(t,\theta)}{\partial \theta} = 0,\ \forall t \geq 0, \forall \theta \in [-r,0], \\ \dfrac{\partial v(t,0)}{\partial \theta} - Bv(t,0) = \widehat{L}(v(t,.)), \forall t \geq 0, \\ v(0,.) = \phi \in \mathscr{C}. \end{cases} \tag{1.4.6}$$

The first way (see Webb [356, 357] and Travis and Webb [340, 341]) to give an abstract formulation for this problem is to incorporate the boundary condition into the definition of the domain. More precisely, consider the linear operator $\widehat{A} : D(\widehat{A}) \subset \mathscr{C} \to \mathscr{C}$ defined by

$$\widehat{A}\phi = \phi'$$

with the boundary condition incorporated into the domain

$$D(\widehat{A}) = \left\{ \phi \in C^1([-r,0],\mathbb{R}^n) : \phi'(0) - B\phi(0) = \widehat{L}(\phi) \right\}.$$

Then the PDE (1.4.6) can be formulated as an abstract Cauchy problem

$$\frac{du(t)}{dt} = \widehat{A}u(t) \text{ for } t \geq 0 \text{ with } u(0) = \phi \in \mathscr{C}. \tag{1.4.7}$$

In order to define the solutions of (1.4.7), we use the theory of linear strongly continuous semigroups, and the solutions (1.4.7) satisfy

$$\int_0^t u(s)ds \in D(\widehat{A})$$

and

$$u(t) = \phi + \widehat{A}\int_0^t u(s)ds, \ \forall t \geq 0.$$

In fact we will see that $\widehat{A}$ is the infinitesimal generator of $\left\{\widehat{T}(t)\right\}_{t\geq 0}$, a strongly continuous semigroup of bounded linear operators. Of course one needs to establish a relationship between $\left\{\widehat{T}(t)\right\}_{t\geq 0}$ and the solutions of (1.4.5), and we will see that

$$x_t = \widehat{T}(t)\phi, \ \forall t \geq 0.$$

In particular the spectrum defined in section 1.2.3 is the spectrum of the linear operator $\widehat{A}$. But now, we can use the spectral theory of linear operators and can also compute the projectors on the generalized eigenspaces. This part becomes important when one needs to project on the eigenspace.

(c) Relationship between $\widehat{A}$ and $A+L$. Next we can observe that $\widehat{A}$ is $(A+L)_{\overline{D(A)}}$, the part of $A+L : D(A) \subset X \to X$ in $\overline{D(A)}$. Indeed the linear operator $(A+L)_{\overline{D(A)}}$ is defined by

$$(A+L)_{\overline{D(A)}}x = (A+L)\begin{pmatrix} 0 \\ \phi \end{pmatrix}, \quad \forall \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in D\left((A+L)_{\overline{D(A)}}\right)$$

and

$$D\left((A+L)_{\overline{D(A)}}\right) = \left\{x \in D(A) : (A+L)x \in \overline{D(A)}\right\}.$$

So this is equivalent to

$$(A+L)_{\overline{D(A)}}\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi'(0) + B\phi(0) + \widehat{L}(\phi) \\ \phi' \end{pmatrix}$$

and

$$D\left((A+L)_{\overline{D(A)}}\right) = \left\{ \begin{array}{l} \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in \{0\} \times C^1([-r,0],\mathbb{R}^n) : \\ \qquad \begin{pmatrix} -\phi'(0) + B\phi(0) + \widehat{L}(\phi) \\ \phi' \end{pmatrix} \in \{0\} \times C([-r,0],\mathbb{R}^n) \end{array} \right\}.$$

Therefore,

$$(A+L)_{\overline{D(A)}}\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi' \end{pmatrix}$$

and

$$D\left((A+L)_{\overline{D(A)}}\right)=\left\{\begin{pmatrix}0\\ \phi\end{pmatrix}\in\{0\}\times C^1\left([-r,0],\mathbb{R}^n\right):-\phi'(0)+B\phi(0)+\widehat{L}(\phi)=0\right\}.$$

Thus

$$(A+L)_{\overline{D(A)}}\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ \widehat{A}\phi\end{pmatrix}$$

and

$$D\left((A+L)_{\overline{D(A)}}\right)=\{0\}\times D(\widehat{A}).$$

(d) Nonlinear RFDE. Consider the nonlinear RFDE

$$\begin{cases}\dfrac{dx(t)}{dt}=Bx(t)+\widehat{L}(x_t)+G(x_t),\ \forall t\geq 0,\\ x_0=\phi\in\mathscr{C}.\end{cases}\tag{1.4.8}$$

As before we can consider the PDE associated to this problem

$$\begin{cases}\dfrac{\partial v(t,\theta)}{\partial t}-\dfrac{\partial v(t,\theta)}{\partial\theta}=0,\ \forall t\geq 0,\forall\theta\in[-r,0],\\ \dfrac{\partial v(t,0)}{\partial\theta}-Bv(t,0)=\widehat{L}(v(t,.))+G(v(t,.)),\ \forall t\geq 0,\\ v(0,.)=\phi\in\mathscr{C}.\end{cases}\tag{1.4.9}$$

In that case, the first attempt is to formalize this problem as an abstract Cauchy problem which was done by Travis and Webb [340, 341], and the idea is again to use the nonlinear semigroup theory. In order to do this, the idea is to incorporate the boundary condition into the definition of the domain. More precisely, they showed that the nonlinear semiflow (or nonlinear semigroup) $\{U(t)\}_{t\geq 0}$ defined by

$$U(t)\phi=x_{\phi t}$$

is generated by the nonlinear unbounded operator $\widehat{A}_N:D(\widehat{A}_N)\subset\mathscr{C}\to\mathscr{C}$ defined by

$$\widehat{A}_N\phi=\phi'$$

with

$$D(\widehat{A}_N)=\left\{\phi\in C^1\left([-r,0],\mathbb{R}^n\right):\phi'(0)-B\phi(0)=\widehat{L}(\phi)+G(\phi)\right\}.$$

As we have seen, if G is C^1, $G(0)=0$ and $DG(0)=0$, then

$$T(t)\widehat{\phi}=\partial_\phi U(t)(0)\widehat{\phi}.$$

This property can be sufficient to establish the stability or the unstability properties of the equilibrium solution 0. We refer to Desch and Schappacher [94] for more results about that. Unfortunately, this property does not seem to be sufficient to built a complete bifurcation theory, in particular if we would like to compute the reduced system.

1.4.2 Age-structured Models

We consider the age-structured model (1.3.1). Let $X = \mathbb{R} \times L^1(0,\infty)$ with the usual product norm. Let $A : D(A) \subset X \to X$ be the linear operator on X defined by

$$A \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi(0) \\ -\phi' - \mu\phi \end{pmatrix} \tag{1.4.10}$$

with

$$D(A) = \{0\} \times W^{1,1}(0,+\infty), \ \overline{D(A)} = \{0\} \times L^1(0,+\infty) = X_0 \neq X.$$

Define the map $F : X_0 \to X$ by

$$F\left(\begin{pmatrix} 0 \\ \phi \end{pmatrix}\right) = \begin{pmatrix} \alpha h(\int_0^\infty \beta(a)\phi(a)da) \\ 0_{L^1} \end{pmatrix}. \tag{1.4.11}$$

Denote

$$v(t) = \begin{pmatrix} 0 \\ u(t,\cdot) \end{pmatrix} \in X_0.$$

Then the age-structured model can be reformulated as the abstract Cauchy problem with nondense domain

$$\begin{cases} \dfrac{dv}{dt} = Av(t) + F(v(t)), \ t \geq 0 \\ v(0) = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in \overline{D(A)}. \end{cases} \tag{1.4.12}$$

1.4.3 Size-structured Models

Consider the system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = \varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} - \mu u(t,x), \ t \geq 0, x \geq 0, \\ -\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + u(t,0) = \alpha h(\displaystyle\int_0^\infty \gamma(x)u(t,x)dx), \\ u(0,\cdot) = \phi \in L^1_+(0,+\infty), \end{cases} \tag{1.4.13}$$

where $u(t,x)$ represents the population density of certain species at time t with size x, $\varepsilon > 0$, $\mu > 0$, $\alpha > 0$, $\gamma \in L^\infty_+(0,+\infty) \setminus \{0\}$, and $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = xe^{-\xi x},\ x \in \mathbb{R}.$$

Let

$$X = \mathbb{R} \times L^1(0,+\infty), \quad \left| \begin{pmatrix} \alpha \\ \phi \end{pmatrix} \right| = |\alpha| + \|\phi\|_{L^1(0,+\infty)}.$$

Define the operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \varepsilon^2 \phi'(0) - \phi(0) \\ \varepsilon^2 \phi'' - \phi' - \mu \phi \end{pmatrix}$$

with domain

$$D(A) = \{0\} \times W^{2,1}(0,+\infty).$$

Then we have

$$\overline{D(A)} = X_0 = \{0\} \times L^1(0,\infty) \neq X.$$

Define the map $F : X_0 \to X$ by

$$F \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \alpha h(\int_0^\infty \gamma(x)\phi(x)dx) \\ 0 \end{pmatrix},$$

and denote

$$v(t) = \begin{pmatrix} 0 \\ u(t,\cdot) \end{pmatrix}.$$

Then, we obtain system (1.4.12).

1.4.4 Partial Functional Differential Equations

Taking the interactions of spatial diffusion and time delay into account, a single species population model can be described by a partial differential equation with time delay as follows:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d\dfrac{\partial^2 u(t,x)}{\partial x^2} - au(t-r,x)[1+u(t,x)], \ \ t > 0, \ \ x \in [0,\pi], \\ \dfrac{\partial u(t,x)}{\partial x} = 0, \ \ x = 0, \pi, \\ u(0,.) = u_0 \in C([0,\pi], \mathbb{R}), \end{cases} \tag{1.4.14}$$

where $u(t,x)$ denotes the density of the species at time t and location x, $d > 0$ is the diffusion rate of the species, $r > 0$ is the time delay constant, and $a > 0$ is a constant.

Consider the Banach space $Y = C([0,\pi], \mathbb{R})$ endowed with the usual supremum norm. Define the operator $B : D(B) \subset Y \to Y$ by

$$B\phi = d\phi''$$

with

$$D(B) = \left\{\phi \in C^2\left([0,\pi],\mathbb{R}\right) : \phi'(0) = \phi'(\pi) = 0\right\}.$$

Denote

$$\widehat{L}(\phi) = -a\phi(-r), \;\; f(\phi) = -a\phi(0)\phi(-r).$$

Equation (1.4.14) can be written as an abstract partial functional differential equations (PFDE) (see, for example, Travis and Webb [340, 341], Wu [374] and Faria [133]):

$$\begin{cases} \dfrac{dy(t)}{dt} = By(t) + \widehat{L}(y_t) + f(y_t), \; \forall t \geq 0, \\ y_0 = \phi \in C_B, \end{cases} \tag{1.4.15}$$

where

$$C_B := \{\phi \in C\left([-r,0];Y\right) : \;\; \phi(0) \in \overline{D(B)}\},$$

$y_t \in C_B$ satisfies $y_t(\theta) = y(t+\theta), \theta \in [-r,0]$, $\widehat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map. As in subsection 1.4.1, system (1.4.15) can be formulated as an abstract Cauchy problem with nondense domain.

1.5 Remarks and Notes

(a) Ordinary Differential Equations. Fundamental theories of ordinary differential equations can be found in many classical textbooks, such as Hartman [180] and Hale [171]. The classical center manifold theory was first established by Pliss [289] and Kelley [208] and was developed and completed in Carr [56], Sijbrand [319], Vanderbauwhede [343], etc. There are two classical methods to prove the existence of center manifolds. The Hadamard (Hadamard[167]) method (the graph transformation method) is a geometric approach which is based on the construction of graphs over linearized spaces, see Hirsch et al. [188] and Chow et al. [65, 66]. The Liapunov-Perron (Liapunov [228], Perron [286]) method (the variation of constants method) is more analytic in nature, which obtains the manifold as a fixed point of a certain integral equation. The technique originated in Krylov and Bogoliubov [220] and was further developed by Hale [169, 171], see also Ball [36], Chow and Lu [67], Yi [378], etc. The smoothness of center manifolds can be proved by using the contraction mapping in a scale of Banach spaces (Vanderbauwhede and van Gils [344]), the Fiber contraction mapping technique (Hirsch et al. [188]), the Henry lemma (Henry [183], Chow and Lu [68]), among other methods (Chow et al. [64]). For further results and references on center manifolds, we refer to the monographs of Carr [56], Chow and Hale [62], Chow et al. [63], Sell and You [314], Wiggins [373], and the survey papers of Bates and Jones [39], Vanderbauwhede [343] and Vanderbauwhede and Iooss [345].

A normal form theorem was obtained first by Poincaré [291, 290] and later by Siegel [317] for analytic differential equations. Simpler proofs of Poincaré's theorem and Siegel's theorem were given in Arnold [32], Meyer [267], Moser [272], and

Zehnder [382]. For more results about normal form theory and its applications see, for example, the monographs by Arnold [32], Chow and Hale [62], Guckenheimer and Holmes [155], Meyer and Hall [**?**], Siegel and Moser [318], Chow et al. [63], Kuznetsov [223], and others.

The Hopf bifurcation theorem was proved by several researchers (see Andronov *et al.* [18], Hopf [191], Friedrichs [145], Hale [171]) and has been used to study bifurcations in many applied subjects (see Marsden and McCraken [257] and Hassard et al. [181]). Golubitsky and Rabinowitz [151] gave a nice commentary on Hopf bifurcation theorem and provided more references.

(b) Functional Differential Equations. The fundamental theories can be found in Hale and Verduyn Lunel [175] and Diekmann et al. [106]. The center manifold theorem in functional differential equations has been studied in Diekmann and van Gils [104, 105], Hupkes and Verduyn Lunel [193]), etc. Normal form theory has been extended to functional differential equations in Faria and Magalhães [136, 137]. Hopf bifurcation theorem can be found in the monographs of Hale and Verduyn Lunel [175], Hassard et al. [181], Diekmann et al. [106], Guo and Wu [159].

(c) Age-Structured Models. The first linear age-structured model described by a first-order hyperbolic equation was proposed by McKendrick [263] in 1926 to study problems in medicine, namely various transitions in epidemology. In the famous series of three papers on mathematical epidemiology published from 1927 to 1933, Kermack and McKendrick [209, 210, 211] used systems of age-structured equations to develop a general theory of infectious disease transmission. The first nonlinear age-structured model in population dynamics was due to Gurtin and MacCamy [162]. Since then, age-structued models have been studied extensively. We refer to the monographs of Webb [362], Metz and Diekmann [266], Iannelli [195], Busenberg and Cook [50], Cushing [79], Anita [19], and Inaba [199] on the theories of age-sructured models. To investigate age-structured models, one can use the classical method, that is, to use solutions integrated along the characteristics and work with nonlinear Volterra equations. We refer to Webb [362] and Iannelli [195] on this method. A second approach is the variational method, we refer to Anita [19], Ainseba [8] and the references cited therein. One can also regard the problem as a semilinear problem with non-dense domain and use the integrated semigroups method. We refer to Thieme [328, 330, 331], Magal [242], Thieme and Vrabie [339], Magal and Thieme [251], Thieme and Vosseler [338] for more details on this approach.

(d) Abstract Semilinear Formulation. Various types of equations, such as functional differential equations (Hale and Verduyn Lunel [175]), age-structured models (Webb [362]), size-structured models (Webb [364]), parabolic partial differential equations (Henry [183], Lunardi [240]), and partial functional differential equations (Wu [374]) can be written as abstract semilinear equations in Banach spaces. Semigroup theory then can be used to study such abstract semilinear equations (Arendt et al. [22], Cazenave and Haraux [58], Chicone and Latushkin [60], Engel and Nagel [126], Henry [183], Pazy [281], Tanabe [325], van Neerven [346], Yagi [376]).

Chapter 2
Semigroups and Hille-Yosida Theorem

The aim of this chapter is to introduce the basic concepts and results about semigroups, resolvents, infinitesimal generators for linear operators and to present the Hille-Yosida theorem for strongly continuous semigroups.

2.1 Semigroups

Let $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$ be two Banach spaces. Denote by $\mathscr{L}(X,Y)$ the space of bounded linear operators from X into Y endowed with the usual norm

$$\|L\|_{\mathscr{L}(X,Y)}=\sup_{x\in X:\|x\|_X\leq 1}\|L(x)\|_Y$$

and denote by $\mathscr{L}(X)=\mathscr{L}(X,X)$ if $X=Y$. We will study the existence and uniqueness of solutions for the Cauchy problem

$$\frac{du}{dt}=Au(t),\ t\geq 0;\ \ u(0)=x\in\overline{D(A)}, \tag{2.1.1}$$

where $A:D(A)\subset X\to X$ is a linear operator on a Banach space $(X,\|.\|)$.

First, we introduce a basic definition.

Definition 2.1.1. A family $\{T(t)\}_{t\geq 0}\subset\mathscr{L}(X)$ is a *semigroup* of bounded linear operators on a Banach space X if the following properties are satisfied

$$T(t)T(s)=T(t+s),\ \forall t,s\geq 0,\ \text{and } T(0)=I.$$

2.1.1 Bounded Case

When A is bounded (i.e. $A \in \mathscr{L}(X)$) the solution of (2.1.1) is uniquely determined (in the sense that there exists a unique C^1-function satisfying (2.1.1) for each $t \geq 0$) and is given by

$$u(t) = e^{At}x, \ \forall t \geq 0,$$

where

$$e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}, \ \forall t \geq 0.$$

The family of bounded linear operators $\left\{e^{At}\right\}_{t\geq 0}$ satisfies the semigroup properties

$$e^{At} \circ e^{As} = e^{A(t+s)}, \ \forall t,s \geq 0, \text{ and } e^{A0} = I.$$

Definition 2.1.2. Let $\{T(t)\}_{t\geq 0}$ be a semigroup of bounded linear operators on a Banach space X. Then $\{T(t)\}_{t\geq 0}$ is said to be *uniformly continuous* (or *operator norm continuous*) if the map $t \to T(t)$ is continuous from $[0,+\infty)$ into $\mathscr{L}(X)$; that is,

$$\lim_{t\to s} \|T(t) - T(s)\|_{\mathscr{L}(X)} = 0, \ \forall s \geq 0.$$

Note that for each $t \geq 0$,

$$\left\|e^{At} - I\right\|_{\mathscr{L}(X)} \leq \sum_{n=1}^{\infty} \frac{\|A\|^n_{\mathscr{L}(X)} |t|^n}{n!} \leq e^{\|A\|_{\mathscr{L}(X)} t} - 1.$$

So

$$\left\|e^{At} - e^{As}\right\|_{\mathscr{L}(X)} \to 0 \text{ as } t \to s.$$

It follows that the semigroup $\left\{e^{At}\right\}_{t\geq 0}$ is uniformly continuous. Actually the converse is also true.

Lemma 2.1.3. *Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous semigroup of bounded linear operators on a Banach space X. Then there exists $A \in \mathscr{L}(X)$ such that*

$$T(t) = e^{At}, \ \forall t \geq 0.$$

Proof. As $\{T(t)\}_{t\geq 0}$ is uniformly continuous, we can find $h > 0$ (small enough) such that

$$\left\| h^{-1} \int_0^h T(l)dl - I \right\|_{\mathscr{L}(X)} < 1.$$

So

$$L_1 = \int_0^h T(l)dl = h\left[I + h^{-1}\int_0^h T(l)dl - I\right]$$

is invertible. Set

$$A = (T(h) - I)L_1^{-1}.$$

Then

$$\begin{aligned}(T(t)-I)L_1 &= T(t)\int_0^h T(l)dl - \int_0^h T(l)dl \\ &= \int_0^h T(l+t)dl - \int_0^h T(l)dl \\ &= \int_t^{t+h} T(l)dl - \int_0^h T(l)dl \\ &= \int_h^{t+h} T(l)dl - \int_0^t T(l)dl \\ &= \int_0^t T(l)dl\,(T(h)-I).\end{aligned}$$

Hence,

$$T(t)-I = \int_0^t T(l)dlA,\ \forall t \geq 0.$$

This implies that $t \to T(t)$ is operator norm differentiable and

$$\frac{dT(t)}{dt} = T(t)A,\ \forall t \geq 0.$$

Let $t > 0$ be fixed. Consider

$$L(s) = T(s)e^{A(t-s)},\ \forall s \in [0,t].$$

Then

$$\frac{dL(s)}{ds} = T(s)Ae^{A(t-s)} - T(s)Ae^{A(t-s)} = 0.$$

So

$$T(t) = L(t) = L(0) = e^{At}.$$

This completes the proof. □

After Lemma 2.1.3, it becomes clear that when A is unbounded, the semigroup $\{T(t)\}_{t\geq 0}$ must satisfy a weaker time continuity condition.

2.1.2 Unbounded Case

When $A : D(A) \subset X \to X$ is unbounded, the formula

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$$

makes no sense for $t \geq 0$. Also if $\{T(t)\}_{t\geq 0}$ is an extension in someway of the notion of the exponential of A, it is clear from Lemma 2.1.3 that we need a weaker notion of time continuity of the family $\{T(t)\}_{t\geq 0}$. The appropriate notion of time continuity is as follows.

Definition 2.1.4. Let $J \subset \mathbb{R}$ be an interval. Let X and Y be two Banach spaces. A family $\{L(t)\}_{t\in J} \subset \mathscr{L}(X,Y)$ is *strongly continuous* if for each $x \in X$ the map $t \to L(t)x$ is continuous from J into Y.

Definition 2.1.5. A family $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ of bounded linear operators on X is a *strongly continuous semigroup* of bounded linear operators on X (or for short a *linear C_0-semigroup* on X) if the following assertions are satisfied:

(i) $\{T(t)\}_{t\geq 0}$ is a semigroup;
(ii) $\{T(t)\}_{t\geq 0}$ is strongly continuous.

The strong continuity can be expressed by saying that for every $x \in X$ the map

$$t \to T(t)x$$

is continuous from $\mathbb{R}_+$ to X.

Definition 2.1.6. A map f from $[0,+\infty)$ into a Banach space $(X, \|.\|)$ is said to be *exponentially bounded* if there exist two constants, $M \geq 0$ and $\omega \geq 0$, such that

$$\|f(t)\| \leq Me^{\omega t}, \ \forall t \geq 0.$$

For a strongly continuous semigroup $\{T(t)\}_{t\geq 0} \subset \mathscr{L}(X)$ of bounded linear operators on X, the orbit $\{T(t)x : \ t \in [0,t_0]\}$ is the continuous image of a closed interval $[0,t_0]$. Thus, it is bounded for each $x \in X$. Uniform boundedness principle implies that each strongly continuous semigroup is uniformly bounded on each closed interval, which in turn implies exponential boundedness of the strongly continuous semigroup in $\mathbb{R}_+$.

Proposition 2.1.7. *Let $\{T(t)\}_{t\geq 0}$ be a linear C_0-semigroup on a Banach space X. Then $\{T(t)\}_{t\geq 0}$ is exponentially bounded and the map $(t,x) \to T(t)x$ is continuous from $[0,+\infty) \times X$ into X.*

Proof. Since $\{T(t)\}_{t\geq 0}$ is strongly continuous, we deduce that

$$\sup_{t\in[0,1]} \|T(t)x\| < +\infty, \ \forall x \in X.$$

So by the principle of uniform boundedness, we deduce that

$$M = \sup_{t\in[0,1]} \|T(t)\|_{\mathscr{L}(X)} < +\infty.$$

Since $T(0) = I$, we have $M \geq 1$. Set $\omega = \ln(M) > 0$. Then we have

$$\|T(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \ \forall t \in [0,1],$$

and for each integer $n \geq 0$ and each $t \in [n, n+1]$,

$$\begin{aligned}\|T(t)\|_{\mathscr{L}(X)} &\leq \|T(t-n)\|_{\mathscr{L}(X)} \|T(1)\|^n_{\mathscr{L}(X)} \\ &\leq M^n M e^{\omega(t-n)} = M e^{\omega t}.\end{aligned}$$

We deduce that $\{T(t)\}_{t\geq 0}$ is exponentially bounded. The last part of the proposition is now an immediate consequence of continuity of $x \to T(t)x$ uniform with respect to t in bounded sets of $[0, +\infty)$. $\square$

To further describe the relationship between $\{T(t)\}_{t\geq 0}$ and A, we will see that $t \to T(t)x$ turns to be a *mild solution* (or *an integrated solution*) of the Cauchy problem (2.1.1); that is,

$$\int_0^t T(l)xdl \in D(A),\ \forall t \geq 0,\ \forall x \in X,$$

and

$$T(t)x = x + A\int_0^t T(l)xdl,\ \forall t \geq 0,\ \forall x \in X.$$

It is important to note that in general we have

$$T(t)x \notin D(A)$$

when $x \notin D(A)$ and $t \geq 0$. Nevertheless, when $x \in D(A)$ we will see that

$$T(t)x \in D(A),\ \forall t \geq 0,$$

the map $t \to AT(t)x$ is continuous from $[0, +\infty)$ into X, the map $t \to T(t)x$ is continuously differentiable, and

$$\frac{dT(t)x}{dt} = AT(t)x,\ \forall t \geq 0.$$

So when $x \in D(A)$, the map $t \to T(t)x$ is the so-called *classical solution* of the Cauchy problem (2.1.1).

2.2 Resolvents

Assume A is an $n \times n$ symmetric matrix. Then A has n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with respect to algebraic multiplicity) and there is an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{R}^n$ such that e_i is an eigenvector corresponding to λ_i. To generalize the eigenvalue problems of linear algebra to operators on Banach spaces, we introduce the concept of resolvent.

Definition 2.2.1. Let $A : D(A) \subset X \to X$ be a linear operator on a $\mathbb{K}$-Banach space X with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *resolvent set* $\rho(A)$ of A is the set of all points $\lambda \in \mathbb{K}$ such that $\lambda I - A$ is a bijection from $D(A)$ into X and the inverse $(\lambda I - A)^{-1}$, called the *resolvent* of A, is a bounded linear operator from X into itself.

Definition 2.2.2. A linear operator $A : D(A) \subset X \to X$ on a Banach space X is *closed* if and only if the *graph*

$$G(A) := \{(x, Ax) : x \in D(A)\}$$

of A is a closed subspace of $X \times X$ endowed with the usual product norm.

Lemma 2.2.3. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. If $\rho(A) \neq \emptyset$, then A is closed.*

Proof. Consider two sequences $\{x_n\} \subset D(A)$ with $x_n \to x$ and $\{y_n\} \subset X$ with $y_n \to y$. Assume that

$$y_n = Ax_n, \ \forall n \geq 0.$$

Let $\lambda \in \rho(A)$ be given. Then

$$\lambda x_n - y_n = \lambda x_n - Ax_n, \ \forall n \geq 0 \Leftrightarrow (\lambda I - A)^{-1}(\lambda x_n - y_n) = x_n, \ \forall n \geq 0.$$

Now since $(\lambda I - A)^{-1}$ is bounded, when n goes to $+\infty$ we have

$$(\lambda I - A)^{-1}(\lambda x - y) = x.$$

So $x \in D(A)$ and $y = Ax$. $\square$

Let $L : D(L) \subset X \to Y$ be a linear operator from a Banach space X into a Banach space Y. Define the *null space* (or *kernel*) of L by

$$\mathcal{N}(L) = \{x \in D(L) : Lx = 0\}$$

and the *range* of L by

$$\mathcal{R}(L) = \{y \in Y : \exists x \in D(L) \text{ satisfying } y = Lx\}.$$

Lemma 2.2.4. *Let $A : D(A) \subset X \to X$ be a closed linear operator on a Banach space X. Then $\lambda \notin \rho(A)$ if and only if $\mathcal{N}(\lambda I - A) \neq \{0\}$ or $\mathcal{R}(\lambda I - A) \neq X$.*

Proof. We first observe that if $\mathcal{N}(\lambda I - A) \neq \{0\}$ or $\mathcal{R}(\lambda I - A) \neq X$, then $\lambda I - A$ is not a bijection from $D(A)$ into X. So $\lambda \notin \rho(A)$.

Conversely, assume that $\mathcal{N}(\lambda I - A) = \{0\}$ and $\mathcal{R}(\lambda I - A) = X$. We can consider $\lambda I - A : D(A) \subset X_0 \to X$ as a linear operator from $X_0 = \overline{D(A)}$ into X. Then $\lambda I - A$ is closed and densely defined in X_0. Moreover, by the assumption that $\mathcal{N}(\lambda I - A) = \{0\}$ and $\mathcal{R}(\lambda I - A)$ is closed, Brezis [47, Theorem II.20] implies that there exists a constant $C > 0$ such that

$$\|x\| \leq C \|(\lambda I - A)x\|, \ \forall x \in D(A).$$

Thus, setting $y=(\lambda I-A)x$ and using the fact that $\mathscr{R}(\lambda I-A)=X$, we obtain

$$\left\|(\lambda I-A)^{-1}y\right\| \leq C\|y\|, \ \forall y\in X.$$

So $\lambda \in \rho(A)$. $\square$

From the definition of $(\lambda I-A)^{-1}$ we have the *resolvent formula*.

Proposition 2.2.5 (Resolvent Formula). *Whenever $\lambda,\mu \in \rho(A)$, we have*

$$(\lambda I-A)^{-1}-(\mu I-A)^{-1}=(\mu-\lambda)(\lambda I-A)^{-1}(\mu I-A)^{-1}. \tag{2.2.1}$$

Proof. By applying $\lambda I-A$ on both sides of equation (2.2.1) we obtain

$$\begin{aligned}
&I-(\lambda I-A)(\mu I-A)^{-1}=(\mu-\lambda)(\mu I-A)^{-1}\\
&\quad\Leftrightarrow I-((\lambda-\mu)I+\mu I-A)(\mu I-A)^{-1}=(\mu-\lambda)(\mu I-A)^{-1}\\
&\quad\Leftrightarrow I-(\lambda-\mu)(\mu I-A)^{-1}-I=(\mu-\lambda)(\mu I-A)^{-1}.
\end{aligned}$$

The resolvent formula is proved. $\square$

One may also observe from the resolvent formula that $(\lambda I-A)^{-1}$ and $(\mu I-A)^{-1}$ commute; that is,

$$(\lambda I-A)^{-1}(\mu I-A)^{-1}=(\mu I-A)^{-1}(\lambda I-A)^{-1}, \ \forall \lambda,\mu\in\rho(A).$$

Another consequence of the resolvent formula is the following result.

Lemma 2.2.6. *Let $A:D(A)\subset X\to X$ be a linear operator on a $\mathbb{K}$-Banach space X with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Then the resolvent set $\rho(A)$ is an open set. Moreover, if $\rho(A)\neq\emptyset$, $\mu\in\rho(A)$, and $\lambda\in\mathbb{R}$ lies in the interval (or $\lambda\in\mathbb{C}$ lies in the disk)*

$$|\lambda-\mu|<\frac{1}{\|(\mu I-A)^{-1}\|},$$

then

$$(\lambda I-A)^{-1}=\sum_{n=0}^{+\infty}(\mu-\lambda)^n(\mu I-A)^{-(n+1)}.$$

Proof. If $\rho(A)=\emptyset$, it is trivial. Assume that $\rho(A)\neq\emptyset$. Let $\mu\in\rho(A)$. Set

$$L_\lambda=\sum_{n=0}^{+\infty}(\mu-\lambda)^n(\mu I-A)^{-(n+1)}$$

when $|\lambda-\mu|\,\|(\mu I-A)^{-1}\|<1$. Then

$$L_\lambda=\left[I-(\mu-\lambda)(\mu I-A)^{-1}\right]^{-1}(\mu I-A)^{-1}$$

$$= (\mu I - A)^{-1}\left[I - (\mu - \lambda)(\mu I - A)^{-1}\right]^{-1}$$

and

$$(\lambda I - A)(\mu I - A)^{-1} = (\lambda - \mu)(\mu I - A)^{-1} + I = I - (\mu - \lambda)(\mu I - A)^{-1}.$$

It follows that

$$(\lambda I - A) L_\lambda x = x, \ \forall x \in X$$

and

$$L_\lambda (\lambda I - A) x = x, \ \forall x \in D(A).$$

The result follows. □

As an immediate consequence of the previous lemma one has the following result.

Lemma 2.2.7. *Let $A : D(A) \subset X \to X$ be a linear operator on a real Banach space X. Assume $\rho(A) \neq \emptyset$. Then $\lambda \to (\lambda I - A)^{-1}$ belongs to $C^\infty(\rho(A), \mathscr{L}(X))$ and*

$$\frac{d^n}{d\lambda^n}(\lambda I - A)^{-1} = (-1)^n n!\,(\lambda I - A)^{-(n+1)}, \ \forall \lambda \in \rho(A).$$

We now turn to non-densely defined linear operators. We consider some easy consequences for the part of a linear operator.

Definition 2.2.8. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X and let Y be a subspace of X. The *part of A in Y* is the linear operator $A_Y : D(A_Y) \subset Y \to Y$ defined by

$$A_Y x = Ax, \ \forall x \in D(A_Y) := \{x \in D(A) \cap Y : Ax \in Y\}.$$

Lemma 2.2.9. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X with $\rho(A) \neq \emptyset$. Assume that*

$$(\lambda I - A)^{-1} Y \subset Y$$

for some $\lambda \in \rho(A)$. Then $\lambda \in \rho(A_Y)$ and

$$D(A_Y) = (\lambda I - A)^{-1} Y, \ (\lambda I - A_Y)^{-1} = (\lambda I - A)^{-1}|_Y.$$

Proof. We have

$$\begin{aligned} D(A_Y) :&= \{x \in D(A) \cap Y : Ax \in Y\} \\ &= \{x \in D(A) \cap Y : (\lambda I - A)x \in Y\} \\ &= \left\{x \in D(A) \cap Y : x \in (\lambda I - A)^{-1} Y\right\} \\ &= (\lambda I - A)^{-1} Y. \end{aligned}$$

Moreover, if $x \in D(A_Y)$ and $y \in Y$, we have

$$(\lambda I - A_Y)x = y \Leftrightarrow x = (\lambda I - A)^{-1} y.$$

The proof is completed. $\square$

In the non-dense case, we consider a linear operator $A : D(A) \subset X \to X$ which satisfies

$$X_0 := \overline{D(A)} \neq X.$$

We will be especially interested in $A_0 := A_{\overline{D(A)}}$, the part of A in $\overline{D(A)}$, which is defined by

$$A_0 x = Ax, \ \forall x \in D(A_0) = \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\}.$$

From Lemma 2.2.9 we have $\lambda \in \rho(A)$. Thus,

$$\begin{gathered} \lambda \in \rho(A_0), \\ D(A_0) = (\lambda I - A)^{-1} \overline{D(A)}, \\ (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} \mid_{\overline{D(A)}}. \end{gathered}$$

The following result shows that $\rho(A_0)$ and $\rho(A)$ are in fact equal when $\rho(A) \neq \emptyset$.

Lemma 2.2.10. *Let $(X, \|.\|)$ be a $\mathbb{K}$-Banach space (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $A : D(A) \subset X \to X$ be a linear operator. Assume that $\rho(A) \neq \emptyset$. Then*

$$\rho(A_0) = \rho(A).$$

Moreover, we have the following:

(i) For each $\lambda \in \rho(A_0) \cap \mathbb{K}$ and each $\mu \in \rho(A) \cap \mathbb{K}$,

$$(\lambda I - A)^{-1} = (\mu - \lambda)(\lambda I - A_0)^{-1} (\mu I - A)^{-1} + (\mu I - A)^{-1};$$

(ii) For each $\lambda \in \rho(A) \cap \mathbb{K}$,

$$D(A_0) = (\lambda I - A)^{-1} X_0 \text{ and } (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} \mid_{X_0}.$$

Proof. Without loss of generality we can assume that X is a complex Banach space. Assume that $\lambda \in \rho(A_0)$, $\mu \in \rho(A) \cap \mathbb{K}$, and set

$$L = (\mu - \lambda)(\lambda I - A_0)^{-1} (\mu I - A)^{-1} + (\mu I - A)^{-1}.$$

Then one can check that

$$Lx \in D(A), \ (\lambda I - A) Lx = x, \ \forall x \in X,$$

and

$$L(\lambda I - A)x = x, \ \forall x \in D(A).$$

Thus, $\lambda I - A$ is invertible and $(\lambda I - A)^{-1} = L$ is bounded, so $\lambda \in \rho(A)$. This implies that (i) and $\rho(A_0) \subset \rho(A)$ hold. To prove the converse inclusion, let $\lambda \in \rho(A)$. Since $(\lambda I - A)^{-1} X \subset D(A)$, we can apply Lemma 2.2.9 with $Y = \overline{D(A)}$ and (ii) follows, and we also deduce that $\rho(A) \subset \rho(A_0)$. $\square$

The following lemma basically provides necessary and sufficient conditions to ensure that

$$\overline{D(A_0)} = \overline{D(A)}.$$

Lemma 2.2.11. *Let $A : D(A) \subset X \to X$ be a linear operator on a real Banach space X. Assume that there exist two constants, $\omega \in \mathbb{R}$ and $M > 0$, such that $(\omega, +\infty) \subset \rho(A)$ and*

$$\left\| \lambda (\lambda I - A_0)^{-1} \right\|_{\mathscr{L}(X_0)} = \left\| \lambda (\lambda I - A)^{-1} \right\|_{\mathscr{L}\left(\overline{D(A)}\right)} \leq M, \ \forall \lambda > \omega.$$

Then the following properties are equivalent:

(i) $\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \ \forall x \in \overline{D(A)};$

(ii) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X;$

(iii) $\overline{D(A_0)} = \overline{D(A)}.$

Proof. (i)$\Rightarrow$(ii). Let $\lambda, \mu > \omega$ and $x \in D(A)$. Set $y = (\mu I - A)x$. Then by using the resolvent formula, we have

$$(\lambda I - A)^{-1} (\mu I - A)^{-1} y = \frac{1}{\lambda - \mu} (\mu I - A)^{-1} y - \frac{1}{\lambda - \mu} (\lambda I - A)^{-1} y,$$

so

$$\lambda (\lambda I - A)^{-1} x = \frac{\lambda}{\lambda - \mu} x - \frac{\lambda}{\lambda - \mu} (\lambda I - A)^{-1} y.$$

It follows that

$$\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \ \forall x \in D(A) \Leftrightarrow \lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X.$$

This equivalence first implies that (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i). Conversely assume that (ii) holds. We have

$$\left[\lambda (\lambda I - A)^{-1} - I \right] (\mu I - A)^{-1} = \frac{\lambda}{\mu - \lambda} \left[(\lambda I - A)^{-1} - (\mu I - A)^{-1} \right] - (\mu I - A)^{-1}.$$

It implies that

$$\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \ \forall x \in D(A).$$

Let $x \in \overline{D(A)}$ be fixed and let $\{x_n\}_{n \geq 0} \subset D(A) \to x$. Then we have for each $n \geq 0$ that

$$\left\|\lambda\,(\lambda I-A)^{-1}x-x\right\| \leq \left\|\lambda\,(\lambda I-A)^{-1}(x-x_n)\right\| \\ +\left\|\lambda\,(\lambda I-A)^{-1}x_n-x_n\right\|+\|x-x_n\|.$$

It follows that

$$\limsup_{\lambda\to+\infty}\left\|\lambda\,(\lambda I-A)^{-1}x-x\right\|\leq[M+1]\,\|x-x_n\|,\ \forall n\geq 0,$$

and the result follows as $n\to+\infty$.

(i)⇒(iii). Recall that

$$D(A_0)=(\lambda I-A)^{-1}\,\overline{D(A)},\ \forall\lambda>\omega.$$

So it is clear that (i)⇒(iii).

It remains to prove (iii)⇒(i). By applying the same argument as above to A_0, we have

$$\lim_{\lambda\to+\infty}\lambda\,(\lambda I-A_0)^{-1}x=x,\ \forall x\in D(A_0)\Leftrightarrow\lim_{\lambda\to+\infty}(\lambda I-A_0)^{-1}x=0,\ \forall x\in\overline{D(A)}.$$

But by the assumption on A_0 we have

$$\left\|\lambda\,(\lambda I-A_0)^{-1}\right\|_{\mathscr{L}\left(\overline{D(A)}\right)}=\left\|\lambda\,(\lambda I-A)^{-1}\right\|_{\mathscr{L}\left(\overline{D(A)}\right)}\leq M,\ \forall\lambda>\omega,$$

so

$$\lim_{\lambda\to+\infty}\lambda\,(\lambda I-A_0)^{-1}x=x,\ \forall x\in D(A_0).$$

Let $x\in\overline{D(A)}$. Now since $\overline{D(A_0)}=\overline{D(A)}$, as in the proof of (ii)⇒(i) we can find a sequence $\{x_n\}_{n\geq 0}\subset D(A_0)\to x$ such that

$$\limsup_{\lambda\to+\infty}\left\|\lambda\,(\lambda I-A)^{-1}x-x\right\|\leq[M+1]\,\|x-x_n\|,\ \forall n\geq 0.$$

Since $x_n\to x\in\overline{D(A)}$, (i) follows. □

For some applications the operator A is not explicitly known. Nevertheless, if we know some pseudo-resolvent $\{J_\lambda\}_{\lambda\in\Delta}$, i.e. a family of bounded linear operators satisfying the resolvent formula, then it becomes important to know that there exists some linear operator A on X such that

$$(\lambda I-A)^{-1}=J_\lambda,\ \forall\lambda\in\Delta.$$

Definition 2.2.12. Let X be a $\mathbb{K}$-Banach space with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $\Delta\subset\mathbb{K}$. A family of bounded linear operators $\{J_\lambda\}_{\lambda\in\Delta}$ on X is called a *pseudo-resolvent* if the following property is satisfied

$$J_\lambda-J_\mu=(\mu-\lambda)\,J_\lambda J_\mu,\ \forall\lambda,\mu\in\Delta. \tag{2.2.2}$$

Lemma 2.2.13. *Let $\{J_\lambda\}_{\lambda\in\Delta}$ be a pseudo-resolvent on a Banach space X. Then*

$$J_\lambda J_\mu = J_\mu J_\lambda,\ \forall \lambda,\mu \in \Delta. \tag{2.2.3}$$

The null space $\mathscr{N}(J_\lambda)$ and the range $\mathscr{R}(J_\lambda)$ are independent of $\lambda \in \Delta$. The subspace $\mathscr{N}(J_\lambda)$ is closed.

Proof. Commutativity of J_λ and J_μ follows from (2.2.2). Rewritting (2.2.2) as

$$J_\lambda = J_\mu\left[I + (\mu-\lambda)J_\lambda\right]. \tag{2.2.4}$$

Let $y \in \mathscr{R}(J_\lambda)$. Then $y = J_\lambda x$ for some $x \in X$ and

$$y = J_\mu\left[x + (\mu-\lambda)J_\lambda x\right].$$

So $y \in \mathscr{R}(J_\mu)$, and we deduce that

$$\mathscr{R}(J_\lambda) \subset \mathscr{R}(J_\mu).$$

By symmetry we also have $\mathscr{R}(J_\mu) \subset \mathscr{R}(J_\lambda)$. Thus, $\mathscr{R}(J_\mu) = \mathscr{R}(J_\lambda)$.

Let $x \in \mathscr{N}(J_\lambda)$. Then $J_\lambda x = 0$, and by (2.2.4) we have

$$0 = J_\lambda x = J_\mu\left[x + (\mu-\lambda)J_\lambda x\right] = J_\mu x.$$

So $J_\mu x = 0$ and $\mathscr{N}(J_\lambda) \subset \mathscr{N}(J_\mu)$. Again by symmetry we obtain $\mathscr{N}(J_\lambda) = \mathscr{N}(J_\mu)$. □

Proposition 2.2.14. *Let $\omega \in \mathbb{R}$ and let $\{J_\lambda\}_{\lambda\in(\omega,+\infty)}$ be a pseudo-resolvent on a Banach space X. Then J_λ is the resolvent of a unique closed linear operator $A : D(A) \subset X \to X$ if and only if $\mathscr{N}(J_\lambda) = \{0\}$.*

Proof. It is clear that if J_λ is the resolvent of a linear operator $A : D(A) \subset X \to X$, then we must have $\mathscr{N}(J_\lambda) = \{0\}$. Conversely, assume that $\mathscr{N}(J_\lambda) = \{0\}$. The map J_λ is one to one. Let $\lambda_0 \in (\omega,+\infty)$. Define

$$Ax := \left(\lambda_0 I - J_{\lambda_0}^{-1}\right)x,\ \forall x \in D(A) := \mathscr{R}\left(J_{\lambda_0}\right). \tag{2.2.5}$$

The operator A is linear and closed. From (2.2.5) we have

$$(\lambda_0 I - A)J_{\lambda_0}x = J_{\lambda_0}^{-1}J_{\lambda_0}x = x,\ \forall x \in X,$$

and

$$J_{\lambda_0}(\lambda_0 I - A)x = J_{\lambda_0}J_{\lambda_0}^{-1}x = x,\ \forall x \in D(A).$$

Therefore, $J_{\lambda_0} = (\lambda_0 I - A)^{-1}$. If $\lambda \in (\omega,+\infty)$ and $x \in X$,

$$\begin{aligned}(\lambda I - A)J_\lambda x &= (\lambda I - \lambda_0 I + \lambda_0 I - A)J_\lambda x\\ &= (\lambda-\lambda_0)J_\lambda x + (\lambda_0 I - A)J_{\lambda_0}\left[I + (\lambda_0-\lambda)J_\lambda\right]x\end{aligned}$$

$$= (\lambda - \lambda_0) J_\lambda x + [I + (\lambda_0 - \lambda) J_\lambda] x$$
$$= x.$$

Similarly, if $\lambda \in (\omega, +\infty)$ and $x \in D(A)$,

$$\begin{aligned} J_\lambda (\lambda I - A) x &= J_\lambda (\lambda I - \lambda_0 I + \lambda_0 I - A) x \\ &= (\lambda - \lambda_0) J_\lambda x + [I + (\lambda_0 - \lambda) J_\lambda] J_{\lambda_0} (\lambda_0 I - A) x \\ &= (\lambda - \lambda_0) J_\lambda x + [I + (\lambda_0 - \lambda) J_\lambda] x \\ &= x. \end{aligned}$$

Therefore, $J_\lambda = (\lambda I - A)^{-1}$, $\forall \lambda \in (\omega, +\infty)$. □

Corollary 2.2.15. *Let $\omega \in \mathbb{R}$ and let $\{J_\lambda\}_{\lambda \in (\omega, +\infty)}$ be a pseudo-resolvent on a Banach space X. Assume that there exists a closed subspace X_0 of X such that*

(a) $\mathscr{N}(J_\lambda) \subset X_0$ and $\mathscr{R}(J_\lambda) \subset X_0$ for some $\lambda \in (\omega, +\infty)$;
(b) There exists a sequence $\{\lambda_n\}_{n \geq 0} \subset (\omega, +\infty)$ such that $\lambda_n \to +\infty$ as $n \to +\infty$, and

$$\lim_{n \to +\infty} \lambda_n J_{\lambda_n} x = x, \ \forall x \in X_0.$$

Then J_λ is the resolvent of a unique closed linear operator $A : D(A) \subset X \to X$ with $\overline{D(A)} = X_0$.

Proof. By (a), (b), and Lemma 2.2.13, it follows that $\mathscr{N}(J_\lambda) = \{0\}$. The result follows from Proposition 2.2.14. Moreover, by Lemma 2.2.13 and (b) it follows that $D(A) = \mathscr{R}(J_\lambda)$ is dense in X_0. □

Example 2.2.16 (Pseudo-resolvent for an age-structured model). Let $t \to A(t)$ be continuous from $[0,1]$ into $M_n(\mathbb{R})$. Consider a family of matrices $\{U(t,s)\}_{1 \geq t \geq s \geq 0}$ which are the solutions of the nonautonomous differential equation

$$\frac{dU(t,s)}{dt} = A(t) U(t,s), \ 1 \geq t \geq s, \ \ U(s,s) = I$$

for each $s \in [0,1]$. Then $U(t,s)$ satisfies the properties of nonautonomous semiflows

$$\begin{aligned} &U(t,r)U(r,s) = U(t,s) \text{ for } t \geq r \geq s, \\ &U(s,s) = I, \\ &(t,s) \to U(t,s) \text{ is continuous.} \end{aligned}$$

For each $\lambda \in \mathbb{R}$, consider the linear operator J_λ on $L^1(0,1)$ defined by

$$J_\lambda(\varphi)(x) = \int_0^x e^{-\lambda(x-s)} U(x,s) \varphi(s) ds.$$

We can check that the family $\{J_\lambda\}_{\lambda \in \mathbb{R}}$ is a pseudo-resolvent. Indeed,

$$(\lambda - \mu) J_\lambda J_\mu(\varphi)(x) = (\lambda - \mu) \int_0^x e^{-\lambda(x-s)} U(x,s) \int_0^s e^{-\mu(s-l)} U(s,l) \varphi(l) dl ds$$

$$\begin{aligned}
&= (\lambda - \mu) \int_0^x \int_0^s e^{-\lambda(x-s)} e^{-\mu(s-l)} U(x,l)\varphi(l) dl ds \\
&= e^{-\lambda x} \int_0^x \int_l^x (\lambda - \mu) e^{(\lambda-\mu)s} ds e^{\mu l} U(x,l)\varphi(l) dl \\
&= e^{-\lambda x} \int_0^x \left[e^{(\lambda-\mu)x} - e^{(\lambda-\mu)l} \right] e^{\mu l} U(x,l)\varphi(l) dl \\
&= \int_0^x e^{-\mu(x-l)} U(x,l)\varphi(l) dl - \int_0^x e^{-\lambda(x-l)} U(x,l)\varphi(l) dl \\
&= J_\mu(\varphi)(x) - J_\lambda(\varphi)(x).
\end{aligned}$$

Remark 2.2.17. As an exercise, one can prove that Corollary 2.2.15 applies to the family $\{J_\lambda\}_{\lambda\in\mathbb{R}}$.

2.3 Infinitesimal Generators

By Lemma 2.1.3 we know that every uniformly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space can be characterized as an exponential $T(t) = e^{At}$ for a linear operator A and all $t \geq 0$. Now we define the analogue of A for strongly continuous semigroups, called the generator of a semigroup.

Definition 2.3.1. Let $\{T(t)\}_{t\geq 0}$ be a linear C_0-semigroup on a Banach space X. The *infinitesimal generator* of $\{T(t)\}_{t\geq 0}$ is a linear operator $A : D(A) \subset X \to X$ satisfying the following properties

$$D(A) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}, \ \forall x \in D(A).$$

Lemma 2.3.2. *Let $\{T(t)\}_{t\geq 0}$ be a linear C_0-semigroup on a Banach space X. A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ if and only if the following properties are satisfied:*

(i) $\int_0^t T(s)xds \in D(A),\ \forall t \geq 0, \forall x \in X$;
(ii) $T(t)x = x + A\int_0^t T(s)xds,\ \forall t \geq 0, \forall x \in X$;
(iii) $A\int_0^t T(s)xds = \int_0^t T(s)Axds,\ \forall t \geq 0,\ \forall x \in D(A)$;
(iv) *A is closed.*

Proof. Let $x \in X$ be fixed. For each $h > 0$, we have by using the semigroup property that

$$(T(h) - I)\int_0^t T(s)xds = \int_0^t T(h)T(s)xds - \int_0^t T(s)xds$$

$$= \int_0^t T(h+s)xds - \int_0^t T(s)xds$$
$$= \int_0^{t+h} T(s)xds - \int_0^h T(s)xds - \int_0^t T(s)xds.$$

So
$$(T(h)-I)\int_0^t T(s)xds = \int_t^{t+h} T(s)xds - \int_0^h T(s)xds$$
and we deduce that
$$\lim_{h \searrow 0} \frac{(T(h)-I)}{h} \int_0^t T(s)xds = T(t)x - x.$$

By the definition of $D(A)$, we have
$$\int_0^t T(s)xds \in D(A),\ \forall t \geq 0,$$
and
$$A\int_0^t T(s)xds = T(t)x - x,\ \forall t \geq 0.$$
Let $x \in D(A)$ be fixed. Then for each $h > 0$ we have
$$(T(h)-I)\int_0^t T(s)xds = \int_0^t T(s)\,(T(h)-I)\,xds$$
Dividing by h, we obtain (iii) as $h \searrow 0$.

Let $\{x_n\}_{n\geq 0} \subset D(A)$ be a sequence such that $x_n \to x$ and $y_n = Ax_n \to y$. Using (ii) and (iii), we have
$$T(t)x_n = x_n + \int_0^t T(s)y_n ds,\ \forall t \geq 0, \forall n \geq 0.$$
When $n \to +\infty$, we obtain
$$T(t)x = x + \int_0^t T(s)yds,\ \forall t \geq 0.$$
Thus
$$\lim_{t \searrow 0} \frac{T(t)x - x}{t} = y.$$
By using the definition of the infinitesimal generator, we deduce that
$$x \in D(A) \text{ and } y = Ax.$$
It follows that A is closed.

Conversely, assume that (i)-(iv) are satisfied. Let $B : D(B) \subset X \to X$ be the infinitesimal generator of $\{T(t)\}_{t\geq 0}$. Let $x \in D(B)$ be fixed. Then by using (i) and (ii),

we have
$$\frac{T(h)x - x}{h} = A\left(\frac{1}{h}\int_0^h T(s)xds\right),\ \forall h > 0, \forall x \in X.$$
Notice that A is closed, since the limit
$$\lim_{h \searrow 0} \frac{T(h)x - x}{h}$$
exists, we have $x \in D(A)$ and $Bx = Ax$. It follows that Graph$(B) \subset$ Graph(A).

Let $x \in D(A)$ be fixed. By using (ii) and (iii), we deduce for each $x \in D(A)$ that
$$\lim_{h \searrow 0} \frac{T(h)x - x}{h} = Ax.$$
Now by the definition of B, we have $x \in D(B)$ and $Bx = Ax$. So Graph$(A) \subset$ Graph(B) and the proof is completed. $\square$

Lemma 2.3.3. *Let $\{T(t)\}_{t \geq 0}$ be a linear C_0-semigroup on a Banach space X and let $A : D(A) \subset X \to X$ be its infinitesimal generator. Then $D(A)$ is dense in X.*

Proof. From (i) in Lemma 2.3.2, we know that
$$\frac{1}{h}\int_0^h T(s)xds \in D(A),\ \forall h > 0, \forall x \in X.$$
But since $t \to T(t)x$ is continuous, we have
$$\lim_{h \searrow 0} \frac{1}{h}\int_0^h T(s)xds = x.$$
So $D(A)$ is dense in X. $\square$

The following result provides another characterization for the infinitesimal generator. This definition is closely related to the definition of the generator for integrated semigroups introduced by Thieme [328].

Proposition 2.3.4. *Let $\{T(t)\}_{t \geq 0}$ be a linear C_0-semigroup on a Banach space X. A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of $\{T(t)\}_{t \geq 0}$ if and only if*
$$x \in D(A) \text{ and } y = Ax \Leftrightarrow T(t)x = x + \int_0^t T(s)yds,\ \forall t \geq 0. \tag{2.3.1}$$

Proof. Assume first that $A : D(A) \subset X \to X$ is the infinitesimal generator of $\{T(t)\}_{t \geq 0}$. Then by Lemma 2.3.2, we have for each $x \in D(A)$ that
$$T(t)x = x + \int_0^t T(s)yds,\ \forall t \geq 0, \tag{2.3.2}$$

with $y = Ax$. Conversely, assume that (2.3.2) is satisfied for some $x, y \in X$. Then

$$\lim_{t \to 0^+} \frac{T(t)x - x}{t} = y.$$

So $x \in D(A)$ and $y = Ax$.

Now assume that (2.3.1) is satisfied. Let $B : D(B) \subset X \to X$ be the infinitesimal generator of $\{T(t)\}_{t \geq 0}$. Let $x \in D(A)$. Then (2.3.1) implies that

$$\lim_{t \to 0^+} \frac{T(t)x - x}{t} = Ax.$$

So $x \in D(B)$ and $Ax = Bx$. We deduce that $\mathrm{Graph}(A) \subset \mathrm{Graph}(B)$.

Let $x \in D(B)$ be fixed. Then by Lemma 2.3.2, we have

$$T(t)x = x + \int_0^t T(s)yds, \forall t \geq 0,$$

with $y = Bx$. So by using (2.3.1), we deduce that $x \in D(A), y = Ax$, and $\mathrm{Graph}(B) \subset \mathrm{Graph}(A)$. So we conclude that $A = B$. $\square$

Corollary 2.3.5. *Let $\{T(t)\}_{t \geq 0}$ be a linear C_0-semigroup on a Banach space X and let $A : D(A) \subset X \to X$ be its infinitesimal generator. Then for each $x \in D(A)$, the map $t \to T(t)x$ is continuously differentiable, $T(t)x \in D(A), \forall t \geq 0$, and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \ \forall t \geq 0.$$

Proof. Let $x \in D(A)$. Using (ii) and (iii) of Lemma 2.3.2, we have

$$T(t)x = x + \int_0^t T(s)Axds.$$

So $t \to T(t)x$ is continuously differentiable and

$$\frac{d}{dt}T(t)x = T(t)Ax, \ \forall t \geq 0.$$

Moreover, by (ii) of Lemma 2.3.2, we have

$$\frac{T(t+h)x - T(t)x}{h} = A\left(\frac{1}{h}\int_t^{t+h} T(s)xds\right).$$

Since A is closed, taking the limit when $h \searrow 0$ on both sides, it follows that $T(t)x \in D(A), \forall t \geq 0$, and

$$\frac{d}{dt}T(t)x = AT(t)x, \forall t \geq 0.$$

This completes the proof. $\square$

In the following we will show that if a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is exponentially bounded, then the resolvent of its generator has a very nice integral representation.

Proposition 2.3.6 (Integral Resolvent Formula (for Operators)). *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space $(X, \|.\|)$. Assume that there are two constants, $\omega \in \mathbb{R}$ and $M \geq 1$, such that*

$$\|T(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \; \forall t \geq 0.$$

Then $(\omega, +\infty) \subset \rho(A)$ and

$$(\lambda I - A)^{-1} x = \int_0^{+\infty} e^{-\lambda s} T(s) x ds, \; \forall \lambda > \omega, \forall x \in X.$$

Proof. For each $\lambda > \omega$, set

$$L_\lambda x = \int_0^{+\infty} e^{-\lambda s} T(s) x ds, \; \forall x \in X.$$

Let $\lambda > \omega$ be fixed. Then we have

$$\int_0^t T(l) L_\lambda x dl = L_\lambda \int_0^t T(l) x dl, \; \forall t \geq 0,$$

and

$$\begin{aligned}
\lambda \int_0^t T(s) L_\lambda x ds &= \lambda \int_0^{+\infty} e^{-\lambda s} \int_0^t T(l) T(s) x dl ds \\
&= \lambda \int_0^{+\infty} e^{-\lambda s} \int_s^{s+t} T(l) x dl ds \\
&= \left[-e^{-\lambda s} \int_s^{s+t} T(l) x dl \right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda s} \left[T(t+s)x - T(s)x \right] ds.
\end{aligned}$$

So

$$\lambda \int_0^t T(s) L_\lambda x ds = \int_0^t T(l) x dl + T(t) L_\lambda x - L_\lambda x.$$

Thus, for each $t > 0$,

$$\frac{T(t) L_\lambda x - L_\lambda x}{t} = \frac{1}{t} \left[\lambda \int_0^t T(s) L_\lambda x ds - \int_0^t T(l) x dl \right].$$

Therefore, when $t \searrow 0$ we obtain

$$L_\lambda x \in D(A), \forall x \in X,$$

$$(\lambda I - A) L_\lambda x = x, \forall x \in X,$$

and

$$L_\lambda (\lambda I - A) x = x, \forall x \in D(A).$$

It follows that $\lambda I - A$ is invertible and $(\lambda I - A)^{-1} = L_\lambda$. $\square$

Suggested by the above result we turn to the Laplace transform. Let $(X, \|.\|)$ be a Banach space. Let $f : [0,+\infty) \to X$ be a continuous map. Set

$$\omega(f) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0} e^{-\omega t}\|f(t)\| < +\infty\right\}.$$

Notice that $\omega(f)$ can be regarded as the growth bound of the map f and $\omega(f) = -\infty$ may occur. If $\omega(f) < +\infty$, we can define the *Laplace transform* of f by

$$\widehat{f}(\lambda) := \int_0^{+\infty} e^{-\lambda s} f(s)ds, \ \forall \lambda > \omega(f).$$

Recall that the *Gamma function* is defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \ \forall x > 0.$$

Then

$$\Gamma(x+1) = x\Gamma(x), \ \forall x > 0,$$

so

$$\Gamma(n+1) = n!, \ \forall n \in \mathbb{N}.$$

We have the *Stirling's formula*

$$\lim_{x\to+\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1.$$

We now state a version of the Post-Widder theorem and refer to Arendt et al. [22, Theorem 1.7.7, p. 43] for a more general version of this result.

Theorem 2.3.7 (Post-Widder). *Let $(X, \|.\|)$ be a Banach space. Let $f : [0,+\infty) \to X$ be an exponentially bounded and continuous map. Then*

$$f(t) = \lim_{n\to+\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right), \ \forall t > 0,$$

where $\widehat{f}^{(n)}$ is the nth derivative of $\widehat{f}$.

Proof. Let $t > 0$ be fixed. We have for each integer $n \geq 0$ and each $\lambda > \omega(f)$ that

$$\widehat{f}^{(n)}(\lambda) = (-1)^n \int_0^{+\infty} s^n e^{-\lambda s} f(s)ds.$$

Consider an integer $n_0 > \max(0, t\omega(f))$. We have for each $n \geq n_0$ that

$$\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right) = \int_0^{+\infty} \rho_n(s) f(s)ds,$$

where

$$\rho_n(s) := \left(\frac{n}{t}\right) \frac{\left(\frac{ns}{t}\right)^n e^{-\frac{ns}{t}}}{n!}.$$

So

$$\int_0^{+\infty} \rho_n(s)\,ds = \frac{\int_0^{+\infty} l^n e^{-l} dl}{n!} = \frac{\Gamma(n+1)}{n!} = 1$$

and

$$\frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)}\left(\frac{n}{t}\right) - f(t) = \int_0^{+\infty} \rho_n(s)\,[f(s) - f(t)]\,ds.$$

It is sufficient to consider

$$I_n := \frac{n^{n+1}}{n!} \int_0^{+\infty} \left(re^{-r}\right)^n [f(rt) - f(t)]\,dr. \tag{2.3.3}$$

Given $\varepsilon > 0$, we choose $0 < a < 1 < b < +\infty$ such that

$$\|f(rt) - f(t)\| \le \varepsilon \text{ when } r \in [a,b].$$

Then we break the integral on the right-hand side of (2.3.3) into three integrals I_n^1, I_n^2, and I_n^3 on the intervals $[0,a]$, $[a,b]$, and $[b,+\infty)$, respectively. Notice that re^{-r} is monotonely non-decreasing on $[0,1]$ and monotonely non-increasing on $[1,+\infty)$, we have

$$\begin{aligned}
\left\|I_n^1\right\| &\le \frac{n^{n+1}}{n!}\left(ae^{-a}\right)^n \int_0^a \|f(rt) - f(t)\|\,dr,\\
\left\|I_n^2\right\| &\le \varepsilon\frac{n^{n+1}}{n!} \int_a^b \left(re^{-r}\right)^n dr \le \varepsilon,\\
\left\|I_n^3\right\| &\le C\frac{n^{n+1}}{n!} \int_b^{+\infty} \left[e^{\omega rt} + 1\right]\left(re^{-r}\right)^n dr
\end{aligned}$$

for some constants $C > 0$ and $\omega > 0$. Note that $ae^{-a} < e^{-1}$. Set $\delta = \frac{ae^{-a}}{e^{-1}} < 1$, we deduce by using Stirling's formula that

$$\begin{aligned}
\limsup_{n\to+\infty} \left\|I_n^1\right\| &\le \lim_{n\to+\infty} \frac{n^{n+1}\delta^n e^{-n}}{\sqrt{2\pi n}n^n e^{-n}} \int_0^a \|f(rt) - f(t)\|\,dr\\
&= \lim_{n\to+\infty} \frac{n^{1/2}\delta^n}{\sqrt{2\pi}} \int_0^a \|f(rt) - f(t)\|\,dr = 0
\end{aligned}$$

and

$$\left\|I_n^3\right\| \le 2C\frac{n^{n+1}}{n!} \int_b^{+\infty} e^{k_0 r}\left(re^{-r}\right)^n dr,$$

where k_0 is an integer such that $k_0 > \omega t$. We have

$$\int_b^{+\infty} e^{k_0 r}\left(re^{-r}\right)^n dr = \int_b^{+\infty} r^n e^{-(n-k_0)r} dr$$

$$= \left(\frac{n+1}{n-k_0}\right)^{n+1} \int_b^{+\infty} l^n e^{-(n+1)l} dl$$
$$\leq \left(\frac{n+1}{n-k_0}\right)^{n+1} \int_b^{+\infty} e^{-l} dl \left(be^{-b}\right)^n.$$

Since
$$\lim_{n\to+\infty} \left(\frac{n+1}{n-k_0}\right)^{n+1} = \lim_{n\to+\infty} \left(\frac{1+1/n}{1-k_0/n}\right)^{n+1} = e^{1-k_0},$$
also $be^{-b} < e^{-1}$, $\widehat{\delta} = \frac{be^{-b}}{e^{-1}} < 1$, we obtain

$$\limsup_{n\to+\infty} \left\|I_n^3\right\| \leq e^{1-k_0} \limsup_{n\to+\infty} 2C \frac{n^{n+1} e^{-n} \widehat{\delta}^n}{n!} \int_b^{+\infty} e^{-l} dl = 0.$$

From the above estimates we deduce that

$$\limsup_{n\to+\infty} \left\| \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \widehat{f}^{(n)} \left(\frac{n}{t}\right) - f(t) \right\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is chosen arbitrarily, the result follows. □

Corollary 2.3.8 (Uniqueness of Laplace Transform). *Let $(X, \|.\|)$ be a Banach space. Let $f : [0,+\infty) \to X$ and $g : [0,+\infty) \to X$ be two exponentially bounded continuous maps. Assume that there exists $\lambda_0 > \max(\omega(f), \omega(g))$ such that*

$$\int_0^{+\infty} e^{-\lambda s} f(s) ds = \int_0^{+\infty} e^{-\lambda s} g(s) ds, \ \forall \lambda \geq \lambda_0.$$

Then $g = f$.

The following theorem is due to Arendt [20], which provides a Laplace transform characterization for the infinitesimal generator of a strongly continuous semigroup of bounded linear operators.

Theorem 2.3.9 (Arendt). *Let $\{T(t)\}_{t\geq 0}$ be an exponentially bounded and strongly continuous family of bounded linear operators on a Banach space X and let $A : D(A) \subset X \to X$ be a linear operator. Then $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup and A is its infinitesimal generator if and only if there exists $\omega \in \mathbb{R}$ such that*

$$\sup_{t\geq 0} e^{-\omega t} \|T(t)\|_{\mathscr{L}(X)} < +\infty,$$

$$(\omega, +\infty) \subset \rho(A),$$

and
$$(\lambda I - A)^{-1} x = \int_0^{+\infty} e^{-\lambda s} T(s) x ds, \ \forall \lambda > \omega, \ \forall x \in X.$$

Proof. For $\mu > \lambda > \omega$, we have

$$\begin{aligned}
&\frac{(\lambda I-A)^{-1}x-(\mu I-A)^{-1}x}{\mu-\lambda}\\
&=\int_0^{+\infty}e^{(\lambda-\mu)t}(\lambda I-A)^{-1}xdt-\frac{1}{\mu-\lambda}\int_0^{+\infty}e^{(\lambda-\mu)t}e^{-\lambda t}T(t)xdt\\
&=\int_0^{+\infty}e^{(\lambda-\mu)t}\int_0^{+\infty}e^{-\lambda s}T(s)xdsdt-\int_0^{+\infty}e^{(\lambda-\mu)t}\int_0^{t}e^{-\lambda s}T(s)xdsdt\\
&=\int_0^{+\infty}e^{(\lambda-\mu)t}\int_t^{+\infty}e^{-\lambda s}T(s)xdsdt\\
&=\int_0^{+\infty}e^{-\mu t}\int_t^{+\infty}e^{-\lambda(s-t)}T(s)xdsdt\\
&=\int_0^{+\infty}e^{-\mu t}\int_0^{+\infty}e^{-\lambda l}T(l+t)xdldt.
\end{aligned}$$

On the other hand,

$$(\mu I-A)^{-1}(\lambda I-A)^{-1}x=\int_0^{+\infty}e^{-\mu t}\int_0^{+\infty}e^{-\lambda s}T(t)T(s)xdsdt.$$

So by the uniqueness of the Laplace transform (first for μ and next for λ) we deduce that

$$T(s+t)=T(t)T(s),\ \forall s,t\geq 0.$$

From the semigroup property we deduce that $T(0)$ is a projection. Moreover, if $T(0)x=0$, then $T(t)x=T(t)T(0)x=0, \forall t\geq 0$, so $(\lambda I-A)^{-1}x=0, \forall\lambda>\omega$. Thus $x=0$. It implies that $T(0)=Id$. We deduce that $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup. Denote its generator by B. Then

$$(\lambda I-B)^{-1}=\int_0^{+\infty}e^{-\lambda s}T(s)ds=(\lambda I-A)^{-1}, \forall\lambda>\omega.$$

Hence $A=B$. This proves one implication. The other implication follows from Proposition 2.3.6. □

As an immediate consequence of Theorem 2.3.9 we have the following result.

Corollary 2.3.10. *Let $A: D(A)\subset X\to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. Then for each $\alpha\in\mathbb{R}$, $A+\alpha I$ is the infinitesimal generator of the C_0-semigroup $\{e^{\alpha t}T(t)\}_{t\geq 0}$.*

Combining the Post-Widder theorem and the Arendt theorem, we obtain the following exponential formula for the strongly continuous semigroup.

Corollary 2.3.11 (Exponential Formula). *Let $A: D(A)\subset X\to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. Then*

$$\lim_{n\to+\infty}\left(\frac{n}{t}\right)^n\left(\frac{n}{t}I-A\right)^{-n}x=T(t)x,\ \forall t>0,\ \forall x\in X.$$

Proof. First it is easy to see that

$$\lambda\left(\lambda I-A\right)^{-1}x=\lambda\int_0^{+\infty}e^{-\lambda t}T(t)xdt\to x \text{ as } \lambda\to+\infty.$$

Moreover, we have

$$\frac{d^n\left(\lambda I-A\right)^{-1}}{d\lambda^n}=(-1)^n n!\left(\lambda I-A\right)^{-(n+1)}.$$

Applying the Post-Widder theorem to the resolvent, we deduce that $\forall t>0$,

$$\left(\frac{n}{t}\right)^{(n+1)}\left(\frac{n}{t}I-A\right)^{-(n+1)}x\to T(t)x \text{ as } n\to+\infty.$$

Note that

$$\begin{aligned}&\left\|\left(\frac{n}{t}\right)^{n}\left(\frac{n}{t}I-A\right)^{-n}x-\left(\frac{n}{t}\right)^{(n+1)}\left(\frac{n}{t}I-A\right)^{-(n+1)}x\right\|\\ &\leq\left\|\left(\frac{n}{t}\right)^{n}\left(\frac{n}{t}I-A\right)^{-n}\right\|\left\|x-\left(\frac{n}{t}\right)\left(\frac{n}{t}I-A\right)^{-1}x\right\|\\ &\leq M\left(\frac{\frac{n}{t}}{\frac{n}{t}-\omega}\right)^{n}\left\|x-\left(\frac{n}{t}\right)\left(\frac{n}{t}I-A\right)^{-1}x\right\|\to 0 \text{ as } n\to+\infty.\end{aligned}$$

The result follows. □

2.4 Hille-Yosida Theorem

The goal of this section is to prove the Hille-Yosida theorem which provides the relationship between a strongly continuous semigroup and its generator. First, we introduce the notion of a Hille-Yosida operator.

Definition 2.4.1. A linear operator $A: D(A)\subset X\to X$ on a Banach space $(X,\|.\|)$ (densely defined or not) is called a *Hille-Yosida operator* if there exist two constants, $\omega\in\mathbb{R}$ and $M\geq 1$, such that

$$(\omega,+\infty)\subset\rho(A)$$

and

$$\left\|(\lambda I-A)^{-n}\right\|_{\mathscr{L}(X)}\leq\frac{M}{(\lambda-\omega)^n},\ \forall\lambda>\omega,\ \forall n\geq 1.$$

Proposition 2.4.2. *Let $A: D(A)\subset X\to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space $(X,\|.\|)$. Then A is a Hille-Yosida operator and there exists a norm $|.|$ on X, which is equivalent to $\|.\|$, such that*

$$\left|(\lambda I-A)^{-1}\right|_{\mathscr{L}(X)}\leq\frac{1}{\lambda-\omega},\ \forall\lambda>\omega.$$

Proof. Let $\omega > 0$ be given such that

$$\|T(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \; \forall t \geq 0.$$

Then the map $|.| : X \to \mathbb{R}_+$

$$|x| = \sup_{t \geq 0} e^{-\omega t} \|T(t)x\|, \; \forall x \in X$$

defines a norm on X. Moreover,

$$\|x\| \leq |x| \leq M \|x\|$$

and

$$|T(t)x| \leq e^{\omega t} |x|, \; \forall t \geq 0, \forall x \in X,$$

So

$$\left|(\lambda I - A)^{-1}\right|_{\mathscr{L}(X)} \leq \frac{1}{\lambda - \omega}, \; \forall \lambda > \omega,$$

and

$$\begin{aligned}\left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}(X)} &\leq M\left|(\lambda I - A)^{-n}\right|_{\mathscr{L}(X)} \\ &\leq M\left|(\lambda I - A)^{-1}\right|^{n}_{\mathscr{L}(X)} \\ &\leq \frac{M}{(\lambda - \omega)^n}, \; \forall \lambda > \omega.\end{aligned}$$

This completes the proof. □

Before stating and proving the Hille-Yosida theorem, we give two lemmas.

Lemma 2.4.3. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator (densely defined or not). Then there exists a norm $|.|$ on X such that*

$$\left|(\lambda I - A)^{-1} x\right| \leq \frac{|x|}{\lambda - \omega}, \; \forall \lambda > \omega, \forall x \in X,$$

and

$$\|x\| \leq |x| \leq M \|x\|, \; \forall x \in X,$$

where M is introduced in Definition 2.4.1.

Proof. Replacing A by $A - \omega I$, we can always assume that $\omega = 0$. For each $\mu > 0$, set

$$\|x\|_\mu = \sup_{n \geq 0} \left\|\mu^n (\mu I - A)^{-n} x\right\|.$$

Then

$$\|x\| \leq \|x\|_\mu \leq M \|x\|, \; \forall x \in X,$$

and

$$\left\| \mu (\mu I - A)^{-1} x \right\|_{\mu} \leq \|x\|_{\mu}, \; \forall x \in X.$$

Moreover, if $\mu \geq \lambda > 0$ and $x \in X$, we have by the resolvent formula that

$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} + (\mu - \lambda)(\mu I - A)^{-1} (\lambda I - A)^{-1}.$$

So

$$\left\| (\lambda I - A)^{-1} x \right\|_{\mu} \leq \left\| (\mu I - A)^{-1} x \right\|_{\mu} + (\mu - \lambda) \left\| (\mu I - A)^{-1} (\lambda I - A)^{-1} x \right\|_{\mu}$$

and

$$\left\| (\lambda I - A)^{-1} x \right\|_{\mu} \leq \frac{\|x\|_{\mu}}{\mu} + \left(1 - \frac{\lambda}{\mu} \right) \left\| (\lambda I - A)^{-1} x \right\|_{\mu}.$$

It follows that

$$\left\| (\lambda I - A)^{-1} x \right\|_{\mu} \leq \frac{\|x\|_{\mu}}{\lambda}, \; \forall \mu \geq \lambda > 0, \forall x \in X$$

and

$$\left\| \lambda^n (\lambda I - A)^{-n} x \right\| \leq \left\| \lambda (\lambda I - A)^{-1} x \right\|_{\mu} \leq \|x\|_{\mu}, \; \forall \mu \geq \lambda > 0, \forall x \in X, \forall n \geq 0.$$

Therefore,

$$\|x\|_{\lambda} \leq \|x\|_{\mu}, \; \forall \mu \geq \lambda > 0, \forall x \in X.$$

Setting

$$|x| = \lim_{\mu \to +\infty} \|x\|_{\mu}, \forall x \in X,$$

the result follows. □

Lemma 2.4.4. *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator with dense domain. Then*

$$\lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} x = x, \; \forall x \in X.$$

Proof. This lemma follows from the fact that $\overline{D(A)} = X$ and Lemma 2.2.9. □

Theorem 2.4.5 (Hille-Yosida). *A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geq 0}$ if and only if A is a Hille-Yosida operator with dense domain (i.e. $\overline{D(A)} = X$). Moreover, if M and ω are the constants introduced in Definition 2.4.1, we must have*

$$\|T(t)\| \leq M e^{\omega t}, \; \forall t \geq 0.$$

Proof. From Lemma 2.3.3 and Proposition 2.4.2 we already know that the condition is necessary. So it remains to prove the sufficient part of the theorem.

First, assume that $A : D(A) \subset X \to X$ is a densely defined linear operator such that $(0, +\infty) \subset \rho(A)$ and

$$\left\|\lambda\left(\lambda I-A\right)^{-1}\right\|\leq 1,\ \forall\lambda>0.$$

Set

$$A_{\lambda}=\lambda A\left(\lambda I-A\right)^{-1}.$$

We have

$$A_{\lambda}=\lambda\left(A-\lambda I+\lambda I\right)\left(\lambda I-A\right)^{-1}=\lambda^{2}\left(\lambda I-A\right)^{-1}-\lambda I.$$

So

$$e^{A_{\lambda}t}=e^{\left[\lambda^{2}(\lambda I-A)^{-1}-\lambda I\right]t}=e^{-\lambda t}e^{\lambda^{2}(\lambda I-A)^{-1}t}.$$

Thus

$$\left\|e^{A_{\lambda}t}\right\|\leq e^{-\lambda t}\left\|e^{\lambda^{2}(\lambda I-A)^{-1}t}\right\|\leq e^{-\lambda t}e^{\left\|\lambda^{2}(\lambda I-A)^{-1}\right\|t},$$

which implies that

$$\left\|e^{A_{\lambda}t}\right\|\leq 1,\ \forall t\geq 0.$$

Set

$$H(s)=e^{tsA_{\lambda}}e^{t(1-s)A_{\mu}},\ \forall s\in\mathbb{R}.$$

Then the map H is continuously differentiable from $\mathbb{R}$ into $\mathscr{L}\left(X\right)$. Since A_{λ} and A_{μ} commute, we have

$$\begin{aligned}e^{tA_{\lambda}}-e^{tA_{\mu}}&=H(1)-H(0)\\&=\int_{0}^{1}H'(s)ds\\&=\int_{0}^{1}t\left(A_{\lambda}-A_{\mu}\right)e^{tsA_{\lambda}}e^{t(1-s)A_{\mu}}ds.\end{aligned}$$

Hence

$$\left\|e^{tA_{\lambda}}-e^{tA_{\mu}}\right\|\leq t\left\|A_{\lambda}-A_{\mu}\right\|.$$

Let $x\in D(A)$. We have

$$\left\|e^{tA_{\lambda}}x-e^{tA_{\mu}}x\right\|\leq t\left\|A_{\lambda}x-A_{\mu}x\right\|\leq t\left[\left\|A_{\lambda}x-Ax\right\|+\left\|Ax-A_{\mu}x\right\|\right].$$

But Lemma 2.4.4 implies that

$$\lim_{\lambda\to+\infty}A_{\lambda}x=\lim_{\lambda\to+\infty}\lambda\left(\lambda I-A\right)^{-1}Ax=Ax.$$

It follows that $t\to e^{tA_{\lambda}}x$ converges, as $\lambda\to+\infty$, uniformly in t on bounded intervals of $[0,+\infty)$. Since $D(A)$ is dense in X and $\left\|e^{A_{\lambda}t}\right\|\leq 1,\forall t\geq 0$, it follows for each $x\in X$ that

$$\lim_{\lambda\to+\infty}e^{A_{\lambda}t}x=T(t)x,$$

where the limit is uniform on bounded intervals of $[0,+\infty)$. Therefore, $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on X and $\|T(t)\| \leq 1, \forall t \geq 0$.

Next we prove that A is the generator of $\{T(t)\}_{t\geq 0}$. First, note that for each $x \in D(A)$ and each $t \geq 0$, we have

$$T(t)x - x = \lim_{\lambda\to+\infty} e^{A_\lambda t}x - x = \lim_{\lambda\to+\infty} \int_0^t e^{A_\lambda s} A_\lambda x ds = \int_0^t T(s)Axds.$$

So

$$T(t)x - x = \int_0^t T(s)Axds, \ \forall x \in D(A), \ \forall t \geq 0. \tag{2.4.1}$$

Since $(\mu I - A)^{-1}$ commutes with $e^{A_\lambda t}$, we also deduce that

$$(\mu I - A)^{-1} T(t)x = T(t)(\mu I - A)^{-1} x, \ \forall x \in X, \forall t \geq 0, \forall \mu > 0. \tag{2.4.2}$$

We now apply Proposition 2.3.4. Assume first that $x \in D(A)$ and $y = Ax$. Then from (2.4.1) it follows that

$$T(t)x - x = \int_0^t T(s)yds, \ \forall t \geq 0. \tag{2.4.3}$$

Assume now that (2.4.3) is satisfied for some x and y in X. Let $\mu > 0$ be fixed. Then from (2.4.2) and (2.4.3), we have for each $t \geq 0$ that

$$(\mu I - A)^{-1} T(t)x - (\mu I - A)^{-1} x = \int_0^t T(s)(\mu I - A)^{-1} yds.$$

By (2.4.1) and (2.4.2), we have

$$\begin{aligned}(\mu I - A)^{-1} T(t)x - (\mu I - A)^{-1} x &= T(t)(\mu I - A)^{-1} x - (\mu I - A)^{-1} x \\ &= \int_0^t T(s)A(\mu I - A)^{-1} xds.\end{aligned}$$

So we deduce that

$$\int_0^t T(s)A(\mu I - A)^{-1} xds = \int_0^t T(s)(\mu I - A)^{-1} yds, \ \forall t \geq 0.$$

It follows that

$$T(t)A(\mu I - A)^{-1} x = T(t)(\mu I - A)^{-1} y, \ \forall t \geq 0,$$

and for $t = 0$ we obtain

$$A(\mu I - A)^{-1} x = (\mu I - A)^{-1} y.$$

Since

$$A(\mu I - A)^{-1} x = \mu(\mu I - A)^{-1} x - x,$$

it follows that
$$x = (\mu I - A)^{-1} [\mu x - y],$$
so $x \in D(A)$ and
$$Ax = y.$$

Therefore,

$$x \in D(A) \text{ and } Ax = y \Leftrightarrow T(t)x = x + \int_0^t T(s)yds, \forall t \geq 0.$$

By applying Proposition 2.3.4, we conclude that A is the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ and
$$\|T(t)\| \leq 1, \ \forall t \geq 0.$$

To complete the proof, it is sufficient to note that by Lemma 2.4.3, we can find a norm $|.|$ on X such that

$$|(\lambda I - A)x| \leq \frac{1}{\lambda - \omega}, \ \forall \lambda > \omega,$$

and
$$\|x\| \leq |x| \leq M \|x\|, \ \forall x \in X.$$

So when X is endowed with the norm $|.|$, the linear operator $A - \omega I$ satisfies the assumptions of the first part of the proof, the result follows from Corollary 2.3.10. □

When the domain of A of is not dense in X, the following result will be useful.

Corollary 2.4.6. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. The part $A_{\overline{D(A)}}$ of A in $\overline{D(A)}$ is the infinitesimal generator of a C_0-semigroup $\left\{T_{A_{\overline{D(A)}}}(t)\right\}_{t\geq 0}$ on $\overline{D(A)}$ if and only if the following two conditions are satisfied:*

(a) $(\lambda I - A)^{-1} x \to 0$ *as* $\lambda \to +\infty$, $\forall x \in X$;
(b) *There exist $\omega \in \mathbb{R}$ and $M > 0$ such that*

$$\left\|(\lambda I - A_0)^{-n}\right\|_{\mathscr{L}\left(\overline{D(A)}\right)} = \left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}\left(\overline{D(A)}\right)} \leq \frac{M}{(\lambda - \omega)^n}, \ \forall \lambda > \omega, \forall n \geq 1.$$

Proof. This corollary is an immediate consequence of the Hille-Yosida theorem and Lemmas 2.2.10-2.2.11. □

2.5 Nonhomogeneous Cauchy problem

Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|.\|)$. Let $I \subset [0, +\infty)$ be an interval. Define

$$C\left(I,D(A)\right)=\left\{\begin{array}{r}\varphi\in C\left(I,X\right):\varphi(t)\in D(A),\ \forall t\in I,\text{ and the map}\\ t\to A\varphi(t)\text{ is continuous from }I\text{ into }X\end{array}\right\}$$

and

$$L^1_{\text{Loc}}\left(I,X\right)=\left\{\varphi:I\to X:\ \forall a,b\in I\text{ with }a<b,\ \varphi\mid_I\in L^1\left(\left(a,b\right),X\right)\right\}.$$

In this section we consider the nonhomogeneous Cauchy problem

$$\begin{cases}\dfrac{du(t)}{dt}=Au(t)+f(t),\ t\in[0,\tau),\\ u(0)=x\in\overline{D(A)},\end{cases}\tag{2.5.1}$$

where $\tau\in(0,+\infty]$ and $f\in L^1_{Loc}\left([0,\tau),X\right)$.

The Cauchy problem (2.5.1) is said to be *densely defined* if

$$\overline{D(A)}=X,$$

and *non-densely defined* otherwise.

Definition 2.5.1. (a) Assume that $f\in C\left([0,\tau),X\right)$. Then a function $u\in C\left([0,\tau),X\right)$ is called a *classical solution* of (2.5.1) if $u\in C^1\left([0,\tau),X\right)\cap C\left([0,\tau),D(A)\right)$, and satisfies

$$\frac{du(t)}{dt}=Au(t)+f(t),\forall t\in[0,\tau);\ u(0)=x.$$

(b) Assume that $f\in L^1_{\text{Loc}}\left([0,\tau),X\right)$. A function $u\in C\left([0,\tau),X\right)$ is called an *integrated solution* (or a *mild solution*) of (2.5.1) if

$$\int_0^t u(s)ds\in D(A),\ \forall t\in[0,\tau)$$

and

$$u(t)=x+A\int_0^t u(s)ds+\int_0^t f(s)ds,\ \forall t\in[0,\tau).$$

The following result describes the relationships between classical and integrated solutions.

Lemma 2.5.2. *Assume that A is closed and $f\in C\left([0,\tau),X\right)$. If u is a classical solution of (2.5.1) then u is an integrated solution of (2.5.1). Conversely, if u is an integrated solution of (2.5.1) and $u\in C^1\left([0,\tau),X\right)$ or $u\in C\left([0,\tau),D(A)\right)$, then*

$$u\in C^1\left([0,\tau),X\right)\cap C\left([0,\tau),D(A)\right)$$

and u is a classical solution of (2.5.1).

Proof. Assume first that u is classical solution. Then by integrating (2.5.1) we obtain

$$u(t)=x+\int_0^t Au(s)ds+\int_0^t f(s)ds,\ \forall t\in[0,\tau).$$

Moreover, since A is closed, we have

$$\int_0^t u(s)ds \in D(A) \text{ and } \int_0^t Au(s)ds = A\int_0^t u(s)ds, \forall t \in [0,\tau).$$

It follows that u is an integrated solution of (2.5.1).

Assume that u is an integrated solution of (2.5.1). Suppose first that $u \in C^1([0,\tau), X)$, then

$$A\frac{1}{h}\int_t^{t+h} u(s)ds = \frac{1}{h}[u(t+h) - u(t)] - \frac{1}{h}\int_t^{t+h} f(s)ds.$$

By using the fact that A is closed, we deduce when $h \searrow 0$ that

$$u(t) \in D(A), \ \forall t \in [0,\tau), \text{ and } Au(t) = u'(t) - f(t), \forall t \in [0,\tau).$$

So $u \in C^1([0,\tau), X) \cap C([0,\tau), D(A))$ and u is a classical solution.

Assume that $u \in C([0,\tau), D(A))$. Since u is an integrated solution and A is closed, we deduce that

$$u(t) = x + \int_0^t Au(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau).$$

So $u \in C^1([0,\tau), X)$ and the result follows. □

Lemma 2.5.3 (Uniqueness). *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. Let $u : [0,\tau) \to X$ be continuous such that*

$$\int_0^t u(s)ds \in D(A)$$

and

$$u(t) = A\int_0^t u(s)ds, \ \forall t \in [0,\tau).$$

Then

$$u(t) = 0, \ \forall t \in [0,\tau).$$

Proof. Let $t \in [0,\tau)$ be fixed. We have for each $r \in [0,t]$ that

$$\frac{d}{dr}\left(T(t-r)\int_0^r u(s)ds\right) = -T(t-r)A\int_0^r u(s)ds + T(t-r)u(r) = 0.$$

So by integrating this equation from 0 to t, we obtain

$$\int_0^t u(s)ds = 0.$$

By differentiating we obtain $u = 0$.

As an immediate consequence of the previous lemma we obtain the following theorem.

Theorem 2.5.4. *A strongly continuous semigroup of bounded linear operators is uniquely determined by its infinitesimal generator.*

Theorem 2.5.5. *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a linear C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X. Then for each $f \in L^1_{\text{Loc}}([0,\tau),X)$ and for each $x \in X$, there exists at most one integrated solution $u \in C([0,\tau),X)$ of (2.5.1). Moreover,*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \ \forall t \in [0,\tau).$$

Furthermore, u is a classical solution of (2.5.1) if

$$u \in C^1([0,\tau),X) \text{ or } u \in C([0,\tau),D(A)).$$

In particular if $x \in D(A)$ and either $f \in C([0,\tau),D(A))$ or $f \in C^1([0,\tau),X)$, then u is a classical solution of (2.5.1).

Proof. The uniqueness follows from Lemma 2.5.3. To prove the existence it is sufficient to prove that

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \ \forall t \in [0,\tau),$$

is an integrated solution. Set

$$v(t) = T(t)x \text{ and } w(t) = \int_0^t T(t-s)f(s)ds, \ \forall t \in [0,\tau).$$

We already know that

$$v(t) = x + A\int_0^t v(s)ds, \ \forall t \in [0,\tau).$$

So it is sufficient to prove that w satisfies

$$w(t) = A\int_0^t w(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau). \tag{2.5.2}$$

Indeed, we have

$$\begin{aligned}\int_0^t w(l)dl &= \int_0^t\int_0^l T(l-s)f(s)dsdl \\ &= \int_0^t\int_s^t T(l-s)f(s)dlds \\ &= \int_0^t\int_0^{t-s} T(l)f(s)dlds,\end{aligned}$$

so $\int_0^t w(l)dl \in D(A)$ and

$$\begin{aligned} A\int_0^t w(l)dl &= \int_0^t A\int_0^{t-s} T(l)f(s)dlds \\ &= \int_0^t T(t-s)f(s)ds - \int_0^t f(s)ds \\ &= w(t) - \int_0^t f(s)ds. \end{aligned}$$

Hence, w satisfies (2.5.2) and it is follows that $u(t) = T(t)x + \int_0^t T(t-s)f(s)ds$ is an integrated solution of (2.5.1). By using Lemma 2.5.2 it follows that u is a classical solution whenever $u \in C^1([0,\tau),X)$ or $u \in C([0,\tau),D(A))$. Finally, if $x \in D(A)$ and $f \in C([0,\tau),D(A))$, we have

$$u(t) = (\lambda - A)^{-1}\left[T(t)(\lambda - A)x + \int_0^t T(t-s)(\lambda - A)f(s)ds\right], \forall t \in [0,\tau),$$

so $u \in C([0,\tau),D(A))$. If $x \in D(A)$ and $f \in C^1([0,\tau),X)$, we have

$$u(t) = T(t)x + \int_0^t T(s)f(t-s)ds.$$

Thus, $u \in C^1([0,\tau),X)$, and

$$u'(t) = AT(t)x + T(t)f(0) + \int_0^t T(s)f'(t-s)ds, \forall t \in [0,\tau).$$

By using Lemma 2.5.2 the result follows. □

Combining the Hille-Yosida Theorem (Theorem 2.4.5) and Theorem 2.5.5, we know that the nonhomogeneous Cauchy problem (2.5.1) is well-posed with respect to the integrated or mild solution whenever the linear operator A is a densely defined Hille-Yosida operator. In practice the following result can be useful in obtaining integrated solutions of a Cauchy problem.

Lemma 2.5.6. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Assume that $\rho(A) \cap \mathbb{R} \neq \emptyset$ and $f \in C([0,\tau),X)$. Then $u \in C([0,\tau),X)$ is an integrated solution solution of (2.5.1) if and only if there exists $\lambda \in \rho(A) \cap \mathbb{R}$ such that $u_\lambda := (\lambda I - A)^{-1} u \in C^1([0,\tau),X)$ and u_λ satisfies the following ordinary differential equation*

$$\begin{cases} \dfrac{du_\lambda(t)}{dt} = \lambda u_\lambda(t) - u(t) + (\lambda I - A)^{-1} f(t), \ \forall t \in [0,\tau), \\ u_\lambda(0) = (\lambda I - A)^{-1} x. \end{cases} \tag{2.5.3}$$

Proof. Assume first that u is an integrated solution of (2.5.1). Then

$$u_\lambda(t) = (\lambda I - A)^{-1} x + A(\lambda I - A)^{-1}\int_0^t u(s)ds + \int_0^t (\lambda I - A)^{-1} f(s)ds, \forall t \in [0,\tau).$$

Since $A(\lambda I - A)^{-1} = -I + \lambda (\lambda I - A)^{-1}$ is bounded, we deduce that $u_\lambda \in C^1([0,\tau), X)$. By differentiation,

$$\frac{du_\lambda(t)}{dt} = A(\lambda I - A)^{-1} u(t) + (\lambda I - A)^{-1} f(t), \forall t \in [0,\tau),$$

and it follows that u_λ satisfies (2.5.3).

Conversely, assume that u_λ satisfies (2.5.3). By integrating (2.5.3) from 0 to t, we obtain for each $\forall t \in [0,\tau)$ that

$$\begin{array}{c}(\lambda I - A)^{-1} u(t) = (\lambda I - A)^{-1} x + A(\lambda I - A)^{-1} \int_0^t u(s)ds \\ + (\lambda I - A)^{-1} \int_0^t f(s)ds.\end{array} \tag{2.5.4}$$

Since $A(\lambda I - A)^{-1} = -I + \lambda (\lambda I - A)^{-1}$, we obtain

$$\int_0^t u(s)ds = (\lambda I - A)^{-1} \left[x - u(t) + \lambda \int_0^t u(s)ds + \int_0^t f(s)ds \right], \forall t \in [0,\tau),$$

so

$$\int_0^t u(s)ds \in D(A), \forall t \in [0,\tau).$$

By using (2.5.4), we obtain

$$(\lambda I - A)^{-1} u(t) = (\lambda I - A)^{-1} \left[x + A \int_0^t u(s)ds + \int_0^t f(s)ds \right], \forall t \in [0,\tau),$$

and since $(\lambda I - A)^{-1}$ is injective, we deduce that u is an integrated solution. □

2.6 Examples

Roughly speaking, the Hille-Yosida theorem says that a linear operator A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space X if and only if the following conditions are satisfied:

(a) A is densely defined, i.e. $\overline{D(A)} = X$;

(b) A is a Hille-Yosida operator; that is, there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that for each $\lambda > \omega$, $\lambda I - A$ is a bijection from $D(A)$ into X, $(\lambda I - A)^{-1}$ is bounded (for short $\lambda \in \rho(A)$ where $\rho(A)$ is the resolvent set of A), and

$$\left\| (\lambda I - A)^{-n} \right\|_{\mathscr{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}, \forall \lambda > \omega.$$

In this section we give some examples to show how to use the Hille-Yosida theorem.

Example 2.6.1. Let $X = L^1((0,1), \mathbb{R})$ be the space of integrable maps from $(0,1)$ into $\mathbb{R}$ endowed with the usual norm

$$\|\varphi\|_{L^1((0,1),\mathbb{R})} = \int_0^1 |\varphi(x)|\, dx.$$

Consider the Cauchy problem

$$\frac{du(t)}{dt} = \chi u(t), \ t \geq 0, \ \ u(0) = \varphi \in L^1((0,1),\mathbb{R}), \tag{2.6.1}$$

where $\chi \in C([0,1),\mathbb{R})$, and assume that

$$\lim_{x \nearrow 1} \chi(x) = -\infty.$$

Set

$$\overline{\chi} := \sup_{x \in [0,1)} \chi(x) < +\infty.$$

Here the linear operator $A : D(A) \subset X \to X$ is defined by

$$A(\varphi)(x) = \chi(x)\varphi(x) \text{ for almost every } x \in (0,1)$$

with

$$D(A) = \left\{ \varphi \in L^1((0,1),\mathbb{R}) : \chi\varphi \in L^1((0,1),\mathbb{R}) \right\}.$$

A natural formula for the semigroup solution of (2.6.1) is

$$T(t)(\varphi)(x) = e^{\chi(x)t}\varphi(x), \ \forall t \geq 0.$$

Clearly, the family $\{T(t)\}_{t \geq 0}$ defines a semigroup of bounded linear operators on X.

We prove that $\{T(t)\}_{t \geq 0}$ is strongly continuous. Let $s \geq 0$ and $\varphi \in L^1((0,1),\mathbb{R})$ be fixed. Since $C_c((0,1),\mathbb{R})$ (the space of continuous functions with compact support in $(0,1)$) is dense in $L^1((0,1),\mathbb{R})$, given a sequence $\varphi_n \in C_c((0,1),\mathbb{R}) \to \varphi$ in $L^1((0,1),\mathbb{R})$, we have for each $n \geq 0$ that

$$\lim_{t \to s} \|T(t)\varphi_n - T(s)\varphi_n\|_{L^1((0,1),\mathbb{R})} = 0.$$

Let $\varepsilon > 0$ be fixed. Then we can find $n_0 \geq 0$ such that

$$e^{\max(\overline{\chi},0)(s+1)} \|\varphi - \varphi_n\|_{L^1((0,1),\mathbb{R})} < \frac{\varepsilon}{2}, \ \forall n \geq n_0.$$

Then for each $t \in [0, s+1]$ and each $n \geq n_0$, we have

$$\begin{aligned}
&\|T(t)\varphi - T(s)\varphi\|_{L^1((0,1),\mathbb{R})} \\
&\quad \leq \|T(t)[\varphi - \varphi_n]\|_{L^1((0,1),\mathbb{R})} \\
&\qquad + \|T(t)\varphi_n - T(s)\varphi_n\|_{L^1((0,1),\mathbb{R})} + \|T(s)[\varphi - \varphi_n]\|_{L^1((0,1),\mathbb{R})} \\
&\quad \leq \varepsilon + \|T(t)\varphi_n - T(s)\varphi_n\|_{L^1((0,1),\mathbb{R})}.
\end{aligned}$$

Thus,
$$\limsup_{t\to s}\|T(t)\varphi - T(s)\varphi\|_{L^1((0,1),\mathbb{R})} \leq \varepsilon.$$
It follows that $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on X.

We now turn to the linear operator A. It is easy to prove that $D(A)$ endowed with the norm
$$\|\varphi\|_A = \|\varphi\|_{L^1((0,1),\mathbb{R})} + \|\chi\varphi\|_{L^1((0,1),\mathbb{R})}$$
is a Banach space. So A is closed. Moreover, $D(A)$ is dense in X since
$$C_c((0,+\infty),\mathbb{R}) \subset D(A) \subset L^1((0,1),\mathbb{R}),$$
and for each $\lambda > \overline{\chi}$,
$$(\lambda I - A)\varphi = \psi \Leftrightarrow (\lambda - \chi(x))\varphi(x) = \psi(x) \Leftrightarrow \varphi(x) = \frac{\psi(x)}{\lambda - \chi(x)}$$
and
$$0 \leq \frac{1}{\lambda - \chi(x)} \leq \frac{1}{\lambda - \overline{\chi}}.$$
So for each $\lambda > \overline{\chi}$, the linear operator $\lambda I - A$ is one-to-one and onto from $D(A)$ into X. Moreover, the resolvent $(\lambda I - A)^{-1} : X \to X$ defined by
$$(\lambda I - A)^{-1}(\psi)(x) = \frac{\psi(x)}{\lambda - \chi(x)}$$
is a bounded linear operator and
$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda - \overline{\chi}}, \ \forall \lambda > \overline{\chi}.$$
It follows that
$$\begin{aligned}
\|(\lambda I - A)^{-n}\| &= \left\|\left((\lambda I - A)^{-1}\right)^n\right\| \\
&\leq \left\|(\lambda I - A)^{-1}\right\|^n \\
&\leq \frac{1}{(\lambda - \overline{\chi})^n}, \ \forall \lambda > \overline{\chi}, \ \forall n \geq 0.
\end{aligned}$$
So A satisfies the conditions of the Hille-Yosida theorem. Furthermore, A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$; that is,
$$D(A) = \left\{\varphi \in X : \lim_{t\searrow 0} \frac{T(t)\varphi - \varphi}{t} \text{ exists}\right\}$$
and

$$A\varphi = \lim_{t \searrow 0} \frac{T(t)\varphi - \varphi}{t}, \forall \varphi \in D(A).$$

Example 2.6.2. Let $X = BC([0,1), \mathbb{R})$ be the space of bounded and continuous maps from $[0,1)$ into $\mathbb{R}$ endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{x \in [0,1)} |\varphi(x)|.$$

Then it is well known that $(X, \|.\|_\infty)$ is a Banach space. As an exercise, one can prove the following statements:

(a) $\{T(t)\}_{t \geq 0}$ defined by

$$T(t)(\varphi)(x) := e^{\chi(x)t}\varphi(x), \forall t \geq 0,$$

is a semigroup of bounded linear operators on X;

(b) $\{T(t)\}_{t \geq 0}$ is not strongly continuous (*Hint*: Consider $\varphi(x) = 1, \forall x \in [0,1)$);

(c) Define

$$X_0 = \left\{ \varphi \in C_b([0,1), \mathbb{R}) : \lim_{x \nearrow 1} \varphi(x) = 0 \right\}.$$

Consider the family of bounded linear operators $\{T_0(t)\}_{t \geq 0}$ on X_0 defined by

$$T_0(t) := T(t)\,|_{X_0}, \forall t \geq 0.$$

Then $\{T_0(t)\}_{t \geq 0}$ is a strongly continuous semigroup on X_0 (*Hint*: Observe that $C_c([0,1), \mathbb{R})$ is dense in X_0).

Example 2.6.3. Let $X = C([-1,0], \mathbb{R})$ be the space of continuous maps from $[-1,0]$ into $\mathbb{R}$ endowed with the usual supremum norm

$$\|\varphi\|_\infty = \sup_{\theta \in [-1,0]} |\varphi(\theta)|.$$

Consider the partial differential equation

$$\begin{cases} \dfrac{\partial u(t,\theta)}{\partial t} - \dfrac{\partial u(t,\theta)}{\partial \theta} = 0, \ t > 0, \ \theta \in (-1,0) \\ \dfrac{\partial u(t,0)}{\partial \theta} = 0, \ t > 0 \\ u(0,x) = \varphi \in X. \end{cases}$$

Consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A\varphi = \varphi', \ \forall \varphi \in D(A)$$

with

$$D(A) = \left\{ \varphi \in C^1([-1,0], \mathbb{R}) : \varphi'(0) = 0 \right\}.$$

Then for each $\lambda > 0$,

$$\begin{aligned}
&\lambda\varphi - A\varphi = \psi \text{ and } \varphi \in D(A) \Leftrightarrow \lambda\varphi - \varphi' = \psi \text{ and } \varphi \in D(A)\\
&\Leftrightarrow \varphi(\theta) = e^{\lambda\left(\theta-\widehat{\theta}\right)}\varphi(\widehat{\theta}) - \textstyle\int_{\widehat{\theta}}^{\theta} e^{\lambda(\theta-s)}\psi(s)ds,\ \forall\theta,\widehat{\theta}\in[-1,0] \text{ with } \theta \geq \widehat{\theta}\\
&\quad\text{and } \lambda\varphi(0) = \psi(0)\\
&\Leftrightarrow \varphi(\widehat{\theta}) = e^{\lambda\widehat{\theta}}\varphi(0) + \textstyle\int_{\widehat{\theta}}^{0} e^{\lambda\left(\widehat{\theta}-s\right)}\psi(s)ds,\ \forall\widehat{\theta}\in[-1,0] \text{ and } \lambda\varphi(0) = \psi(0)\\
&\Leftrightarrow \varphi(\widehat{\theta}) = e^{\lambda\widehat{\theta}}\tfrac{\psi(0)}{\lambda} + \textstyle\int_{\widehat{\theta}}^{0} e^{\lambda\left(\widehat{\theta}-s\right)}\psi(s)ds,\ \forall\widehat{\theta}\in[-1,0].
\end{aligned}$$

So for each $\lambda > 0$, the linear operator $\lambda I - A : D(A) \to X$ is one to one and onto. Moreover, the resolvent $(\lambda I - A)^{-1} : X \to X$ is defined by

$$(\lambda I - A)^{-1}(\psi)(\widehat{\theta}) = e^{\lambda\widehat{\theta}}\frac{\psi(0)}{\lambda} + \int_{\widehat{\theta}}^{0} e^{\lambda\left(\widehat{\theta}-s\right)}\psi(s)ds, \forall\widehat{\theta}\in[-1,0].$$

Note that

$$\begin{aligned}
\left\|(\lambda I - A)^{-1}(\psi)\right\|_{\infty} &\leq \sup_{\widehat{\theta}\in[-1,0]}\left[\frac{e^{\lambda\widehat{\theta}}}{\lambda} + \int_{\widehat{\theta}}^{0} e^{\lambda\left(\widehat{\theta}-s\right)}ds\right]\|\psi\|_{\infty}\\
&= \sup_{\widehat{\theta}\in[-1,0]}\left[\frac{e^{\lambda\widehat{\theta}}}{\lambda} + \int_{\widehat{\theta}}^{0} e^{\lambda l}dl\right]\|\psi\|_{\infty}\\
&= \frac{\|\psi\|_{\infty}}{\lambda},
\end{aligned}$$

so

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda},\ \forall\lambda > 0.$$

It follows that

$$\left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda^n},\ \forall\lambda > 0,\ \forall n \geq 1.$$

Consider now a family of bounded linear operators $\{T(t)\}_{t\geq 0}$ on X defined by

$$T(t)(\varphi)(\theta) = \begin{cases} \varphi(0) & \text{if } t+\theta \geq 0,\\ \varphi(t+\theta) & \text{if } t+\theta \leq 0.\end{cases}$$

Then it can be checked that $\{T(t)\}_{t\geq 0}$ is a linear C_0-semigroup on X. Moreover, A is the infinitesimal generator of $\{T(t)\}_{t\geq 0}$.

Example 2.6.4. Note that here we incorporate the boundary condition into the definition of the domain (i.e. $\varphi(0) = 0$ in the definition of $D(A)$). It is also possible to proceed differently. Consider the space

$$X = \mathbb{R} \times C([-1,0],\mathbb{R})$$

endowed with the usual product norm. Consider

$$A\begin{pmatrix} 0\\ \varphi\end{pmatrix} = \begin{pmatrix} -\varphi'(0)\\ \varphi'\end{pmatrix}$$

with

$$D(A) = \{0\} \times C^1([-1,0],\mathbb{R})$$

and A_0, the part of A in $\overline{D(A)} = \{0\} \times C([-1,0],\mathbb{R})$, that is,

$$A_0 x = Ax, \ \forall x \in D(A_0),$$

where

$$\begin{aligned} D(A_0) &= \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\} \\ &= \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0\} \times C^1([-1,0],\mathbb{R}) : \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix} \in \{0\} \times C([-1,0],\mathbb{R}) \right\}. \end{aligned}$$

Thus,

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0\} \times C^1([-1,0],\mathbb{R}) : \varphi'(0) = 0 \right\}$$

and

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}.$$

One can prove that A_0 is the infinitesimal generator of the linear C_0-semigroup $\left\{T_{A_0}(t)\right\}_{t \geq 0}$ on $\overline{D(A)}$. Actually, $T_{A_0}(t)$ is defined by

$$T_{A_0}(t) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{A_0}(t)\varphi \end{pmatrix},$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} \varphi(0) & \text{if } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases}$$

Example 2.6.5. Consider $X = L^1((0,+\infty),\mathbb{R})$ endowed with the usual norm and consider the PDE

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = 0, \ t > 0, \ x > 0, \\ u(t,0) = 0, \ t > 0, \\ u(0,.) = \varphi \in L^1((0,+\infty),\mathbb{R}). \end{cases}$$

Let $A : D(A) \subset X \to X$ be the linear operator defined by

$$A\varphi = -\varphi'$$

with

$$D(A) = \left\{ \varphi \in W^{1,1}((0,+\infty),\mathbb{R}) : \varphi(0) = 0 \right\}.$$

The space $W^{1,1}((0,+\infty),\mathbb{R})$ can be identified to the space of absolutely continuous maps, which turns to be differentiable almost everywhere (see Rudin [303]).

The above PDE can be reformulated as the following abstract Cauchy problem

$$\frac{du}{dt} = Au(t),\ t \geq 0;\ \ u(0) = \varphi \in X.$$

By using similar computations as before, we obtain for each $\lambda > 0$ that

$$\begin{aligned}
&(\lambda I - A)\,\varphi = \psi \\
&\quad \Leftrightarrow \lambda\,\varphi(x) + \varphi'(x) = \psi(x) \text{ for almost every } x \geq 0 \text{ and } \varphi(0) = 0 \\
&\quad \Leftrightarrow \varphi(x) = \int_0^x e^{-\lambda(x-s)}\psi(s)ds \text{ for almost every } x \geq 0.
\end{aligned}$$

So for $\lambda > 0$, the map $\lambda I - A$ is one-to-one and onto from $D(A)$ into X, and

$$(\lambda I - A)^{-1}\,\psi = \varphi \Leftrightarrow \varphi(x) = \int_0^x e^{-\lambda(x-s)}\psi(s)ds.$$

It follows that

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda},\ \forall \lambda > 0.$$

So A satisfies the conditions of the Hille-Yosida theorem. Thus A is the infinitesimal generator of the linear C_0-semigroup $\{T(t)\}_{t \geq 0}$ on X. Moreover, $T(t)$ is defined by

$$T(t)(\varphi)(x) = \begin{cases} \varphi(t-x) & \text{if } t \geq x, \\ 0 & \text{if } t \leq x. \end{cases}$$

Example 2.6.6. One can consider the same problem in L^p.

Example 2.6.7. Consider $X = UBC(\mathbb{R},\mathbb{R})$ the space of uniformly continuous and bounded maps endowed with the usual supremum norm

$$\|\varphi\|_\infty = \sup_{x \in \mathbb{R}} |\varphi(x)|$$

Then $UBC(\mathbb{R},\mathbb{R})$ is a Banach space. Consider now the diffusion equation in this space:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2},\ t > 0,\ x \in \mathbb{R}, \\ u(0,.) = \varphi \in UBC(\mathbb{R},\mathbb{R}). \end{cases}$$

Let $A : D(A) \subset X \to X$ be defined by

$$A\varphi = \varphi''$$

with

$$D(A) = \left\{\varphi \in C^2(\mathbb{R},\mathbb{R}) \cap UBC(\mathbb{R},\mathbb{R}) : \varphi', \varphi'' \in UBC(\mathbb{R},\mathbb{R})\right\}.$$

As before we can rewrite the above PDE as an abstract Cauchy problem

$$\frac{du}{dt} = Au(t),\ t \geq 0;\ u(0) = \varphi \in UBC(\mathbb{R},\mathbb{R}).$$

It is well known that $D(A)$ is dense in X. Let $\lambda > 0$. Then

$$(\lambda I - A)\,\varphi = \psi \Leftrightarrow \lambda\varphi - \varphi'' = \psi$$

Set $\widehat{\varphi} = \varphi'$. Then

$$\begin{aligned}(\lambda I - A)\,\varphi = \psi &\Leftrightarrow \varphi' = \widehat{\varphi} \text{ and } \widehat{\varphi}' = \lambda\varphi - \psi \\ &\Leftrightarrow \begin{cases} \sqrt{\lambda}\varphi' + \widehat{\varphi}' = \sqrt{\lambda}\left(\sqrt{\lambda}\varphi + \widehat{\varphi}\right) - \psi \\ \sqrt{\lambda}\varphi' - \widehat{\varphi}' = -\sqrt{\lambda}\left(\sqrt{\lambda}\varphi - \widehat{\varphi}\right) + \psi. \end{cases}\end{aligned}$$

Define

$$w = \left(\sqrt{\lambda}\varphi + \widehat{\varphi}\right) \text{ and } \widehat{w} = \left(\sqrt{\lambda}\varphi - \widehat{\varphi}\right).$$

We obtain

$$(\lambda I - A)\,\varphi = \psi \Leftrightarrow \begin{cases} w' = \sqrt{\lambda}w - \psi \\ \widehat{w}' = -\sqrt{\lambda}\widehat{w} + \psi. \end{cases} \tag{2.6.2}$$

The first equation of system (2.6.2) is equivalent to

$$e^{-\sqrt{\lambda}x}w(x) = e^{-\sqrt{\lambda}y}w(y) - \int_y^x e^{-\sqrt{\lambda}l}\psi(l)dl, \ \forall x \geq y.$$

So when $x \to +\infty$ we obtain (since w is bounded)

$$w(y) = \int_y^{+\infty} e^{\sqrt{\lambda}(y-l)}\psi(l)dl = \int_0^{+\infty} e^{-\sqrt{\lambda}s}\psi(s+y)dl.$$

Similarly, the second equation of system (2.6.2) is equivalent to

$$\widehat{w}(x) = e^{-\sqrt{\lambda}(x-y)}\widehat{w}(y) + \int_y^x e^{-\sqrt{\lambda}(x-l)}\psi(l)dl, \ \forall x \geq y.$$

So when $y \to -\infty$ we obtain

$$\widehat{w}(x) = \int_{-\infty}^x e^{-\sqrt{\lambda}(x-l)}\psi(l)dl = \int_{-\infty}^0 e^{\sqrt{\lambda}s}\psi(s+x)dl.$$

Since

$$w + \widehat{w} = 2\sqrt{\lambda}\varphi$$

and

$$\int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|s|}\psi(s+x)ds = \int_x^{+\infty} e^{\sqrt{\lambda}(x-l)}\psi(l)dl + \int_{-\infty}^x e^{-\sqrt{\lambda}(x-l)}\psi(l)dl,$$

we have for each $\lambda > 0$ that

$$(\lambda I - A)^{-1}(\psi)(x) = \frac{1}{2\sqrt{\lambda}}\int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|s|}\psi(s+x)ds$$

$$= \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|x-l|} \psi(l) dl.$$

One may observe that $(\lambda I - A)^{-1}(\psi)$ is defined by a convolution operator and it follows that

$$(\lambda I - A)^{-1} BC(\mathbb{R}, \mathbb{R}) \subset UBC(\mathbb{R}, \mathbb{R})$$

where $BC(\mathbb{R}, \mathbb{R})$ is the space of bounded and continuous maps from $\mathbb{R}$ into itself. But $UBC(\mathbb{R}, \mathbb{R})$ is not dense in $BC(\mathbb{R}, \mathbb{R})$. So in order to obtain $\overline{D(A)} = X$ we need to use the space UBC.

In summary we deduce that for each $\lambda > 0$, the linear operator $\lambda I - A$ is one-to-one and onto from $D(A)$ into X, and

$$(\lambda I - A)^{-1}(\psi)(x) = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} e^{-\sqrt{\lambda}|s|} \psi(s+x) ds.$$

Moreover,

$$\left\| (\lambda I - A)^{-1} \psi \right\| \leq \frac{1}{\lambda} \|\psi\|, \forall \lambda > 0.$$

It follows that A is the infinitesimal generator of a C_0-semigroup $\{T_A(t)\}_{t \geq 0}$. Furthermore, by using Fourier transform, one can prove that A is the infinitesimal generator of

$$T(t)(\varphi)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \varphi(y) dy.$$

Example 2.6.8. As an exercise one can consider the same problem in $L^p(\mathbb{R}, \mathbb{R})$.

2.7 Remarks and Notes

Linear semigroup theory started with Hille-Yosida generation theorem in 1948 (see Hille [186] and Yosida [380]). Since then, there have been many monographs presenting various aspects of this theory, we refer to Hille and Phillips [187], Davies [87], Yosida [381], Pazy [281], Goldstein [150], Engel and Nagel [126, 128], and Arendt [22] for more results on semigroup theory. In this chapter we discussed the notions of strongly continuous semigroups, resolvents, and pseudo-resolvents, as well as the Hille-Yosida theorem. All these results of this chapter are well-known. Section 2.3 perhaps is the most original part with respect to the literature in which we discussed various equivalent relationships between mild solutions and the infinitesimal generator. The main idea of the part is to prepare for the chapter devoted to integrated semigroup theory where similar ideas will be used.

There are several important concepts and results in semigroup theory that we would like to briefly mention here.

(a) Lumer-Phillips Theorem. One can find a complete description of the Lumer-Phillips theorem for example in the book of Pazy [281].

Definition 2.7.1. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Then A is said to be *dissipative* if

$$\|(\lambda I - A)x\| \geq \lambda \|x\|, \ \forall x \in D(A), \forall \lambda > 0.$$

Definition 2.7.2. A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is a *semigroup of contractions* if and only if

$$\|T(t)\|_{\mathscr{L}(X)} \leq 1, \ \forall t \geq 0.$$

Theorem 2.7.3 (Lumer-Phillips [239]). *Let $A : D(A) \subset X \to X$ be a linear operator on Banach space X. Then A generates a semigroup of contractions if and only if*

- *$D(A)$ is dense in X;*
- *A is closed;*
- *A is dissipative;*
- *$\lambda I - A$ is surjective for some $\lambda > 0$.*

(b) Sectorial Linear Operators. For parabolic equations, it is not convenient to use the Hille-Yosida condition, but it is possible to use the notion of sectorial operators instead. We refer for instance to Friedman [146], Tanabe [325], Henry [183], Pazy [281], Temam [327], Lunardi [240], Cholewa and Dlotko [61], Engel and Nagel [126] for more results on the subject.

Definition 2.7.4. Let $L : D(L) \subset X \to X$ a linear operator on a Banach space X. L is said to be a *sectorial* operator if there are constants $\widehat{\omega} \in \mathbb{R}$, $\theta \in \left]\pi/2, \pi\right[$, and $\widehat{M} > 0$ such that

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$,

(ii) $\left\|(\lambda I - L)^{-1}\right\| \leq \dfrac{\widehat{M}}{|\lambda - \widehat{\omega}|}, \forall \lambda \in S_{\theta,\widehat{\omega}}$.

Definition 2.7.5. A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is said to be an *analytic semigroup* if the function $t \to T(t)$ is analytic in $\left(0, +\infty\right[$ with values in $\mathscr{L}(X)$ (i.e. $T(t) = \sum_{n=0}^{+\infty} (t - t_0)^n L_n$ for $|t - t_0|$ small enough).

Theorem 2.7.6 (Sectorial Linear Operator Theorem). *Assume that $L : D(L) \subset X \to X$ is a linear operator on a Banach space X and is sectorial. Then L is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X. Moreover,*

$$T_L(t) = \frac{1}{2\pi i} \int_{\widehat{\omega} + \gamma_{r,\eta}} (\lambda I - L)^{-1} e^{\lambda t} d\lambda, \ t > 0, \text{ and } T_L(0)x = x, \ \forall x \in X,$$

where $r > 0, \eta \in (\pi/2, \theta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta, |\lambda| = r\}$ oriented counterclockwise.

One may realize that both Lumer-Phillips Theorem and the above Sectorial Linear Operator Theorem provide an alternative method to prove the existence of a strongly continuous semigroup of bounded linear operators without using the Hille-Yosida condition.

(c) Linear Perturbations. The following theorem is proved for example in Pazy [281].

Theorem 2.7.7 (Perturbation Theorem). *Let $A : D(A) \subset X \to X$ be a Hille-Yosida operator on a Banach space X. Let $B \in \mathscr{L}(X)$ be a bounded linear operator on X. Then $A+B : D(A) \subset X \to X$ is a Hille-Yosida operator. Let $\{T_A(t)\}_{t\geq 0}$ (respectively $\{T_{A+B}(t)\}_{t\geq 0}$) be the semigroup generated by A (respectively generated by $A+B$), then for each $x \in X$, the map $t \to T_{A+B}(t)x$ is the unique continuous function satisfying (the fixed point problem)*

$$T_{A+B}(t)x = T_A(t)x + \int_0^t T_A(t-s)BT_{A+B}(s)xds, \ \forall t \geq 0.$$

Theorem 3.5.1 in Chapter 3 is a generalization of this perturbation theorem in the non-densely defined case. We refer to Desch-Schappacher [95] for a perturbation theorem with B unbounded. When A is sectorial, we refer to Pazy [281] for a perturbation theorem whenever B is unbounded and composed with some fractional power of the resolvent of A being bounded (we will present the notion of fractional power in Chapter 9).

Chapter 3
Integrated Semigroups and Cauchy Problems with Non-dense Domain

The goal of this chapter is to introduce the integrated semigroup theory and use it to investigate the existence and uniqueness of integrated (mild) solutions of the nonhomogeneous Cauchy problems when the domain of the linear operator A is not dense in the state space and A is not a Hille-Yosida operator.

3.1 Preliminaries

Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Consider the nonhomogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = x \in \overline{D(A)}, \tag{3.1.1}$$

where $f \in L^1_{\text{Loc}}((0, \tau), X)$ for some $\tau > 0$. Recall that $u \in C([0, \tau], X)$ is an integrated solution (or a mild solution) of (3.1.1) if u satisfies

$$\int_0^t u(s)ds \in D(A), \ \ \forall t \in [0, \tau],$$

and

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \ \forall t \in [0, \tau].$$

From the results in Sections 2.4 and 2.5 we know that when the domain of A is dense in X and A is a Hille-Yosida operator, the integrated solution of (3.1.1) is given by

$$u(t) = T_A(t)x + \int_0^t T_A(t-s)f(s)ds,$$

where $\{T_A(t)\}_{t \geq 0}$ is the linear C_0-semigroup generated by A.

When $\overline{D(A)} \neq X$, in order to define such an integrated solution we start by considering the special case

$$\frac{du}{dt} = Au(t) + x \text{ for } t \geq 0 \text{ and } u(0) = 0, \tag{3.1.2}$$

where $x \in X$. Define

$$X_0 := \overline{D(A)}.$$

Assume that A_0, the part of A in X_0, is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0 and the resolvent set $\rho(A)$ of A is nonempty. We will see that the unique integrated solution of (3.1.2) is given by

$$u(t) = S_A(t)x,$$

where $\{S_A(t)\}_{t\geq 0}$ is a strongly continuous family of bounded linear operators on X, which is the *integrated semigroup* generated by A, and

$$S_A(t)x := (\lambda I - A_0)\int_0^t T_{A_0}(s)ds(\lambda I - A)^{-1}x,$$

in which $\lambda \in \rho(A)$.

We will study the relationship between $\{S_A(t)\}_{t\geq 0}$, A and $\rho(A)$ as well as the relationship between $\{S_A(t)\}_{t\geq 0}$ and the integrated solution of (3.1.2); that is,

$$S_A(t)x = A\int_0^t S_A(l)xdl + tx,\ \forall t \geq 0,\ \forall x \in X,$$

with

$$\int_0^t S_A(l)xdl \in D(A).$$

We will see that the integrated solution of (3.1.1) (when it exists) is given by

$$u(t) = T_{A_0}(t)x + \frac{d}{dt}\int_0^t S_A(t-s)f(s)ds$$

whenever the map $t \to (S_A * f)(t) := \int_0^t S_A(t-s)f(s)ds$ is continuously differentiable. Then we will study the properties of $A+B$ when $B : \overline{D(A)} \to X$ is a bounded linear operator from $\overline{D(A)}$ into X. Finally, we will derive some estimates on $\frac{d}{dt}\int_0^t S_A(t-s)f(s)ds$ based on some growth rate estimation on the semigroup $\{T_{A_0}(t)\}_{t\geq 0}$.

In Section 3.8 we will consider the following example.

Example 3.1.1 (Abstract Age-structured Model in L^p). Let $p, q \in [1, +\infty)$. Consider the PDE associated to this problem

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -\mu(a)v(t,a) \text{ for } t \geq 0 \text{ and } a \geq 0, \\ v(t,0) = h(t), \\ v(0,.) = \varphi \in L^p((0,+\infty);\mathbb{R}), \end{cases} \tag{3.1.3}$$

where

$$h \in L^q((0,\tau);\mathbb{R}).$$

Assume that

$$\mu \in L^\infty_+(0,+\infty).$$

Usually, when the solution exists, it is given by

$$v(t,a) = \begin{cases} \exp\left(-\int_{a-t}^{a} \mu(s)ds\right)\varphi(a-t) & \text{if } a-t \geq 0 \\ \exp\left(-\int_{0}^{a} \mu(s)ds\right)h(t-a) & \text{if } a-t \leq 0. \end{cases}$$

This suggests that in order to obtain the existence of solutions in L^p (in some sense which still need to be specified) we will need to assume that $q \geq p$.

Consider the Banach space

$$X := \mathbb{R} \times L^p\left((0,+\infty);\mathbb{R}\right)$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^p}$$

and the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with domain

$$D(A) = \{0_{\mathbb{R}}\} \times W^{1,p}\left((0,+\infty);\mathbb{R}\right).$$

The domain of A is not dense in X and one can show that A is a Hille-Yosida operator if and only if $p = 1$. By identifying

$$u(t) := \begin{pmatrix} 0_{\mathbb{R}} \\ v(t,\cdot) \end{pmatrix},$$

the PDE (3.1.3) can be rewritten as an abstract nonhomogeneous Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi \end{pmatrix} \in \overline{D(A)},$$

where

$$f(t) := \begin{pmatrix} h(t) \\ 0_{L_p} \end{pmatrix}.$$

3.2 Integrated Semigroups

In this section we first give the definition of an integrated semigroup.

Definition 3.2.1. Let $\{S(t)\}_{t\geq 0}$ be a family of bounded linear operators on a Banach space $(X, \|.\|)$. We say that $\{S(t)\}_{t\geq 0}$ is an *integrated semigroup* on X if the following properties are satisfied:

(i) $S(0) = 0$;
(ii) $t \to S(t)x$ is continuous from $[0, +\infty)$ into X for each $x \in X$;
(iii) $S(t)$ satisfies

$$S(t)S(s) = \int_0^t [S(r+s) - S(r)]dr, \; \forall t, s \in [0, +\infty). \tag{3.2.1}$$

Remark 3.2.2. (i) Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $(X, \|.\|)$. Define

$$S(t) = \int_0^t T(s)ds.$$

Then $\{S(t)\}_{t\geq 0}$ is an integrated semigroup on $X \times X$. In other words, the integration of a strongly continuous semigroup is an integrated semigroup.

(ii) We have

$$\int_0^{t+s} S(r)dr - \int_0^t S(r)dr - \int_0^s S(r)dr = \int_0^t [S(r+s) - S(r)]dr.$$

Therefore, by using (3.2.1) it follows that

$$S(t)S(s) = S(s)S(t), \; \forall t, s \in [0, +\infty). \tag{3.2.2}$$

Set

$$X_k = \left\{ x \in X : S(.)x \in C^k\left([0, +\infty), X\right) \right\}, \forall k \geq 1.$$

It is clear that X_1 is a subspace of X, and we have the following properties.

Lemma 3.2.3. *Let $\{S(t)\}_{t\geq 0}$ be an integrated semigroup on a Banach space $(X, \|.\|)$. The following properties are satisfied*

(i) $S(t)X \subset X_1, \forall t \geq 0$;
(ii) $S'(t)S(s)x = S(t+s)x - S(t)x, \forall t, s \geq 0, \forall x \in X$;
(iii) $S'(t)S(s)x = S(s)S'(t)x, \forall t, s \geq 0, \forall x \in X_1$;
(iv) $S'(t)X_1 \subset X_1, \forall t \geq 0$;
(v) $S'(s)S'(t)x = S'(t+s)x, \forall t, s \geq 0, \forall x \in X_1$.

Proof. (i) and (ii) are immediate consequences of (3.2.1). (iii) follows from (3.2.2), the property (i) and the fact that $S(s)$ is bounded. To prove (iv) and (v), it is sufficient to note that

$$S(s)S'(t)x = S(t+s)x - S(t)x, \forall x \in X_1, \forall t, s \geq 0.$$

Since the second member of the above equation is differentiable with respect to s, we obtain (iv) and (v). $\square$

Definition 3.2.4. An integrated semigroup $\{S(t)\}_{t\geq 0}$ is *non-degenerate* if

$$S(t)x = 0, \forall t \geq 0 \Rightarrow x = 0.$$

Lemma 3.2.5. *Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$. Then $S'(0) = I_{X_1}$*

Proof. We have

$$S(s)S'(t)x = S(t+s)x - S(t)x, \forall t, s \geq 0, \forall x \in X_1.$$

So for $t = 0$, we obtain

$$S(s)\left[S'(0)x - x\right] = 0, \forall s \geq 0, \forall x \in X_1.$$

Since $\{S(t)\}_{t\geq 0}$ is non-degenerate, it follows that $S'(0) = I_{X_1}$. $\square$

Definition 3.2.6. Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$. A linear operator $A : D(A) \subset X \to X$ is said to be the *generator* of $\{S(t)\}_{t\geq 0}$ if and only if

$$x \in D(A) \text{ and } y = Ax \Leftrightarrow S(t)x = tx + \int_0^t S(s)y ds, \forall t \geq 0,$$

or equivalently

$$\text{Graph}(A) = \left\{(x,y) \in X \times X : S(t)x = tx + \int_0^t S(s)y ds, \forall t \geq 0\right\}. \tag{3.2.3}$$

It can be readily checked that

$$G = \left\{(x,y) \in X \times X : S(t)x = tx + \int_0^t S(s)y ds, \forall t \geq 0\right\}$$

is a closed subspace of $X \times X$. Moreover, if $\{S(t)\}_{t\geq 0}$ is non-degenerate, then G is the graph of a linear operator A. Indeed, we define

$$D(A) = \{x \in X : (x,y) \in G \text{ for some } y \in X\}.$$

Assume that $x \in D(A)$ and there exist $y \in X$ and $z \in X$ such that

$$(x,y) \in G \text{ and } (x,z) \in G.$$

Then

$$S(t)x = x + \int_0^t S(s)y ds = x + \int_0^t S(s)z ds, \ \forall t \geq 0,$$

so

$$S(t)(y-z) = 0, \forall t \geq 0.$$

Since $\{S(t)\}_{t\geq 0}$ is non-degenerate, we deduce that

$$y = z.$$

Thus, G is the graph of a map A from $D(A)$ into X. Moreover, since G is a closed linear subspace of $X \times X$, it follows that $A : D(A) \subset X \to X$ is a closed linear operator on X.

From the above observation we have the following lemma.

Lemma 3.2.7. *The generator of a non-degenerate integrated semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space $(X, \|.\|)$ is uniquely determined.*

We also remark that

$$\text{Graph}(A) = \left\{(x,y) \in X_1 \times X : S'(t)x = x + S(t)y, \forall t \geq 0\right\}. \tag{3.2.4}$$

Lemma 3.2.8. *Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$ and let A be its generator. Then A is a closed linear operator.*

Proof. Assume that $(x_n, y_n) \in \text{Graph}(A) \longrightarrow (x,y)$. Then for each $n \in \mathbb{N}$ and each $t \geq 0$, we have

$$S(t)x_n = tx_n + \int_0^t S(s)y_n ds.$$

Taking the limit when n goes to $+\infty$ (for t fixed), we obtain that

$$S(t)x = tx + \int_0^t S(s)yds, \ \ \forall t \geq 0.$$

Therefore, $(x,y) \in \text{Graph}(A)$. $\square$

Lemma 3.2.9. *Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$ and let A be its generator. For each $x \in X$ and each $t \geq 0$, we have*

$$\int_0^t S(s)xds \in D(A) \text{ and } S(t)x = A\int_0^t S(s)xds + tx.$$

Proof. By using (3.2.1), we obtain

$$S(t)\int_0^s S(\sigma)xd\sigma = \int_0^s S(t)S(\sigma)xd\sigma = \int_0^s \int_0^t [S(r+\sigma) - S(r)]\, xdrd\sigma$$

and by Fubini's theorem we obtain

$$S(t)\int_0^s S(\sigma)xd\sigma = \int_0^t \int_0^s [S(r+\sigma) - S(r)]\, xd\sigma dr.$$

Hence, $\int_0^s S(\sigma)xd\sigma \in X_1$, and

$$\begin{aligned}
\frac{dS(t)}{dt}\int_0^s S(\sigma)xd\sigma &= \int_0^s [S(t+\sigma)-S(t)]\,xd\sigma \\
&= \int_0^s [S(t+\sigma)-S(\sigma)]\,xd\sigma + \int_0^s S(\sigma)xd\sigma - sS(t)x \\
&= S(t)(S(s)x-sx) + \int_0^s S(\sigma)xd\sigma.
\end{aligned}$$

Setting $\widehat{x} = \int_0^s S(\sigma)xd\sigma$ and $\widehat{y} = S(s)x - sx$, we obtain

$$S'(t)\widehat{x} - \widehat{x} = S(t)\widehat{y}, \;\; \forall t \geq 0,$$

and integrating the last equation yields that

$$S(t)\widehat{x} - t\widehat{x} = \int_0^t S(\sigma)\widehat{y}d\sigma, \;\; \forall t \geq 0.$$

Thus,

$$\widehat{x} = \int_0^t S(l)xdl \in D(A) \text{ and } \widehat{y} = A\widehat{x}.$$

It remains to observe that

$$\widehat{y} = A\widehat{x} \Leftrightarrow S(s)x - sx = A\int_0^s S(\sigma)xd\sigma.$$

This completes the proof. □

Lemma 3.2.10. *Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$ and let A be its generator. We have the following properties*

(i) $D(A) \subset X_1$;
(ii) $S(t)X_1 \subset D(A), \forall t \geq 0$, *and* $S'(t)x - x = AS(t)x, \forall t \geq 0, \forall x \in X_1$;
(iii) $AS(t)x = S(t)Ax, \forall t \geq 0, \forall x \in D(A)$.

Proof. (i) Assume that $x \in D(A)$. By the definition of $D(A)$, we can find $y \in X$ such that

$$S(t)x - tx = \int_0^t S(l)ydl, \;\; \forall t \geq 0.$$

Now by taking the derivative of the last expression and by using the fact that $y = Ax$, we have

$$S'(t)x = x + S(t)Ax, \forall t \geq 0, \;\; \forall x \in D(A). \tag{3.2.5}$$

Let $x \in X_1$ be fixed. Then from Lemma 3.2.9 we have

$$S(t)x - tx = A\int_0^t S(l)xdl, \forall t \geq 0,$$

so

$$\frac{[S(t+h)-S(t)]\,x}{h} - x = A\frac{1}{h}\int_t^{t+h} S(l)xdl, \;\; \forall t \geq 0, \forall h > 0.$$

Since A is closed, we deduce when $h \searrow 0$ that

$$S(t)x \in D(A) \text{ and } S'(t)x - x = AS(t)x, \ \forall t \geq 0. \tag{3.2.6}$$

(ii) and (iii) follow from (3.2.5) and (3.2.6). □

The following lemma provides the uniqueness of mild solutions whenever A generates an integrated semigroup.

Theorem 3.2.11 (Uniqueness). *Let $\{S(t)\}_{t\geq 0}$ be a non-degenerate integrated semigroup on a Banach space $(X, \|.\|)$ and let A be its generator. Let $\tau \in (0, +\infty]$. Assume that $u : [0, \tau) \to X$ is continuous such that*

$$\int_0^t u(\sigma)d\sigma \in D(A), \forall t \in [0, \tau),$$

and

$$u(t) = A\int_0^t u(\sigma)d\sigma, \forall t \in [0, \tau).$$

Then $u(t) = 0, \forall t \in [0, \tau)$.

Proof. Since $D(A) \subset X_1$, we have $\int_0^s u(\sigma)d\sigma \in X_1, \forall s \in [0, \tau)$. By Lemma 3.2.10 we have

$$\frac{d}{ds}\left(S(t-s)\int_0^s u(\sigma)d\sigma\right) = -S'(t-s)\int_0^s u(\sigma)d\sigma + S(t-s)u(s)$$

and by using (3.2.5) we obtain

$$\frac{d}{ds}\left(S(t-s)\int_0^s u(\sigma)d\sigma\right) = -\int_0^s u(\sigma)d\sigma - S(t-s)A\int_0^s u(\sigma)d\sigma + S(t-s)u(s).$$

By using the fact that u is a mild solution we deduce that

$$\frac{d}{ds}\left(S(t-s)\int_0^s u(\sigma)d\sigma\right) = -\int_0^s u(\sigma)d\sigma.$$

Integrating the map $s \to \frac{d}{ds}\left(S(t-s)\int_0^s u(\sigma)d\sigma\right)$ from 0 to t and noting that $S(0) = 0$, we obtain

$$0 = -\int_0^t \int_0^s u(\sigma)d\sigma ds, \ \forall t \in [0, \tau).$$

Differentiating twice we obtain $u = 0$. □

As an immediate consequence of Lemma 3.2.9 and Theorem 3.2.11 we obtain the following theorem.

Theorem 3.2.12. *A non-degenerate integrated semigroup is uniquely determined by its generator.*

Proof. Assume that A generates two non-degenerates integrated semigroups, $\{S(t)\}_{t\geq 0}$ and $\left\{\widehat{S}(t)\right\}_{t\geq 0}$. Then by Lemma 3.2.9 we have

$$\int_0^t \left[S(s)-\widehat{S}(s)\right] xds \in D(A) \text{ and } \left[S(t)-\widehat{S}(t)\right] x = A\int_0^t \left[S(s)-\widehat{S}(s)\right] xds.$$

Applying Theorem 3.2.11, we deduce that

$$S(t)=\widehat{S}(t), \forall t \geq 0.$$

This proves the theorem. □

Another consequence of Theorem 3.2.11 is the uniqueness of mild (or integrated) solutions of the Cauchy problem (3.1.1).

Theorem 3.2.13. *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Assume that A generates a non-degenerate integrated semigroup $\{S_A(t)\}_{t\geq 0}$. Then the Cauchy problem (3.1.1) has at most one integrated solution; that is, there exists at most one continuous function $u : [0,\tau) \to X$ such that*

$$\int_0^t u(s)ds \in D(A), \ \forall t \in [0,\tau),$$

and

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau).$$

Theorem 3.2.14. *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous family of bounded linear operators on a Banach space X and let $A : D(A) \subset X \to X$ be a closed linear operator. Then $\{S(t)\}_{t\geq 0}$ is a non-degenerate integrated semigroup and A is its generator if and only if the following two conditions are satisfied:*

(i) For all $x \in D(A)$ and $t > 0$, $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;
(ii) For all $x \in X$ and $t > 0$, $\int_0^t S(s)xds \in D(A)$ and $A\int_0^t S(s)xds = S(t)x - tx$.

Proof. The "only if" part follows from Lemma 3.2.9 and Lemma 3.2.10. Assume now that conditions (i) and (ii) are satisfied. Then $\{S(t)\}_{t\geq 0}$ is a non-degenerate family. Indeed, assume that

$$S(t)x = 0, \forall t \geq 0.$$

Then from (ii) we have

$$0 = A0 = -tx, \forall t \geq 0.$$

So

$$x = 0.$$

To prove that $\{S(t)\}_{t\geq 0}$ is an integrated semigroup, we note that (ii) implies

$$S(0) = 0. \tag{3.2.7}$$

Moreover, by combining (i) and (ii) and the closeness of A, we deduce that for all $x \in D(A)$ the map $t \to S(t)x$ is continuously differentiable, and

$$S'(t)x = x + AS(t)x = x + S(t)Ax, \forall t \geq 0.$$

By using this fact and the same argument as in the proof of Theorem 3.2.11, we deduce that for each $\tau \in (0,+\infty]$, if $u \in C([0,\tau),X)$ and satisfies

$$\int_0^t u(l)dl \in D(A) \text{ and } u(t) = A\int_0^t u(l)dl, \forall t \in [0,\tau).$$

Then $u = 0$.

It remains to prove that

$$v_1(t) := S(t)S(r)x \text{ and } v_2(t) := \int_0^t [S(r+l) - S(l)]\,xdl$$

are equal. To prove this we will show that

$$v_j(t) = A\int_0^t v_j(l)dl + tS(r)x \text{ for } t \geq 0 \text{ and } j = 1,2. \tag{3.2.8}$$

Then $u(t) = v_1(t) - v_2(t)$ satisfies

$$\int_0^t u(l)dl \in D(A), \text{ and } u(t) = A\int_0^t u(l)dl, \ \forall t \in [0,+\infty)$$

which implies that $u = 0$.

To prove (3.2.8), we have

$$\int_0^t v_1(l)dl = \int_0^t S(l)S(r)xdl \in D(A)$$

and

$$\begin{aligned} A\int_0^t v_1(l)dl &= A\int_0^t S(l)S(r)xdl \\ &= S(t)S(r)x - tS(r)x = v_1(t) - tS(r)x. \end{aligned}$$

We also have

$$v_2(t) = \int_0^{t+r} S(l)xdl - \int_0^t S(l)xdl - \int_0^r S(l)xdl \in D(A),$$

$$Av_2(t) = S(t+r)x - S(t)x - S(r)x,$$

and, since A is closed,

$$\begin{aligned} A\int_0^t v_2(l)dl &= \int_0^t S(l+r)x - S(l)xdl - tS(r)x \\ &= v_2(t) - tS(r)x. \end{aligned}$$

It follows that $\{S(t)\}_{t\geq 0}$ is a non-degenerate integrated semigroup.

It remains to prove that A is the generator of $\{S(t)\}_{t\geq 0}$. Let $B : D(B) \subset X \to X$ be the generator of $\{S(t)\}_{t\geq 0}$. We have

$$\text{Graph}(B) = \left\{ (x,y) \in X \times X : S(t)x = tx + \int_0^t S(s)yds, \forall t \geq 0 \right\}.$$

By using (i) and (ii), and the fact that A is closed, we have

$$S(t)x = tx + \int_0^t S(s)Axds, \forall t \geq 0, \forall x \in D(A).$$

It follows that $\text{Graph}(A) \subset \text{Graph}(B)$.

Conversely, let $x \in D(B)$. We have

$$A\int_0^t S(l)xdl = S(t)x - tx = \int_0^t S(s)Bxds,$$

so

$$A\frac{1}{h}\int_t^{t+h} S(l)xdl = \frac{1}{h}\int_t^{t+h} S(l)Bxdl.$$

Now since A is closed, we deduce when $h \searrow 0$ that

$$S(t)x \in D(A) \text{ and } AS(t)x = S(t)Bx. \tag{3.2.9}$$

Hence

$$tx = S(t)x - \int_0^t S(s)Bxds \in D(A)$$

and

$$\begin{aligned} tAx &= AS(t)x - A\int_0^t S(s)Bxds \\ &= AS(t)x - S(t)Bx + tBx. \end{aligned}$$

By using (3.2.9) we obtain

$$Ax = Bx.$$

So $\text{Graph}(B) \subset \text{Graph}(A)$ and the proof is complete. □

3.3 Exponentially Bounded Integrated Semigroups

Definition 3.3.1. An integrated semigroup $\{S(t)\}_{t\geq 0}$ is *exponentially bounded* if and only if there exist two constants, $M > 0$ and $\omega > 0$, such that

$$\|S(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \ \forall t \geq 0.$$

Proposition 3.3.2. *Let A be the generator of an exponentially bounded non-degenerate integrated semigroup $\{S(t)\}_{t\geq 0}$. Then for $\lambda > \omega$, $\lambda I - A$ is invertible and*

$$(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda t} S(t) x dt, \forall x \in X.$$

Proof. Set

$$R_\lambda x = \lambda \int_0^{+\infty} e^{-\lambda t} S(t) x dt, \forall x \in X.$$

We first show that $R_\lambda X \subset D(A)$ and $(\lambda I - A) R_\lambda = I$. By integrating by parts we have

$$R_\lambda x = \lambda^2 \int_0^{+\infty} e^{-\lambda t} \int_0^t S(l) x dl dt, \forall x \in X.$$

Observe that

$$A \int_0^t S(l) x dl = S(t)x - tx, \forall x \in X,$$

hence the map $t \to \int_0^t S(l) x dl$ belongs to $C([0,+\infty), D(A))$. Since A is closed, it follows that

$$R_\lambda x \in D(A), \forall x \in X,$$

and

$$\begin{aligned} AR_\lambda x &= \lambda^2 \int_0^{+\infty} e^{-\lambda t} A \int_0^t S(l) x dl dt \\ &= \lambda^2 \int_0^{+\infty} e^{-\lambda t} S(t) x dt - \lambda^2 \int_0^{+\infty} t e^{-\lambda t} x dt \\ &= \lambda R_\lambda x - x, \end{aligned}$$

so

$$(\lambda I - A) R_\lambda x = x, \forall x \in X.$$

Now let $x \in D(A)$. As $S(t)$ commutes with A by Lemma 3.2.10, we have

$$R_\lambda A x = \lambda \int_0^{+\infty} e^{-\lambda t} S(t) A x dt = \lambda \int_0^{+\infty} e^{-\lambda t} A S(t) x dt.$$

Since A is closed, we deduce that

$$R_\lambda A x = A R_\lambda x.$$

Hence

$$R_\lambda (\lambda I - A) x = x, \forall x \in D(A),$$

and the result follows. □

Recall that the dual space X^* of X consists of the bounded linear forms $x^* : X \to \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$)

$$x^*(x) = \langle x^*, x \rangle \text{ at } x \in X,$$

where $\langle .,. \rangle$ is the scalar product for the duality X^*, X. We introduce the following definition.

Definition 3.3.3. Let $A : D(A) \subset X \to Y$. Define

$$G^*(A) := \{(y^*, x^*) \in Y^* \times X^* : \langle y^*, Ax \rangle = \langle x^*, x \rangle, \forall x \in D(A)\}.$$

Lemma 3.3.4. *Let $A : D(A) \subset X \to Y$ be a closed linear operator. Then*

$$(x,y) \in \text{Graph}(A) \Leftrightarrow \langle y^*, y \rangle = \langle x^*, x \rangle, \forall (y^*, x^*) \in G^*(A).$$

Proof. The implication $(\Rightarrow)$ is an immediate consequence of the definition of $G^*(A)$. To prove $(\Leftarrow)$, assume that

$$(\widehat{x}, \widehat{y}) \notin \text{Graph}(A).$$

Since Graph(A) is a closed subspace by the Hahn-Banach theorem, we can find a bounded linear functional f on $X \times Y$ such that

$$f(\widehat{x}, \widehat{y}) \neq 0 \text{ and } f(x,y) = 0, \forall (x,y) \in \text{Graph}(A).$$

Setting $x^*(x) = f(x,0)$ and $y^*(y) = -f(0,y)$, then we have

$$\langle y^*, \widehat{y} \rangle \neq \langle x^*, \widehat{x} \rangle, \text{ and } \langle y^*, y \rangle = \langle x^*, x \rangle, \forall (x,y) \in \text{Graph}(A).$$

So we obtain

$$\langle y^*, \widehat{y} \rangle \neq \langle x^*, \widehat{x} \rangle \text{ and } (y^*, x^*) \in G^*(A).$$

This completes the proof. $\square$

When we restrict ourselves to the class of non-degenerate exponentially bounded integrated semigroups, Thieme's notion of generator [329] is equivalent to the one introduced by Arendt [21]. More precisely, combining Theorem 3.1 in Arendt [21] and Proposition 3.10 in Thieme [329], one has the following result.

Theorem 3.3.5 (Arendt-Thieme). *Let $\{S(t)\}_{t\geq 0}$ be an exponentially bounded and strongly continuous family of bounded linear operators on a Banach space $(X, \|.\|)$. Let $A : D(A) \subset X \to X$ be a linear operator on X. Then $\{S(t)\}_{t\geq 0}$ is a non-degenerate integrated semigroup and A is its generator if and only if there exist two constants $\omega > 0$ and $M > 0$ such that*

$$(\omega, +\infty) \subset \rho(A),$$

$$\|S(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \ \forall t \geq 0,$$

and

$$(\lambda I - A)^{-1} x = \lambda \int_0^\infty e^{-\lambda s} S(s) x ds, \ \ \forall \lambda > \omega.$$

Proof. We apply Theorem 3.2.14. To verify assertion (i) of Theorem 3.2.14 it is sufficient to show that $\langle y^*, S(t)Ax \rangle = \langle x^*, S(t)x \rangle, \forall (y^*, x^*) \in G^*(A)$. Let $x \in D(A)$. We have for each $\lambda > \omega$ and each $(y^*, x^*) \in G^*(A)$ that

$$\int_0^{+\infty} e^{-\lambda t} \langle x^*, S(t)x \rangle dt = \left\langle x^*, \int_0^{+\infty} e^{-\lambda t} S(t) x dt \right\rangle = \left\langle x^*, \frac{1}{\lambda} (\lambda I - A)^{-1} x \right\rangle$$

$$= \left\langle y^*, A\frac{1}{\lambda}(\lambda I - A)^{-1}x \right\rangle = \left\langle y^*, \frac{1}{\lambda}(\lambda I - A)^{-1}Ax \right\rangle$$
$$= \int_0^{+\infty} e^{-\lambda t} \langle y^*, S(t)Ax \rangle \, dt.$$

By the uniqueness properties of the Laplace transform, we deduce that

$$\langle x^*, S(t)x \rangle = \langle y^*, S(t)Ax \rangle, \forall t \geq 0, \forall (y^*, x^*) \in G^*(A).$$

So by Lemma 3.3.4 we have that

$$S(t)x \in D(A) \text{and } AS(t)x = S(t)Ax, \forall t \geq 0.$$

We now prove assertion (ii) of Theorem 3.2.14. By using Lemma 3.3.4 it is sufficient to prove that

$$\left\langle x^*, \int_0^t S(l)x dl \right\rangle = \langle y^*, S(t)x - tx \rangle, \forall t \geq 0, \forall (y^*, x^*) \in G^*(A).$$

In fact,

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} \left\langle x^*, \int_0^t S(l)x dl \right\rangle dt &= \frac{1}{\lambda}\int_0^{+\infty} e^{-\lambda t} \langle x^*, S(t)x \rangle \, dt \\ &= \frac{1}{\lambda^2}\left\langle x^*, (\lambda I - A)^{-1}x \right\rangle = \frac{1}{\lambda^2}\left\langle y^*, A(\lambda I - A)^{-1}x \right\rangle \\ &= -\frac{1}{\lambda^2}\langle y^*, x \rangle + \frac{1}{\lambda}\left\langle y^*, (\lambda I - A)^{-1}x \right\rangle \\ &= \int_0^{+\infty} e^{-\lambda t} \langle y^*, -tx \rangle \, dt + \int_0^{+\infty} e^{-\lambda t} \langle y^*, S(t)x \rangle \, dt.\end{aligned}$$

Thus,

$$\int_0^{+\infty} e^{-\lambda t} \left\langle x^*, \int_0^t S(l)x dl \right\rangle dt = \int_0^{+\infty} e^{-\lambda t} \langle y^*, S(t)x - tx \rangle \, dt.$$

Once again by the uniqueness of the Laplace transform, we obtain

$$\left\langle x^*, \int_0^t S(l)x dl \right\rangle = \langle y^*, S(t)x - tx \rangle, \forall t \geq 0, \forall (y^*, x^*) \in G^*(A).$$

By Lemma 3.3.4 it follows that

$$\int_0^t S(l)x dl \in D(A) \text{ and } A\int_0^t S(l)x dl = S(t)x - tx, \forall t \geq 0.$$

This completes the proof. □

Corollary 3.3.6. *Let* $\{S_A(t)\}_{t\geq 0}$ *be an exponentially bounded non-degenerate integrated semigroup on a Banach space* X *with generator* $A : D(A) \subset X \to X$. *Then for each* $\mu \in \mathbb{R}$, $A + \mu I$ *generates an exponentially bounded non-degenerate integrated* *semigroup* $\{S_{A+\mu I}(t)\}_{t\geq 0}$ *and*

$$S_{A+\mu I}(t) = e^{\mu t} S_A(t) - \mu \int_0^t e^{\mu l} S_A(l) dl.$$

Proof. We have

$$(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t) dt, \ \forall \lambda > \widehat{\omega}.$$

So for each $\lambda > \widehat{\omega} + \mu$,

$$(\lambda I - (A + \mu I))^{-1} = ((\lambda - \mu) I - A)^{-1} = (\lambda - \mu) \int_0^{+\infty} e^{-(\lambda - \mu)t} S_A(t) dt,$$

$$(\lambda I - (A + \mu I))^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} e^{\mu t} S_A(t) dt - \mu \int_0^{+\infty} e^{-\lambda t} e^{\mu t} S_A(t) dt.$$

By integrating by parts the last integral, we obtain

$$(\lambda I - (A + \mu I))^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} \left[e^{\mu t} S_A(t) - \mu \int_0^t e^{\mu l} S_A(l) dl \right] dt.$$

The result follows from Theorem 3.3.5. □

3.4 Existence of Mild Solutions

Let $(X, \|.\|_X)$ be a Banach space. Let $A : D(A) \subset X \to X$ be a linear operator. In this section we study the Cauchy problem (3.1.1) with $f \in L^1((0,\tau), X)$. From here on, set

$$X_0 = \overline{D(A)}.$$

We denote by A_0 the part of A in X_0. Recall that $A_0 : D(A_0) \subset X_0 \to X_0$ is the linear operator on X_0 defined by

$$A_0 x = Ax, \ \forall x \in D(A_0) = \{y \in D(A) : Ay \in X_0\}.$$

Assume that $(\omega_A, +\infty) \subset \rho(A)$. Then from Lemma 2.2.9 we know that for each $\lambda > \omega$,

$$D(A_0) = (\lambda I - A)^{-1} X_0 \text{ and } (\lambda I - A_0)^{-1} = (\lambda I - A)^{-1} |_{X_0}.$$

Moreover, from Lemma 2.2.10, we know that

$$\rho(A) \neq \emptyset \Rightarrow \rho(A) = \rho(A_0).$$

Motivated by some examples (see the example considered in Section 3.1 when $p > 1$) we do not assume that A is a Hille-Yosida operator. Instead, we first make the following assumption.

Assumption 3.4.1. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X satisfying the following properties:

(a) There exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and $\forall \lambda > \omega_A,\ \forall n \geq 1$,

$$\left\|(\lambda I - A_0)^{-n}\right\|_{\mathscr{L}(\overline{D(A)})} = \left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}(\overline{D(A)})} \leq \frac{M_A}{(\lambda - \omega_A)^n};$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \forall x \in X.$

Note that Assumption 3.4.1-(b) is equivalent to

$$\overline{D(A_0)} = \overline{D(A)}.$$

So by using the Hille-Yosida theorem we obtain the following lemma.

Lemma 3.4.2. *Assumption 3.4.1 is satisfied if and only if $\rho(A) \neq \emptyset$, and A_0 is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{A_0}(t)\right\}_{t \geq 0}$ of bounded linear operators on X_0 with*

$$\left\|T_{A_0}(t)\right\|_{\mathscr{L}(X_0)} \leq M_A e^{\omega_A t},\ \ \forall t \geq 0.$$

Proposition 3.4.3. *Let Assumption 3.4.1 be satisfied. Then A generates a uniquely determined non-degenerate exponentially bounded integrated semigroup $\{S_A(t)\}_{t \geq 0}$. Moreover, for each $x \in X$, each $t \geq 0$, and each $\mu > \omega_A$, $S_A(t)$ is given by*

$$S_A(t) = (\mu I - A_0) \int_0^t T_{A_0}(s) ds \, (\mu I - A)^{-1}, \tag{3.4.1}$$

or equivalently

$$S_A(t)x = \mu \int_0^t T_{A_0}(s) \, (\mu I - A)^{-1} x ds + \left[I - T_{A_0}(t)\right] (\mu I - A)^{-1} x. \tag{3.4.2}$$

Furthermore, for each $\gamma > \max(0, \omega_A)$, there exists $M_\gamma > 0$ such that

$$\|S_A(t)\| \leq M_\gamma e^{\gamma t}, \forall t \geq 0. \tag{3.4.3}$$

Finally, the map $t \to S_A(t)x$ is continuously differentiable if and only if $x \in X_0$ and

$$\frac{dS_A(t)x}{dt} = T_{A_0}(t)x,\ \ \forall t \geq 0,\ \forall x \in X_0. \tag{3.4.4}$$

Proof. Since

$$A_0 \int_0^t T_{A_0}(s)ds = T_{A_0}(t) - I,$$

it is clear that (3.4.1) is equivalent to (3.4.2). Moreover, from (3.4.2) we deduce that for each $\gamma > \max(0, \omega_A)$, there exists $M_\gamma > 0$ such that

$$\|S_A(t)\| \leq M_\gamma e^{\gamma t}, \ \forall t \geq 0.$$

Furthermore, the map

$$t \to S_A(t)x = \mu \int_0^t T_{A_0}(s)(\mu I - A)^{-1} x ds + \left[I - T_{A_0}(t)\right](\mu I - A)^{-1} x$$

is differentiable if and only if the map

$$t \to T_{A_0}(t)(\mu I - A)^{-1} x$$

is differentiable. This is equivalent to say that

$$(\mu I - A)^{-1} x \in D(A_0) \Leftrightarrow x \in \overline{D(A)}.$$

In order to prove that $\{S_A(t)\}_{t\geq 0}$ defined by (3.4.1) is the integrated semigroup generated by A, we apply Theorem 3.3.5. Let $\lambda > \max(0, \omega_A)$ and let $\mu > \omega_A$. Set

$$S_\mu(t) := (\mu I - A_0) \int_0^t T_{A_0}(s)ds(\mu I - A)^{-1}.$$

Since $\mu I - A_0$ is closed, we have

$$\lambda \int_0^{+\infty} e^{-\lambda t} S_\mu(t)dt = (\mu I - A_0)\lambda \int_0^{+\infty} e^{-\lambda t} \int_0^t T_{A_0}(s)ds(\mu I - A)^{-1} dt.$$

By integrating by parts

$$\begin{aligned}\lambda \int_0^{+\infty} e^{-\lambda t} S_\mu(t)dt &= (\mu I - A_0)\int_0^{+\infty} e^{-\lambda t} T_{A_0}(t)(\mu I - A)^{-1} dt \\ &= (\mu I - A_0)(\lambda I - A_0)^{-1}(\mu I - A)^{-1} \\ &= (\mu I - A_0)(\mu I - A_0)^{-1}(\lambda I - A)^{-1},\end{aligned}$$

we have

$$(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S_\mu(t)dt, \forall \lambda > \max(0, \omega_A).$$

From Theorem 3.3.5 it follows that $\left\{S_\mu(t)\right\}_{t\geq 0}$ is a non-degenerate integrated semigroup and A is its generator. Moreover, by Theorem 3.2.12, since an integrated semigroup is uniquely determined by its generator, it follows that $S_\mu(t)$ is independent of μ. □

Now since

$$S_A(t)x \in X_0, \ \forall t \geq 0, \forall x \in X,$$

from (iii) in Definition 3.2.1 which is

$$S_A(t)S_A(s) = \int_0^t S_A(r+s) - S_A(r)dr, \ \forall t, s \in [0, +\infty),$$

we obtain that

$$T_{A_0}(t)S_A(s) = S_A(t+s) - S_A(t), \ \forall t, s \geq 0. \tag{3.4.5}$$

From (3.4.1) or (3.4.2), we know that $S_A(t)$ commutes with $(\lambda I - A)^{-1}$; that is,

$$S_A(t)(\lambda I - A)^{-1} = (\lambda I - A)^{-1} S_A(t), \forall t \geq 0, \forall \lambda > \omega_A, \tag{3.4.6}$$

and

$$S_A(t)x = \int_0^t T_{A_0}(l)xdl, \ \forall t \geq 0, \ \forall x \in X_0.$$

Hence, $\forall x \in X$, $\forall t \geq 0$, $\forall \mu \in (\omega_A, +\infty)$,

$$(\mu I - A)^{-1} S_A(t)x = S_A(t)(\mu I - A)^{-1} x = \int_0^t T_{A_0}(s)(\mu I - A)^{-1} xds.$$

We have the following result.

Lemma 3.4.4. *Let Assumption 3.4.1 be satisfied and let $\tau_0 > 0$ be fixed. For each $f \in C^1([0, \tau_0], X)$, set*

$$(S_A * f)(t) = \int_0^t S_A(s)f(t-s)ds, \ \forall t \in [0, \tau_0].$$

Then we have the following:

(i) The map $t \to (S_A * f)(t)$ is continuously differentiable on $[0, \tau_0]$, and

$$\frac{d}{dt}(S_A * f)(t) = S_A(t)f(0) + \int_0^t S_A(s)f'(t-s)ds;$$

(ii) $(S_A * f)(t) \in D(A)$, $\forall t \in [0, \tau_0]$;

(iii) If we set $u(t) = \frac{d}{dt}(S_A * f)(t)$, then

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0, \tau_0]; \tag{3.4.7}$$

(iv) For each $\lambda \in (\omega, +\infty)$ and each $t \in [0, \tau_0]$, we have

$$(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds. \tag{3.4.8}$$

Proof. Let $f \in C^1([0, \tau_0], X)$. Then

$$\frac{d}{dt}(S_A * f)(t) = \frac{d}{dt}\int_0^t S_A(s)f(t-s)ds = S_A(t)f(0) + \int_0^t S_A(s)f'(t-s)ds.$$

Indeed, by using Fubini's theorem we have

$$\begin{aligned}\int_0^t \int_0^s S_A(r) f'(s-r) dr ds &= \int_0^t \int_r^t S_A(r) f'(s-r) ds dr \\ &= \int_0^t \int_0^{t-r} S_A(r) f'(l) dl dr \\ &= \int_0^t S_A(r) \int_0^{t-r} f'(l) dr dl \\ &= \int_0^t S_A(r)(f(t-r) - f(0)) dr dl.\end{aligned}$$

So

$$(S_A * f)(t) = \int_0^t S_A(s) f(0) ds + \int_0^t \int_0^{t-l} S_A(r) f'(l) dr dl.$$

By using the above formula and Lemma 3.2.9, we have that

$$\int_0^t S_A(s) x ds \in D(A) \text{ and } S_A(t)x = A \int_0^t S_A(s) x ds + tx.$$

By using the fact that A is closed, we deduce that

$$\int_0^t S_A(t-s) f(s) ds \in D(A), \;\; \forall t \in [0, \tau_0], \; \forall x \in X$$

and

$$\begin{aligned} A \int_0^t S_A(t-s) f(s) ds &= A \int_0^t S_A(s) f(0) ds + A \int_0^t \int_0^{t-l} S_A(r) f'(l) dr dl \\ &= A \int_0^t S_A(s) f(0) ds + \int_0^t A \int_0^{t-l} S_A(r) f'(l) dr dl \\ &= S_A(t) f(0) - t f(0) + \int_0^t [S_A(t-l) f'(l) - (t-l) f'(l)] dl \\ &= \frac{d}{dt} \int_0^t S_A(t-l) f(l) dl - t f(0) - \int_0^t (t-l) f'(l) dl \\ &= \frac{d}{dt} \int_0^t S_A(t-l) f(l) dl - \int_0^t f(l) dl. \end{aligned}$$

Therefore,

$$\frac{d}{dt} \int_0^t S_A(t-l) f(l) dl = A \int_0^t S_A(t-l) f(l) dl + \int_0^t f(l) dl, \;\; \forall t \in [0, T].$$

Moreover, we have for $\lambda \in (\omega, +\infty)$ that

$$(\lambda I - A)^{-1} (S_A * f)(t) = \int_0^t S_A(t-s) (\lambda I - A)^{-1} f(s) ds$$

$$= \int_0^t \int_0^{t-s} T_{A_0}(l)\,(\lambda I - A)^{-1} f(s)dlds,$$

which implies that

$$\frac{d}{dt}(\lambda I - A)^{-1}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)\,(\lambda I - A)^{-1} f(s)ds.$$

This completes the proof. □

The following lemma can be used to obtain explicit solutions.

Lemma 3.4.5. *Let Assumption 3.4.1 be satisfied. Let* $v \in C([0,\tau_0], X_0)$, $f \in L^1([0,\tau_0], X)$, *and* $\lambda \in (\omega_A, +\infty)$. *Assume that*

(i) $(\lambda I - A)^{-1} v \in W^{1,1}([0,\tau_0], X)$ *and for almost every* $t \in [0,\tau_0]$,

$$\frac{d}{dt}(\lambda I - A)^{-1} v(t) = \lambda (\lambda I - A)^{-1} v(t) - v(t) + (\lambda I - A)^{-1} f(t);$$

(ii) $t \to (S_A * f)(t)$ *is continuously differentiable on* $[0,\tau_0]$.

Then v is an integrated solution of (3.1.1) and

$$v(t) = T_{A_0}(t)v(0) + \frac{d}{dt}(S_A * f)(t),\ \forall t \in [0,\tau_0].$$

Proof. We have for almost every $t \in [0,\tau_0]$ that

$$\begin{aligned}
&\frac{d}{dt}(\lambda I - A)^{-1} v(t)\\
&\quad = \lambda (\lambda I - A)^{-1} v(t) - (\lambda I - A)(\lambda I - A)^{-1} v(t) + (\lambda I - A)^{-1} f(t)\\
&\quad = A_0 (\lambda I - A)^{-1} v(t) + (\lambda I - A)^{-1} f(t).
\end{aligned}$$

So

$$(\lambda I - A)^{-1} v(t) = T_{A_0}(t)(\lambda I - A)^{-1} v(0) + \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds.$$

By (ii), we have

$$\begin{aligned}
(\lambda I - A)^{-1}\frac{d}{dt}(S_A * f)(t) &= \frac{d}{dt}(\lambda I - A)^{-1}(S_A * f)(t)\\
&= \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds,
\end{aligned}$$

so we have for all $t \in [0,\tau_0]$ that

$$(\lambda I - A)^{-1} v(t) = (\lambda I - A)^{-1}\left[T_{A_0}(t)v(0) + \frac{d}{dt}(S_A * f)(t)\right].$$

Since $(\lambda I - A)^{-1}$ is injective, the result follows. □

In order to obtain the existence of mild (or integrated) solutions, we make the following assumption.

Assumption 3.4.6. Let $\tau_0 > 0$ be fixed. Let $Z \subset L^1((0,\tau_0), X)$ be a Banach space endowed with some norm $\|.\|_Z$. Assume that $C^1([0,\tau_0], X) \cap Z$ is dense in $(Z, \|.\|_Z)$ and the embedding from $(Z, \|.\|_Z)$ into $\left(L^1((0,\tau_0), X), \|.\|_{L^1}\right)$ is continuous. Also assume that there exists a continuous map $\Gamma : [0,\tau_0] \times Z \to [0,+\infty)$ such that

(a) $\Gamma(t,0) = 0$, $\forall t \in [0,\tau_0]$, and the map $f \to \Gamma(t,f)$ is continuous at 0 uniformly in $t \in [0,\tau_0]$;
(b) $\forall t \in [0,\tau_0]$, $\forall f \in C^1([0,\tau_0], X) \cap Z$, we have that

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq \Gamma(t,f).$$

We now state and prove the main result in this section.

Theorem 3.4.7. *Let Assumptions 3.4.1 and 3.4.6 be satisfied. Then for each $f \in Z$ the map $t \to (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0,\tau_0]$, and if we denote $u(t) = \frac{d}{dt}(S_A * f)(t)$, then*

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau_0]$$

and

$$\|u(t)\| \leq \Gamma(t,f), \ \forall t \in [0,\tau_0]. \tag{3.4.9}$$

Moreover, for each $\lambda \in (\omega, +\infty)$, we have

$$(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds. \tag{3.4.10}$$

Proof. Consider the linear operator

$$L_{\tau_0} : \left(C^1([0,\tau_0], X) \cap Z, \|.\|_Z\right) \to \left(C([0,\tau_0], X), \|.\|_{\infty,[0,\tau_0]}\right)$$

defined by

$$L_{\tau_0}(f)(t) = \frac{d}{dt}(S_A * f)(t), \ \forall t \in [0,\tau_0], \ \forall f \in C^1([0,\tau_0], X) \cap Z.$$

Then

$$\sup_{t\in[0,\tau_0]} \left\| L_{\tau_0}(f)(t) \right\| \leq \sup_{t\in[0,\tau_0]} \Gamma(t,f).$$

Since $C^1([0,\tau_0], X) \cap Z$ is dense in Z, using assumptions (a) and (b), we know that L_{τ_0} has a unique extension $\widehat{L}_{\tau_0}$ on Z and

$$\left\|\widehat{L}_{\tau_0}(f)(t)\right\| \leq \Gamma(t,f),\ \forall t\in[0,\tau_0],\ \forall f\in Z.$$

By construction, $\widehat{L}_{\tau_0} : (Z, \|.\|_Z) \to \left(C([0,\tau_0],X), \|.\|_{\infty,[0,\tau_0]}\right)$ is continuous.

Let $f \in Z$ and let $\{f_n\}_{n\geq 0}$ be a sequence in $C^1([0,\tau_0],X)\cap Z$, such that $f_n \to f$ in Z. We have for each $n \geq 0$ and each $t \in [0,\tau_0]$ that

$$\int_0^t \widehat{L}_{\tau_0}(f_n)(s)ds = \int_0^t L_{\tau_0}(f_n)(s)ds = \int_0^t S_A(t-s)f_n(s)ds.$$

Since the embedding from $(Z,\|.\|_Z)$ into $\left(L^1((0,\tau_0),X), \|.\|_{L^1}\right)$ is continuous, we have that $f_n \to f$ in $L^1((0,\tau_0),X)$ and when $n \to +\infty$,

$$\int_0^t \widehat{L}_{\tau_0}(f)(s)ds = \int_0^t S_A(t-s)f(s)ds,\ \forall t\in[0,\tau_0].$$

Thus, the map $t \to (S_A * f)(t)$ is continuously differentiable and

$$\widehat{L}_{\tau_0}(f)(t) = \frac{d}{dt}\int_0^t S_A(t-s)f(s)ds,\ \forall t\in[0,\tau_0].$$

Finally, by Lemma 3.4.4, we have for each $n \geq 0$ and each $t \in [0,\tau_0]$ that

$$\widehat{L}_{\tau_0}(f_n)(t) = A\int_0^t \widehat{L}_{\tau_0}(f_n)(s)ds + \int_0^t f_n(s)ds,$$

the result follows from the fact that A is closed. $\square$

In the proof of Theorem 3.4.7, we basically followed the same method as Kellermann and Hieber [207] used to prove the result of Da Prato and Sinestrari [85] (see also Arendt et al. [22, Theorem 4.5.2, p.145]) for Hille-Yosida operators and with $Z = L^1((0,\tau_0),X)$.

As a consequence of (3.4.10) we obtain the following approximation formula.

Proposition 3.4.8 (Approximation formula). *Let Assumptions 3.4.1 and 3.4.6 be satisfied. Let $f \in Z$. For each $t \in [0,\tau]$ we have that*

$$\frac{d}{dt}(S_A * f)(t) = \lim_{\lambda\to+\infty}\int_0^t T_{A_0}(t-l)\lambda(\lambda I - A)^{-1}f(l)dl. \tag{3.4.11}$$

Proof. Let $f \in Z$ and $t \in [0,\tau]$ be fixed. Since

$$\frac{d}{dt}(S_A * f)(t) \in X_0,$$

we have

$$\lim_{\lambda\to+\infty}\lambda(\lambda I - A)^{-1}\frac{d}{dt}(S_A * f)(t) = \frac{d}{dt}(S_A * f)(t).$$

But by using formula (3.4.10), we have for each $\lambda > 0$ large enough that

$$\lambda (\lambda I - A)^{-1} \frac{d}{dt} (S_A * f)(t) = \int_0^t T_{A_0}(t-s)\lambda (\lambda I - A)^{-1} f(s)ds$$

and the result follows. □

From this approximation formulation, we deduce the following result.

Corollary 3.4.9. *Let Assumptions 3.4.1 and 3.4.6 be satisfied. Let $f \in Z$. For each $t,s \in [0,\tau]$ with $s \leq t$ we have*

$$\frac{d}{dt} (S_A * f)(t) = T_{A_0}(t-s) \frac{d}{dt} (S_A * f)(s) + \frac{d}{dt} (S_A * f(.+s))(t-s). \tag{3.4.12}$$

Proof. Indeed by using the approximation formula we have

$$\begin{aligned} \tfrac{d}{dt} (S_A * f)(t) &= \lim_{\lambda \to +\infty} \int_0^t T_{A_0}(t-l)\lambda (\lambda I - A)^{-1} f(l)dl \\ &= \lim_{\lambda \to +\infty} \Big[\int_0^s T_{A_0}(t-l)\lambda (\lambda I - A)^{-1} f(l)dl \\ &\qquad + \int_s^t T_{A_0}(t-l)\lambda (\lambda I - A)^{-1} f(l)dl \Big] \\ &= \lim_{\lambda \to +\infty} \Big[T_{A_0}(t-s) \int_0^s T_{A_0}(s-l)\lambda (\lambda I - A)^{-1} f(l)dl \\ &\qquad + \int_0^{t-s} T_{A_0}(t-s-l)\lambda (\lambda I - A)^{-1} f(l+s)dl \Big] \end{aligned}$$

the result follows. □

By Lemma 3.4.2 and Theorem 3.4.7, we obtain the following result.

Corollary 3.4.10. *Let Assumptions 3.4.1 and 3.4.6 be satisfied. Then for each $x \in X_0$ and each $f \in Z$, the Cauchy problem (3.1.1) has a unique mild solution $u \in C([0,\tau_0], X_0)$ given by*

$$u(t) = T_{A_0}(t)x + \frac{d}{dt} (S_A * f)(t), \ \forall t \in [0,\tau_0]. \tag{3.4.13}$$

Moreover, we have

$$\|u(t)\| \leq M_A e^{\omega_A t} \|x\| + \Gamma(t,f), \ \forall t \in [0,\tau_0]. \tag{3.4.14}$$

3.5 Bounded Perturbation

In this section we investigate the properties of $A+L : D(A) \subset X \to X$, where L is a bounded linear operator from X_0 into X. If A is a Hille-Yosida operator, it is well known that $A+L$ is also a Hille-Yosida operator (see Arendt et al. [22, Theorem 3.5.5]).

The following theorem is closely related to Desch and Schappacher's theorem (see [95] or Engel and Nagel [126, Theorem 4.1, p. 183]). This is in fact an integrated semigroup formulation of this result.

Theorem 3.5.1. *Let Assumptions 3.4.1 and 3.4.6 be satisfied. Assume in addition that $C([0,\tau_0],X) \subset Z$ and there exists a constant $\delta > 0$ such that*

$$\Gamma(t,f) \le \delta \sup_{s\in[0,t]} \|f(s)\|, \ \forall f \in C([0,\tau_0],X), \ \forall t \in [0,\tau_0].$$

Let $L \in \mathscr{L}(X_0,X)$ and assume that

$$\|L\|_{\mathscr{L}(X_0,X)} \delta < 1.$$

Then $A+L : D(A) \subset X \to X$ satisfies Assumptions 3.4.1 and 3.4.6. More precisely, if we denote by $\{S_{A+L}(t)\}_{t\ge 0}$ the integrated semigroup generated by $A+L$, then $\forall f \in C^1([0,\tau_0],X)$, we have

$$\left\| \frac{d}{dt}(S_{A+L} * f)(t) \right\| \le \frac{1}{1-\|L\|_{\mathscr{L}(X_0,X)}\delta} \sup_{s\in[0,t]} \Gamma(s,f), \ \forall t \in [0,\tau_0]. \tag{3.5.1}$$

Proof. We first prove that there exists $\widehat{\omega} \in \mathbb{R}$ such that $(\widehat{\omega},+\infty) \subset \rho(A+L)$. We have for $x \in D(A)$ and $y \in X$ that

$$\begin{aligned}(\lambda I-(A+L))x = y &\Leftrightarrow (\lambda I - A)x = y + Lx \\ &\Leftrightarrow x = (\lambda I-A)^{-1} y + (\lambda I - A)^{-1} Lx.\end{aligned}$$

So $\lambda I-(A+L)$ is invertible if $\left\|(\lambda I-A)^{-1}L\right\|_{\mathscr{L}(X_0,X)} < 1$. Since $\{S_A(t)\}_{t\ge 0}$ is exponentially bounded, by Proposition 3.3.2, we have for all $\lambda > \widetilde{\omega}$ that

$$(\lambda I-A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t)xdt, \ \forall x \in X.$$

We obtain that

$$(\lambda I-A)^{-1} Lx = \lambda \int_{\tau_0}^{+\infty} e^{-\lambda t} S_A(t)Lxdt + \lambda \int_0^{\tau_0} e^{-\lambda t} S_A(t)Lxdt.$$

Since $S_A(t)y = \frac{d}{dt}\int_0^t S_A(t-s)yds, \forall y \in X$, from the assumption we have

$$\|S_A(t)y\| \le \delta \|y\|, \ \forall t \in [0,\tau_0], \ \forall y \in X.$$

Thus,

$$\left\| \lambda \int_0^{\tau_0} e^{-\lambda t} S_A(t)Lxdt \right\| \le \lambda \int_0^{\tau_0} e^{-\lambda t}dt \, \|L\|_{\mathscr{L}(X_0,X)} \delta \|x\|$$

and

$$\lambda \int_0^{\tau_0} e^{-\lambda t}dt = 1 - e^{-\lambda\tau_0} \to 1 \text{ as } \lambda \to +\infty.$$

Moreover

$$\left\| \lambda \int_{\tau_0}^{+\infty} e^{-\lambda t} S_A(t) L x dt \right\| \to 0 \text{ as } \lambda \to +\infty.$$

So we obtain

$$\limsup_{\lambda \to +\infty} \left\| (\lambda I - A)^{-1} L \right\|_{\mathscr{L}(X_0, X)} \leq \|L\|_{\mathscr{L}(X_0,X)} \delta < 1.$$

We know that there exists $\widehat{\omega} \in \mathbb{R}$ such that

$$\left\| (\lambda I - A)^{-1} L \right\|_{\mathscr{L}(X_0,X)} < \frac{\|L\|_{\mathscr{L}(X_0,X)} \delta + 1}{2}, \ \forall \lambda \in (\widehat{\omega}, +\infty).$$

Hence, for all $\lambda \in (\widehat{\omega}, +\infty)$, $\lambda I - (A+L)$ is invertible,

$$(\lambda I - (A+L))^{-1} y = \sum_{k=0}^{+\infty} \left[(\lambda I - A)^{-1} L \right]^k (\lambda I - A)^{-1} y$$

and for each $y \in X$,

$$\left\| (\lambda I - (A+L))^{-1} y \right\| \leq \frac{1}{1 - \frac{\|L\|_{\mathscr{L}(X_0,X)} \delta + 1}{2}} \left\| (\lambda I - A)^{-1} y \right\| \to 0 \text{ as } \lambda \to +\infty.$$

To prove Assumption 3.4.1 it remains to show that $(A+L)_0$, the part of $A+L$ in X_0, is a Hille-Yosida operator. Let $x \in X_0$. Define $\Pi, \Psi_x : C([0,\tau_0], X_0) \to C([0,\tau_0], X_0)$ for each $v \in C([0,\tau_0], X_0)$ by

$$\Pi(v)(t) = \frac{d}{dt} (S_A * Lv)(t) \text{ and } \Psi_x(v)(t) = T_{A_0}(t)x + \Pi(v)(t), \ \forall t \in [0, \tau_0].$$

Then from the assumptions it is clear that Ψ_x is an $\|L\|_{\mathscr{L}(X_0,X)} \delta-$contraction. It implies that Ψ_x has a unique fixed point given by

$$U(t)x = \sum_{k=0}^{\infty} \Pi^k \left(T_{A_0}(.)x \right)(t), \ \forall t \in [0, \tau_0].$$

In particular,

$$\|U(t)x\| \leq \frac{1}{1 - \|L\|_{\mathscr{L}(X_0,X)} \delta} M_A e^{\omega_A t} \|x\|, \ \forall t \in [0, \tau_0].$$

Thus, we obtain $\{U(t)\}_{0 \leq t \leq \tau_0}$, a family of bounded linear operators on X_0, such that for each $x \in X_0$, $t \to U(t)x$ is the unique solution of

$$U(t)x = x + A \int_0^t U(s)x ds + \int_0^t LU(s)x ds, \ \forall t \in [0, \tau_0].$$

Therefore,

$$U(0)=I \text{ and } U(t+s)=U(t)U(s),\ \forall t,s\in[0,\tau_0] \text{ with } t+s\leq\tau_0.$$

We can define for each integer $k\geq 0$ and each $t\in[k\tau_0,(k+1)\tau_0]$ that

$$U(t)=U(t-k\tau_0)U(\tau_0)^k,$$

which yields a C_0-semigroup of X_0 and

$$U(t)x=x+A\int_0^t U(s)xds+\int_0^t LU(s)xds,\ \forall t\geq 0.$$

It remains to show that $(A+L)_0$ is the generator of $\{U(t)\}_{t\geq 0}$. Let $B:D(B)\subset X_0\to X_0$ be the generator of $\{U(t)\}_{t\geq 0}$. Since $U(t)x$ is the unique solution of

$$U(t)x=x+(A+L)\int_0^t U(s)xds,\ \forall t\geq 0,\forall x\in X_0,$$

we know that $(\lambda I-(A+L))^{-1}$ and $U(t)$ commute, in particular $(\lambda I-(A+L))^{-1}$ and $(\lambda I-B)^{-1}$ commute. On the other hand, we also have

$$B\int_0^t U(s)x=(A+L)\int_0^t U(s)xds,\ \forall t\geq 0,\forall x\in X_0.$$

Now since $(\lambda I-(A+L))^{-1}$ and $(\lambda I-B)^{-1}$ are commuting we deduce that

$$(\lambda I-(A+L))^{-1}\int_0^t U(s)x=(\lambda I-B)^{-1}\int_0^t U(s)xds,\ \forall t\geq 0,\forall x\in X_0.$$

By computing the derivative of the last expression at $t=0$, we obtain for sufficiently large $\lambda\in\mathbb{R}$ that

$$(\lambda I-(A+L))^{-1}x=(\lambda I-B)^{-1}x,\ \forall x\in X_0.$$

Therefore, $B=(A+L)_0$ and $A+L$ satisfies Assumption 3.4.1.

Now by using Proposition 3.4.3 we know that $A+L$ generates an integrated semigroup $\{S_{A+L}(t)\}_{t\geq 0}$ and

$$S_{A+L}(t)x=(A+L)\int_0^t S_{A+L}(t)x+\int_0^t xds,\ \forall t\geq 0,\ \forall x\in X.$$

So

$$S_{A+L}(t)x=S_A(t)x+\frac{d}{dt}\left(S_A*LS_{A+L}(.)x\right)(t),\ \forall t\in[0,\tau_0],\ \forall x\in X$$

and for each $f\in L^1([0,\tau_0],X),\forall t\in[0,\tau_0],\ \forall x\in X,$

$$\int_0^t S_{A+L}(t-s)f(s)ds=\int_0^t S_A(t-s)f(s)ds+\int_0^t W(t-s)f(s)ds,$$

where $W(t)x := \frac{d}{dt}\left(S_A * LS_{A+L}(.)x\right)(t)$.

Also notice that

$$\begin{aligned}
\int_0^t \int_0^l W(l-s)f(s)dsdl &= \int_0^t \int_s^t W(l-s)f(s)dlds \\
&= \int_0^t \int_0^{t-s} W(l)f(s)dlds \\
&= \int_0^t \left(S_A * LS_{A+L}(.)f(s)\right)(t-s)ds \\
&= \int_0^t \int_0^{t-s} S_A(t-s-l)LS_{A+L}(l)f(s)dlds \\
&= \int_0^t \int_s^t S_A(t-l)LS_{A+L}(l-s)f(s)dlds \\
&= \int_0^t \int_0^l S_A(t-l)LS_{A+L}(l-s)f(s)dsdl \\
&= \int_0^t S_A(t-l)\int_0^l LS_{A+L}(l-s)f(s)dsdl,
\end{aligned}$$

we then have

$$\int_0^t W(t-s)f(s)ds = \frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(.)\right)(t).$$

Thus,

$$\left(S_{A+L} * f\right)(t) = (S_A * f)(t) + \frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(.)\right)(t),\ \forall t \in [0, \tau_0].$$

Let $f \in C^1\left([0, \tau_0], X\right)$. The map $t \to L\left(S_{A+L} * f\right)(.)$ is continuously differentiable and

$$\frac{d}{dt}\left(S_A * L\left(S_{A+L} * f\right)(.)\right)(t) = S_A(t)L\left(S_{A+L} * f\right)(0) + \left(S_A * \frac{d}{dt}L\left(S_{A+L} * f\right)(.)\right)(t),$$

so

$$\frac{d}{dt}\left(S_{A+L} * f\right)(t) = \frac{d}{dt}(S_A * f)(t) + \frac{d}{dt}\left(S_A * L\frac{d}{dt}\left(S_{A+L} * f\right)(.)\right)(t).$$

Therefore, for each $t \in [0, \tau_0]$, we have

$$\left\| \frac{d}{dt}\left(S_{A+L} * f\right)(t) \right\| \leq \Gamma(t, f) + \|L\|_{\mathscr{L}(X_0, X)}\, \delta \sup_{s \in [0,t]} \left\| \frac{d}{dt}\left(S_{A+L} * f\right)(s) \right\|$$

and

$$\sup_{s \in [0,t]} \left\| \frac{d}{dt}\left(S_{A+L} * f\right)(s) \right\| \leq \frac{1}{1 - \|L\|_{\mathscr{L}(X_0, X)}\, \delta} \sup_{s \in [0,t]} \Gamma(s, f).$$

This completes the proof. $\square$

By Theorem 3.5.1, we can make an alternative for Assumption 3.4.6.

Assumption 3.5.2. Assume that there exist a real number $\tau^* > 0$ and a non-decreasing map $\delta : [0,\tau] \to [0,+\infty)$ such that for each $f \in C^1([0,\tau^*],X)$,

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \; \forall t \in [0,\tau^*],$$

and

$$\lim_{t\to 0^+} \delta(t) = 0.$$

One may observe that the inequality in Assumption 3.5.2 plays a crutial role in obtaining some estimations for the solutions (see Magal and Ruan [247, Proposition 2.14]). Moreover, by using the results of Thieme [335], Assumption 3.5.2 is equivalent to the fact that there exists $\tau > 0$ such that

$$V^\infty(S_A,0,\tau) < +\infty, \forall \tau > 0, \text{ and } \lim_{t(>0)\to 0} V^\infty(S_A,0,t) = 0,$$

where $V^\infty(S_A,0,\tau)$ is the semi-variation of $\{S_A(t)\}_{t\geq 0}$ on $[0,\tau]$ defined by

$$V^\infty(S_A,0,\tau) := \sup\left\{ \left\| \sum_{i=1}^{n} \big(S_A(t_i) - S_A(t_{i-1})\big)x_i \right\| \right\} < +\infty,$$

in which the supremum is taken over all partitions $0 = t_0 < .. < t_n = \tau$ of the interval $[a,b]$ and over any $(x_1,..,x_n) \in X^n$ with $\|x_i\|_X \leq 1$, $\forall i = 1,..,n$.

Thus, under the Assumptions 3.4.1 and 3.5.2, the conclusions of Theorem 3.4.7 hold with (3.4.9) replaced by

$$\|u(t)\| \leq \delta(t) \sup_{s\in[0,t]} \|f(s)\|, \; \forall t \in [0,\tau] \tag{3.5.2}$$

and that of Theorem 3.5.1 hold with (3.5.1) replaced by

$$\left\| \frac{d}{dt}(S_{A+L} * f) \right\| \leq \frac{\delta(t)}{1-\delta(\tau_L)\|L\|_{\mathscr{L}(X_0,X)}} \sup_{s\in[0,t]} \|f(s)\|, \tag{3.5.3}$$

for all $t \in [0,\tau_L]$, where $\tau_L > 0$ is fixed such that

$$\delta(\tau_L)\|L\|_{\mathscr{L}(X_0,X)} < 1.$$

In the following it will be convenient to use the following notation. For each $\widehat{\tau} > 0$ and each $f \in C([0,\widehat{\tau}],X)$, set

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t), \;\; \forall t \in [0,\widehat{\tau}].$$

The following proposition is one of the main tools in studying semilinear problems.

Proposition 3.5.3. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. Let $\varepsilon > 0$ be fixed. Then for each $\gamma > \omega_A$, there exists $C(\varepsilon,\gamma) > 0$, such that for each $f \in C(\mathbb{R}_+, X)$ and $t \geq 0$,*

$$e^{-\gamma t} \left\| (S_A \diamond f)(t) \right\| \leq C(\varepsilon,\gamma) \sup_{s\in[0,t]} e^{-\gamma s} \left\| f(s) \right\|. \tag{3.5.4}$$

More precisely, for each $\varepsilon > 0$, if $\tau_\varepsilon > 0$ is such that $M_A \delta(\tau_\varepsilon) \leq \varepsilon$, then the inequality (3.5.4) holds with

$$C(\varepsilon,\gamma) = \frac{2\varepsilon \max\left(1, e^{-\gamma\tau_\varepsilon}\right)}{1 - e^{(\omega_A-\gamma)\tau_\varepsilon}}, \ \forall \gamma > \omega.$$

Proof. Let $\varepsilon > 0$, $f \in C(\mathbb{R}_+, X)$, and $\gamma > \omega_A$ be fixed. Let $\tau_\varepsilon = \tau_\varepsilon(\varepsilon) \in (0,\tau]$ be such that $M_A \delta(\tau_\varepsilon) \leq \varepsilon$. By Assumption 3.5.2, we have

$$\left\| (S_A \diamond f)(t) \right\| \leq \varepsilon \sup_{s\in[0,t]} \left\| f(s) \right\|, \ \forall t \in [0, \tau_\varepsilon]. \tag{3.5.5}$$

Let $\gamma > \omega_A$ be fixed. Set

$$\varepsilon_1 = \varepsilon \max\left(1, e^{-\gamma\tau_\varepsilon}\right).$$

Let $k \in \mathbb{N}$ and $t \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$ be fixed. First, notice that if $\gamma \geq 0$, we have

$$\begin{aligned}
\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \left\| f(s) \right\| &= \varepsilon \sup_{s\in[k\tau_\varepsilon,t]} e^{\gamma s} e^{-\gamma s} \left\| f(s) \right\| \\
&\leq \varepsilon e^{\gamma t} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\| \\
&= \varepsilon_1 e^{\gamma t} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\|.
\end{aligned}$$

Moreover, if $\gamma < 0$, we have

$$\begin{aligned}
\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \left\| f(s) \right\| &= \varepsilon \sup_{s\in[k\tau_\varepsilon,t]} e^{\gamma s} e^{-\gamma s} \left\| f(s) \right\| \\
&\leq \varepsilon e^{\gamma k\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\| \\
&= \varepsilon e^{\gamma t} e^{-\gamma t} e^{\gamma k\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\| \\
&= e^{\gamma t} \varepsilon e^{-\gamma(t-k\tau_\varepsilon)} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\| \\
&\leq e^{\gamma t} \varepsilon e^{-\gamma\tau_\varepsilon} \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\| \\
&= e^{\gamma t} \varepsilon_1 \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\|.
\end{aligned}$$

Therefore, for each $k \in \mathbb{N}$, each $t \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$, and each $\gamma \in \mathbb{R}$, we obtain

$$\varepsilon \sup_{s\in[k\tau_\varepsilon,t]} \left\| f(s) \right\| \leq e^{\gamma t} \varepsilon_1 \sup_{s\in[k\tau_\varepsilon,t]} e^{-\gamma s} \left\| f(s) \right\|. \tag{3.5.6}$$

It follows from (3.5.5) and (3.5.6) that for all $t \in [0, \tau_\varepsilon]$,

$$\|(S_A \diamond f)(t)\| \leq \varepsilon \sup_{s\in[0,t]} \|f(s)\| = e^{\gamma t}\varepsilon_1 \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|. \tag{3.5.7}$$

Using (3.4.12) with $s = \tau_\varepsilon$, we have for all $t \in [\tau_\varepsilon, 2\tau_\varepsilon]$ that

$$(S_A \diamond f)(t) = T_0(t-\tau_\varepsilon)(S_A \diamond f)(\tau_\varepsilon) + (S_A \diamond f(\tau_\varepsilon + .))(t-\tau_\varepsilon).$$

Using (3.5.5)-(3.5.7), we have

$$\begin{aligned}
\|(S_A \diamond f)(t)\| &\leq e^{\omega_A(t-\tau_\varepsilon)}|(S_A \diamond f)(\tau_\varepsilon)| + |(S_A \diamond f(\tau_\varepsilon + .))(t-\tau_\varepsilon)| \\
&\leq e^{\omega_A(t-\tau_\varepsilon)}e^{\gamma\tau_\varepsilon}\varepsilon_1 \sup_{s\in[0,\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| + \varepsilon \sup_{s\in[\tau_\varepsilon,t]}\|f(s)\| \\
&\leq e^{\omega_A(t-\tau_\varepsilon)}e^{\gamma\tau_\varepsilon}\varepsilon_1 \sup_{s\in[0,\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| \\
&\quad + e^{\gamma t}\varepsilon_1 \sup_{s\in[\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon_1 e^{\gamma t}\left(e^{(\omega_A-\gamma)(t-\tau_\varepsilon)} + 1\right) \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

Similarly, for all $t \in [2\tau_\varepsilon, 3\tau_\varepsilon]$,

$$(S_A \diamond f)(t) = T_{A_0}(t-2\tau_\varepsilon)(S_A \diamond f)(2\tau_\varepsilon) + (S_A \diamond f(2\tau_\varepsilon + .))(t-2\tau_\varepsilon)$$

and

$$\begin{aligned}
\|(S_A \diamond f)(t)\| &\leq e^{\omega_A(t-2\tau_\varepsilon)}\varepsilon_1 e^{\gamma 2\tau_\varepsilon}\left(e^{(\omega_A-\gamma)\tau_\varepsilon} + 1\right) \sup_{s\in[0,2\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| \\
&\quad + \varepsilon \sup_{s\in[2\tau_\varepsilon,t]}\|f(s)\| \\
&\leq e^{\omega_A(t-2\tau_\varepsilon)}\varepsilon_1 e^{\gamma 2\tau_\varepsilon}\left(e^{(\omega_A-\gamma)\tau_\varepsilon} + 1\right) \sup_{s\in[0,2\tau_\varepsilon]} e^{-\gamma s}\|f(s)\| \\
&\quad + \varepsilon_1 e^{\gamma t} \sup_{s\in[2\tau_\varepsilon,t]} e^{-\gamma s}\|f(s)\| \\
&\leq \varepsilon_1 e^{\gamma t}\left[e^{(\omega_A-\gamma)(t-2\tau_\varepsilon)}\left(e^{(\omega_A-\gamma)\tau_\varepsilon} + 1\right) + 1\right] \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\|.
\end{aligned}$$

By induction, we obtain $\forall k \in \mathbb{N}$ with $k \geq 1, \forall t \in [k\tau_\varepsilon, (k+1)\tau_\varepsilon]$, and for each $\gamma > \omega_A$ that

$$\begin{aligned}
\|(S_A \diamond f)(t)\| &\leq \varepsilon_1 e^{\gamma t} \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\| \left[e^{(\omega_A-\gamma)(t-k\tau_\varepsilon)} \sum_{n=0}^{k-1}(e^{(\omega_A-\gamma)\tau_\varepsilon})^n + 1\right] \\
&\leq \varepsilon_1 e^{\gamma t} \sup_{s\in[0,t]} e^{-\gamma s}\|f(s)\| \left[\sum_{n=0}^{\infty}\left(e^{(\omega_A-\gamma)\tau_\varepsilon}\right)^n + 1\right].
\end{aligned}$$

Since $\gamma > \omega_A$, we have for each $t \geq 0$ that

$$e^{-\gamma t} \left\| (S_A \diamond f)(t) \right\| \leq e^{-\gamma t} \left| (S_A \diamond f)(t) \right| \leq \frac{2\varepsilon_1}{1 - e^{(\omega_A - \gamma)\tau_\varepsilon}} \sup_{s \in [0,t]} e^{-\gamma s} \left\| f(s) \right\|.$$

This completes the proof. □

We have the following lemma.

Lemma 3.5.4. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. Then $t \to S_A(t)$ is operator norm continuous from $[0,+\infty)$ into $\mathscr{L}(X)$ and*

$$\lim_{\lambda \to +\infty} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} = 0.$$

Proof. By Theorem 3.3.5 , we have for each $\lambda > \max(0, \omega_A)$ that

$$(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t) x dt.$$

Note that

$$S_A(t)x = \frac{d}{dt} \int_0^t S_A(t-s) x ds,$$

so by Assumption 3.5.2, we have

$$\left\| S_A(t)x \right\| \leq V^\infty(S_A, 0, t) \left\| x \right\|, \ \forall t \geq 0.$$

But

$$S_A(t+h) - S_A(t) = T_{A_0}(t) S_A(h),$$

it follows that $t \to S_A(t)$ is operator norm continuous. Let $\varepsilon > 0$ be fixed and let $\tau_\varepsilon > 0$ be such that $V^\infty(S_A, 0, \tau_\varepsilon) \leq \varepsilon$. Choose $\gamma > \max(0, \omega_A)$ and $M_\gamma > 0$ such that

$$\left\| S_A(t)x \right\| \leq M_\gamma e^{\gamma t}, \forall t \geq 0.$$

Then we have for each $\lambda > \gamma$ that

$$\left\| (\lambda I - A)^{-1} x \right\| \leq \lambda \left[M_\gamma \int_{\tau_\varepsilon}^{+\infty} e^{(\gamma - \lambda)t} dt + \varepsilon \int_0^{\tau_\varepsilon} e^{-\lambda t} dt \right] \left\| x \right\|.$$

Thus,

$$\limsup_{\lambda \to +\infty} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} \leq \varepsilon.$$

This proves the lemma. □

Let $J \subset [0,+\infty)$ be an interval. Set $s := \inf J \geq 0$. For each $\gamma \geq 0$, define

$$BC^\gamma(J,Y) := \left\{ \varphi \in C(J,Y) : \sup_{l \in J} e^{-\gamma(l-s)} \left\| \varphi(l) \right\|_Y < +\infty \right\}$$

and

$$\|\varphi\|_{BC^{\gamma}(J,Y)} := \sup_{l \in J} e^{-\gamma(l-s)} \|\varphi(l)\|_Y .$$

It is well known that $BC^{\gamma}(J,Y)$ endowed with the norm $\|.\|_{BC^{\gamma}(J,Y)}$ is a Banach space.

By using Proposition 3.5.3 we obtain the following result.

Lemma 3.5.5. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. For each $s \geq 0$ and each $\sigma \in (s,+\infty]$, define a linear operator $\mathscr{L}_s : C([s,\sigma),X) \to C([s,\sigma),X_0)$ by*

$$\mathscr{L}_s(\varphi)(t) = (S_A \diamond \varphi(.+s))(t-s), \ \forall t \in [s,\sigma), \ \forall \varphi \in C([s,\sigma),X).$$

Then for each $\gamma > \omega_A$, $\mathscr{L}_s$ is a bounded linear operator from $BC^{\gamma}([s,\sigma),X)$ into $BC^{\gamma}([s,\sigma),X_0)$. Moreover, for each $\varepsilon > 0$ and each $\tau_\varepsilon > 0$ such that $M_A\delta(\tau_\varepsilon) \leq \varepsilon$,

$$\|\mathscr{L}_s(\varphi)\|_{\mathscr{L}(BC^{\gamma}([s,\sigma),X),BC^{\gamma}([s,\sigma),X_0))} \leq C(\varepsilon,\gamma).$$

Proof. Let $\varphi \in BC^{\gamma}([s,\sigma),X)$ be fixed. By using Proposition 3.5.3, we have for $t \in [s,\sigma)$ that

$$\begin{aligned} e^{-\gamma(t-s)} \|(S_A \diamond \varphi(.+s))(t-s)\| &\leq C(\varepsilon,\gamma) \sup_{l \in [0,t-s]} e^{-\gamma l} \|\varphi(l+s)\| \\ &= C(\varepsilon,\gamma) \sup_{r \in [s,t]} e^{-\gamma(r-s)} \|\varphi(r)\| \\ &\leq C(\varepsilon,\gamma) \sup_{r \in [s,\sigma)} e^{-\gamma(r-s)} \|\varphi(r)\| \end{aligned}$$

and the result follows. □

3.6 The Hille-Yosida Case

In this section we assume that A is a Hille-Yosida operator. This assumption corresponds to the case where Assumption 3.4.1 is verified in the L^1-space. Hence, we fix

$$Z = L^1((0,\tau_0),X) \text{ and } \Gamma(t,f) = M_A \left\| e^{\omega_A(t-.)} f(.) \right\|_{L^1((0,t),X)}$$

in Assumption 3.4.6.

Recall that A is a Hille-Yosida operator if the following hold.

Assumption 3.6.1. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X, so that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that

$$(\omega_A,+\infty) \subset \rho(A)$$

and

$$\left\| (\lambda I - A)^{-k} \right\|_{\mathscr{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \ \forall \lambda > \omega_A, \ \forall k \geq 1;$$

The following result is due to Kellermann and Hieber [207].

Theorem 3.6.2 (Kellermann-Hieber). *Let Assumption 3.6.1 be satisfied. Let $\tau_0 > 0$. Then for each $f \in L^1((0,\tau_0), X)$ the map $t \to (S_A * f)(t)$ is continuously differentiable, $(S_A * f)(t) \in D(A), \forall t \in [0,\tau_0]$, and if we denote $u(t) = \frac{d}{dt}(S_A * f)(t)$, then*

$$u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \ \forall t \in [0,\tau_0]$$

and

$$\|u(t)\| \leq M_A \int_0^t e^{\omega_A (t-s)} \|f(s)\| ds, \ \forall t \in [0,\tau_0].$$

Moreover, for each $\lambda \in (\omega_A, +\infty)$, we have

$$(\lambda I - A)^{-1} \frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds.$$

Proof. First it is clear that Assumption 3.4.1 is satisfied whenever A is a Hille-Yosida operator. So it remains to prove that Assumption 3.4.6 is satisfied with

$$\Gamma(t,f) = M_A \left\| e^{\omega_A (t-.)} f(.) \right\|_{L^1((0,t),X)}.$$

Now note that by Lemma 2.4.3, we can find a norm $|.|$ on X, such that

$$\left| (\lambda I - A)^{-1} \right| \leq \frac{1}{\lambda - \omega_A}, \ \forall \lambda > \omega_A,$$

and

$$\|x\| \leq |x| \leq M_A \|x\|, \ \forall x \in X. \tag{3.6.1}$$

It follows that for each $t \geq 0$ and each $h \geq 0$,

$$|[S_A(t+h) - S_A(t)]x| = \lim_{\lambda \to +\infty} \left| \int_t^{t+h} T_{A_0}(l) \lambda (\lambda I - A)^{-1} x dl \right| \leq \int_t^{t+h} e^{\omega_A l} dl \, |x|.$$

So

$$|S_A(t+h) - S_A(t)| \leq \int_t^{t+h} e^{\omega_A l} dl, \ \forall t,h \geq 0.$$

Let $\tau_0 > 0$ and let $f \in C^1([0,\tau_0], X)$ be fixed. We have

$$\begin{aligned} &\frac{d}{dt}(S_A * f)(t) \\ &\quad = \lim_{h \searrow 0} h^{-1} \left[\int_0^{t+h} S_A(t+h-s) f(s) ds - \int_0^t S_A(t-s) f(s) ds \right] \\ &\quad = \lim_{h \searrow 0} h^{-1} \left[\int_t^{t+h} S_A(t+h-s) f(s) ds + \int_0^t [S_A(t+h-s) - S_A(t-s)] f(s) ds \right]. \end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\frac{d}{dt}\left(S_A * f\right)(t)\right| \\
&\quad \leq \lim_{h \searrow 0} h^{-1}\left[\int_t^{t+h}\int_0^{t+h-s} e^{\omega_A l} dl\,|f(s)|\,ds + \int_0^t \int_{t-s}^{t+h-s} e^{\omega_A l} dl\,|f(s)|\,ds\right] \\
&\quad \leq \int_0^t e^{\omega_A (t-s)}\,|f(s)|\,ds.
\end{aligned}$$

Now by using (3.6.1) we obtain

$$\left\|\frac{d}{dt}\left(S_A * f\right)(t)\right\| \leq M_A \int_0^t e^{\omega_A (t-s)}\,\|f(s)\|\,ds,$$

and the results follows from Theorem 3.4.7. □

Consider the nonhomogeneous Cauchy problem (3.1.1). As an immediate consequence of Corollary 3.4.10, we have the following result.

Corollary 3.6.3. *Let Assumption 3.6.1 be satisfied. Let* $\tau_0 > 0$. *Then for each* $x \in X_0$ *and each* $f \in L^1((0,\tau_0), X)$*, the Cauchy problem (3.1.1) has a unique integrated solution* $u \in C([0,\tau_0], X_0)$ *given by*

$$u(t) = T_{A_0}(t)x + \frac{d}{dt}\left(S_A * f\right)(t),\ \forall t \in [0,\tau_0].$$

Moreover, we have

$$\|u(t)\| \leq M_A e^{\omega_A t}\,\|x\| + M_A \int_0^t e^{\omega_A (t-s)}\,\|f(s)\|\,ds,\ \forall t \in [0,\tau_0].$$

3.7 The Non-Hille-Yosida Case

In this section we investigate the case when

$$Z = L^p((0,\tau_0), X) \text{ and } \Gamma(t,f) = \widehat{M}\left\|e^{\widehat{\omega}(t-.)} f(.)\right\|_{L^p((0,t),X)},$$

where $p \in [1,+\infty)$, $\widehat{M} > 0$, $\widehat{\omega} \in \mathbb{R}$, and $(X, \|.\|)$ is a Banach space. From now on, for any Banach space $(Y, \|.\|_Y)$ we denote by Y^* the space of continuous linear functionals on Y. We recall a result which will be used in the following (see Diestel and Uhl [107, p.97-98]).

Proposition 3.7.1. *Let* Z *be a Banach space and* $J \subset \mathbb{R}$ *be a non-empty open interval. Assume* $p,q \in [1,+\infty]$ *with* $1/p + 1/q = 1$.

(i) For each $q \in [1,+\infty]$ *and each* $\psi \in L^q(J, Z^*) \cap C(J, Z^*)$,

$$\|\psi\|_{L^q(J,Z^*)} = \sup_{\substack{\varphi \in C_c^\infty(J,Z) \\ \|\varphi\|_{L^p(J,Z)} \leq 1}} \int_J \psi(s)(\varphi(s))\,ds;$$

(ii) For each $p \in [1,+\infty)$ and for each $\varphi \in L^p(J,Z)$,

$$\|\varphi\|_{L^p(J,Z)} = \sup_{\substack{\psi \in C_c^\infty(J,Z^*) \\ \|\psi\|_{L^q(J,Z)^*} \leq 1}} \int_J \psi(s)(\varphi(s))\,ds.$$

Before proving Proposition 3.7.1, we make some comments about the main point in Proposition 3.7.1. We will use $C_c^\infty(I,Y^*)$ instead of $L^p((0,c),Y)^*$ because if $L^p((0,c),Y)^*$ is used, we would need a representation theorem for $L^p((0,c),Y)^*$ with $1 \leq p < \infty$. But we know (see Diestel and Uhl [107], Theorem 1 on p.98) that $L^p((0,c),Y)^* = L^q((0,c),Y^*)$ (with $p \in [1,+\infty)$ and $1/p+1/q=1$) if and only if Y^* has the Radon-Nikodym property. Recall that a Banach space Z has the Radon-Nikodym property if and only if every absolutely continuous function $F:\mathbb{R}_+ \to Y$ is differentiable almost everywhere (see Arendt et al. [22, p.19]). When Y is reflexive, Y^* has the Radon-Nikodym property. In practice if we take for example $Y = L^1((0,1),\mathbb{R})$, then $Y^* = L^\infty((0,1),\mathbb{R})$, but $L^\infty((0,1),\mathbb{R})$ does not have the Radon-Nikodym property (see Arendt et al. [22, Example 1.2.8 b) p.20]).

Proof. (i) Let $\psi \in C_c^0(J,Z^*)$ be fixed. By Holder inequality, we have

$$\|\psi\|_{L^p(J,Z)^*} = \sup_{\|\phi\|_{L^p(J,Z)}=1} \int_J \psi(\theta)(\phi(\theta))d\theta \leq \|\psi\|_{L^q(J,Z^*)}.$$

If $q=+\infty$, then there exists $t_0 \in J$ such that $\|\psi\|_{L^\infty(J,Z^*)} = \|\psi(t_0)\|_{Z^*}$. Let $\{e_n\}_{n\geq 0} \subset Z$ be a sequence such that $\|e_n\|=1, \forall n \geq 0$, and $\psi(t_0)(e_n) \to \|\psi(t_0)\|_{Z^*}$ as $n \to +\infty$. Let $\{\rho_n\}_{n\geq 0} \subset C_c^\infty(\mathbb{R},\mathbb{R})$ be a mollifiers (i.e., support$(\rho_n) \subset [-1/n,+1/n]$, $\int_{\mathbb{R}} \rho_n(s)ds = 1$, and $\rho_n \geq 0$). If we set

$$\varphi_n(s) = \rho_n(t_0 - s)e_n,\ \forall s \in J,\ \forall n \geq 0,$$

then $\|\varphi_n\|_{L^1(J,Z)} = 1$ for all $n \geq 0$ large enough, and

$$\int_J \psi(s)\varphi_n(s)ds = \int_J \rho_n(t_0-s)\psi(s)e_n ds \to \|\psi(t_0)\|_{Z^*} \text{ as } n \to +\infty.$$

If $q \in [1,+\infty)$, let $a,b \in J$ be fixed such that $a<b$ and support$(\psi) \subset [a,b]$. Set

$$t_k^n = a + k\frac{b-a}{n+1},\ \forall n \geq 0, \forall k = 0,...,n+1$$

and

$$\psi^n(s) = \sum_{k=0}^{n} \psi(t_k^n)1_{[t_k^n,t_{k+1}^n)}(s).$$

We have
$$\|\psi^n\|_{L^q(J,Z^*)} \to \|\psi\|_{L^q(J,Z^*)} \text{ as } n \to +\infty$$
and
$$\|\psi^n\|_{L^p(J,Z)^*} \to \|\psi\|_{L^p(J,Z)^*} \text{ as } n \to +\infty.$$
So it is sufficient to show that
$$\|\psi^n\|_{L^q(J,Z^*)} = \|\psi^n\|_{L^p(J,Z)^*}, \ \forall n \geq 0.$$
Let $n \geq 0$ be fixed. For all $\varphi \in L^p(J,Z) : \|\varphi\|_{L^p(J,Z)} \leq 1$, we have
$$\int_J \psi^n(s)(\varphi(s))\, ds = \sum_{k=0}^{n} \psi(t_k^n) \int_{t_k^n}^{t_{k+1}^n} \varphi(s) ds.$$
For each $k = 0, ..., n$, let $\{e_k^l\}_{l\geq 0} \subset Z$ such that
$$\psi(t_k^n) e_k^l \to \|\psi(t_k^n)\|_{Z^*} \text{ as } l \to +\infty.$$
We can assume that $\psi^n \neq 0$ and set
$$\varphi^l(s) = \frac{\sum_{\substack{k=0 \\ \psi(t_k^n)\neq 0}}^{n} e_k^l \, 1_{[t_k^n, t_{k+1}^n)}(s)}{\sum_{\substack{k=0 \\ \psi(t_k^n)\neq 0}}^{n} \left(t_{k+1}^n - t_k^n\right)} \quad \text{if } q = 1$$
and
$$\varphi^l(s) = \frac{\sum_{\substack{k=0 \\ \psi(t_k^n)\neq 0}}^{n} e_k^l \left\|\psi(t_k^n)\right\|_{Z^*}^{q-1} 1_{[t_k^n, t_{k+1}^n)}(s)}{\left(\sum_{\substack{k=0 \\ \psi(t_k^n)\neq 0}}^{n} \left(t_{k+1}^n - t_k^n\right)\left(\left\|\psi(t_k^n)\right\|_{Z^*}^{(q-1)}\right)^p\right)^{1/p}} \quad \text{if } q \in (1, +\infty).$$
Then
$$\left\|\varphi^l\right\|_{L^p(J,Z)} = 1, \ \forall l \geq 0$$
and
$$\int_J \psi^n(s)\left(\varphi^l(s)\right) ds \to \|\psi\|_{L^q(J,Z^*)} \text{ as } l \to +\infty.$$
(ii) Let $\varphi \in L^p(J,Z)$ be fixed. By using (i), it is sufficient to prove that

$$\|\varphi\|_{L^p(J,Z)} = \sup_{\substack{\psi \in C_c^\infty(J,Z^*) \\ \|\psi\|_{L^q(J,Z^*)} \leq 1}} \int_J \psi(s)\,(\varphi(s))\,ds =: |\varphi|_p\,.$$

Using Holder inequality, it is clear that

$$|\varphi|_p \leq \|\varphi\|_{L^p(J,Z)}\,,\ \forall \varphi \in L^p\,(J,Z)$$

and

$$\Big||\varphi|_p - |\widehat{\varphi}|_p\Big| \leq |\varphi - \widehat{\varphi}|_p \leq \|\varphi - \widehat{\varphi}\|_{L^p(J,Z)}\,,\ \forall \varphi, \widehat{\varphi} \in L^p\,(J,Z)$$

because $|\varphi + \widehat{\varphi}|_p \leq |\varphi|_p + |\widehat{\varphi}|_p\,, \forall \varphi, \widehat{\varphi} \in L^p\,(J,Z)$. Since $p \in [1,+\infty)$, we know that $C_c^0\,(J,Z)$ is dense in $L^p\,(J,Z)$, and it is sufficient to prove that for each $\varphi \in C_c^0\,(J,Z)$,

$$\|\varphi\|_{L^p(J,Z)} = |\varphi|_p\,.$$

Let $\varphi \in C_c^0\,(J,Z)$ be fixed. Let $a,b \in J$ with $a<b$, such that support$\,(\varphi) \subset [a,b]$. Set $t_k^n = a + k\frac{(b-a)}{n+1}$, $\forall k = 0,...,n+1$, and

$$\varphi^n(s) = \sum_{k=0}^{n} \varphi(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(s).$$

Then $\|\varphi - \varphi^n\|_{L^p(J,Z)} \to 0$ as $n \to +\infty$. So it is sufficient to prove for $\varphi(s) = \sum_{k=0}^n y_k 1_{[t_k^n, t_{k+1}^n)}(s)$ with $y_k \in Z, \forall k = 0,...,n$, that

$$\|\varphi\|_{L^p(J,Z)} = |\varphi|_p\,.$$

But

$$\|\varphi\|_{L^p(J,Z)} = \left(\sum_{k=0}^{n} \left(t_{k+1}^n - t_k^n\right) \|y_k\|^p \right)^{1/p} = \left(\sum_{k=0}^{n} \left(\left(t_{k+1}^n - t_k^n\right)^{1/p} \|y_k\| \right)^p \right)^{1/p}$$

and for each $\psi \in C_c^\infty\,(J,Z^*)$,

$$\int_J \psi(s)\varphi(s)ds = \sum_{k=0}^{n} \int_{t_k^n}^{t_{k+1}^n} \psi(s) y_k ds.$$

Let $\chi(s) = \sum_{k=0}^n z_k 1_{[t_k^n, t_{k+1}^n)}(s)$ with $z_k \in Z, \forall k = 0,...,n$. We have

$$\|\chi\|_{L^q(J,Z^*)} = \left(\sum_{k=0}^{n} \left(\left(t_{k+1}^n - t_k^n\right)^{1/q} \|z_k\|_{Z^*} \right)^q \right)^{1/q}$$

and

$$\int_J \chi(s)\varphi(s)ds = \sum_{k=0}^{n} \left(t_{k+1}^n - t_k^n\right) z_k\left(y_k\right) = \sum_{k=0}^{n} \left(t_{k+1}^n - t_k^n\right)^{1/q} z_k\left(\left(t_{k+1}^n - t_k^n\right)^{1/p} y_k\right).$$

Applying the Hahn-Banach theorem in $E = Y^{n+1}$ endowed with the norm $\|y\|_p = \left(\sum_{k=0}^{n} \left(\|y_k\|_Z\right)^p\right)^{1/p}$, and $E^* = (Y^*)^{n+1}$ endowed with the norm $\|z\|_q = \left(\sum_{k=0}^{n} \left(\|z_k\|_{Z^*}\right)^q\right)^{1/q}$, we can find $\chi(s) = \sum_{k=0}^{n} z_k 1_{\left[t_k^n, t_{k+1}^n\right)}(s)$ with $\|\chi\|_{L^q(J,Z^*)} = 1$, such that

$$\|\varphi\|_{L^p(J,Z)} = \int_J \chi(s)\varphi(s)ds.$$

On the other hand, if we set

$$\chi^n(s) = \sum_{k=0}^{n} z_k \int_{-\infty}^{+\infty} \rho_n(s-t) 1_{\left[t_k^n, t_{k+1}^n\right)}(t)dt, \ \forall s \in J$$

where $\{\rho_n\}_{n\geq 0}$ is a mollifiers, then for all $n > 0$ large enough, $\chi^n \in C_c^\infty\left(J, Z^*\right)$,

$$\|\chi^n - \chi\|_{L^p(J,Z^*)} \to 0 \text{ as } n \to +\infty$$

and

$$\int_J \chi^n(s)\varphi(s)ds \to \int_J \chi(s)\varphi(s)ds = \|\varphi\|_{L^p(J,Z)} \text{ as } n \to +\infty.$$

Hence, we have $\varphi^n = \frac{1}{\|\chi^n\|_{L^p(J,Z^*)}}\chi^n$ for all $n \geq 0$ large enough and

$$\int_J \varphi^n(s)\varphi(s)ds \to \|\varphi\|_{L^p(J,Z)} \text{ as } n \to +\infty.$$

The result follows. □

From now on, denote

$$abs\left(f\right) := \inf\left\{\delta > 0 : e^{-\delta .} f(.) \in L^1\left((0,+\infty), X\right)\right\} < +\infty$$

and define the *Laplace transform* of f by

$$\mathscr{L}\left(f\right)\left(\lambda\right) = \int_0^{+\infty} e^{-\lambda s} f(s)ds$$

when $\lambda > abs\left(f\right)$. We first give a necessary condition for the L^p case when $p \in [1,+\infty]$.

Lemma 3.7.2. *Let Assumption 3.4.1 be satisfied and let $p, q \in [1,+\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that there exist $\widehat{M} > 0$ and $\widehat{\omega} \in \mathbb{R}$, so that $\forall t \geq 0$, $\forall f \in C^1\left([0,t], X\right)$,*

$$\|(S_A \diamond f)(t)\| \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.)\right\|_{L^p((0,t),X)}. \tag{3.7.1}$$

Then there exists a subspace $E \subset X_0^*$, for each $x^* \in E$ there exists $V_{x^*} \in L^q([0,+\infty), X^*) \cap C([0,+\infty), X^*)$ such that

$$x^*\left(\left(\lambda I - (A - \widehat{\omega} I)\right)^{-1} x\right) = \int_0^{+\infty} e^{-\lambda s} V_{x^*}(s) x ds, \tag{3.7.2}$$

when $\lambda > 0$ is sufficiently large,

$$x^*\left(S_{(A-\widehat{\omega}I)}(t)x\right) = \int_0^t V_{x^*}(s) x ds,\ \forall t \geq 0, \tag{3.7.3}$$

$$\sup_{x^* \in E : \|x^*\|_{X_0^*} \leq 1} \|V_{x^*}\|_{L^q([0,+\infty),X^*)} \leq \widehat{M},\ \forall t \geq 0,$$

and

$$\|x\| \leq \sup_{x^* \in E : \|x^*\|_{X_0^*} \leq M_A} x^*(x), \forall x \in X_0, \tag{3.7.4}$$

where $M_A > 0$ is the constant introduced in Assumption 3.4.1.

Proof. Set

$$B = \left\{ (\lambda - \omega)^2 y^* \circ (\lambda I - A_0)^{-2} : y^* \in X_0^*,\ \|y^*\|_{X_0^*} \leq 1, \text{ and } \lambda > \omega \right\}.$$

From Assumption 3.6.1, we obtain $\sup\left\{ \|x^*\|_{X_0^*} : x^* \in B \right\} \leq M$ and

$$\lim_{\lambda \to +\infty} (\lambda - \omega)^2 (\lambda I - A_0)^{-2} x = x, \forall x \in X_0.$$

Using the Hahn-Banach Theorem, we have

$$\|x\| \leq \sup_{x^* \in B} x^*(x).$$

Let E be the subspace of X_0^* generated by B. Then

$$\|x\| \leq \sup_{x^* \in B} x^*(x) \leq \sup_{x^* \in E : \|x^*\|_{X_0^*} \leq M} x^*(x)$$

and (3.7.4) is satisfied.

Let $y^* \in X_0^*$ be fixed such that $\|y^*\|_{X_0^*} \leq 1$ and let $\mu > \omega$. Set

$$x^* := (\mu - \omega)^2 y^* \circ (\mu I - A_0)^{-2}.$$

Then for $\lambda > \widehat{\omega} + \max(0, \omega)$, we have for each $x \in X$ that

$$\begin{aligned} &x^*((\lambda I - (A - \widehat{\omega} I))^{-1} x) \\ &\quad = (\mu - \omega)^2 y^* \left((\mu I - A_0)^{-1} (\lambda - (A_0 - \widehat{\omega} I))^{-1} (\mu I - A)^{-1} x \right) \end{aligned}$$

$$= (\mu-\omega)^2 y^* \left((\mu I - A_0)^{-1} \int_0^{+\infty} e^{-(\lambda+\widehat{\omega})t} T_{A_0}(t) (\mu I - A)^{-1} x dt \right).$$

So

$$x^*((\lambda I - (A - \widehat{\omega} I))^{-1} x) = \int_0^{\infty} e^{-\lambda t} V_{x^*}(t) x dt$$

with

$$V_{x^*}(t) = e^{-\widehat{\omega} t} (\mu-\omega)^2 y^* \circ (\mu I - A_0)^{-1} \circ T_{A_0}(t) \circ (\mu I - A)^{-1}, \forall t \geq 0.$$

Since

$$T_{A_0}(t)x = x + A_0 \int_0^t T_{A_0}(l) x dl$$

and $A_0 (\mu I - A_0)^{-1}$ is bounded, it follows that $t \to (\mu I - A_0)^{-1} T_{A_0}(t)$ is continuous from $[0,+\infty)$ into $\mathscr{L}(X_0)$ and is exponentially bounded. Thus $t \to V_{x^*}(t)$ is Bochner measurable from $[0,+\infty)$ into X^* and belongs to $L^1_{\text{Loc}}([0,+\infty), X^*)$. Moreover, for each $f \in C^1([0,t], X)$, we have

$$\begin{aligned}
&x^*((S_A \diamond f)(t)) \\
&\quad = (\mu-\omega)^2 \int_0^t x^* \circ (\mu I - A_0)^{-1} \circ T_{A_0}(t-s) \circ (\mu I - A)^{-1} (f(s)) ds \\
&\quad = \int_0^t V_{x^*}(t-s) e^{\widehat{\omega}(t-s)} f(s) ds.
\end{aligned} \tag{3.7.5}$$

Since E is the set of all finite linear combinations of elements of B, it follows that (3.7.2), (3.7.3) and (3.7.5) are satisfied for each $x^* \in E$. Let $x^* \in E$ with $\|x^*\|_{X_0^*} \leq 1$. We have from (3.7.1) that

$$\int_0^t V_{x^*}(t-s) e^{\widehat{\omega}(t-s)} f(s) ds = x^*((S_A \diamond f)(t)) \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.) \right\|_{L^p((0,t),X)}.$$

Using Proposition 3.7.1-(i), we have

$$\|V_{x^*}\|_{L^q((0,t),X^*)} \leq \widehat{M}, \forall t \geq 0.$$

This completes the proof. □

Theorem 3.7.3. *Let Assumption 3.4.1 be satisfied. Let $B : \overline{D(A)} \to Y$ be a bounded linear operator from $\overline{D(A)}$ into a Banach space $(Y, \|.\|_Y)$ and $\chi : (0,+\infty) \to \mathbb{R}$ a non-negative measurable function with abs$(\chi) < +\infty$. Then the following assertions are equivalent:*

(i) $\|B(S_A \diamond f)(t)\| \leq \int_0^t \chi(t-s) \|f(s)\| ds, \forall t \geq 0, \forall f \in C^1([0,+\infty), X)$;
(ii) $\left\| B(\lambda I - A)^{-n} \right\|_{\mathscr{L}(X,Y)} \leq \frac{1}{(n-1)!} \int_0^{+\infty} s^{n-2} e^{-\lambda s} \chi(s) ds$, $\forall \lambda > \delta$, $\forall n \geq 1$;
(iii) $\|B[S_A(t+h) - S_A(t)]\|_{\mathscr{L}(X,Y)} \leq \int_t^{t+h} \chi(s) ds$, $\forall t, h \geq 0$.

Moreover, if one of the above three conditions is satisfied, $\chi \in L^q_{\text{Loc}}([0,+\infty), \mathbb{R})$ for some $q \in [1,+\infty]$ and $p \in [1,+\infty)$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then for each $\tau > 0$ and each

$f \in L^p((0,\tau),X)$, the map $t \to B(S_A * f)(t)$ is continuously differentiable and

$$\left\| \frac{d}{dt} B(S_A * f)(t) \right\| \le \int_0^t \chi(t-s) \|f(s)\| \, ds, \; \forall t \in [0,\tau].$$

Proof. (i)⇒(ii). Let $x \in X$ be fixed. From the formula

$$(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda l} S_A(l) x dl, \;\; \forall \lambda > \delta$$

one deduces that

$$n! (\lambda I - A)^{-(n+1)} x = (-1)^n \frac{d^n (\lambda I - A)^{-1}}{d\lambda^n} = \int_0^{+\infty} \left[\lambda l^n - n l^{n-1}\right] e^{-\lambda l} S_A(l) x dl.$$

We also remark that

$$-\int_0^t l^n e^{-\lambda l} S_A(l) x dl = \int_0^t S_A(l) f(t, t-l) \, dl = (S_A * f(t,.))(t),$$

where

$$f(t,s) = h(t-s) x \text{ with } h(l) = -l^n e^{-\lambda l}.$$

It follows that

$$-t^n e^{-\lambda t} S_A(t) = \frac{d}{dt} \left[(S_A * f(t,.))(t)\right] = (S_A \diamond f(t,.))(t) + \left(S_A * \frac{\partial f(t,.)}{\partial t}\right)(t),$$

so for all $\lambda > 0$ large enough

$$\lim_{t \to +\infty} (S_A \diamond f(t,.))(t) = -\lim_{t \to +\infty} \left(S_A * \frac{\partial f(t,.)}{\partial t}\right)(t).$$

But

$$\left(S_A * \frac{\partial f(t,.)}{\partial t}\right)(t) = \int_0^t S_A(l) h'(t-(t-l)) \, dl = \int_0^t \left[\lambda l^n - n l^{n-1}\right] e^{-\lambda l} S_A(l) x dl,$$

so we have

$$n! (\lambda I - A)^{-(n+1)} = \lim_{t \to +\infty} \left(S_A * \frac{\partial f(t,.)}{\partial t}\right)(t) = -\lim_{t \to +\infty} (S_A \diamond f(t,.))(t).$$

Now by using (i), it follows that

$$\begin{aligned} \left\| n! B (\lambda I - A)^{-(n+1)} x \right\| &= \lim_{t \to +\infty} \|B(S_A \diamond f(t,.))(t)\| \\ &\le \lim_{t \to +\infty} \int_0^t \chi(l) \|f(t, t-l)\| \, dl \end{aligned}$$

$$= \int_0^{+\infty} l^{n-1} e^{-\lambda l} \chi(l)\, dl\, \|x\|$$

and (ii) follows.

(ii)⇒(i). Let $f \in C^1([0,+\infty), X)$ be fixed. Without loss of generality we assume that f is exponentially bounded. Note that

$$\begin{aligned}(\lambda I - A)^{-1} \mathscr{L}(f)(\lambda) &= \lambda \int_0^{+\infty} e^{-\lambda l} S_A(l) dl \int_0^{+\infty} e^{-\lambda l} f(l) dl \\ &= \lambda \int_0^{+\infty} e^{-\lambda l} (S_A * f)(l) dl.\end{aligned}$$

Integrating by parts we obtain that

$$\int_0^{+\infty} e^{-\lambda l} (S_A \diamond f)(l) dl = (\lambda I - A)^{-1} \mathscr{L}(f)(\lambda).$$

Then

$$\frac{d^n}{d\lambda^n} \int_0^{+\infty} e^{-\lambda l} (S_A \diamond f)(l) dl = \sum_{k=0}^{n} C_n^k \frac{d^{n-k} (\lambda I - A)^{-1}}{d\lambda^{n-k}} \frac{d^k}{d\lambda^k} \mathscr{L}(f)(\lambda)$$

and

$$\begin{aligned}&\left\| \frac{d^n}{d\lambda^n} \int_0^{+\infty} e^{-\lambda l} B(S_A \diamond f)(l) dl \right\| \\ &\quad \leq \sum_{k=0}^{n} C_n^k \left\| \frac{d^{n-k} B(\lambda I - A)^{-1}}{d\lambda^{n-k}} \frac{d^k \mathscr{L}(f)(\lambda)}{d\lambda^k} \right\| \\ &\quad = \sum_{k=0}^{n} C_n^k (n-k)! \left\| B(\lambda I - A)^{-(n-k+1)} \right\| (-1)^k \frac{d^k \mathscr{L}(\|f\|)(\lambda)}{d\lambda^k}.\end{aligned}$$

Now using (ii) it follows that

$$\begin{aligned}&\left\| \frac{d^n}{d\lambda^n} \int_0^{+\infty} e^{-\lambda l} B(S_A \diamond f)(l) dl \right\| \\ &\quad \leq (-1)^n \sum_{k=0}^{n} C_n^k \frac{d^{n-k} \mathscr{L}(\chi)(\lambda)}{d\lambda^{n-k}} \frac{d^k \mathscr{L}(\|f\|)(\lambda)}{d\lambda^k} \\ &\quad = (-1)^n \frac{d^n}{d\lambda^n} \int_0^{+\infty} e^{-\lambda l} (\chi * \|f\|)(l) dl\end{aligned}$$

and by the Post-Widder Theorem (see Arendt et al. [22]) we obtain

$$\|B(S_A \diamond f)(t)\| \leq (\chi * \|f\|)(t), \forall t \geq 0.$$

So we obtain (i) for all the maps f in $C^1([0,+\infty), X)$.

(iii)⇒(ii). First assume that $n = 1$. We have

$$B(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda s} BS(s)x ds.$$

Using (iii), we obtain

$$\left\| B(\lambda I - A)^{-1} \right\| \leq \lambda \int_0^{+\infty} e^{-\lambda s} \int_0^s \chi(l)\, dl ds$$

and by integrating by parts (ii) follows. Next assume that $n \geq 2$. We have

$$\begin{aligned}
B(\lambda I - A)^{-n} &= B(\lambda I - A_0)^{-(n-1)} (\lambda I - A)^{-1} \\
&= \frac{(-1)^{n-2}}{(n-2)!} \lambda B \left(\frac{d^{n-2} (\lambda I - A_0)^{-1}}{d\lambda^{n-2}} \right) \int_0^{+\infty} e^{-\lambda s} S_A(s) ds \\
&= \frac{\lambda}{(n-2)!} B \int_0^{+\infty} s^{n-2} e^{-\lambda s} T_{A_0}(s) ds \int_0^{+\infty} e^{-\lambda s} S_A(s) ds \\
&= \frac{\lambda}{(n-2)!} B \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} T_{A_0}(s-l) S_A(l) dl ds.
\end{aligned}$$

But $T_{A_0}(s-l)S_A(l) = S_A(s) - S_A(s-l)$, so

$$B(\lambda I - A)^{-n} = \frac{\lambda}{(n-2)!} \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} \left[BS_A(s) - BS_A(s-l) \right] dl ds.$$

From (iii), we obtain

$$\left\| B(\lambda I - A)^{-n} \right\|_{\mathscr{L}(X)} \leq \frac{\lambda}{(n-2)!} \int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} \int_{s-l}^s \chi(r)\, dr dl ds.$$

Notice that

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-2} \int_{s-l}^s \chi(r)\, dr dl ds \\
&\quad = \int_0^{+\infty} e^{-\lambda s} \int_0^s l^{n-2} \int_l^s \chi(r)\, dr dl ds \\
&\quad = \int_0^{+\infty} e^{-\lambda s} \int_0^s \int_0^r l^{n-2} dl \chi(r)\, dr ds \\
&\quad = \frac{1}{n-1} \int_0^{+\infty} e^{-\lambda s} \int_0^s r^{n-1} \chi(r)\, dr ds,
\end{aligned}$$

integrating by parts, we have

$$\int_0^{+\infty} e^{-\lambda s} \int_0^s (s-l)^{n-1} \int_{s-l}^s \chi(r)\, dr dl ds = \frac{1}{(n-1)\lambda} \int_0^{+\infty} s^{n-1} \chi(s) e^{-\lambda s} ds.$$

It follows that

$$\left\| B\left(\lambda I - A\right)^{-n} \right\|_{\mathscr{L}(X)} \leq \frac{1}{(n-1)!} \int_0^{+\infty} s^{n-1} \chi\left(s\right) e^{-\lambda s} ds.$$

(i)⇒(iii). Let $h > 0$ and $t > h$ be fixed. We have

$$\begin{aligned}
\frac{d}{dt}\left(S_A * 1_{[0,h]}(.)x\right)(t) &= \frac{d}{dt}\int_0^t S_A(t-s)1_{[0,h]}(s)xds \\
&= \frac{d}{dt}\int_0^h S_A(t-s)xds = \frac{d}{dt}\int_{t-h}^t S_A(s)xds \\
&= S_A(t)x - S_A(t-h)x.
\end{aligned}$$

Let $\{\phi_n\}_{n\geq 0} \subset C^1\left(\mathbb{R}_+, \mathbb{R}\right)$ be a sequence of non-increasing functions such that

$$\phi_n(t) \begin{cases} = 1 & \text{if } t \in [0,h], \\ \in [0,h] & \text{if } t \in \left[h, h+\frac{1}{n+1}\right], \\ = 0 & \text{if } t \geq h + \frac{1}{n+1}. \end{cases}$$

We can always assume that $\phi_{n+1} \leq \phi_n, \forall n \geq 0$. Then we have

$$\begin{aligned}
\frac{d}{dt}\left(S_A * \phi_n(.)x\right)(t) &= \frac{d}{dt}\int_0^t S_A(s)\phi_n(t-s)xds \\
&= S_A(t)\phi_n(0)x + \int_0^t S_A(s)\phi_n^{'}(t-s)xds \\
&= S_A(t)x + \int_0^t S_A(t-s)\phi_n^{'}(s)xds \\
&= S_A(t)x + \int_0^{h+\frac{1}{n+1}} S_A(t-s)\phi_n^{'}(s)xds.
\end{aligned}$$

By the continuity of $t \to S_A(t)x$, it follows that

$$\lim_{n\to+\infty} \frac{d}{dt}\left(S_A * \phi_n(.)x\right)(t) = S_A(t)x - S_A(t-h)x.$$

On the other hand, we have $\chi\mid_{[0,t]} \in L^1\left((0,t), \mathbb{R}\right)$, and $s \to \chi(t-s)\phi_n(s)$ is a non-increasing sequence in $L^1\left((0,t), \mathbb{R}\right)$. So by the Beppo-Levi (Monotone Convergence) Theorem, we obtain

$$\begin{aligned}
\lim_{n\to+\infty}\int_0^t \chi(t-s)\phi_n(s)ds &= \int_0^t \chi(t-s)1_{[0,h]}(s)ds = \int_0^h \chi(t-s)ds \\
&= \int_{t-h}^t \chi(l)dl,
\end{aligned}$$

and (iii) follows from (i). The proof of the last part of the theorem is similar to the proof of Theorem 3.4.7. □

Remark 3.7.4. When $B = I$ and $X_1 = X$, the previous theorem provides an extension of the Hille-Yosida case. Unfortunately, this kind property is not satisfied in the context of age-structured models. Because if the property (iii) were satisfied for some function $\chi \in L^q_{\text{Loc}}([0,+\infty),\mathbb{R})$, this implies that $t \to S_A(t)$ is locally of bounded L^q-variation from $[0,+\infty)$ into $\mathscr{L}(X)$.

Following Bochner and Taylor [44], we now consider functions of bounded L^p-variation. Let J be an interval in $\mathbb{R}$ with interior $\overset{\circ}{J}$. Let $H : \overset{\circ}{J} \to X$ be a map. If $p \in [1,+\infty)$, set

$$VL^p(J,H) = \sup_{\substack{t_0<t_1<\ldots<t_n \\ t_i \in \overset{\circ}{J}, \forall i=1,\ldots,n}} \left\{ \left(\sum_{i=1}^{n} \frac{\|H(t_i) - H(t_{i-1})\|^p}{|t_i - t_{i-1}|^{p-1}} \right)^{1/p} \right\},$$

where the supremum is taken over all finite strictly increasing sequences in $\overset{\circ}{J}$. If $p = +\infty$, set

$$VL^\infty(J,H) = \sup_{t,s \in \overset{\circ}{J}} \left\{ \frac{\|H(t) - H(s)\|}{|t-s|} \right\}.$$

We say that H is of *bounded L^p-variation* on J if $VL^p(J,H) < +\infty$.

Let $(Y, \|.\|_Y)$ be a Banach space. Let $H : J \to \mathscr{L}(X,Y)$ and $f : J \to X$. If π is a finite sequence $t_0 < t_1 < \ldots < t_n$ in $\overset{\circ}{J}$ and $s_i \in [t_{i-1}, t_i]$ $(i = 1, \ldots, n)$, we denote by

$$\begin{aligned} S(dH, f, \pi) &= \sum_{i=1}^{n} (H(t_i) - H(t_{i-1})) f(s_i), \\ S(H, df, \pi) &= \sum_{i=1}^{n} H(s_i) [f(t_i) - f(t_{i-1})], \\ |\pi| &= \max_{i=0,\ldots,n} |t_i - t_{i-1}|. \end{aligned}$$

We say that f is *Riemann-Stieltjes integrable* with respect to H if

$$\int_a^b dH(t) f(t) := \lim_{|\pi| \to 0, t_0 \to a, t_n \to b} S(dH, f, \pi) \text{ exists}$$

and H is *Riemann-Stieltjes integrable* with respect to f if

$$\int_a^b H(t) df(t) := \lim_{|\pi| \to 0, t_0 \to a, t_n \to b} S(H, df, \pi) \text{ exists.}$$

We say that f is *Riemann integrable* on $[a,b]$ if f is Riemann-Stieltjes integrable with respect to $H(t) = tId_X$, and we write

$$\int_a^b f(t) dt := \int_a^b dH(t) f(t) = \lim_{|\pi| \to 0, t_0 \to a, t_n \to b} S(dH, f, \pi).$$

Note that

$$\begin{aligned}
S(dH,f,\pi) &= \sum_{i=1}^{n} \left(H\left(t_i\right) - H\left(t_{i-1}\right)\right) f\left(s_i\right) \\
&= \left(H\left(t_n\right) - H\left(t_{n-1}\right)\right) f\left(s_n\right) + ... + \left(H\left(t_1\right) - H\left(t_0\right)\right) f\left(s_1\right) \\
&= H(t_n)f(t_n) - H(t_0)f(t_0) \\
&\quad - \begin{bmatrix} H\left(t_n\right)\left(f(t_n) - f\left(s_n\right)\right) + H\left(t_{n-1}\right)\left(f\left(s_n\right) - f\left(s_{n-1}\right)\right) + ... \\ ... + H(t_0)\left[f\left(s_1\right) - f\left(t_0\right)\right] \end{bmatrix}
\end{aligned}$$

so

$$S(dH,f,\pi) = H(t_n)f(t_n) - H(t_0)f(t_0) - S(H,df,\widehat{\pi}) \tag{3.7.6}$$

with $\widehat{\pi} = \{t_0, s_1, s_2, ..., s_n, t_n\}$.

By using (3.7.6) we immediately deduce the following result.

Lemma 3.7.5. *Let $f : [a,b] \to X$ and $H : [a,b] \to \mathscr{L}(X,Y)$ be two maps. Then the following assertions are equivalent:*

(a) f is Riemann-Stieltjes integrable with respect to H;
(b) H is Riemann-Stieltjes integrable with respect to f.

Moreover, if (a) or (b) is satisfied, we have

$$\int_a^b dH(t)f(t) = H(b)f(b) - H(a)f(a) - \int_a^b H(t)df(t).$$

We have the following result (see Section 1.9 in Arendt et al. [22] and Section III.4.3 in Hille and Phillips [187] for more details).

Lemma 3.7.6. *Assume $p,q \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let $f \in C^1([a,b],X)$. Let $H : [a,b] \to \mathscr{L}(X,Y)$ be a bounded and strongly continuous map. Then f is Riemann-Stieltjes integrable with respect to H and*

$$\int_a^b dH(t)f(t) = H(b)f(b) - H(a)f(a) - \int_a^b H(t)f'(t)dt,$$

where the last integral is a Riemann integral.

Proof. Since H is bounded and strongly continuous, the map $t \to H(t)\left(f'(t)\right)$ is continuous, so the integral

$$\int_a^b H(t)f'(t)dt$$

is well defined as a Riemann integral. It remains to prove that

$$\lim_{|\pi|\to 0, t_0 \to a, t_n \to b} \left[S(d\left(tId_Y\right), H(.)f'(.), \pi) - S(H(.), df(.), \pi)\right] = 0.$$

We have

$$S(d\left(tId_Y\right), H(.)f'(.), \pi) - S(H(.), df(.), \pi)$$

$$= \sum_{i=1}^{n} (t_i - t_{i-1}) H(s_i) f'(s_i) - \sum_{i=1}^{n} H(s_i) [f(t_i) - f(t_{i-1})]$$
$$= \sum_{i=1}^{n} H(s_i) \left[(t_i - t_{i-1}) f'(s_i) - \int_{t_{i-1}}^{t_i} f'(l) \, dl \right].$$

Let $\varepsilon > 0$ be fixed. Set $C := \sup_{t \in [a,b]} \|H(t)\|_{\mathscr{L}(X,Y)}$. Since f' is continuous, f' is uniformly continuous on $[a,b]$, there exists $\eta > 0$ such that $|t-s| \leq \eta \Rightarrow \|f'(t) - f'(s)\| \leq \frac{\varepsilon}{(C+1)(b-a)}$, and we obtain that

$$|\pi| \leq \eta \Rightarrow \left\| S(d(tId_Y), H(.)f'(.), \pi) - S(H(.), df(.), \pi) \right\| \leq \varepsilon.$$

This completes the proof. □

Lemma 3.7.7. *Let $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in C^1([a,b], X)$. Let $H : [a,b] \to \mathscr{L}(X,Y)$ be a bounded and strongly continuous map. Assume in addition that H is of bounded L^q-variation on $[a,b]$. Then*

$$\left\| \int_a^b dH(t) f(t) \right\| \leq VL^q([a,b], H) \|f\|_{L^p((a,b),X)}.$$

Proof. Assume for simplicity that $q \in (1, +\infty)$, the case $q = 1$ or $q = +\infty$ is similar. We have

$$\begin{aligned} \|S(dH, f, \pi)\| &\leq \sum_{i=1}^{n} \|H(t_i) - H(t_{i-1})\|_{\mathscr{L}(X,Y)} \|f(s_i)\| \\ &= \sum_{i=1}^{n} \frac{\|H(t_i) - H(t_{i-1})\|_{\mathscr{L}(X,Y)}}{|t_i - t_{i-1}|^{1-\frac{1}{q}}} |t_i - t_{i-1}|^{\frac{1}{p}} \|f(s_i)\| \\ &\leq \left(\sum_{i=1}^{n} \frac{\|H(t_i) - H(t_{i-1})\|_{\mathscr{L}(X,Y)}^q}{|t_i - t_{i-1}|^{q-1}} \right)^{1/q} \left(\sum_{i=1}^{n} |t_i - t_{i-1}| \|f(s_i)\|^p \right)^{1/p}. \end{aligned}$$

So we obtain

$$\|S(dH, f, \pi)\| \leq VL^q([a,b], H) \left(\sum_{i=1}^{n} |t_i - t_{i-1}| \|f(s_i)\|^p \right)^{1/p}$$

and the result follows when $|\pi| \to 0$. □

Motivated by Lemma 3.7.2, we introduce the following definition.

Definition 3.7.8. Let $(Y, \|.\|_Y)$ be a Banach space. Let E be a subspace of Y^*. E is called a *norming space* of Y if the map $|.|_E : Y \to R_+$ defined by

$$|y|_E = \sup_{\substack{y^* \in E \\ \|y^*\|_{Y^*} \leq 1}} y^*(y), \ \forall y \in Y$$

is a norm equivalent to $\|.\|_Y$.

The main result of this section is the following theorem.

Theorem 3.7.9. *Let Assumption 3.6.1 be satisfied. Let $p,q \in [1,+\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\widehat{\omega} \in \mathbb{R}$. Then the following properties are equivalent:*

(i) There exists $\widehat{M}>0$, such that for each $\tau_0 \geq 0$, $\forall f \in C^1([0,\tau_0],X)$,

$$\|(S_A \diamond f)(t)\| \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.) \right\|_{L^p((0,t),X)}, \ \forall t \in [0,\tau_0];$$

(ii) There exists a norming space E of X_0, such that for each $x^* \in E$ the map $t \to x^* \circ S_{A+\omega I}(t)$ is of bounded L^q-variation from $[0,+\infty)$ into X^* and

$$\sup_{x^* \in E: \|x^*\|_{X_0^*} \leq 1} \lim_{t \to +\infty} VL^q([0,t], x^* \circ S_{A-\widehat{\omega}I}(.)) < +\infty; \tag{3.7.7}$$

(iii) There exists a norming space E of X_0, such that for each $x^* \in E$ there exists $\chi_{x^*} \in L^q_+((0,+\infty),\mathbb{R})$,

$$\|x^* \circ S_{A-\widehat{\omega}I}(t+h) - x^* \circ S_{A-\widehat{\omega}I}(t)\|_{X^*} \leq \int_t^{t+h} \chi_{x^*}(s)\,ds, \ \forall t,h \geq 0 \tag{3.7.8}$$

and

$$\sup_{x^* \in E: \|x^*\|_{X_0^*} \leq 1} \|\chi_{x^*}\|_{L^q((0,+\infty),\mathbb{R})} < +\infty. \tag{3.7.9}$$

Proof. (i)⇒(iii) is an immediate consequence of Lemma 3.7.2. (iii)⇒(ii) is an immediate consequence of the fact that (iii) implies

$$VL^q([0,t], x^* \circ S_{A-\widehat{\omega}I}(.)) \leq \|\chi_{x^*}\|_{L^q((0,t),\mathbb{R})}, \forall t \geq 0.$$

So it remains to prove (ii)⇒(i). Let $x^* \in E$ and $f \in C^1((0,\tau_0),X)$ be fixed. By Lemma 3.4.4, we have for each $t \in [0,\tau_0]$ that

$$\frac{d}{dt}(S_A * f)(t) = S_A(t)f(0) + \int_0^t S_A(s)f'(t-s)ds = \int_0^t dS_A(s)f(t-s)ds.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}(S_A * f)(t) &= \lim_{\lambda \to +\infty} \lambda (\lambda I - A_0)^{-1} \frac{d}{dt}(S_A * f)(t) \\
&= \lim_{\lambda \to +\infty} \lambda \int_0^t T_{A_0}(t-s)(\lambda I - A)^{-1} f(s)ds \\
&= \lim_{\lambda \to +\infty} \lambda \int_0^t T_{A_0-\widehat{\omega}I}(t-s)(\lambda I - A)^{-1} e^{\widehat{\omega}(t-s)} f(s)ds \\
&= \frac{d}{dt}\left(S_{A-\widehat{\omega}I} * e^{\widehat{\omega}(t-.)} f(.)\right)(t)
\end{aligned}$$

$$= \int_0^t dS_{A-\widehat{\omega}I}(s)e^{\widehat{\omega}(t-s)}f(t-s)ds.$$

By using the last part of Lemma 3.7.7, we have

$$\begin{aligned} x^*\left(\frac{d}{dt}(S_A * f)(t)\right) &= \int_0^t d\left(x^* \circ S_{A-\widehat{\omega}I}\right)(s)e^{\widehat{\omega}(t-s)}f(t-s) \\ &\leq VL^q\left([0,t],\left(x^* \circ S_{A-\widehat{\omega}I}\right)(.)\right)\left\|e^{\widehat{\omega}(t-.)}f(.)\right\|_{L^p((0,t),X_1)}. \end{aligned}$$

Hence, $\forall t \in [0,\tau_0]$ we have

$$x^*\left(\frac{d}{dt}(S_A * f)(t)\right) \leq VL^q\left([0,+\infty),\ \left(x^* \circ S_{A-\widehat{\omega}I}\right)(.)\right)\left\|e^{\widehat{\omega}(t-.)}f\right\|_{L^p((0,t),X)}$$

and the result follows from the fact that E is a norming space. $\square$

Remark 3.7.10. (a) We can use Theorem 3.7.3 to replace (iii) by the equivalent condition

$$\left\|x^* \circ (\lambda-(A-\widehat{\omega}I))^{-n}\right\|_{X^*} \leq \frac{1}{(n-1)!}\int_0^{+\infty} s^{n-1}e^{-\lambda s}\chi_{x^*}(s)\,ds, \forall \lambda > \delta, \forall n \geq 1. \tag{3.7.10}$$

(b) We know that

$$(\lambda I - A)^{-1}x = \lambda \int_0^{+\infty} e^{-\lambda s}S_A(s)xds$$

for $\lambda > 0$ sufficiently large. So we can also apply the results of Weis [370] to verify assertion (iii) of Theorem 3.7.9.

(c) In the Hille-Yosida case, assertions (ii) and (iii) of Theorem 3.7.9 are satisfied for $q=+\infty$, $E=X_0^*$, and $\chi_{x^*}(s)=M, \forall s \geq 0$.

(d) In the context of age-structured models in L^p spaces the property (iii) holds. But in some cases we have

$$\|S_{A-\omega I}(t+h)-S_{A-\omega I}(t)\|_{\mathscr{L}(X)} \geq \left(\int_t^{t+h} e^{p\omega l}dl\right)^{1/p},\ \ \forall t,h \geq 0.$$

So $t \to S_{A-\omega I}(t)$ is not of bounded L^q-variation. Nevertheless, we will see that assertion (iii) in Theorem 3.7.9 is satisfied. This shows that a dual approach is necessary in general.

3.8 Applications to a Vector Valued Age-Structured Model in L^p

Let $p \in [1,+\infty)$ and $a_0 \in (0,+\infty]$ be fixed. We are now interested in solutions $v \in C\left([0,\tau_0], L^p\left((0,a_0),Y\right)\right)$ of the following problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial a} = A(a)v(t,a) + g(t,a), \ a \in (0,a_0), \\ v(t,0) = h(t), \\ v(0,\cdot) = \psi \in L^p((0,a_0),Y), \end{cases} \tag{3.8.1}$$

where

$$h \in L^p((0,\tau_0),Y) \text{ and } g \in L^1((0,\tau_0),L^p((0,a_0),Y)).$$

In order to apply the results obtained in Sections 3.2-3.7 to study the age structured problem (3.8.1) in L^p, as in Thieme [330, 331], we assume that the family of linear operators $\{A(a)\}_{0\leq a\leq a_0}$ generates an exponentially bounded evolution family $\{U(a,s)\}_{0\leq s\leq a<a_0}$. We refer to Kato and Tanabe [206], Acquistapace and Terreni [2], Acquistapace [1], and the monograph of Chicone and Latushkin [60] for further information on evolution families. Then we introduce a closed bounded operator B based on $\{U(a,s)\}_{0\leq s\leq a<a_0}$. Next we rewrite system (3.8.1) as a Cauchy problem with the linear operator B and show that B generates an integrated semigroup $\{S_B(t)\}_{t\geq 0}$. Now the results in the previous sections can be applied to the problem.

Definition 3.8.1. A family of bounded linear operators $\{U(a,s)\}_{0\leq s\leq a<a_0}$ on Y is called an *exponentially bounded evolution family* if the following conditions are satisfied:

(a) $U(a,a) = Id_Y$ if $0 \leq a < a_0$;
(b) $U(a,r)U(r,s) = U(a,s)$ if $0 \leq s \leq r \leq a < a_0$;
(c) For each $y \in Y$, the map $(a,s) \to U(a,s)y$ is continuous from $\{(a,s) : 0 \leq s \leq a < a_0\}$ into Y;
(d) There exist two constants, $M \geq 1$ and $\omega \in \mathbb{R}$, such that $\|U(a,s)\| \leq Me^{\omega(a-s)}$ if $0 \leq s \leq a < a_0$.

Remark 3.8.2. In the Example 3.1.1 we have $Y = \mathbb{R}$ and $A(a) := \mu(a)$. So we can just use

$$U(a,s) = \exp(\int_s^a \mu(r)dr), \forall a \geq s \geq 0.$$

For an n-dimension system we can assume that

$$A(a) := -M(a) + N(a),$$

where

$$a \to M(a) := \mathrm{diag}(\mu_1(\mathrm{a}), \mu_2(\mathrm{a}), ..., \mu_\mathrm{n}(\mathrm{a})) \in \mathrm{L}^1_{\mathrm{loc},+}((0,\mathrm{a}_0), \mathrm{M}_\mathrm{n}(\mathbb{R}))$$

and

$$a \to N(a) \in L^\infty_+((0,a_0), M_n(\mathbb{R})).$$

Let $s \in [0,a_0)$ be fixed. Then we define $a \to U(a,s)$ as the fixed point solution of

$$U(a,s) := V(a,s) + \int_s^a V(a,r)N(r)U(r,s)dr, \forall a \in [s,a_0),$$

where

$$V(a,s) := \mathrm{diag}(\exp(-\int_{\mathrm{s}}^{\mathrm{a}} \mu_1(\mathrm{r})\mathrm{dr}), ..., \exp(-\int_{\mathrm{s}}^{\mathrm{a}} \mu_{\mathrm{n}}(\mathrm{r})\mathrm{dr}))$$

whenever $a \geq s \geq 0$.

From now on, set

$$X = Y \times L^p\left((0,a_0),Y\right) \text{ and } X_0 = \{0_Y\} \times L^p\left((0,a_0),Y\right)$$

endowed with the product norm

$$\left\| \begin{pmatrix} y \\ \psi \end{pmatrix} \right\| = \|y\|_Y + \|\psi\|_{L^p((0,a_0),Y)}.$$

Define for each $\lambda > \omega$ a linear operator $J_\lambda : X \to X_0$ by

$$J_\lambda \begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \Leftrightarrow$$
$$\varphi(a) = e^{-\lambda a} U(a,0)y + \int_0^a e^{-\lambda(a-s)} U(a,s)\psi(s)ds, \ a \in (0,a_0).$$

Lemma 3.8.3. *Assume that $\{A(a)\}_{0\leq a\leq a_0}$ generates an exponentially bounded evolution family $\{U(a,s)\}_{0\leq s\leq a<a_0}$. Then there exists a unique closed linear operator $B : D(B) \subset X \to X$ such that $(\omega,+\infty) \subset \rho(B)$, $J_\lambda = (\lambda I - B)^{-1}, \forall \lambda > \omega$, and $\overline{D(B)} = X_0$.*

Proof. It is readily to check that J_λ is a pseudo resolvent on $(\omega,+\infty)$ (i.e. $J_\lambda - J_\mu = (\mu-\lambda)J_\lambda J_\mu$, $\forall \lambda,\mu \in (\omega,+\infty)$). By construction we have $\mathscr{R}(J_\lambda) \subset X_0$. Moreover, let $x = \begin{pmatrix} y \\ \psi \end{pmatrix} \in X$ and assume that $J_\lambda x = 0$. Then, for $a \in (0,a_0)$

$$I_a := \frac{1}{a}\int_0^a \left\| e^{-\lambda\xi} U(\xi,0)y + \int_0^\xi e^{-\lambda(\xi-s)} U(\xi,s)\psi(s)ds \right\| d\xi = 0$$

and

$$\lim_{a\to 0^+} I_a = \|y\|.$$

So $y = 0$ and $\mathscr{N}(J_\lambda) \subset X_0$. Moreover, using Young's inequality, we have for all $\lambda > \omega$ that

$$\begin{aligned}\left\| J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| &\leq M \left\| \left(e^{(-\lambda+\omega).} * \|\psi(.)\| \right)(.) \right\|_{L^p((0,a_0),\mathbb{R})} \\ &\leq M \left\| e^{(-\lambda+\omega).} \right\|_{L^1((0,a_0),\mathbb{R})} \|\psi\|_{L^p((0,a_0),Y)},\end{aligned}$$

so

$$\left\| J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} \right\| \leq \frac{M}{\lambda-\omega} \|\psi\|_{L^p((0,a_0),Y)}.$$

Moreover, we can prove that $\forall \psi \in C_c^0\left((0,a_0),Y\right)$,

$$\lim_{\lambda\to+\infty} \lambda J_\lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$

By the density of $C_c^0\left((0,a_0),Y\right)$ in $L^p\left((0,a_0),Y\right)$, we obtain that

$$\lim_{\lambda\to+\infty} \lambda J_\lambda x = x, \forall x \in X_0.$$

By using Corollary 2.2.13, the result follows. □

Consider equation (3.8.1) as the following Cauchy problem

$$\frac{du}{dt} = Bu(t) + f(t), \quad t \geq 0, \;\; u(0) = x \in X_0 \tag{3.8.2}$$

with

$$f \in L^p\left((0,\tau_0),X\right).$$

Lemma 3.8.4. *Assume that* $\{A(a)\}_{0\leq a\leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\leq s\leq a<a_0}$. *Then* B *satisfies Assumption 3.4.1*

Proof. One can check that

$$\left\| (\lambda I - B)^{-1} \begin{pmatrix} y \\ 0 \end{pmatrix} \right\| \leq \frac{M}{p^{1/p}\left(\lambda-\omega\right)^{1/p}} \left\|y\right\|, \; \forall \lambda > \omega.$$

Using the Young inequality we have

$$\left\| (\lambda I - B)^{-k} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \right\| \leq \frac{M}{\left(\lambda-\omega\right)^k} \left\|\varphi\right\|_{L^p((0,a_0),Y)}, \; \forall \lambda > \omega, \, \forall k \geq 1.$$

This completes the proof. □

Now we can claim that B_0 (the part of B in X_0) generates a C_0-semigroup $\left\{T_{B_0}(t)\right\}_{t\geq 0}$ and B generates an integrated semigroup $\{S_B(t)\}_{t\geq 0}$.

Lemma 3.8.5. *Assume that* $\{A(a)\}_{0\leq a\leq a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0\leq s\leq a<a_0}$. *Then* $\left\{T_{B_0}(t)\right\}_{t\geq 0}$, *the* C_0*-semigroup generated by* B_0 *(the part of* B *in* X_0*), is defined by*

$$T_{B_0}(t) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{B_0}(t)\varphi \end{pmatrix}$$

with

$$\widehat{T}_{B_0}(t)\left(\varphi\right)(a) = \begin{cases} 0 & \text{if } a \in [0,t], \\ U(a,a-t)\varphi(a-t) & \text{if } a \geq t. \end{cases}$$

Moreover, $\{S_B(t)\}_{t\geq 0}$, *the integrated semigroup generated by* B, *is defined by*

$$S_B(t)\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ W(t)y + \int_0^t \widehat{T}_{B_0}(s)\varphi ds \end{pmatrix}$$

with

$$W(t)(y)(a) = \begin{cases} U(a,0)y & \text{if } a \le t, \\ 0 & \text{if } a \ge t. \end{cases}$$

Proof. If $T_{B_0}(t)$ and $S_B(t)$ are defined by the above formulas, then it is readily to check that

$$\frac{d}{dt}(\lambda I - B)^{-1} T_{B_0}(t)x = \lambda (\lambda I - B)^{-1} T_{B_0}(t)x - T_{B_0}(t)x$$

and

$$\frac{d}{dt}(\lambda I - B)^{-1} S_B(t)x = \lambda (\lambda I - B)^{-1} S_B(t)x - S_B(t)x + (\lambda I - B)^{-1} x,$$

and the result follows. □

Define $P : X \to X$ by

$$P\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} y \\ 0_{L^p} \end{pmatrix},$$

and set

$$X_1 = Y \times \left\{0_{L^p((0,a_0),Y)}\right\}.$$

We obtain the following theorem.

Theorem 3.8.6. *Assume that* $\{A(a)\}_{0 \le a \le a_0}$ *generates an exponentially bounded evolution family* $\{U(a,s)\}_{0 \le s \le a < a_0}$. *Then for each* $f \in L^p((0,\tau_0), X_1) \oplus L^1((0,\tau_0), X_0)$ *and each* $x \in \overline{D(B)}$, *there exists* $u \in C([0,\tau_0], \overline{D(B)})$, *a unique integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Bu(t) + f(t),\ t \in [0,\tau_0],\ u(0) = x, \tag{3.8.3}$$

given by

$$u(t) = T_{B_0}(t)x + \frac{d}{dt}(S_B * f)(t),\ \forall t \in [0,\tau_0], \tag{3.8.4}$$

which satisfies for a certain $\widehat{M} > 0$ *independent of* τ_0 *that*

$$\|u(t)\| \le Me^{\omega t}\|x\| + \widehat{M}\left(\int_0^t \left(e^{\omega(t-s)}\|Pf(s)\|\right)^p ds\right)^{1/p}$$
$$+M\int_0^t e^{\omega(t-s)}\|(I-P)f(s)\|ds,\ \ \forall t \in [0,\tau_0]$$

Moreover,

$$u(t) = T_{B_0}(t)x + \begin{pmatrix} 0 \\ w(t) \end{pmatrix},\ \forall t \in [0,\tau_0] \tag{3.8.5}$$

with

$$w(t)(a) = \begin{cases} U(a,0)Pf(t-a) + \left(\int_0^t \widehat{T}_0(t-s)(I-P)f(s)ds\right)(a) & \textit{if } a \le t, \\ \left(\int_0^t \widehat{T}_0(t-s)(I-P)f(s)ds\right)(a) & \textit{if } a \ge t. \end{cases}$$

Proof. Let $\psi \in C_c^\infty((0,a_0), Y^*)$ be fixed. Define $x^* \in X_0^*$ by

$$x^* \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \int_0^{a_0} \psi(s)(\varphi(s))ds.$$

Let $x = \begin{pmatrix} y \\ \varphi \end{pmatrix} \in X$ be given. We have

$$x^*\left((\lambda I - B)^{-1} x\right) = x^*\left((\lambda I - B)^{-1} Px\right) + x^*\left((\lambda I - B)^{-1}(I-P)x\right)$$

and

$$x^*(\lambda I - B)^{-1}(I-P)x = \int_0^{+\infty} e^{(-\lambda+\omega)t} x^*\left(e^{-\omega t} T_{B_0}(t)(I-P)x\right)dt,$$

and for each $\lambda > \omega$ that

$$\begin{aligned} x^*\left((\lambda I - B)^{-1} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right) &= \int_0^{a_0} e^{-\lambda a} \psi(a)(U(a,0)y)\,da \\ &= \int_0^{+\infty} e^{(-\lambda+\omega)t} W_{x^*}(t)(y)\,dt \end{aligned}$$

with

$$W_{x^*}(t)(y) = \begin{cases} e^{-\omega t}\psi(t)U(t,0)y & \text{if } 0 \le t < a_0, \\ 0 & \text{if } t \ge a_0. \end{cases}$$

$$\begin{aligned} x^*\left((\lambda I - B)^{-n} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right) &= \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} x^*\left((\lambda I - B)^{-1} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right) \\ &= \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{(-\lambda+\omega)t} W_{x^*}(t)(y)\,dt. \end{aligned}$$

So

$$\left| x^*\left((\lambda I - B)^{-n} P \begin{pmatrix} y \\ \varphi \end{pmatrix}\right)\right| \le \frac{1}{(n-1)!} \int_0^{+\infty} t^{n-1} e^{-\lambda t} \chi_{x^*}(t)\,dt\, \|y\|_Y,$$

where

$$\chi_{x^*}(t) = \begin{cases} M\|\psi(t)\|_{Y^*} & \text{if } t \in (0,a_0) \\ 0 & \text{otherwise.} \end{cases}$$

The result follows by applying Theorem 3.7.9. $\square$

3.9 Remarks and Notes

In Section 3.2 we recalled some results on integrated semigroup theory which were taken from Thieme [329]. The representation Theorem 3.3.5 was taken from combines Theorem 3.1 in Arendt [21] and Proposition 3.10 in Thieme [329]. More results can be found in Weis [370]. In Section 3.4 we discussed the existence of mild when A is not necessarily a Hille-Yosida operator. In Section 3.5 we proved a bounded linear perturbation result. The results of Sections 3.4 and 3.5 were taken from Magal and Ruan [245]. The Section 3.6 was devoted to the existence of mild solutions in the Hille-Yosida case which was proved by Kellermann and Hieber [207]. For the Hille-Yosida case we also refer to the book of Arendt et al. [22] for more results. Section 3.7 focused to the existence of mild solutions in the non-Hille-Yosida case which was taken from Magal and Ruan [245]. This problem was reconsidered by Thieme [335].

(a) Commutative Sum of Operators. For the commutative sum of operators (Da Prato and Gisvard [82], Favini and Yagi [138]), an integrated semigroup approach has been developed by Thieme [331, 335]. This problem has been reconsidered more recently in Ducrot and Magal [114].

Assumption 3.9.1. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 3.4.1 and 3.5.2 and let $B : D(B) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T_B(t)\}_{t\geq 0}$ on X. We assume in addition that the linear operators A and B commute in the sense that one has

$$(\lambda I - A)^{-1} (\mu I - B)^{-1} = (\mu I - B)^{-1} (\lambda I - A)^{-1}, \forall \lambda, \mu \in \rho(A) \cap \rho(B).$$

Theorem 3.9.2. *Let Assumptions 3.9.1 be satisfied. Then the linear operator $A + B : D(A) \cap D(B) \to X$ is closable, and its closure $\overline{A+B} : D(\overline{A+B}) \subset X \to X$ satisfies Assumptions 3.4.1 and 3.5.2. More precisely the following properties hold:*

(i) The linear operator $\left(\overline{A+B}\right)_0 : D(\left(\overline{A+B}\right)_0) \subset \overline{D(A)} \to \overline{D(A)}$ defined as the part of $\overline{A+B}$ in $X_0 := \overline{D(A)}$ is the infinitesimal generator of a C_0-semigroup $\left\{T_{\left(\overline{A+B}\right)_0}(t)\right\}_{t\geq 0}$ on X_0 and

$$T_{\left(\overline{A+B}\right)_0}(t)x = T_B(t)T_{A_0}(t)x, \forall x \in X_0, \forall t \geq 0.$$

In addition one has

$$\omega_0\left(\left(\overline{A+B}\right)_0\right) \leq \omega_0(A_0) + \omega_0(B).$$

(ii) The linear operator $\overline{A+B}$ generates an exponentially bounded (non-degenerate) integrated semigroup $\left\{S_{\overline{A+B}}(t)\right\}_{t\geq 0}$ of bounded linear operators on X, given by

$$S_{\overline{A+B}}(t)x = (S_A \diamond T_B(t-.)x)(t), \forall x \in X, \forall t \geq 0,$$

and

$$V^{\infty}(S_{\overline{A+B}},0,t) \leq \sup_{s\in[0,t]} \|T_B(s)\| V^{\infty}(S_A,0,t), \forall t \geq 0.$$

(iii) The following inclusions hold

$$\begin{aligned}
&D(A_0)\cap D(B) \subset D\left(\left(\overline{A+B}\right)_0\right) \subset \overline{D(A)},\\
&D(A)\cap D(B) \subset D\left(\overline{A+B}\right) \subset \overline{D(A)},\\
&\overline{D\left(\left(\overline{A+B}\right)_0\right)} = \overline{D\left(\overline{A+B}\right)} = \overline{D(A)}.
\end{aligned}$$

(iv) The equality

$$-\left(\overline{A+B}\right)x = y \text{ and } x \in D(\overline{A+B})$$

holds if and only if

$$\left[(\mu - B)^{-1} + (\lambda - A)^{-1}\right] x = (\mu - B)^{-1}(\lambda - A)^{-1}[y + (\lambda+\mu)x]$$

for some $\lambda \in \rho(A)$ and $\mu \in \rho(B)$.

Remark 3.9.3. In the above theorem the fact that B is densely defined is necessary (see for example the following item).

(b) Abstract Cauchy Problems as a Commutative Sum of Operators. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 3.4.1. Reconsider the abstract Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = x \in \overline{D(A)}, \tag{3.9.1}$$

where $f \in L^1((0,\tau),X)$. Let

$$\mathscr{X} := X \times L^1((0,\tau),X)$$

be the Banach space endowed with the usual product norm. Let $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{X} \to \mathscr{X}$ be the linear operator defined by

$$\mathscr{A}\begin{pmatrix} x \\ f(.) \end{pmatrix} = \begin{pmatrix} Ax \\ Af(.) \end{pmatrix}$$

with

$$D(\mathscr{A}) = D(A) \times L^1(0,\tau;D(A)).$$

Let $\mathscr{B} : D(\mathscr{B}) \subset \mathscr{X} \to \mathscr{X}$ be the linear operator defined by

$$\mathscr{B}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' \end{pmatrix}$$

and

$$D(\mathscr{B}) = \{0_X\} \times W^{1,1}(0,\tau;X).$$

Da Prato and Sinestrari [85] reformulated the Cauchy problem (3.9.1) as the commutative sum of operators

$$\mathscr{B}\begin{pmatrix} 0_X \\ u \end{pmatrix} + \mathscr{A}\begin{pmatrix} 0_X \\ u \end{pmatrix} + \begin{pmatrix} x \\ f \end{pmatrix} = 0_{\mathscr{X}}.$$

In order to verify that $\mathscr{A}$ and $\mathscr{B}$ are commutative, we observe that the resolvent of $\mathscr{A}$ is defined for each $\lambda > \omega_A$ by

$$(\lambda I - \mathscr{A})^{-1})\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} (\lambda I - A)^{-1} y \\ (\lambda I - A)^{-1}\varphi(.) \end{pmatrix},$$

so $\mathscr{A}$ satisfies the same assumption as A. The resolvent of $\mathscr{B}$ is defined for each $\lambda > 0$ by

$$(\lambda I - \mathscr{B})^{-1}\begin{pmatrix} y \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_X \\ \psi \end{pmatrix} \Leftrightarrow \psi(s) = e^{-\lambda s} y + \int_0^s e^{-\lambda(s-r)}\varphi(r)dr.$$

It is clear that the two resolvents commute. But since both $\mathscr{A}$ and $\mathscr{B}$ are not densely defined we cannot apply Theorem 3.9.2. We refer to Da Prato and Sinestrarie [85] for more results about this subject. In their work they investigated several notions of solutions for such problems.

(c) Nonautonomous Cauchy Problems. Let $t_0, t_{\max} \in \mathbb{R}$ with $t_0 < t_{\max}$. Let $\{A(t)\}_{t\in[t_0,t_{\max}]}$ be a time parameterized family of linear operators on X. Consider the nonhomogeneous Cauchy problem

$$\frac{du}{dt} = A(t)u(t) + f(t) \text{ for } t \in [t_0, t_{\max}], \text{ and } u(t_0) = x \in \overline{D(A)}, \tag{3.9.2}$$

where f belongs to a subspace of $L^1((t_0, t_{\max}), X)$.

Assumption 3.9.4. Let X be a Banach space with a norm $\|.\|$. Let $D \subset X$ be a subspace of X which is a Banach space endowed with the norm $\|.\|_D$. Let $\{A(t)\}_{t\in[t_0,t_{\max}]}$ be a time parameterized family of linear operators on X with domain D. Assume that

i) There exists a constant $c_0 > 0$

$$c_0^{-1}\|x\|_D \le \|x\| + \|A(t)x\| \le c_0\|x\|_D, \forall [t_0, t_{\max}], \forall x \in X;$$

ii) $A(.) \in C([0,\tau], \mathscr{L}(D,E))$;

iii) There exist two constants $M_A \ge 1$ and $\omega_A \in \mathbb{R}$ such that

$$\|(\lambda I - A(t_n))^{-1}(\lambda I - A(t_{n-1}))^{-1} \dots (\lambda I - A(t_1))^{-1}\| \le \frac{M_A}{(\lambda - \omega_A)^n}$$

whenever $0 \le t_1 \le t_2 \dots \le t_n \le t_{\max}$ and $\lambda > \omega_A$.

Assuming that f is continuous, one may consider the implicit approximation scheme

$$\begin{cases} \dfrac{u(t_{i+1})-u(t_i)}{\Delta t} = A(t_{i+1})u(t_{i+1}) + f(t_{i+1}), \forall i = 0,\ldots,N, \\ u(0) = x, \end{cases}$$

where $\Delta t = \dfrac{1}{N}$ and $t_i = i\Delta t, \forall i = 0,\ldots,N$. The above assumption means that whenever Δt is small enough we can solve the implicit approximation scheme which becomes equivalent to

$$\begin{cases} u(t_{i+1}) = (I - \Delta t A(t_{i+1}))^{-1}[u(t_i) + \Delta t f(t_{i+1})], \forall i = 0,\ldots,N-1, \\ u(0) = x. \end{cases}$$

We refer to DaPrato and Sinestrari [86], Pazy [281], Kobayashi et al. [214, 215] and references therein for more results about this topic. The above approximation scheme has also been successfully used in the context of nonlinear semigroups (see Barbu [38], Goldstein [150], Pavel [282]).

(d) Approximation Formula for a Nonautonomous Bounded Linear Perturbation. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumptions 3.4.1 and 3.5.2 and let $\{B(t)\}_{t\in\mathbb{R}} \subset \mathscr{L}(\overline{D(A)}, X)$ is a locally bounded and strongly continuous family of bounded linear operators. Consider

$$\frac{du}{dt} = Au(t) + B(t)u(t) + f(t) \text{ for } t \geq t_0, \text{ and } u(t_0) = x \in \overline{D(A)}, \tag{3.9.3}$$

where $f \in C(\mathbb{R}, X)$.

Assumption 3.9.5. Let $\{B(t)\}_{t\in\mathbb{R}} \subset \mathscr{L}(\overline{D(A)}, X)$. Assume that $t \to B(t)$ is strongly continuous from $\mathbb{R}$ into $\mathscr{L}(X_0, X)$; that is, for each $x \in X_0$ the map $t \to B(t)x$ is continuous from $\mathbb{R}$ into X. Assume that for each integer $n \geq 1$

$$\sup_{t\in[-n,n]} \|B(t)\|_{\mathscr{L}(X_0,X)} < +\infty.$$

Define

$$\Delta := \left\{(t,s) \in \mathbb{R}^2 : t \geq s\right\},$$

and recall the notion of an evolution family.

Definition 3.9.6. Let $(Z, \|\cdot\|)$ be a Banach space. A two-parameter family of bounded linear operators on Z, $\{U(t,s)\}_{(t,s)\in\Delta}$ is an *evolution family* if

(i) For each $t, r, s \in \mathbb{R}$ with $t \geq r \geq s$

$$U(t,t) = I_{\mathscr{L}(Z)} \quad \text{and} \quad U(t,r)U(r,s) = U(t,s);$$

(ii) For each $x \in Z$, the map $(t,s) \to U(t,s)x$ is continuous from Δ into Z.

If in addition there exist two constants $\widehat{M} \geq 1$ and $\widehat{\omega} \in \mathbb{R}$ such that

$$\|U(t,s)\|_{\mathscr{L}(Z)} \leq \widehat{M} e^{\widehat{\omega}(t-s)}, \ \forall (t,s) \in \Delta,$$

we say that $\{U(t,s)\}_{(t,s)\in\Delta}$ is an *exponentially bounded evolution family*.

Consider the following homogeneous equation for each $t_0 \in \mathbb{R}$

$$\frac{du(t)}{dt} = (A + B(t))u(t) \text{ for } t \geq t_0 \text{ and } u(t_0) = x \in \overline{D(A)}. \tag{3.9.4}$$

By using Theorem 5.2.7 and Proposition 5.4.1 we obtain the following Proposition.

Proposition 3.9.7. *Let Assumptions 3.4.1, 3.5.2 and 3.9.5 be satisfied. Then the homogeneous Cauchy problem (3.9.4) generates a unique evolution family* $\{U_B(t,s)\}_{(t,s)\in\Delta} \subset \mathscr{L}(\overline{D(A)})$*. Moreover,* $U_B(\cdot,t_0)x_0 \in C([t_0,+\infty),\overline{D(A)})$ *is the unique solution of the fixed point problem*

$$U_B(t,t_0)x_0 = T_{A_0}(t-t_0)x_0 + \frac{d}{dt}\int_{t_0}^{t} S_A(t-s)B(s)U_B(s,t_0)x_0 ds, \ \forall t \geq t_0.$$

If we assume in addition that

$$\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathscr{L}(X_0,X)} < +\infty,$$

then the evolution family $\{U_B(t,s)\}_{(t,s)\in\Delta}$ *is exponentially bounded.*

The following theorem provides an approximation formula of the solutions of equation (3.9.3). This is the first main result.

Theorem 3.9.8 (Approximation Formula). *Let Assumptions 3.4.1, 3.5.2 and 3.9.5 be satisfied. Then for each* $t_0 \in \mathbb{R}$*, each* $x_0 \in \overline{X_0}$*, and each* $f \in C([t_0,+\infty],X)$*, the unique integrated solution* $u_f \in C([t_0,+\infty],\overline{D(A)})$ *of (3.9.3) is given by*

$$u_f(t) = U_B(t,t_0)x_0 + \lim_{\lambda\to+\infty}\int_{t_0}^{t} U_B(t,s)\lambda(\lambda I - A)^{-1} f(s) ds, \ \forall t \geq t_0, \tag{3.9.5}$$

where the limit exists in $\overline{D(A)}$*. Moreover, the convergence in (3.9.5) is uniform with respect to* $t,t_0 \in I$ *for each compact interval* $I \subset \mathbb{R}$*.*

Remark 3.9.9. Under Assumptions 3.4.1 and 3.5.2 we may have

$$\limsup_{\lambda\to+\infty} \|\lambda(\lambda I - A)^{-1}\| = +\infty.$$

Theorem 3.9.8 was proved first by Guhring and Rabiger [156] when A is a Hille-Yosida operator by using the extrapolation method to define the mild solutions. Theorem 3.9.8 was proved in Magal and Seydi [250].

(e) Extrapolation Method. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 3.4.1. The extrapolation theory has been developped for Hille-Yosida operators only. We now adapt some ideas from the Hille-Yosida case in Da Prato and Grisvard [82], Amann [12], Thieme [329] and Nagel and Sinestrari [275] to the non-Hille-Yosida case. Consider the norm on X

$$\|x\|_\lambda := \|(\lambda I - A)^{-1}x\|$$

for $\lambda > \omega_A$.

Lemma 3.9.10. *Let Assumption 3.4.1 be satisfied. Then the following properties are satisfied:*

(i) For each $\lambda, \mu > \omega_A$, the norms $\|.\|_\lambda$ and $\|.\|_\mu$ are equivalent;
(ii) For each $\lambda > \omega_A$ there exists a constant $c > 0$ such that

$$\|x\|_\lambda \le c\|x\|, \forall x \in X;$$

(iii) For each $\mu > \omega_A$ and each $x \in X$

$$\lim_{\lambda \to +\infty} \|\lambda(\lambda I - A)^{-1}x - x\|_\mu = 0.$$

Proof. We have

$$\begin{aligned}
\|x\|_\lambda &= \|(\lambda I - A)^{-1}x\| \\
&\le \|(\lambda I - A)^{-1}x - (\mu I - A)^{-1}x\| + \|x\|_\mu \\
&= \|(\mu - \lambda)(\lambda I - A_0)^{-1}(\mu I - A)^{-1}x\| + \|x\|_\mu \\
&\le [\frac{|\mu - \lambda|}{\lambda - \omega_A} + 1]\|x\|_\mu
\end{aligned}$$

and (i) follows. Moreover, from the above inequality we have

$$\|x\|_\lambda \le [\frac{|\mu - \lambda|}{\lambda - \omega_A} + 1]\|(\mu I - A)^{-1}\|_{\mathscr{L}(X)}\|x\|$$

and (ii) follows. Next we observe that

$$\begin{aligned}
\|\lambda(\lambda I - A)^{-1}x - x\|_\mu &= \|(\mu I - A)^{-1}[\lambda(\lambda I - A)^{-1}x - x]\| \\
&= \|\lambda(\lambda I - A_0)^{-1}(\mu I - A)^{-1}x - (\mu I - A)^{-1}x\|
\end{aligned}$$

and since $(\mu I - A)^{-1}x \in \overline{D(A)}$ the property (iii) follows. $\square$

In the following we introduce the completion space. For completeness we now recall how the space is constructed.

Completion space of $(X, \|.\|_\lambda)$. Let $\lambda > \omega_A$ be fixed. Recall some results from Lang [224, Section 4 p. 71]. Consider the collection $C(X)$ of all Cauchy sequences of X endowed with the norm $\|.\|_\lambda$. Define the relation $\sim$ on $C(X)$ by

$$\{x_n\}_{n\in\mathbb{N}} \sim \{y_n\}_{n\in\mathbb{N}} \Leftrightarrow \lim_{n\to+\infty} \|x_n - y_n\|_\lambda = 0.$$

Then $\sim$ is an equivalence relation on $C(X)$. Define X_{-1} the completion space of $(X, \|.\|_\lambda)$ as the space of equivalent classes for $\sim$. That is, X_{-1} is the space composed by elements

$$\widehat{\{x_n\}}_{n\in\mathbb{N}} := \{\{y_n\}_{n\in\mathbb{N}} \in C(X) : \{x_n\}_{n\in\mathbb{N}} \sim \{y_n\}_{n\in\mathbb{N}}\}.$$

Define the norm on X_{-1} by

$$\|\widehat{\{x_n\}}_{n\in\mathbb{N}}\|_{-1} := \lim_{n\to+\infty} \|x_n\|_\lambda \tag{3.9.6}$$

for each element $\widehat{\{x_n\}}_{n\in\mathbb{N}} \in X_{-1}$.

Observe that the limit exists in (3.9.6) since $l_n := \|x_n\|_\lambda$ is a Cauchy sequence. Indeed, we have for each $n, p \in \mathbb{N}$ that

$$|l_n - l_{n+p}| = |\|x_n\|_\lambda - \|x_m\|_\lambda| \leq \|x_n - x_m\|_\lambda$$

and by construction $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, so is $\{l_n\}_{n\in\mathbb{N}}$. Hence, the limit exists in (3.9.6).

To show that this norm is well defined, we consider two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in the class $\widehat{\{x_n\}}_{n\in\mathbb{N}}$. We have by definition of the class that

$$\lim_{n\to+\infty} \|x_n - y_n\|_\lambda = 0$$

and

$$|\|x_n\|_\lambda - \|y_n\|_\lambda| \leq \|x_n - \widehat{x}_n\|_\lambda.$$

Therefore,

$$\lim_{n\to+\infty} \|x_n\|_\lambda = \lim_{n\to+\infty} \|y_n\|_\lambda$$

and the norm is well defined.

Next define the map $J : X \to X_{-1}$ for each $x \in X$ by

$$J(x) = \widehat{\{x\}}_{n\in\mathbb{N}},$$

where the second member of this equality is the class of the constant sequence with all elements being equal to x.

Lemma 3.9.11. *The map J is isometric from $(X, \|.\|_\lambda)$ into $(X_{-1}, \|.\|_{-1})$. Moreover, $J(X)$ is dense in X_{-1}.*

Proof. The fact that J is isometric is clear. So let us prove that $J(X)$ is dense in X_{-1}. Let $\widehat{\{x_n\}}_{n\in\mathbb{N}} \in X_{-1}$ and $\varepsilon > 0$ be fixed. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we can find $n_0 \in \mathbb{N}$ such that

$$\|x_n - x_{n_0}\|_\lambda \leq \varepsilon, \forall n \geq n_0.$$

Hence

$$\|J(x_{n_0}) - \widehat{\{x_n\}}_{n\in\mathbb{N}}\|_{-1} \leq \varepsilon.$$

This completes the proof. $\square$

Lemma 3.9.12. $(X_{-1}, \|.\|_{-1})$ *is a Banach space.*

Proof. It is sufficient to prove that every Cauchy sequence in $J(X)$ converges in $(X_{-1}, \|.\|_{-1})$. Let $\{x_n\}_{n\in\mathbb{N}} \in X$ be a sequence such that $\{J(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X_{-1}. Since J is isometric, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, \|.\|_\lambda)$. Consider

$$\widehat{x} := \widehat{\{x_n\}}_{n\in\mathbb{N}}.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, \|.\|_\lambda)$, it follows that

$$\lim_{n\to+\infty} \|J(x_n) - \widehat{x}\|_{-1} = 0.$$

This proves the claim. $\square$

As a consequence of the properties (ii) and (iii) of Lemma 3.9.10, we deduce the following lemma.

Lemma 3.9.13. *The map J is a continuous embedding from $(X, \|.\|)$ into $(X_{-1}, \|.\|_{-1})$. Moreover, $J(\overline{D(A)})$ is dense into $(X_{-1}, \|.\|_{-1})$.*

Next we can define a family of linear operators $\{T(t)\}_{t\geq 0}$ on $J(X)$ as follows

$$T(t)J(x) = J(T_{A_0}(t)x),\ \forall x \in \overline{D(A)},$$

or in other words, $T(t)J(x)$ is the equivalent class of the constant sequence

$$\{T_{A_0}(t)x, T_{A_0}(t)x, \ldots\}.$$

Since J is isometric from $(X, \|.\|_\lambda)$ into $(X_{-1}, \|.\|_{-1})$, we deduce that for each $x \in \overline{D(A)}$

$$\begin{aligned}\|T(t)J(x)\|_{-1} &= \|T_{A_0}(t)x\|_\lambda = \|(\lambda I - A)^{-1} T_{A_0}(t)x\| \\ &= \|T_{A_0}(t)(\lambda I - A)^{-1}x\| \leq M_A e^{\omega_A t} \|(\lambda I - A)^{-1}x\|.\end{aligned}$$

Thus, by using the fact that J is isometric we have

$$\|T(t)J(x)\|_{-1} \leq M_A e^{\omega_A t} \|J(x)\|_{-1}.$$

By using the fact that $J(\overline{D(A)})$ is dense in X_{-1}, it follows that $T(t)$ admits a unique extension $T_{-1}(t)$ to the whole space X_{-1}. Moreover, we have the following theorem.

Theorem 3.9.14. *$\{T_{-1}(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $(X_{-1}, \|.\|_{-1})$ and*

$$\|T_{-1}(t)\|_{\mathscr{L}(X_{-1})} = \|T_{A_0}(t)\|_{\mathscr{L}(X_0)}.$$

Proof. Since $J(\overline{D(A)})$ is dense in X_{-1} it follows that

$$\begin{aligned}\|T_{-1}(t)\|_{\mathscr{L}(X_{-1})} &= \sup\{\|T_{-1}(t)\widehat{x}\|_{-1} : \widehat{x} \in X_{-1} \text{ and } \|\widehat{x}\|_{-1} \leq 1\} \\ &= \sup\left\{\|T_{-1}(t)J(x)\|_{-1} : x \in \overline{D(A)} \text{ and } \|J(x)\|_{-1} \leq 1\right\}\end{aligned}$$

Now by using the definitions of the norm $\|.\|_{-1}$, the embedding $J(x)$ and $T_{-1}(t)$, we obtain

$$\begin{aligned}\|T_{-1}(t)\|_{\mathscr{L}(X_{-1})} &= \sup\left\{\|T_{A_0}(t)x\|_\lambda : x \in \overline{D(A)} \text{ and } \|x\|_\lambda \leq 1\right\} \\ &= \sup\left\{\|T_{A_0}(t)y\| : y \in D(A_0) \text{ and } \|y\| \leq 1\right\} \\ &= \|T_{A_0}(t)\|_{\mathscr{L}(X_0)}.\end{aligned}$$

The strong continuity of $\{T_{-1}(t)\}_{t\geq 0}$ is straightforward. $\square$

The following theorem is an analogue of the theorem proved for Hille-Yosida operators by Kellermann and Hiber [207]. This theorem has been proved by Nagel and Sinestrari [275, Proposition 2.1].

Theorem 3.9.15 (Nagel-Sinestrari). *Assume that $A : D(A) \subset X \to X$ is a Hille-Yosida linear operator. For $f \in L^1((0,+\infty),X)$ and $t \geq 0$*

$$(T_{-1} * J(f))(t) := \int_0^t T_{-1}(t-s)J(f(s))ds.$$

Then the following properties are satisfied:

(i) For each $t \geq 0$

$$(T_{-1} * J(f))(t) \in J(X_0);$$

(ii) For each $r > 0$, there exists a constant $M_1 = M_1(r) > 0$ (independent of f) such that for each $t \in [0,r]$,

$$\|(T_{-1} * J(f))(t)\| \leq M_1 \int_0^t \|f(s)\|ds.$$

One may find more information about this topic in Da Prato and Grisvard [82], Amann [11, 12, 14], Thieme [329], Nagel and Sinestrari [275], Sinestrari [321], Nagel [274], Engel and Nagel [126, 127, 128], Arendt et al. [23], DiBlasio [97], Maniar and Rhandi [253], Amir and Maniar [15] and references therein. As far as we know no extrapolation method has been developed for the non-Hille-Yosida case.

(f) Parabolic Problems with Nonhomogeneous Boundary Conditions. Parabolic equations with nonhomogeneous boundary conditions have been studied by using other approaches in the literature. One of the first references on the subject is the book of Lions and Magenes [231]. More recently, another powerful approach has been developed in Denk et al. [92, 93], Meyries and Schnaubelt [269]. See also Both and Prüss [45] for an application to Navier-Stokes equations.

Chapter 4
Spectral Theory for Linear Operators

This chapter covers fundamental results on the spectral theory, including Fredholm alternative theorem and Nussbaum's theorem on the radius of essential spectrum for bounded linear operators; growth bound and essential growth bound of linear operators; the relationship between the spectrum of semigroups and the spectrum of their infinitesimal generators; spectral decomposition of the state space; and asynchronous exponential growth of linear operators. The estimates of growth bound and essential growth bound of linear operators will be used in proving the center manifold theorem in Chapter 6.

4.1 Basic Properties of Analytic Maps

Let $(X, \|.\|)$ be a complex Banach space; that is, X is a $\mathbb{C}$-vector space and $\|.\|$ is a norm on X satisfying

$$\begin{cases} \|x\| = 0 \Leftrightarrow x = 0, \\ \|\lambda x\| = |\lambda| \, \|x\|, \ \forall x \in X, \ \forall \lambda \in \mathbb{C}, \\ \|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in X, \end{cases}$$

and $(X, \|.\|)$ is complete.

When X is a $\mathbb{C}$-Banach space, the dual space X^* is the space of all bounded linear maps x^* from X into $\mathbb{C}$. Of course, if X is a $\mathbb{C}$-Banach space, then X is also a $\mathbb{R}$-Banach space. When X is a $\mathbb{R}$-Banach space, we denote by $X^*_{\mathbb{R}}$ the space of bounded linear functionals from X into $\mathbb{R}$.

Let $x^* \in X^*$ be given. Then

$$x^*(x) = \mathrm{Re}\,(x^*(x)) + i\mathrm{Im}\,(x^*(x)), \ \forall x \in X.$$

It can be seen that

$$\mathrm{Re}\,(x^*(.)) \in X^*_{\mathbb{R}}, \ \ \mathrm{Im}\,(x^*(.)) \in X^*_{\mathbb{R}},$$

and by using the fact that x^* is $\mathbb{C}$-linear (i.e. $x^*(ix) = ix^*(x), \forall x \in X$), we have

$$\mathrm{Re}\,(x^*(x)) = \mathrm{Im}\,(x^*(ix))\,,\ \forall x \in X,$$

or equivalently

$$\mathrm{Re}\,(x^*(ix)) = -\mathrm{Im}\,(x^*(x))\,,\ \forall x \in X.$$

Conversely, if

$$x^*(x) = y^*(ix) + iy^*(x),\ \forall x \in X^*,$$

where $y^* \in X^*_{\mathbb{R}}$, then $x^* \in X^*$. It follows that

$$X^* = \{y^*(i.) + iy^*(.) : y^* \in X^*_{\mathbb{R}}\} = \{z^*(.) - iz^*(i.) : z^* \in X^*_{\mathbb{R}}\}\,.$$

In particular, from this representation of the dual space X^*, it becomes clear that most consequences of the Hahn-Banach theorem for real Banach spaces hold for complex Banach spaces. Based on this fact, one may extend the results on holomorphic maps from $\mathbb{C}$ into $\mathbb{C}$ to maps from $\mathbb{C}$ into a Banach space X.

Now we recall some basic facts about analytic vector valued functions (see Taylor and Lay [326, p.264-272] for more details).

Definition 4.1.1. Let $f : \Omega \subset \mathbb{C} \to X$ be a map from an open subset $\Omega \subset \mathbb{C}$ into a complex Banach space X. We say that f is *holomorphic* on Ω if for each $\lambda_0 \in \Omega$ the limit

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0}$$

exists. We say that f is *analytic* on Ω if for each $\lambda_0 \in \Omega$, there exists a sequence $\{a_n\}_{n\geq 0} = \{a_n^{\lambda_0}\}_{n\geq 0} \subset X$, such that

$$\delta := \limsup_{n\to+\infty} \sqrt[n]{\|a_n\|} > 0$$

and

$$f(\lambda) = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n a_n$$

whenever $|\lambda - \lambda_0| < R := 1/\delta$.

The proofs of the following results are based on the Hahn-Banach theorem applied in complex Banach spaces, by observing that if $f : \Omega \subset \mathbb{C} \to X$ is analytic, then for each $x^* \in X^*$ the map $\lambda \to x^*(f(\lambda))$ is analytic from $\mathbb{C}$ into itself.

Theorem 4.1.2. *Let $f : \Omega \subset \mathbb{C} \to X$ be a map from an open subset $\Omega \subset \mathbb{C}$ into a complex Banach space X. Then f is holomorphic if and only if f is analytic on Ω. Moreover, for each $\lambda_0 \in \Omega$,*

$$f(\lambda) = \sum_{n=0}^{+\infty} \frac{(\lambda - \lambda_0)^n}{n!} f^{(n)}(\lambda_0)$$

whenever $|\lambda - \lambda_0|$ *is small enough and*

$$\frac{f^{(n)}(\lambda_0)}{n!} = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-(n+1)} f(\lambda)\, d\lambda$$

for each $\varepsilon > 0$ *small enough,* $S_{\mathbb{C}}(\lambda_0,\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \varepsilon\}$, *and* $S_{\mathbb{C}}(\lambda_0,\varepsilon)^+$ *is the counterclockwise oriented circumference* $|\lambda - \lambda_0| = \varepsilon$.

Theorem 4.1.3 (Laurent Expansion). *Let* $f : \Omega \subset \mathbb{C} \to X$ *be a map from an open subset* $\Omega \subset \mathbb{C}$ *into a complex Banach space* X. *Assume that* f *is analytic on an annulus* $0 \le r_1 < |\lambda - \lambda_0| < r_2$, *then* f *has a unique Laurent expansion*

$$f(\lambda) = \sum_{n=-\infty}^{+\infty} (\lambda - \lambda_0)^n a_n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-(n+1)} f(\lambda)\, d\lambda$$

for each $\varepsilon \in (r_1, r_2)$, *where* $S_{\mathbb{C}}(\lambda_0,\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \varepsilon\}$, *and* $S_{\mathbb{C}}(\lambda_0,\varepsilon)^+$ *is the counterclockwise oriented circumference* $|\lambda - \lambda_0| = \varepsilon$.

Note that the above integral is a Steiltjes integral of the form

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_0^{2\pi} z(\theta)^{-(n+1)} f(z(\theta) + \lambda_0)\, dz(\theta) \\ &= \frac{1}{2\pi i}\int_0^{2\pi} z(\theta)^{-(n+1)} f(z(\theta) + \lambda_0)\, z'(\theta) d\theta, \end{aligned}$$

where

$$z(\theta) = \varepsilon e^{i\theta}.$$

The following lemma is well known (see Dolbeault [108, Theorem 2.1.2, p. 43]).

Proposition 4.1.4. *Let* $f : \Omega \to X$ *be an analytic map from an open connected subset* $\Omega \subset \mathbb{C}$ *into a Banach space* X. *Let* $z_0 \in \Omega$. *Then the following assertions are equivalent*

(i) $f = 0$ *on* Ω;
(ii) f *is null in a neighborhood of* z_0;
(iii) *For each* $k \in \mathbb{N}, f^{(k)}(z_0) = 0$.

Proof. (i)⇒(ii)⇒(iii) is trivial. We prove (iii)⇒(i). Since Ω is connected, it is sufficient to show that the subset

$$A = \left\{ z \in \Omega : f^{(k)}(z) = 0,\ \forall k \in \mathbb{N} \right\}$$

is non-empty, and both open and closed. Clearly A is non-empty since it contains z_0. Moreover, A is closed since it is the intersection of the closed subsets

$$A_k = \left\{ z \in \Omega : f^{(k)}(z) = 0 \right\}$$

for $k \geq 0$. Furthermore, if $z_1 \in A = \left\{ z \in \Omega : f^{(k)}(z) = 0, \forall k \in \mathbb{N} \right\}$, since

$$f(z) = \sum_{n=0}^{+\infty} (z - z_1) f^{(n)}(z_1)$$

whenever $|z - z_1|$ is small enough, it follows that

$$f(z) = 0$$

whenever $|z - z_1|$ is small enough. So A contains some neighborhood of z_1. Hence, A non-empty, open, and closed in Ω, so $A = \Omega$. $\square$

4.2 Spectra and Resolvents of Linear Operators

Let $A : D(A) \subset X \to X$ be a linear operator on complex Banach space X. Recall that the *resolvent set* $\rho(A)$ of A is the set of all points $\lambda \in \mathbb{C}$, such that $\lambda I - A$ is a bijection from $D(A)$ into X and the inverse $(\lambda I - A)^{-1}$ is a bounded linear operator from X into itself.

Definition 4.2.1. Let $A : D(A) \subset X \to X$ be a linear operator on a complex Banach space X. The *spectrum* of the operator A is defined as the complement of the resolvent set

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

Consider the following three conditions:

(1) $(\lambda I - A)^{-1}$ exists;
(2) $(\lambda I - A)^{-1}$ is bounded;
(3) the domain of $(\lambda I - A)^{-1}$ is dense in X.

The spectrum $\sigma(A)$ can be further decomposed into three disjoint subsets.

(a) The *point spectrum* is the set

$$\sigma_p(A) := \{\lambda \in \sigma(A) : \mathcal{N}(\lambda I - A) \neq \{0\}\}.$$

Elements of the point spectrum $\sigma_p(A)$ are called *eigenvalues*. If $\lambda \in \sigma_p(A)$, elements $x \in \mathcal{N}(\lambda I - A)$ are called *eigenvectors* or *eigenfunctions*. The dimension of $\mathcal{N}(\lambda I - A)$ is the *multiplicity* of λ.

(b) The *continuous spectrum* is the set

$$\sigma_c(A) := \{\lambda \in \sigma(A) : \text{(1) and (3) hold but (2) does not}\}.$$

(c) The *residual spectrum* is the set

$$\sigma_r(A) := \{\lambda \in \sigma(A) : \text{ (1) holds but (3) does not}\}$$
$$= \left\{\lambda \in \sigma(A) : (\lambda I - A)^{-1} \text{ exists but } \overline{\mathscr{R}(\lambda I - A)} \neq X\}\right\}.$$

We have the following spectrum decomposition

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Example 4.2.2 (Point spectrum). Let X be a Banach space with finite dimension and $A : X \to X$ be a linear operator. Since

$$\dim(\mathscr{R}(\lambda I - A)) + \dim(\mathscr{N}(\lambda I - A)) = \dim X,$$

it follows that $\lambda I - A$ is one-to-one if and only if $\mathscr{R}(\lambda I - A) = X$, so the residual spectrum is an empty set, $\sigma_r(A) = \emptyset$. If $\lambda I - A$ is one-to-one and $(\lambda I - A)^{-1}$ exists, since linear operators in a finite dimensional space are countinuous, it follows that the continuous spectrum is also an empty set, $\sigma_c(A) = \emptyset$. Therefore, a finite dimensional space only has point spectrum, $\sigma(A) = \sigma_p(A)$. If we identify A to its matrix $A = (a_{ij})$ into a given basis, then we have

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}.$$

Example 4.2.3 (Continuous spectrum). Let $X = L^2(\mathbb{R})$. Define $A : X \to X$ as follows:

$$Ax(t) = tx(t),\ \forall t \in \mathbb{R}$$

with

$$D(A) = \{x(t) \in L^2(\mathbb{R}) :\ tx(t) \in L^2(\mathbb{R})\}.$$

Consider $(\lambda I - A)x = 0$, that is, $(\lambda - t)x(t) = 0$. We have $x(t) = 0$ for almost every $t \neq \lambda$, so $\lambda I - A$ is one-to-one and the point spectrum is an empty set, $\sigma_p(A) = \emptyset$. Moreover, the range

$$\mathscr{R}(\lambda I - A) = \{y(t) \in L^2(\mathbb{R}) :\ \frac{y(t)}{\lambda - t} \in L^2(\mathbb{R})\}$$

is dense in $L^2(\mathbb{R})$. So the residual spectrum is an empty set, $\sigma_r(A) = \emptyset$. Finally, from $(\lambda - t)x(t) = y(t)$ we can see that if $\lambda \in \mathbb{C}$ and $\mathrm{Im}(\lambda) \neq 0$, then $(\lambda I - A)^{-1}$ is bounded. Thus,

$$\rho(A) = \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \neq 0\}.$$

If $\lambda \in \mathbb{R}$, then $(\lambda I - A)^{-1}$ is unbounded,

$$\sigma(A) = \sigma_c(A) = \{\lambda : \mathrm{Im}\lambda = 0\} = (-\infty, +\infty).$$

Example 4.2.4 (An operator with a spectral value that is not an eigenvalue). Let $X = \ell^2(\mathbb{N}, \mathbb{R})$ the space of real value sequences $x = \{x_n\}_{n \in \mathbb{N}}$ with

$$\|x\|_{\ell^2(\mathbb{N},\mathbb{R})}^2 := \sum_{n\geq 0} |x_n|^2 < +\infty.$$

Consider the right-shift operator $A : \ell^2(\mathbb{N},\mathbb{R}) \to \ell^2(\mathbb{N},\mathbb{R})$ defined by

$$A : (x_1, x_2, \cdots) \to (0, x_1, x_2, \cdots), \ \forall x = (x_1, x_2, \cdots) \in \ell^2(\mathbb{N},\mathbb{R}).$$

Notice that

$$\|Ax\|^2 = \sum_{i=1}^{\infty} |x_i|^2 = \|x\|^2,$$

we have

$$\|A\| = 1.$$

Since $Ax = 0$ implies $x = 0$, it follows that 0 is not an eigenvalue. But the range

$$\mathscr{R}(A) = \{y = \{y_i\} \in \ell^2 : y_1 = 0\}$$

is not dense in $\ell^2(\mathbb{N},\mathbb{R})$, which means that

$$0 \in \sigma_r(A).$$

That is, 0 belongs to the residual spectrum of A but is not an eigenvalue of A.

The following definition was introduced by Browder [49].

Definition 4.2.5. The *essential spectrum* $\sigma_{\text{ess}}(A)$ of A is the set of $\lambda \in \sigma(A)$ such that at least one of the following holds:

(i) $\mathscr{R}(\lambda I - A)$ is not closed;
(ii) λ is a limit point of $\sigma(A)$;
(iii) $\bigcup_{k=1}^{+\infty} \mathscr{N}((\lambda I - A)^k)$ is infinite dimensional.

The *discrete spectrum* is the set $\sigma_d(A) = \sigma(A) \setminus \sigma_{\text{ess}}(A)$.

So we have another spectrum decomposition

$$\sigma(A) = \sigma_d(A) \cup \sigma_{\text{ess}}(A).$$

Definition 4.2.6. Let $A : D(A) \subset X \to X$ be a linear operator on a complex Banach space X. If $\lambda \in \sigma(A)$, then the *generalized eigenspace* of A with respect to λ is defined by

$$\mathscr{N}_\lambda(A) := \overline{\bigcup_{k=1}^{+\infty} \mathscr{N}\left((\lambda I - A)^k\right)}.$$

Lemma 4.2.7. *The resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$. Moreover, if $\lambda_0 \in \rho(A)$, then*

$$(\lambda I - A)^{-1} = (\lambda_0 I - A)^{-1} \sum_{n=0}^{+\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - A)^{-n} \tag{4.2.1}$$

whenever $|\lambda_0 - \lambda| \, \| (\lambda_0 I - A)^{-1} \|_{\mathscr{L}(X)} < 1$.

Proof. Set

$$L_\lambda := (\lambda_0 I - A)^{-1} \sum_{n=0}^{+\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - A)^{-n},$$

which is well defined whenever $|\lambda_0 - \lambda| \, \| (\lambda_0 I - A)^{-1} \|_{\mathscr{L}(X)} < 1$. Then one may easily prove that

$$(\lambda I - A) L_\lambda x = x, \ \forall x \in X$$

and

$$L_\lambda (\lambda I - A) x = x, \ \forall x \in D(A),$$

and the result follows. □

Corollary 4.2.8. *The spectrum* $\sigma(A) \subset \mathbb{C}$ *of a bounded linear operator* A *is a compact set.*

The power series representation (4.2.1) of the resolvent $(\lambda I - A)^{-1}$ enables us to employ the techniques and results on analytic functions of complex variables to analytic functions with values in a Banach space. From the formula (4.2.1) one may observe that for each $\lambda_0 \in \rho(A)$,

$$\lim_{\lambda \to \lambda_0} \frac{(\lambda I - A)^{-1} - (\lambda_0 I - A)^{-1}}{\lambda - \lambda_0} = -(\lambda_0 I - A)^{-2},$$

where the limit is taken in the norm of operators. It follows that $\lambda \to (\lambda I - A)^{-1}$ from $\rho(A)$ into $\mathscr{L}(X)$ is analytic. So if $\lambda_0 \in \sigma(A)$ is isolated in $\sigma(A)$, the resolvent has a Laurent's expansion:

$$(\lambda I - A)^{-1} = \sum_{k=-\infty}^{+\infty} (\lambda - \lambda_0)^k B_k, \tag{4.2.2}$$

where $B_k \in \mathscr{L}(X)$ is given by

$$B_k = \frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0, \varepsilon)^+} (\lambda - \lambda_0)^{-(k+1)} (\lambda I - A)^{-1} d\lambda \tag{4.2.3}$$

for each $\varepsilon > 0$, where $S_{\mathbb{C}}(\lambda_0, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \varepsilon\}$ and $S_{\mathbb{C}}(\lambda_0, \varepsilon)^+$ is the counterclockwise oriented circumference $|\lambda - \lambda_0| = \varepsilon$ for sufficiently small $\varepsilon > 0$ so that $|\lambda - \lambda_0| \le \varepsilon$ does not contain any other point of the spectrum than λ_0.

Definition 4.2.9. A point of the spectrum $\lambda_0 \in \sigma(A)$ is a *pole of the resolvent* $(\lambda I - A)^{-1}$ if λ_0 is an isolated point of the spectrum (i.e. there exists $\varepsilon > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\} \cap \sigma(A) = \emptyset$) and there exists an integer $m \ge 1$ such that

$$B_{-m} \neq 0, \ \ B_{-k} = 0, \ \ \forall k \ge m+1.$$

The integer m is then called the *order* of the pole λ_0.

The following theorem is proved in Yosida [381, Theorems 1 and 2, p.228-229].

Theorem 4.2.10. *Assume that λ_0 is a pole of order m of the resolvent $(\lambda I - A)^{-1}$. Then we have the following properties*

$$
\begin{aligned}
&B_k B_m = 0 \ (k \geq 0, m \leq -1),\\
&B_n = (-1)^n B_0^{n+1} \ (n \geq 1),\\
&B_{-p-q+1} = B_{-p} B_{-q} \ (p, q \geq 1),\\
&B_n = (A - \lambda_0 I) B_{n+1} \ (n \geq 0),\\
&(A - \lambda_0 I) B_{-n} = B_{-(n+1)} = (A - \lambda_0 I)^n B_{-1} \ (n \geq 1),\\
&(A - \lambda_0 I) B_0 = B_{-1} - I.
\end{aligned}
\tag{4.2.4}
$$

Moreover,

$$(A - \lambda_0 I)^{m+k} B_{-1} = 0, \forall k \geq 0. \tag{4.2.5}$$

Note that from the third equation of (4.2.4), we have for each $p \geq 1$ that

$$B_{-p} B_{-1} = B_{-p-1+1} = B_{-p},$$

so B_{-1} is a projector on X. Since

$$(A - \lambda_0 I) B_{-1} = B_{-2},$$

it follows that

$$A B_{-1} = \lambda_0 B_{-1} + B_{-2}.$$

So A restricted to $\mathscr{R}(B_{-1})$ is a bounded linear operator. We also have for each $p \geq 1$ that

$$A B_{-p} = A B_{-1} B_{-p} = \lambda_0 B_{-1} B_{-p} + B_{-2} B_{-p} = \lambda_0 B_{-p} + B_{-p-1}. \tag{4.2.6}$$

Moreover, from (4.2.3) it is clear that B_{-1} commutes with $(\lambda I - A)^{-1}$ for each $\lambda \in \rho(A)$. Thus,

$$\left(\lambda_0 I - A \mid_{B_{-1}(X)}\right)^{-1} = (\lambda_0 I - A)^{-1} \mid_{B_{-1}(X)} .$$

Recall that $B_{-1}(X)$ contains the generalized eigenspace associated to λ_0. Therefore the operator $\lambda_0 I - A$ is invertible from $D(A) \cap (I - B_{-1})(X)$ into $(I - B_{-1})(X)$. Moreover, by using the last equation of (4.2.4), we deduce that $\lambda_0 \notin \sigma\left(A \mid_{(I-B_{-1})(X)}\right)$ and

$$\left(\lambda_0 I - A \mid_{(I-B_{-1})(X)}\right)^{-1} = B_0 \mid_{(I-B_{-1})(X)} .$$

The following result is proved in Yosida [381, Theorem 3, p.229].

Theorem 4.2.11 (Yosida). *Let $A : D(A) \subset X \to X$ be a closed linear operator in the complex Banach space X and let λ_0 be a pole of $(\lambda I - A)^{-1}$ of order $m \geq 1$. Then λ_0 is an eigenvalue of A, and*

$$\mathscr{R}(B_{-1}) = \mathscr{N}((\lambda_0 I - A)^n), \ \mathscr{R}(I - B_{-1}) = \mathscr{R}((\lambda_0 I - A)^n), \ \forall n \geq m,$$

$$X = \mathscr{N}\left((\lambda_0 I - A)^n\right) \oplus \mathscr{R}\left((\lambda_0 I - A)^n\right), \ \forall n \geq m.$$

We already knew that $A\mid_{B_{-1}(X)}$ is bounded. Moreover, if λ_0 is a pole of $(\lambda I - A)^{-1}$ of order $m \geq 1$, we have from the above theorem that

$$\left(\lambda_0 I - A\mid_{B_{-1}(X)}\right)^m = 0.$$

From (4.2.6) for $p = m$, we obtain

$$AB_{-p} = \lambda_0 B_{-p}.$$

Since $B_{-p} \neq 0$, we have $\{\lambda_0\} \subset \sigma\left(A\mid_{B_{-1}(X)}\right)$. To prove the converse inclusion we use the same argument as in the proof of Kato [205, Theorem 6.17, p.178]. For $\lambda \in \mathbb{C}$ and $\varepsilon < |\lambda - \lambda_0|$, set

$$L_\lambda = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} \frac{(\lambda' I - A)^{-1}}{\lambda - \lambda'} d\lambda'.$$

Then we have

$$\begin{aligned}
(\lambda I - A) L_\lambda &= \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda I - A)\frac{(\lambda' I - A)^{-1}}{\lambda - \lambda'} d\lambda' \\
&= \frac{1}{2\pi i}\left[\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} \left(\lambda' I - A\right)^{-1} d\lambda' + \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} \frac{1}{\lambda - \lambda'} d\lambda'\right] \\
&= \frac{1}{2\pi i}\left[\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} \left(\lambda' I - A\right)^{-1} d\lambda'\right] = B_{-1}.
\end{aligned}$$

Similarly, we have

$$L_\lambda (\lambda I - A) x = B_{-1} x, \forall x \in D(A).$$

It follows that for each $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$, $\lambda I - A\mid_{B_{-1}(X)}$ is invertible and

$$\left(\lambda I - A\mid_{B_{-1}(X)}\right)^{-1} = L_\lambda \mid_{B_{-1}(X)}.$$

It implies that

$$\sigma\left(A\mid_{B_{-1}(X)}\right) = \{\lambda_0\}.$$

Furthermore, since $\lambda_0 \notin \sigma\left(A\mid_{(I-B_{-1})(X)}\right)$, we have

$$\sigma\left(A\mid_{(I-B_{-1})(X)}\right) = \sigma(A)\setminus\{\lambda_0\}.$$

Assume that λ_1 and λ_2 are two distinct poles of $(\lambda I - A)^{-1}$. Set for each $i = 1, 2$ that

$$P_i = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_i,\varepsilon)^+} (\lambda I - A)^{-1} d\lambda,$$

where $\varepsilon > 0$ is small enough. It is clear that P_1 commutes with P_2 and

$$P_1P_2 = P_2P_1 = 0.$$

Indeed, let $x \in \mathscr{R}(P_1)$ be fixed. Since P_1 commutes with $(\lambda I - A)^{-1}$ for each $\lambda \in \rho(A)$, we have

$$P_2 x = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_2,\varepsilon)^+} (\lambda I - A)^{-1} x d\lambda = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_2,\varepsilon)^+} \left(\lambda I - A\mid_{P_1(X)}\right)^{-1} x d\lambda.$$

Furthermore, since $\sigma\left(A\mid_{P_1(X)}\right) = \{\lambda_1\}$, it follows from Lemma 2.2.6 that

$$\begin{aligned} P_2 x &= \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_2,\varepsilon)^+} \sum_{n=0}^{\infty} (\lambda-\lambda_2)^n \left[\left(\lambda_2 I - A\mid_{P_1(X)}\right)^{-1}\right]^{n+1} x d\lambda \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{S_{\mathbb{C}}(\lambda_2,\varepsilon)^+} (\lambda-\lambda_2)^n d\lambda \left[\left(\lambda_2 I - A\mid_{P_1(X)}\right)^{-1}\right]^{n+1} x \\ &= 0. \end{aligned}$$

Hence,

$$P_2 x = 0, \ \forall x \in R(P_1).$$

Assumption 4.2.12. Let $(X, \|.\|)$ be a complex Banach space and let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 3.4.1. Assume that there exists $\eta \in \mathbb{R}$ such that

$$\Sigma_\eta := \sigma(A_0) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \eta\}$$

is non-empty, finite, and contains only poles of $(\lambda I - A_0)^{-1}$.

By using Lemma 2.2.10 we know that

$$\sigma(A_0) = \sigma(A),$$

so

$$\Sigma_\eta := \sigma(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \eta\},$$

and for each $\lambda_0 \in \Sigma_\eta$, we set

$$B^0_{\lambda_0,k} = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda-\lambda_0)^{-k-1} (\lambda I - A_0)^{-1} x d\lambda, \ \forall k \in \mathbb{Z}, \ x \in X_0$$

and

$$B_{\lambda_0,k} = \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda-\lambda_0)^{-k-1} (\lambda I - A)^{-1} x d\lambda, \ \forall k \in \mathbb{Z}, \ x \in X_0.$$

The previous Yosida Theorem holds only for densely defined linear operators. Now we extend the projectors $B^0_{\lambda_0,1} \in \mathscr{L}(X_0)$ to the all space X.

Lemma 4.2.13. *Let Assumption 4.2.12 be satisfied. If $\lambda_0 \in \Sigma_\eta$ is a pole of $(\lambda I - A_0)^{-1}$ of order m, then λ_0 is a pole of order m of $(\lambda I - A)^{-1}$ and*

$$B_{\lambda_0,1}x = \lim_{\mu\to+\infty} B^0_{\lambda_0,1}\mu\,(\mu I - A)^{-1}\,x,\ \forall x\in X.$$

Proof. Let $x\in X$ and $k\in\mathbb{Z}$ be fixed. We have $B_{\lambda_0,k}x\in X_0$, so

$$B_{\lambda_0,k}x = \lim_{\mu\to+\infty} \mu\,(\mu I - A)^{-1}\,B_{\lambda_0,k}x.$$

Thus,

$$\begin{aligned}\mu\,(\mu I - A)^{-1}\,B_{\lambda_0,k}x &= \frac{1}{2\pi i}\mu\,(\mu I - A)^{-1}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{-k-1}\,(\lambda I - A)^{-1}\,xd\lambda\\ &= \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{-k-1}\,(\lambda I - A_0)^{-1}\,\mu\,(\mu I - A)^{-1}\,xd\lambda\\ &= \lim_{\mu\to+\infty} B^0_{\lambda_0,k}\mu\,(\mu I - A)^{-1}\,x,\end{aligned}$$

and the result follows. □

From now on, for any isolated pole of the resolvent $\lambda_0\in\sigma(A)$, we denote

$$\Pi_{\lambda_0} := B_{-1} = \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda I - A)^{-1}\,d\lambda$$

whenever $\varepsilon>0$ is small enougth. From the above result, we see that Π_{λ_0} is a projector. In fact, Π_{λ_0} is the projector on the generalized eigenspace of A. Moreover, A is a bounded linear operator on $\Pi_{\lambda_0}(X)$ because A is closed (since its spectrum is non-empty) and

$$A\Pi_{\lambda_0} = \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} A\,(\lambda I - A)^{-1}\,d\lambda = \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}[-I + \lambda\,(\lambda I - A)^{-1}]d\lambda.$$

Furthermore, we have the following result which extends Yosida Theorem for densely defined linear operators to non-densely defined linear operators.

Proposition 4.2.14 (Generalized Yosida). *Let $A: D(A)\subset X\to X$ be a linear operator on a Banach space X. Assume that λ_0 is a pole of the resolvent of A. Then*

$$\Pi_{\lambda_0}\,(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\,\Pi_{\lambda_0}, \forall\lambda\in\rho(A).$$

Moreover, $A_{\Pi_{\lambda_0}(X)}$ and $A_{(I-\Pi_{\lambda_0})(X)}$ are the parts of A in $\Pi_{\lambda_0}(X)$ and $\left(I-\Pi_{\lambda_0}\right)(X)$, respectively, and satisfy

$$\sigma\left(A_{\Pi_{\lambda_0}(X)}\right) = \{\lambda_0\} \text{ and } \sigma\left(A_{\left(I-\Pi_{\lambda_0}\right)(X)}\right) = \sigma(A)\setminus\{\lambda_0\}.$$

Proof. By Theorem 4.2.10, Π_{λ_0} is a projector, and by construction we have

$$(\lambda I - A)^{-1}\,\Pi_{\lambda_0} = \Pi_{\lambda_0}\,(\lambda I - A)^{-1},\ \forall\lambda\in\rho(A). \tag{4.2.7}$$

By assumption λ_0 is a pole of order $m \geq 1$ of the resolvent. Thus, by using (4.2.5) we deduce that

$$(A-\lambda_0 I)^{m+k}\Pi_{\lambda_0} = 0, \ \forall k \geq 0. \tag{4.2.8}$$

In particular it follows that

$$\mathscr{R}\left(\Pi_{\lambda_0}\right) \subset \mathscr{N}\left((\lambda_0 I - A)^m\right).$$

Consider the splitting

$$X = \Pi_{\lambda_0}(X) \oplus \left(I - \Pi_{\lambda_0}\right)(X).$$

Since Π_{λ_0} commutes with the resolvent of A, we have

$$\left(\lambda I - A_{\Pi_{\lambda_0}(X)}\right)^{-1} = (\lambda I - A)^{-1}|_{\Pi_{\lambda_0}(X)},$$

$$\left(\lambda I - A_{\left(I-\Pi_{\lambda_0}\right)(X)}\right)^{-1} = (\lambda I - A)^{-1}|_{\left(I-\Pi_{\lambda_0}\right)(X)}.$$

By using the first equality in equation (4.2.4), we observe that for each $x \in \Pi_{\lambda_0}(X)$ and each $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$ close enough

$$\left(\lambda I - A_{\Pi_{\lambda_0}(X)}\right)^{-1} x = (\lambda I - A)^{-1} B_{-1} x = \sum_{k=-m}^{-1} (\lambda - \lambda_0)^k B_k x.$$

Let $f : \rho(A_{\Pi_{\lambda_0}(X)}) \to \mathscr{L}(X)$ be the map defined by

$$f(\lambda) := (\lambda - \lambda_0)^m \left(\lambda I - A_{\Pi_{\lambda_0}(X)}\right)^{-1}$$

and let $g : \mathbb{C} \to \mathscr{L}(X)$ be the map defined by

$$g(\lambda) := \sum_{k=-m}^{-1} (\lambda - \lambda_0)^{m+k} B_k x.$$

For each $r > 0$, denote

$$B_{\mathbb{C}}(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}.$$

Set

$$r_0 := \sup\left\{r > 0 : B_{\mathbb{C}}(\lambda_0, r)\setminus\{0\} \subset \rho\left(A_{\Pi_{\lambda_0}(X)}\right)\right\}.$$

Assume that $r_0 < +\infty$. Since $\sigma(A_{\Pi_{\lambda_0}(X)})$ is closed, we can find $\lambda_1 \in \sigma(A_{\Pi_{\lambda_0}(X)})$, such that

$$|\lambda_1 - \lambda_0| = r_0.$$

Then f and g are defined on $B_{\mathbb{C}}(\lambda_0, r_0)\setminus\{0\}$ and coincide in some neighborhood of λ_0. Since $B_{\mathbb{C}}(\lambda_0, r_0)\setminus\{0\}$ is open and connected, by applying Proposition 4.1.4 to

$f-g$, we deduce that

$$f(\lambda)=g(\lambda),\ \ \forall\lambda\in B_{\mathbb{C}}(\lambda_0,r_0)\backslash\{0\}.$$

But since g is analytic and defined on $\mathbb{C}$, it follows for each sequence $\{\lambda_n\}\subset B_{\mathbb{C}}(\lambda_0,r_0)\backslash\{0\}\to\lambda_1$ that

$$f(\lambda_n)=g(\lambda_n)\to g(\lambda_1).$$

Now since A is closed (because its resolvent set is not empty), we have

$$\begin{aligned}\left(\lambda_1 I-A_{\Pi_{\lambda_0}(X)}\right)g(\lambda_n) &= (\lambda_n-\lambda_0)^m\left(\lambda_1 I-A_{\Pi_{\lambda_0}(X)}\right)\left(\lambda_n I-A_{\Pi_{\lambda_0}(X)}\right)^{-1}\\ &= (\lambda_n-\lambda_0)^m\left[(\lambda_1-\lambda_n)\left(\lambda_n I-A_{\Pi_{\lambda_0}(X)}\right)^{-1}+I\right]\\ &= (\lambda_1-\lambda_n)\,g(\lambda_n)+(\lambda_n-\lambda_0)^m I\end{aligned}$$

or equivalently

$$(\lambda_1 I-A_{\Pi_{\lambda_0}(X)})g(\lambda_n)=(\lambda_1-\lambda_n)g(\lambda_n)+(\lambda_n-\lambda_0)^m I.$$

Since A is closed and g is analytic on $\mathbb{C}$, we obtain (when $\lambda_n\to\lambda_1$) that

$$\left(\lambda_1 I-A_{\Pi_{\lambda_0}(X)}\right)g(\lambda_1)=(\lambda_1-\lambda_0)^m I.$$

Similarly we have

$$g(\lambda_1)\left(\lambda_1 I-A_{\Pi_{\lambda_0}(X)}\right)=(\lambda_1-\lambda_0)^m I_{\Pi_{\lambda_0}(X)}.$$

It follows that $\lambda_1\in\rho(A_{\Pi_{\lambda_0}(X)})$ and

$$g(\lambda_1)=(\lambda_1-\lambda_0)^m\left(\lambda_1 I-A_{\Pi_{\lambda_0}(X)}\right)^{-1},$$

a contradiction, which implies that $r_0=+\infty$. So $\sigma(A_{\Pi_{\lambda_0}(X)})=\{\lambda_0\}$ and

$$\left(\lambda I-A_{\Pi_{\lambda_0}(X)}\right)^{-1}x=\sum_{k=-m}^{-1}(\lambda-\lambda_0)^k B_k x,\forall\lambda\in\mathbb{C}\backslash\{\lambda_0\}.$$

Now we compute the Laurent's expansion of $(\lambda I-A_{(I-\Pi_{\lambda_0})(X)})^{-1}$ around λ_0. For $k>0$ and $x\in\left(I-\Pi_{\lambda_0}\right)(X)$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{-(-k+1)}\left(\lambda I-A_{\left(I-\Pi_{\lambda_0}\right)(X)}\right)^{-1}xd\lambda\\ &\quad=\left(I-\Pi_{\lambda_0}\right)B_{-k}x=(I-B_{-1})B_{-k}x\end{aligned}$$

$$= (A - \lambda_0 I) B_0 B_{-k} x = 0.$$

It follows that

$$\left(\lambda I - A_{\left(I - \Pi_{\lambda_0}\right)(X)}\right)^{-1} = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n B_n \mid_{\left(I - \Pi_{\lambda_0}\right)(X)},$$

so $\lambda_0 \notin \sigma(A_{(I-\Pi_{\lambda_0})(X)})$ and the result follows. $\square$

4.3 Spectral Theory of Bounded Linear Operators

Let $T \in \mathscr{L}(X)$ be a bounded linear operator. From now on denote

$$T^{n+1} = T \circ T^n, \ \forall n \geq 0, \text{ and } T^0 = I.$$

Definition 4.3.1. Let $T \in \mathscr{L}(X)$. Then the *essential semi-norm* $\|T\|_{\text{ess}}$ of T is defined by

$$\|T\|_{\text{ess}} = \kappa\left(T\left(B_X(0,1)\right)\right),$$

where $B_X(0,1) = \{x \in X : \|x\|_X \leq 1\}$, and for each bounded set $B \subset X$,

$$\kappa(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by a finite number of balls of radius } \leq \varepsilon\}$$

is the *Kuratovsky measure of non-compactness*.

In the rest of this section, we use some properties of the measure of non-compactness on Banach spaces. For various properties of the Kuratowskis measure of non-compactness, we refer to Deimling [89], Martin [258], and Sell and You [314, Lemma 22.2].

Lemma 4.3.2. *Let $(X, \|.\|)$ be a Banach space and $\kappa(.)$ the measure of non-compactness defined as above. Then for any bounded subsets B and $\widehat{B}$ of X, we have the following properties:*

(a) $\kappa(B) = 0$ *if and only if* $\overline{B}$ *is compact;*
(b) $\kappa(B) = \kappa\left(\overline{B}\right)$;
(c) *If* $B \subset \widehat{B}$ *then* $\kappa(B) \leq \kappa\left(\widehat{B}\right)$;
(d) $\kappa\left(B + \widehat{B}\right) \leq \kappa(B) + \kappa\left(\widehat{B}\right)$, *where* $B + \widehat{B} = \left\{x + y : x \in B, y \in \widehat{B}\right\}$.

Proposition 4.3.3. *For each pair of bounded linear operators $T, \widehat{T} \in \mathscr{L}(X)$, we have the following properties:*

(a) $\|T\|_{\text{ess}} = 0$ *if and only if* T *is compact;*
(b) $\|\lambda T\|_{\text{ess}} \leq |\lambda| \, \|T\|_{\text{ess}}, \forall \lambda \in \mathbb{C}$;
(c) $\left\|T + \widehat{T}\right\|_{\text{ess}} \leq \|T\|_{\text{ess}} + \left\|\widehat{T}\right\|_{\text{ess}}$;

(d) $\left\|T\widehat{T}\right\|_{\text{ess}} \leq \|T\|_{\text{ess}} \left\|\widehat{T}\right\|_{\text{ess}}$;
(e) $\|T\|_{\text{ess}} \leq \|T\|_{\mathscr{L}(X)}$.

Lemma 4.3.4. *Let $T \in \mathscr{L}(X)$. Then the limits $\lim_{n\to+\infty} \|T^n\|_{\mathscr{L}(X)}^{1/n}$ and $\lim_{n\to+\infty} \|T^n\|_{\text{ess}}^{1/n}$ exist.*

Proof. We use standard arguments (see Yosida [381, p.212]). Assume that $\{a_n\}_{n\geq 0}$ is a sequence of non-negative real numbers such that

$$(a_{m+p})^{m+p} \leq (a_m)^m (a_p)^p, \ \forall m, p \geq 1.$$

Then

$$\begin{aligned}
a_{mp} &\leq (a_m)^{\frac{m}{mp}} \left(a_{m(p-1)}\right)^{\frac{m(p-1)}{mp}} \\
&\leq (a_m)^{\frac{m}{mp}} \left(\left(a_{m(p-2)}\right)^{\frac{m(p-2)}{m(p-1)}} (a_m)^{\frac{m}{m(p-1)}} \right)^{\frac{m(p-1)}{mp}} \\
&\leq (a_m)^{\frac{2m}{mp}} \left(\left(a_{m(p-2)}\right)\right)^{\frac{m(p-2)}{mp}} \\
&\leq a_m.
\end{aligned}$$

Set $r = \liminf_{n\geq 1} a_n$. Let $\varepsilon > 0$ be fixed. Let $m \geq 0$ be an integer satisfying

$$a_m \leq r + \varepsilon.$$

Then for any integer $n \geq 0$ by using the Euclidian division, we can find an integer $p \geq 0$ and a remainder $0 \leq q \leq m-1$ such that $n = pm + q$. We have

$$a_{mp+q} \leq (a_{mp})^{\frac{mp}{n}} (a_q)^{\frac{q}{n}} \leq (a_m)^{\frac{mp}{n}} (a_q)^{\frac{q}{n}}$$

and

$$\lim_{n\to+\infty} pm/n = 1 \text{ and } \lim_{n\to+\infty} q/n = 0.$$

Therefore, it follows that

$$\limsup_{n\to+\infty} a_n \leq r + \varepsilon.$$

This completes the proof. □

Definition 4.3.5. Let $T \in \mathscr{L}(X)$. The *spectral radius* $r(T)$ of T is defined by

$$r(T) := \lim_{n\to+\infty} \|T^n\|_{\mathscr{L}(X)}^{1/n}$$

and the *essential spectral radius* $r_{\text{ess}}(T)$ of T is defined by

$$r_{\text{ess}}(T) := \lim_{n\to+\infty} \|T^n\|_{\text{ess}}^{1/n}.$$

Observe that for any integer $n \geq 0$ we have $\|T^n\|_{\mathscr{L}(X)}^{1/n} \leq \|T\|_{\mathscr{L}(X)}$. By using Definition 4.3.5 it follows that

$$r(T) \leq \|T\|_{\mathscr{L}(X)}.$$

In fact, the spectral radius describes the range of the spectrum and we have the following result.

Lemma 4.3.6. *For each $T \in \mathscr{L}(X)$, we have*

$$r(T) \geq \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Proof. Let $\gamma > r(T)$ be given. Set

$$|x| = \sup_{n\geq 0} \frac{\|T^n x\|}{\gamma^n}.$$

Then by the definition of the spectral radius, there exists $M \geq 1$ such that

$$\|x\| \leq |x| \leq M\|x\|$$

and

$$|Tx| \leq \gamma |x|.$$

It follows that for any $\lambda \in \mathbb{C}$ such that $|\lambda| > \gamma$, the map $\lambda I - T$ is invertible (since $\left|\lambda^{-1}T\right| < 1$) and $(\lambda I - T)^{-1} = \lambda^{-1}\left(I - \lambda^{-1}T\right)^{-1} = \lambda^{-1}\sum_{k=0}^{+\infty}\left(\lambda^{-1}T\right)^k$. It follows that

$$\sup\{|\lambda| : \lambda \in \sigma(T)\} \leq r(T).$$

This completes the proof. $\square$

Theorem 4.3.7. *For each $T \in \mathscr{L}(X)$, we have*

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Proof. Denote

$$p := \sup\{|\lambda| : \lambda \in \sigma(T)\}, \;\; q := r(T) = \inf_n\{\|T^n\|_{\mathscr{L}(X)}^{1/n}\} = \lim_{n\to+\infty}\|T^n\|_{\mathscr{L}(X)}^{1/n}.$$

By Lemma 4.3.6 we have $p \leq q$. Let us prove the converse inequality. Let $\varepsilon > 0$ and let $\lambda \in \rho(T)$ with $|\lambda| = p + \varepsilon$. Since the resolvent $(\lambda I - T)^{-1}$ is analytic in the resolvent set $\rho(T)$ and since the resolvent set contains the annulus

$$0 < r_1 \leq |\lambda| \leq r_2$$

for any $p < r_1 < p + \varepsilon \leq \max(p+\varepsilon, q) < r_2$, by Lemma 4.3.6, when $|\lambda| > q$ we have

$$(\lambda I - T)^{-1} = \frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^{n} = \sum_{k=1}^{\infty}\lambda^{-k}T^{(k+1)}. \tag{4.3.1}$$

By Theorem 4.1.3 the resolvent operator has a unique Laurent expansion

$$(\lambda I - T)^{-1} = \sum_{n=-\infty}^{+\infty}\lambda^{n}A_n, \tag{4.3.2}$$

where

$$A_n := \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(0,\varepsilon)^+}\lambda^{-(n+1)}(\lambda I - T)^{-1}d\lambda. \tag{4.3.3}$$

By comparing (4.3.1) and (4.3.2) we deduce that

$$A_n = \begin{cases} 0 & \text{if } n \geq 0 \\ T^{(-n+1)} & \text{if } n < 0. \end{cases}$$

By using (4.3.3) and the fact that the resolvent is bounded on the circle $\lambda = r_1 e^{i\theta}$, we obtain for each $n \geq 1$ that

$$\begin{aligned} A_{-n} &= \frac{1}{2\pi i}\int_{S_{\mathbb{C}}(0,r_1)^+}\lambda^{-(n+1)}(\lambda I - T)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_0^{2\pi} r_1^{(n-1)}e^{(n-1)i\theta}(r_1e^{i\theta}I - T)^{-1}\times ir_1e^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r_1^{n}e^{ni\theta}(r_1e^{i\theta}I - T)^{-1}d\theta. \end{aligned}$$

Therefore, we can find a constant $C > 0$ such that

$$\|T^n\| = \|A_{-n+1}\| \leq Cr_1^{n-1}.$$

It follows that

$$\lim_{n\to+\infty}\|\lambda^{-n}T^n\| = 0$$

whenever $|\lambda| = p + \varepsilon > r_1$, and for all n sufficiently large that

$$\|T^n\| \leq |\lambda|^n = (p+\varepsilon)^n,$$

which implies that

$$q = \lim_{n\to+\infty}\|T^n\|_{\mathscr{L}(X)}^{1/n} \leq p + \varepsilon.$$

Since $\varepsilon > 0$ has been chosen arbitrarily, we have $q \leq p$. $\square$

Lemma 4.3.8. *For each $T \in \mathscr{L}(X)$, we have*

(a) $r_{\text{ess}}(T) \leq r(T)$;
(b) $r_{\text{ess}}(T) \leq \|T\|_{\text{ess}}$;
(c) $r(T) \leq \|T\|$.

Proof. From Proposition 4.3.3(e), we have

$$\|T^n\|_{\text{ess}} \leq \|T^n\|_{\mathscr{L}(X)}, \ \forall n \geq 1,$$

and (a) follows. From Proposition 4.3.3(d), we have

$$\|T^n\|_{\text{ess}} \leq \|T\|_{\text{ess}}^n, \ \forall n \geq 1,$$

and (b) follows. The assertion (c) follows from the fact that

$$\|T^n\|_{\mathscr{L}(X)} \leq \|T\|_{\mathscr{L}(X)}^n, \ \forall n \geq 1.$$

This completes the proof. □

Example 4.3.9. In the case $r_{\text{ess}}(T) = r(T)$, the spectrum can take various forms and the situation does not seem to be very clear in general. To illustrate this situation, consider the shift operator $T : BC([0,+\infty), \mathbb{C}) \to BC([0,+\infty), \mathbb{C})$ defined by

$$T(f)(t) = f(t+1), \ \forall t \geq 0.$$

Then the spectrum of T is equal to

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

and

$$r_{\text{ess}}(T) = r(T) = 1.$$

Lemma 4.3.10 (Riesz's Lemma). *Let E be a normed vector space and let F be a closed subspace of E such that*

$$E \neq F.$$

Then for each $\varepsilon \in (0,1)$, there exists $\widehat{x} \in E$, such that $\|\widehat{x}\| = 1$ and $d(\widehat{x}, F) \geq 1 - \varepsilon$.

Proof. Let $x \in E \setminus F$ and $\varepsilon \in (0,1)$ be fixed. Since F is closed, we have $d := d(x,F) > 0$. Fix $y_0 \in F$ such that

$$d \leq \|x - y_0\| \leq \frac{d}{1-\varepsilon}.$$

Then

$$\widehat{x} = \frac{x - y_0}{\|x - y_0\|}$$

satisfies the requirement. Indeed, if $y \in F$, we have

$$\|\widehat{x} - y\| = \left\| \frac{x - y_0}{\|x - y_0\|} - y \right\|$$

since $y_0 + \|x - y_0\| y \in F$. □

As an immediate consequence of the Riesz's lemma we have the following theorem.

Theorem 4.3.11 (Riesz's Theorem). *Let E be a normed vector space. If $B_E(0,1)$ is compact, then* $\dim(E) < +\infty$.

Definition 4.3.12. Let $L : D(L) \subset E \to F$ be a densely defined linear operator from a Banach space E into a Banach space F. Then the *adjoint linear operator* $L^* : D(L^*) \subset F^* \to E^*$ of L is defined by

$$L^* x^*(x) = x^*(Lx), \ \forall x \in D(L),$$

where

$$D(L^*) = \left\{ x^* \in F^* : \ x^* \circ L\,|_{D(L)} \ \text{has a bounded extension to the whole space } E \right\}$$

or equivalently

$$D(L^*) = \left\{ x^* \in F^* : \sup_{x \in D(L): \|x\| \le 1} |x^*(Lx)| < +\infty \right\}.$$

Note that in order for L^* to define a map we need $L^* x^*$ to be uniquely determined. So we need to assure that $x^* \circ L\,|_{D(L)}$ has a unique extension to the whole space E. Thus in the above definition we need $D(L)$ to be dense in E. We also remark that

$$G^*(L) := \{(y^*, x^*) \in Y^* \times X^* : y^*(Lx) = x^*(x), \forall x \in D(L)\}$$

is Graph(L^*), the graph of L^*. Thus, from Lemma 3.3.4 we have

$$(x_0, y_0) \in \mathrm{Graph}(L) \Leftrightarrow \langle y^*, y_0 \rangle = \langle x^*, x_0 \rangle, \ \forall (y^*, x^*) \in \mathrm{Graph}(L^*).$$

By using a similar argument we have

$$(x_0^*, y_0^*) \in \mathrm{Graph}(L^*) \Leftrightarrow \langle y_0^*, y \rangle = \langle x_0^*, x \rangle, \ \forall (x, y) \in \mathrm{Graph}(L).$$

Let $L : E \to F$ be a bounded linear operator from a Banach space E into a Banach space F. Then $L^* : F^* \to E^*$ is simply defined by

$$L^* x^* = x^* \circ L.$$

Lemma 4.3.13. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X. Then we have the following*

(a) $\|T^*\|_{\text{ess}} \le 2\|T\|_{\text{ess}}$;
(b) $\|T\|_{\text{ess}} \le 2\|T^*\|_{\text{ess}}$;
(c) $r_{\text{ess}}(T^*) = r_{\text{ess}}(T)$.

Proof. (a) Let $\varepsilon > \|T\|_{\text{ess}}$. Then by the definition of $\|T\|_{\text{ess}}$ we can find an integer $N \ge 1$ and $y_0, .., y_N \in X$, such that

$$T(B_X(0,1)) \subset \bigcup_{i=1}^{N} B_X(y_i, \varepsilon).$$

Let $\eta > 0$ be given. Assume that

$$\|T^*\|_{\text{ess}} > 2\varepsilon + \eta.$$

Let $x_0^* \in B_{X^*}(0,1)$ be fixed. Since $\|T^*\|_{\text{ess}} > 2\varepsilon + \eta$, we have

$$T^*(B_{X^*}(0,1)) \nsubseteq B_{X^*}(T^*x_0^*, 2\varepsilon + \eta).$$

So we can find $x_1^* \in B_{X^*}(0,1)$ such that

$$T^*x_1^* \notin B_{X^*}(T^*x_0^*, 2\varepsilon + \eta),$$

and since $\|T^*\|_{\text{ess}} > 2\varepsilon + \eta$, we have

$$T^*(B_{X^*}(0,1)) \nsubseteq \bigcup_{i=0}^{1} B_{X^*}(T^*x_i^*, 2\varepsilon + \eta).$$

By induction, we can find a sequence $\{x_n^*\}_{n\geq 0}$ in $B_{X^*}(0,1)$ such that

$$\|T^*x_n^* - T^*x_m^*\|_{X^*} \geq 2\varepsilon + \eta,\ \forall n \neq m.$$

Let $x \in B_X(0,1)$ be fixed. We have

$$(T^*x_n^* - T^*x_m^*)(x) = (x_n^* - x_m^*)(T(x))$$

and $T(x) \in \bigcup\limits_{i=1}^{N} B_X(y_i, \varepsilon)$ that

$$\begin{aligned}|(T^*x_n^* - T^*x_m^*)(x)| &\leq \sup_{\substack{z \in B_X(0,\varepsilon)\\ i=1,\dots,N}} |(x_n^* - x_m^*)(y_i + z)| \\ &\leq \sup_{i=1,\dots,N} |(x_n^* - x_m^*)(y_i)| + 2\varepsilon.\end{aligned}$$

So we obtain

$$2\varepsilon + \eta \leq \sup_{i=1,\dots,N} |(x_n^* - x_m^*)(y_i)| + 2\varepsilon, \forall n \neq m,$$

hence

$$\eta \leq \sup_{i=1,\dots,N} |(x_n^* - x_m^*)(y_i)|, \forall n \neq m.$$

But since $x_n^* \in B_{X^*}(0,1)$, $\forall n \geq 0$, the sequence $\left\{\begin{pmatrix} x_n^*(y_1) \\ \vdots \\ x_n^*(y_N)\end{pmatrix}\right\}_{n\geq 0}$ is bounded. So we can extract a converging subsequence $\left\{\begin{pmatrix} x_{n_p}^*(y_1) \\ \vdots \\ x_{n_p}^*(y_N)\end{pmatrix}\right\}_{n\geq 0}$ such that

$$\sup_{\substack{z\in B_X(y_i,\varepsilon)\\ i=1,\dots,N}} \left| \left(x^*_{n_p} - x^*_{n_{p-1}} \right)(y_i) \right| \to 0 \text{ as } p \to +\infty,$$

and we obtain a contradiction. Therefore,

$$\|T^*\|_{\mathrm{ess}} \le 2\varepsilon, \ \forall \varepsilon > \|T\|_{\mathrm{ess}}.$$

(b) Let $\varepsilon > \|T^*\|_{\mathrm{ess}}$ be given. By the definition of $\|T^*\|_{\mathrm{ess}}$, we can find an integer $N \ge 1$ and $y_0^*,..,y_N^* \in X^*$, such that

$$T^*\left(B_{X^*}(0,1)\right) \subset \bigcup_{i=1}^{N} B_{X^*}\left(y_i^*, \varepsilon\right).$$

Let $\eta > 0$ be a given constant. Assume that

$$\|T\|_{\mathrm{ess}} > 2\varepsilon + \eta.$$

By induction, we can find a sequence $\{x_n\}_{n\ge 0}$ in $B_X(0,1)$ such that

$$\|Tx_n - Tx_m\|_X \ge 2\varepsilon + \eta, \ \forall n \ne m.$$

By the Hahn-Banach theorem we can find $x^*_{m,n} \in X^*$ with $\left\|x^*_{m,n}\right\|_{X^*} = 1$ such that

$$x^*_{m,n}\left(Tx_n - Tx_m\right) = \|Tx_n - Tx_m\|_X \ge 2\varepsilon + \eta.$$

Notice that

$$x^*_{m,n}\left(Tx_n - Tx_m\right) = x^*_{m,n}\left(Tx_n - Tx_m\right) = T^*x^*_{m,n}(x_n) - T^*x^*_{m,n}(x_m)$$

and

$$T^*x^*_{m,n} \in B_{X^*}\left(y^*_{i_0}, \varepsilon\right)$$

for some $i_0 \in \{1,...,N\}$. So

$$T^*x^*_{m,n} = y^*_{i_0} + z^*$$

for some $z^* \in B_{X^*}(0,\varepsilon)$. It follows that

$$x^*_{m,n}\left(Tx_n - Tx_m\right) = y^*_{i_0}\left(x_n - x_m\right) + z^*\left(x_n - x_m\right),$$

so we obtain

$$2\varepsilon + \eta \le \|Tx_n - Tx_m\|_X \le \sup_{i=1,\dots,N} \left|y_i^*\left(x_n - x_m\right)\right| + 2\varepsilon.$$

The result follows by using the same argument as in part (a) of the proof.

(c) We have

$$\frac{1}{2}\|T^{*n}\|_{\mathrm{ess}} \le \|T^n\|_{\mathrm{ess}} \le 2\|T^{*n}\|_{\mathrm{ess}}, \ \forall n \ge 1,$$

so

$$\left(\frac{1}{2}\right)^{1/n} \|T^{*n}\|_{\text{ess}}^{1/n} \leq \|T^n\|_{\text{ess}}^{1/n} \leq 2^{1/n} \|T^{*n}\|_{\text{ess}}^{1/n}, \ \forall n \geq 1,$$

and (c) follows when $n \to +\infty$. □

The following theorem is a direct consequence of the open mapping theorem (see Brezis [48, Corollary 2.7 p. 35] for a proof).

Theorem 4.3.14. *Let E and F be two Banach spaces and let T be a bounded linear operator from E into F that is bijective; i.e. injective (one-to-one) and surjective (onto). Then T^{-1} is bounded from F into E.*

Let Y be a Banach space and Y^* be the space of continuous linear forms on Y. Let E be a subspace Y and F be a subspace of Y^*. Denote the *orthogonal complements* of E and F by

$$\begin{array}{l} E^{\perp} = \{x^* \in Y^* : x^*(x) = 0, \ \forall x \in E\}, \\ F^{\perp} = \{x \in Y : x^*(x) = 0, \ \forall x^* \in F\}. \end{array}$$

We refer to Brezis [47, Corollary II.17 and Theorem II.18] for a proof of the following lemma.

Lemma 4.3.15. *Let $L : D(L) \subset E \to F$ be a closed and densely defined linear operator from a Banach space E into a Banach space F. Then we have the following properties*

(a) $\mathscr{N}(L) = \mathscr{R}(L^*)^{\perp}$;
(b) $\mathscr{N}(L^*) = \mathscr{R}(L)^{\perp}$;
(c) $\mathscr{N}(L^*)^{\perp} = \mathscr{R}(L) \Leftrightarrow \mathscr{R}(L)$ *is closed;*
(d) $\mathscr{N}(L)^{\perp} = \mathscr{R}(L^*) \Leftrightarrow \mathscr{R}(L^*)$ *is closed.*

The first main result of this section is the following theorem.

Theorem 4.3.16. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X and assume that*

$$r_{\text{ess}}(T) < 1.$$

Then there exists an integer $k_0 \geq 0$ such that

(a) $\mathscr{N}((I-T)^{k_0}) = \mathscr{N}((I-T)^{k_0+n}), \ \forall n \geq 1$;
(b) $\dim(\mathscr{N}((I-T)^{k_0})) < +\infty$;
(c) *For each $k \geq 1$, $\mathscr{R}((I-T)^k)$ is closed, and*

$$\mathscr{R}\left((I-T)^k\right) = \mathscr{N}\left((I-T^*)^k\right)^{\perp}.$$

(d) *For each $k \geq 1$, $\mathscr{R}((I-T^*)^k)$ is closed, and*

$$\mathscr{R}\left((I-T^*)^k\right) = \mathscr{N}\left((I-T)^k\right)^{\perp}.$$

Proof. (a) Set
$$E_n := \mathcal{N}\left((I-T)^n\right), \ \forall n \geq 0.$$
Then we have
$$E_n \subset E_{n+1}, \ \forall n \geq 0,$$
and
$$(I-T)\left(E_{n+1}\right) \subset E_n, \ \forall n \geq 1.$$
It follows that
$$(I-T)\left(E_n\right) \subset (I-T)\left(E_{n+1}\right) \subset E_n, \ \forall n \geq 0,$$
hence
$$T\left(E_n\right) \subset E_n, \ \forall n \geq 1.$$
Assume that $E_n \neq E_{n+1}, \forall n \geq 1$. By applying the Riesz's lemma, we can find a sequence $\{u_n\}_{n\geq 1}$, such that
$$u_n \in E_n, \ \|u_n\| = 1, \text{ and } d\left(u_n, E_{n-1}\right) \geq 1/2, \ \forall n \geq 1.$$
Setting
$$v_n := (I-T)\, u_n \in E_{n-1},$$
we have
$$Tu_n = u_n - v_n.$$
Thus,
$$T^2 u_n = T\left(u_n\right) - T\left(v_n\right) = u_n - v_n - T\left(v_n\right)$$
and we obtain by induction that
$$T^k\left(u_n\right) = u_n - \sum_{l=0}^{k-1} T^l\left(v_n\right), \ \forall k \geq 1.$$
Since $T\left(E_{n-1}\right) \subset E_{n-1}, \ \forall n \geq 1$, it follows that
$$z_n^k = \sum_{l=0}^{k-1} T^l\left(v_n\right) \in E_{n-1}, \forall n \geq 1, \forall k \geq 1.$$
Since $\|u_n\| = 1$, we have
$$u_n - z_n^k = T^k\left(u_n\right) \in T^k\left(B_X(0,1)\right).$$
Since $r_{\text{ess}}(T) < 1$, it follows that
$$\kappa\left(T^k\left(B_X(0,1)\right)\right) \to 0 \text{ as } k \to +\infty.$$
Let $k_0 \geq 1$ be given such that

$$\kappa\left(T^{k_0}\left(B_X(0,1)\right)\right) < \left(\frac{1}{2}\right)^3.$$

Since

$$\bigcup_{n\geq 0}\left\{u_n - z_n^{k_0}\right\} \subset T^{k_0}\left(B_X(0,1)\right),$$

we can find a subsequence $\{u_{n_p} - z_{n_p}^{k_0}\}_{p\geq 0}$ and $\widehat{x}\in X$, such that

$$u_{n_p} - z_{n_p}^{k_0} \in B_X\left(\widehat{x}, \left(\frac{1}{2}\right)^3\right),\ \forall p\geq 1.$$

Then

$$\left\|u_{n_p} - z_{n_p}^{k_0} - (u_{n_{p-1}} - z_{n_{p-1}}^{k_0})\right\| \leq \left\|u_{n_p} - z_{n_p}^{k_0} - \widehat{x}\right\| + \left\|\widehat{x} - (u_{n_{p-1}} - z_{n_{p-1}}^{k_0})\right\| \leq \left(\frac{1}{2}\right)^2.$$

Without loss of generality we can assume that $n_{p-1} < n_p, \forall p\geq 0$. We have

$$n_{p-1} \leq n_p - 1, \forall p \geq 0,$$

and

$$u_{n_{p-1}} - z_{n_{p-1}}^{k_0} \in E_{n_{p-1}} \subset E_{n_p - 1},$$

so

$$z_{n_p}^{k_0} + (u_{n_{p-1}} - z_{n_{p-1}}^{k_0}) \in E_{n_p - 1}.$$

It follows that

$$d\left(u_{n_p}, E_{n_p-1}\right) \leq \left(\frac{1}{2}\right)^2,$$

which gives a contradiction to the fact that $d\left(u_{n_p}, E_{n_p-1}\right) \geq 1/2$. From this contradiction it follows that there exists an integer $k_0 \geq 1$ such that

$$\mathcal{N}\left((I-T)^{k_0}\right) = \mathcal{N}\left((I-T)^{k_0+m}\right),\ \forall m\geq 1.$$

(b) We prove $\dim\left(E_{k_0}\right) < +\infty$ by induction. Clearly $E_0 = \{0\}$. Thus,

$$\dim(E_0) = 0.$$

Assume that $\dim(E_k) < +\infty$. Let $u \in B_{E_{k+1}}(0,1)$, then from part (a) of the proof we know that there exists $v \in E_k$ such that

$$Tu = u - v.$$

We have

$$\|v\| \leq (1 + \|T\|) =: \delta$$

and

$$T^k(u) = u - \sum_{l=0}^{k-1} T^l(v).$$

Hence

$$\kappa\left(B_{E_{k+1}}(0,1)\right) \leq \kappa\left(T^k(B_X(0,1)) + B_{E_k}(0,\delta) + TB_{E_k}(0,\delta) + \ldots + T^{k-1}B_{E_k}(0,\delta)\right)$$

and, since $\dim(E_k) < +\infty$, we obtain

$$\kappa\left(B_{E_{k+1}}(0,1)\right) \leq \kappa\left(T^k(B_X(0,1))\right), \ \forall k \geq 1.$$

When k goes to $+\infty$, since $r_{\text{ess}}(T) < 1$, it follows that $\kappa\left(T^k(B_X(0,1))\right) \to 0$. Thus,

$$\kappa\left(B_{E_{k+1}}(0,1)\right) = 0.$$

It implies that $\overline{B_{E_{k+1}}(0,1)}$ is compact. But $(I-T)^{k+1}$ is bounded, we deduce that $E_{k+1} = \mathscr{N}((I-T)^{k+1})$ is closed, so is $B_{E_{k+1}}(0,1)$. Hence, $B_{E_{k+1}}(0,1)$ is compact. Now by applying the Riesz's theorem we obtain that $\dim(E_{k+1}) < +\infty$.

(c) We prove the result by induction. Set

$$X_n = \mathscr{R}((I-T)^n X), \ \forall n \geq 0.$$

We have $X_0 = X$. Assume that X_k is closed. Consider a sequence $\{f_n\} = \{u_n - Tu_n\} \to f$, where $u_n \in X_k$. We want to prove that $f \in \mathscr{R}((I-T)X_k)$. Since

$$\dim\left(\mathscr{N}\left((I-T)\,|_{X_k}\right)\right) < +\infty,$$

we can find $v_n \in \mathscr{N}\left(((I-T)\,|_{X_k}\right)$ such that

$$\|u_n - v_n\| = d\left(u_n, \mathscr{N}\left((I-T)\,|_{X_k}\right)\right).$$

Then we have

$$f_n = (I-T)(u_n - v_n), \ \forall n \geq 0.$$

Assume that $\{u_n - v_n\}_{n \geq 0}$ is unbounded. By extracting a subsequence that we denote with the same index, we have $\|u_n - v_n\| \to +\infty$. Set $w_n := \dfrac{u_n - v_n}{\|u_n - v_n\|}$. We have

$$\begin{aligned}
d\left(w_n, \mathscr{N}\left(((I-T)\,|_{X_k})\right)\right) &= \inf_{x \in N\left((I-T)|_{X_k}\right)} \left\| \frac{u_n - v_n}{\|u_n - v_n\|} - x \right\| \\
&= \inf_{x \in N\left((I-T)|_{X_k}\right)} \left\| \frac{u_n - (v_n + x)}{\|u_n - v_n\|} \right\| \\
&= \inf_{y \in N\left((I-T)|_{X_k}\right)} \left\| \frac{u_n - y}{\|u_n - v_n\|} \right\|.
\end{aligned}$$

So

$$d\left(w_n, \mathscr{N}\left((I-T)\,|_{X_k}\right)\right) = \frac{d\left(u_n, \mathscr{N}\left((I-T)\,|_{X_k}\right)\right)}{\|u_n - v_n\|} = 1.$$

Set $g_n := \frac{f_n}{\|u_n - v_n\|} \to 0$, we have

$$w_n = g_n + Tw_n = \sum_{k=0}^{m} T^k g_n + T^{m+1}(w_n),\ \forall m \geq 0.$$

Denote

$$C = \bigcup_{n\geq 0} \{g_n\} \cup \{0\}.$$

Then C is compact and

$$\kappa\left(\bigcup_{n\geq 0}\{w_n\}\right) \leq \kappa\left(C + TC + \ldots + T^m C + T^{m+1} B_X(0,1)\right),$$

$$\kappa\left(\bigcup_{n\geq 0}\{w_n\}\right) \leq \kappa\left(T^{m+1} B_X(0,1)\right) \to 0,\ m \to +\infty.$$

We deduce that $\bigcup_{n\geq 0}\{w_n\}$ is relatively compact, so we can extract a subsequence $\left\{w_{n_p}\right\} \to w$. Since X_k is closed, we have

$$w \in X_k.$$

Note that

$$g_{n_p} = (I-T)\, w_{n_p},$$

we obtain

$$(I-T)\, w = 0,$$

so

$$w \in \mathscr{N}\left(\left((I-T)\,|_{X_k}\right)\right).$$

Since the map $x \to d\left(x, \mathscr{N}\left(\left((I-T)\,|_{X_k}\right)\right)\right)$ is continuous, we obtain

$$d\left(w, \mathscr{N}\left(\left((I-T)\,|_{X_k}\right)\right)\right) = 1,$$

a contraction. So the sequence $\{u_n - v_n\}_{n\geq 0}$ is bounded.

Now by noting that

$$u_n - v_n = f_n + T\left(u_n - v_n\right)$$

and by using the same arguments as above, we can extract a converging subsequence $\left\{u_{n_p} - v_{n_p}\right\}_{p\geq 0} \to \widehat{w} \in X_k$, and obtain

$$f = (I-T)\,\widehat{w}.$$

Hence, X_{k+1} is closed. Assertion (c) follows by induction because $X_{k+1} = \mathscr{R}((I-T)\mid_{X_k})$. Since $\mathscr{R}((I-T)^k X)$ is closed, it follows from Lemma 4.3.15 (c) that

$$\mathscr{R}\left((I-T)^k X\right) = \mathscr{N}\left((I-T^*)^k\right)^{\perp}, \forall k \geq 0.$$

Finally to prove (d) it is sufficient to observe that $r_{\text{ess}}(T^*) = r_{\text{ess}}(T) < 1$. Therefore, we can apply property (c) to T^* to claim that $\mathscr{R}((I-T^*)^k X)$ is closed. By using Lemma 4.3.15 (d) the result follows. This completes the proof. $\square$

Lemma 4.3.17 (Fredholm Alternative). *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X and assume that*

$$r_{\text{ess}}(T) < 1.$$

Then

$$\mathscr{N}((I-T)) = \{0\} \Leftrightarrow \mathscr{R}((I-T)) = X.$$

Proof. $\Rightarrow$ Assume that

$$\mathscr{N}((I-T)) = \{0\}.$$

Assume by contradiction that

$$E_1 := \mathscr{R}(I-T) \neq X.$$

By Theorem 4.3.16, E_1 is a closed subspace of X, therefore E_1 is a Banach space endowed with the norm of X and we have

$$T(E_1) = T(I-T)(X) = (I-T)(T(X)) \subset (I-T)(X) = E_1$$

and

$$(I-T)(E_1) = (I-T)(I-T)(X) \subset (I-T)(X) = E_1.$$

Since by assumption $I-T$ is one-to-one, by induction and by setting $E_k = (I-T)^k(E_1)$ for all $k \geq 1$, we obtain a decreasing sequence of subspaces $\{E_n\}_{n\geq 1}$, such that $E_{n+1} \neq E_n$, $\forall n \geq 0$. Moreover, since $T(E_1) \subset E_1$, we have

$$TE_n = T(I-T)^n(E_1) = (I-T)^n(TE_1) \subset (I-T)^n(E_1) = E_n,$$

that is,

$$TE_n \subset E_n.$$

By applying the Riesz's lemma we can find a sequence $\{x_n\} \subset E_n$, such that $\|x_n\| = 1$ and $d(x_n, E_{n+1}) \geq \frac{1}{2}$.

Moreover, since $r_{\text{ess}}(T) < 1$, we have $\kappa\left(T^k(B(0,1))\right) \to 0$ as $k \to +\infty$, there exists $k_0 \geq 1$ such that

$$\kappa\left(T^{k_0}(B(0,1))\right) \leq \frac{1}{8}.$$

Hence, we can find a subsequence $\{x_{n_p}\}_{p\geq 0}$ such that

$$\left\| T^{k_0}\left(x_{n_p}\right) - T^{k_0}\left(x_{n_{p-1}}\right) \right\| \leq \frac{1}{4}.$$

We have

$$\begin{aligned} T^{k_0} x_{n_p} &= (I-T) x_{n_p} + T(I-T) x_{n_p} + T^2 (I-T) x_{n_p} + \ldots \\ &\quad + T^{k_0-1} (I-T) x_{n_p} - x_{n_p} \\ &= \sum_{l=0}^{k_0-1} T^l (I-T) x_{n_p} - x_{n_p}. \end{aligned}$$

It follows that

$$T^{k_0} x_{n_p} - T^{k_0} x_{n_{p+1}} = \sum_{l=0}^{k_0-1} T^l (I-T) x_{n_p} - \sum_{l=0}^{k_0-1} T^l (I-T) x_{n_{p+1}} + x_{n_{p+1}} - x_{n_p}.$$

Notice that

$$\sum_{l=0}^{k_0-1} T^l (I-T) x_{n_p} - \sum_{l=0}^{k_0-1} T^l (I-T) x_{n_{p+1}} + x_{n_{p+1}} \in E_{n_p+1},$$

we have

$$\left\| T^{k_0} x_{n_p} - T^{k_0} x_{n_{p+1}} \right\| \geq \frac{1}{2},$$

a contradiction. So $\mathscr{R}(I-T) = X$.

$\Leftarrow$ Conversely, assume that $\mathscr{R}(I-T) = X$. Then by Lemma 4.3.15 (c), we have $\mathscr{N}(I-T^*) = \mathscr{R}(I-T)^{\perp} = X^{\perp} = \{0\}$. Since $r_{\text{ess}}(T^*) = r_{\text{ess}}(T)$, we can apply the previous part of the proof and deduce that $\mathscr{R}(I-T^*) = X^*$. By using Lemma 4.3.15 (d), it follows that $\mathscr{N}(I-T) = \mathscr{R}((I-T^*))^{\perp} = X^{*\perp} = \{0\}$. $\square$

Remark 4.3.18. The Fredholm Alternative Theorem is useful for the solvability of the nonhomogeneous equation $u - Tu = f$: either the nonhomogeneous equation has a unique solution for every $f \in X$ or the nonhomogeneous equation is solvable if and only if f satisfies the orthogonality condition $f \in \mathscr{N}(I-T^*)^{\perp}$.

Lemma 4.3.19. *Under the assumptions of Theorem 4.3.16,* $(I-T)_{\mathscr{R}((I-T)^{k_0})}$*, the part of* $(I-T)$ *in* $\mathscr{R}((I-T)^{k_0})$*, is invertible.*

Proof. Set $X_{k_0} := \mathscr{R}((I-T)^{k_0})$. Then

$$(I-T) X_{k_0} = (I-T)(I-T)^{k_0} X = (I-T)^{k_0} (I-T) X \subset (I-T)^{k_0} X = X_{k_0},$$

so $(I-T) X_{k_0} \subset X_{k_0}$. Moreover, assume that there exists $x \in X_{k_0} \setminus \{0\}$ such that

$$(I-T)x = 0.$$

There exists $y \in X$, such that $x = (I-T)^{k_0} y$, and

$$(I-T)^{k_0} y \neq 0 \text{ and } (I-T)^{k_0+1} y = 0,$$

which implies that $\mathscr{N}((I-T)^{k_0}) \neq \mathscr{N}((I-T)^{k_0+1})$, a contradiction. Since X_{k_0} is closed, it is a Banach space endowed with the norm of X, and by applying Lemma 4.3.17 to $(I-T)_{X_{k_0}}$, we obtain $(I-T)\left(X_{k_0}\right) = X_{k_0}$. Since $(I-T)_{X_{k_0}}$ is bijective and bounded, we know that $(I-T)_{X_{k_0}}^{-1}$ is bounded, the result follows. □

Let E be a subspace of X. Recall that the *quotient space* X/E is defined by

$$X/E := \{\{x+v : v \in E\} : x \in X\}.$$

Set

$$\widehat{x} := \{x+v : v \in E\}, \forall x \in X.$$

Then X/E is a vector space with the addition

$$\widehat{x} + \widehat{y} := \widehat{x+y}$$

and the multiplication by a scalar number

$$\lambda \widehat{x} := \widehat{\lambda x}.$$

If we endow the norm $\|.\|_{X/E}$ defined by

$$\|\widehat{x}\|_{X/E} = \inf_{v \in E} \|x+v\|$$

and if E is a closed subspace of X, then $(X/E, \|.\|_{X/E})$ is a Banach space. Moreover, if $T \in \mathscr{L}(X)$ and E is a subspace of X such that $T(E) \subset E$, then we can define $T_{X/E} : X/E \to X/E$ by

$$T_{X/E}(\widehat{x}) = \widehat{T(x)}.$$

Since $T(E) \subset E$, if $x = y + w$ for some $w \in E$, then

$$\begin{aligned} T_{X/E}(\widehat{x}) &= \{T(x)+v : v \in E\} = \{T(y)+T(w)+v : v \in E\} \\ &= \{T(y)+v : v \in E\} = T_{X/E}(\widehat{y}). \end{aligned}$$

So $T_{X/E}$ defines a map on X/E. Furthermore, it is readily checked that $T_{X/E}$ is linear. The following lemma also provides the boundedness of $T_{X/E}$ and an important estimation for the essential norm of $T_{X/E}$.

Lemma 4.3.20. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X and let E be a subspace of X satisfying*

$$T(E) \subset E.$$

Then we have the following:

(a) $\left\|T_{X/E}\right\| \leq \|T\|_{\mathscr{L}(X)}$;

(b) $\left\|T_{X/E}\right\|_{\mathrm{ess}} \leq \|T\|_{\mathrm{ess}}$;
(c) $r_{\mathrm{ess}}\left(T_{X/E}\right) \leq r_{\mathrm{ess}}(T)$.

Proof. (a) Let $\delta > 0$ be a constant. Let $\widehat{x} = \{x+v : v \in E\} \in B_{X/E}(0,1)$. Since

$$\|\widehat{x}\|_{X/E} = \inf_{v\in E}\|x+v\| \leq 1,$$

there exists $v_0 \in E$ such that

$$\|x+v_0\| \leq 1+\delta$$

and

$$\widehat{x} = \{x+v_0+v : v \in E\}.$$

So we have

$$T_{X/E}(\widehat{x}) = \{T(x+v_0)+v : v \in E\}$$

and

$$\left\|T_{X/E}(\widehat{x})\right\|_{X/E} = \inf_{v\in E}\|T(x+v_0)+v\| \leq \|T(x+v_0)\| \leq \|T\|(1+\delta).$$

Thus,

$$\left\|T_{X|E}\right\|_{\mathscr{L}(X/E)} \leq \|T\|(1+\delta),\ \forall\delta > 0.$$

(b) Let $\varepsilon > \|T\|_{\mathrm{ess}}$. Then we can find an integer $N \geq 1$ and $y_1,\ldots,y_N \in X$ such that

$$T\left(B_X(0,1)\right) \subset \bigcup_{i=1}^{N} B(y_i,\varepsilon).$$

Set

$$\widehat{y_i} = \{y_i+v : v \in E\} \in X/E,\ \forall i = 1,\ldots,N.$$

Let $\eta > 0$ and $\widehat{x} \in B_{X/E}(0,1)$ be fixed. Since $\widehat{x} = \{x+v : v \in E\}$ and

$$\|\widehat{x}\|_{X/E} = \inf_{v\in E}\|x+v\| \leq 1,$$

there exists $v_0 \in E$, such that $\|x+v_0\| \leq 1+\eta$ and $\widehat{x} = \{x+v_0+v : v \in E\}$.
Since there exists $i_0 \in \{1,\ldots,N\}$ such that

$$\left\|T\left(\frac{x+v_0}{1+\eta}\right) - y_{i_0}\right\| \leq \varepsilon$$

and

$$\begin{aligned}\left\|T_{X/E}(\widehat{x}) - (1+\eta)\widehat{y_i}\right\| &= \inf_{w,v\in E}\left\|T(x+v_0)-(1+\eta)y_{i_0}-v\right\| \\ &\leq \left\|T(x+v_0)-(1+\eta)y_{i_0}\right\| \leq (1+\eta)\varepsilon,\end{aligned}$$

it follows that

$$T_{X/E}\left(B_{X/E}(0,1)\right) \subset \bigcup_{i=1}^{N} B\left((1+\eta)\widehat{y}_i,(1+\eta)\varepsilon\right).$$

Thus

$$\left\|T_{X/E}\right\|_{\text{ess}} \leq (1+\eta)\,\varepsilon, \forall\eta>0,\ \forall\varepsilon>\|T\|_{\text{ess}},$$

and (b) follows.

By using the previous part of the proof, we have

$$\left\|T^n_{X/E}\right\|_{\text{ess}} \leq \|T^n\|_{\text{ess}}, \forall n\geq 0,$$

and (c) follows. □

The second main result of this section is the following theorem.

Theorem 4.3.21. *With the same notations and assumptions as in Theorem 4.3.16, we have*

$$X=\mathscr{N}\left((I-T)^{k_0}\right)\oplus\mathscr{R}\left((I-T)^{k_0}\right).$$

Moreover, we have the following properties:

(a) $(I-T)_{\mathscr{R}((I-T)^{k_0})}$ *(the part of* $(I-T)$ *in* $\mathscr{R}((I-T)^{k_0})$*) is invertible;*

(b) The spectrum of $(I-T)_{\mathscr{N}((I-T)^{k_0})}$ *(the part of* $(I-T)$ *in* $\mathscr{N}((I-T)^{k_0})$*) is* $\{0\}$.

Proof. We first prove that $\mathscr{N}((I-T)^{k_0})\cap\mathscr{R}((I-T)^{k_0})=\{0\}$. Assume that there exists $x\in\mathscr{R}((I-T)^{k_0})\setminus\{0\}$ such that

$$(I-T)^{k_0}x=0.$$

Then there exists $y\in X$ with $(I-T)^{k_0}y=x$ such that

$$(I-T)^{k_0}y=x\neq 0 \text{ and } (I-T)^{2k_0}y=0.$$

This implies that $\mathscr{N}((I-T)^{k_0})\neq\mathscr{N}((I-T)^{2k_0})$, which contradicts the fact that $\mathscr{N}((I-T)^{k_0})=\mathscr{N}((I-T)^{k_0+1})$ and is impossible. It follows that the part of $(I-T)$ in $\mathscr{R}((I-T)^{k_0})$ is invertible.

Similarly, assume that $\mathscr{N}((I-T^*)^{k_0})\cap\mathscr{R}((I-T^*)^{k_0})\neq\{0\}$. Then we can find $x\in\mathscr{R}((I-T^*)^{k_0})\setminus\{0\}$ such that

$$(I-T^*)^{k_0}x=0.$$

It implies that $\mathscr{N}((I-T^*)^{k_0})\neq\mathscr{N}((I-T^*)^{2k_0})$ which is impossible.

We now prove that

$$X=\mathscr{N}\left((I-T)^{k_0}\right)\oplus\mathscr{R}\left((I-T)^{k_0}\right).$$

Since $T(\mathcal{N}((I-T)^{k_0}))\subset\mathcal{N}((I-T)^{k_0})$, we can consider the quotient space $X/\mathcal{N}((I-T)^{k_0})$ endowed the norm $\|.\|_{X/\mathcal{N}((I-T)^{k_0})}$ defined as above, and consider

$$T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}:X/\mathcal{N}\left((I-T)^{k_0}\right)\to X/\mathcal{N}\left((I-T)^{k_0}\right).$$

Assume that $\mathcal{N}(I-T_{X/\mathcal{N}((I-T)^{k_0})})\neq\{0\}$. Then there exists

$$\widehat{x}=\left\{x+v:v\in\mathcal{N}\left((I-T)^{k_0}\right)\right\}\in X/\mathcal{N}\left((I-T)^{k_0}\right)\ \text{ with }\ \widehat{x}\neq 0_{X/\mathcal{N}\left((I-T)^{k_0}\right)}$$

such that

$$\left(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}\right)(\widehat{x})=0.$$

So there exist $x\notin\mathcal{N}((I-T)^{k_0})$ and $v,w\in\mathcal{N}((I-T)^{k_0})$ such that

$$(I-T)(x-v)=w\Leftrightarrow(I-T)x=w+(I-T)v.$$

But $(I-T)\mathcal{N}((I-T)^{k_0})\subset\mathcal{N}((I-T)^{k_0})$, we have

$$w+(I-T)v\in\mathcal{N}\left((I-T)^{k_0}\right)\ \text{ and }\ (I-T)^{k_0+1}x=0.$$

Hence, $x\in\mathcal{N}((I-T)^{k_0+1})$, which contradicts $\mathcal{N}((I-T)^{k_0})=\mathcal{N}((I-T)^{k_0+1})$. Thus,

$$\mathcal{N}\left(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}\right)=\{0\}.$$

Now from Lemma 4.3.20 we know that $r_{\text{ess}}(T_{X/\mathcal{N}((I-T)^{k_0})})\leq r_{\text{ess}}(T)<1$. Applying Lemma 4.3.17 to $T_{X/\mathcal{N}((I-T)^{k_0})}$ and noting that $\mathcal{N}(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)})=\{0\}$, we deduce that

$$\mathcal{R}\left(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}\right)=X/\mathcal{N}\left((I-T)^{k_0}\right).$$

It follows that

$$\mathcal{R}\left(\left(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}\right)^{k_0}\right)=X/\mathcal{N}\left((I-T)^{k_0}\right).$$

This is equivalent to say that for each $\widehat{y}=\{y+v:v\in\mathcal{N}((I-T)^{k_0})\}$ with $y\in X$, there exists $\widehat{x}=\{x+v:v\in\mathcal{N}((I-T)^{k_0})\}$ with $x\in X$, such that

$$\widehat{y}=\left(I-T_{X/\mathcal{N}\left((I-T)^{k_0}\right)}\right)^{k_0}(\widehat{x})=\left\{(I-T)^{k_0}(x)+v:v\in\mathcal{N}\left((I-T)^{k_0}\right)\right\}.$$

So for each $y\in X$, there exist $x\in X$ and $v\in\mathcal{N}((I-T)^{k_0})$, such that

$$y=(I-T)^{k_0}x+v.$$

We deduce that

$$X = \mathscr{R}\left((I-T)^{k_0}\right) \oplus \mathscr{N}\left((I-T)^{k_0}\right).$$

To conclude, assertion (a) follows from Lemma 4.3.19 and (b) is immediate since $(I-T)^{k_0}_{\mathscr{N}((I-T)^{k_0})} = 0$. $\square$

By noting that

$$\mathscr{N}((\lambda I - T)) = \mathscr{N}\left(\left(I - \lambda^{-1}T\right)\right)$$

we can apply the previous results to study the spectrum of T contained in $\{\lambda \in \sigma(T) : |\lambda| > r_{\text{ess}}(T)\}$.

As a first consequence we have the following results.

Lemma 4.3.22. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X. Then*

$$\{\sigma(T) : |\lambda| > r_{\text{ess}}(T)\} \subset \sigma_P(T).$$

Proof. Assume that $\lambda \in \sigma(T)$ and $\mathscr{N}(\lambda I - T) = \{0\}$. Applying Lemma 4.3.17 to $\left(I - \lambda^{-1}T\right)$, we deduce that $\mathscr{R}(\lambda I - T) = X$, so $(\lambda I - T)$ is invertible, which is impossible since $\lambda \in \sigma(T)$. $\square$

Lemma 4.3.23. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on X. Assume that $r(T) > r_{\text{ess}}(T)$. Then each $\lambda_0 \in \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{\text{ess}}(T)\}$ is isolated in $\sigma(T)$.*

Proof. Let $\lambda_0 \in \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{\text{ess}}(T)\}$. Replacing T by $\lambda_0^{-1}T$ we can assume (without loss of generality) that $\lambda_0 = 1$. Let $k_0 \geq 1$ be given such that

$$X = \mathscr{N}\left((I-T)^{k_0}\right) \oplus \mathscr{R}\left((I-T)^{k_0}\right)$$

and

$$\mathscr{N}\left((I-T)^{k_0}\right) = \mathscr{N}\left((I-T)^{k_0+1}\right).$$

Let $\Pi \in \mathscr{L}(X)$ be the bounded linear operator of projection such that

$$\mathscr{R}(\Pi) = \mathscr{N}\left((I-T)^{k_0}\right) \text{ and } \mathscr{N}(\Pi) = \mathscr{R}\left((I-T)^{k_0}\right).$$

We have

$$\Pi T = T\Pi.$$

Let $T_{\mathscr{R}(\Pi)}$ be the part of T in $\mathscr{R}(\Pi)$ and $T_{\mathscr{N}(\Pi)}$ be the part T in $\mathscr{N}(\Pi)$. Then from the previous results we known that $\sigma(T_{\mathscr{R}(\Pi)}) = \{1\}$ and $1 \in \rho\left(T_{\mathscr{N}(\Pi)}\right)$ (i.e. $I_{\mathscr{N}(\Pi)} - T_{\mathscr{N}(\Pi)}$ is invertible), and since $\rho\left(T_{\mathscr{N}(\Pi)}\right)$ is open, we can find $\varepsilon > 0$, such that for $\lambda \in B_{\mathbb{C}}(1,\varepsilon)$, $\lambda I_{\mathscr{N}(\Pi)} - T_{\mathscr{N}(\Pi)}$ is invertible. So for each $\lambda \in B_{\mathbb{C}}(1,\varepsilon) \setminus \{1\}$ we have that $\lambda I - T$ is invertible, and

$$(\lambda I - T)^{-1} = \left(\lambda I_{\mathscr{N}(\Pi)} - T_{\mathscr{N}(\Pi)}\right)^{-1}(I - \Pi) + \left(\lambda I_{\mathscr{R}(\Pi)} - T_{\mathscr{R}(\Pi)}\right)^{-1}\Pi.$$

This completes the proof. $\square$

Remark 4.3.24. In Lemma 4.3.23, we assumed $r(T) > r_{\text{ess}}(T)$ only to make sure that $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{\text{ess}}(T)\}$ is nonempty.

Lemma 4.3.25. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on X. Assume that $r(T) > r_{\text{ess}}(T)$. Then for each $\gamma \in (r_{\text{ess}}(T), r(T)]$, the subset*

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \gamma\}$$

is finite.

Proof. Replacing T by $r(T)^{-1}T$, we can assume that $r(T) = 1$. Set

$$\gamma_0 := \inf\{\gamma \in (r_{\text{ess}}(T), 1] : \text{ the subset } \{\lambda \in \sigma(T) : |\lambda| \geq \gamma\} \text{ is finite}\}.$$

Assume that $\gamma_0 > r_{\text{ess}}(T)$. Then for each $\varepsilon \in (0, \gamma_0 - r_{\text{ess}}(T))$, the subset

$$\{\lambda \in \sigma(T) : \gamma_0 + \varepsilon \geq |\lambda| \geq \gamma_0 - \varepsilon\}$$

is infinite.

Hence, we can construct a sequence $\{\lambda_n\}_{n\geq 0} \subset \sigma(T)$ such that

$$|\lambda_n| \to \gamma_0, \ |\lambda_n| > \frac{r_{\text{ess}}(T) + \gamma_0}{2}, \text{ and } \lambda_n \neq \lambda_m \text{ whenever } n \neq m.$$

By taking a subsequence we can assume that $\lambda_n \to \widehat{\lambda}$. Moreover, by Lemma 4.3.22 for each $n \geq 1$, there exists $f_n \in X$ with $\|f_n\| = 1$, such that

$$Tf_n = \lambda_n f_n.$$

Then $\kappa\left(\left\{\lambda_n^{-1}Tf_n : n \geq 0\right\}\right) = \kappa\left(\left\{\lambda_n^{-1}Tf_n : n \geq p\right\}\right)$ and $\lambda_n \to \widehat{\lambda}$. We have

$$\kappa\left(\left\{\lambda_n^{-1}Tf_n : n \geq 0\right\}\right) = \left|\widehat{\lambda}\right|^{-1} \kappa\left(\{Tf_n : n \geq 0\}\right),$$

which implies that

$$\kappa(\{f_n : n \geq 0\}) \leq \gamma_0^{-1} r_{\text{ess}}(T)\, \kappa(\{f_n : n \geq 0\}).$$

It follows that

$$\kappa(\{f_n : n \geq 0\}) = 0.$$

So we can extract a converging subsequence $\left\{f_{n_p}\right\}_{p\geq 0} \to f$, and since T is bounded we obtain that

$$Tf = \widehat{\lambda} f.$$

Thus, we have $\widehat{\lambda} \in \sigma(T)$ and there exists $\{\lambda_n\}_{n\geq 0} \subset \sigma(T) \to \widehat{\lambda}$ with $\lambda_n \neq \widehat{\lambda}, \forall n \geq 0$. It follows that $\widehat{\lambda}$ is not an isolated point of the spectrum, but $\left|\widehat{\lambda}\right| > r_{\text{ess}}(T)$, this is impossible by Lemma 4.3.23. □

Using the definition of the essential spectrum and summarizing the above results, we can state the Nussbaum theorem (Nussbaum [280]) as follows.

Theorem 4.3.26 (Nussbaum). *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X. Then*

$$r_{\text{ess}}(T) = \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\}.$$

The above results can be summarized in the following theorem.

Theorem 4.3.27. *Let $T \in \mathscr{L}(X)$ be a bounded linear operator on a Banach space X. Then*

$$r(T) = \max\left(r_{\text{ess}}(T), \sup_{\lambda \in \sigma(T) \setminus \sigma_{\text{ess}}(T)} |\lambda|\right).$$

Moreover, if $r_{\text{ess}}(T) < r(T)$, then for each $\gamma \in (r_{\text{ess}}(T), r(T)]$ the subset

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq \gamma\}$$

is finite. Furthermore, for each $\lambda_0 \in \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{\text{ess}}(T)\}$, there exists $k_0 \geq 1$ so that

$$X = \mathscr{R}\left((\lambda_0 I - T)^{k_0}\right) \oplus \mathscr{N}\left((\lambda_0 I - T)^{k_0}\right),$$

such that

$$T\left(\mathscr{R}\left((\lambda_0 I - T)^{k_0}\right)\right) \subset \mathscr{R}\left((\lambda_0 I - T)^{k_0}\right),\ T\left(\mathscr{N}\left((\lambda_0 I - T)^{k_0}\right)\right) \subset \mathscr{N}\left((\lambda_0 I - T)^{k_0}\right)$$

with the following properties:

(a) $(\lambda_0 I - T)_{\mathscr{R}((\lambda_0 I - T)^{k_0})}$ (the part of $(\lambda_0 I - T)$ in $\mathscr{R}((\lambda_0 I - T)^{k_0})$) is invertible;
(b) The dimension of $\mathscr{N}((\lambda_0 I - T)^{k_0})$ is finite;
(c) The spectrum of $T_{\mathscr{N}((\lambda_0 I - T)^{k_0})}$ (the part of T in $\mathscr{N}((\lambda_0 I - T)^{k_0})$) is $\{\lambda_0\}$.

4.4 Essential Growth Bound of Linear Operators.

Let $(X, \|.\|)$ be a complex Banach space and $A : D(A) \subset X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T_A(t)\}_{t \geq 0}$ of bounded linear operators on X. In the following lemma we use the convention that

$$e^{-\infty} = 0 \text{ and } \ln(0) = -\infty.$$

Definition 4.4.1. Let $\{T_A(t)\}_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then the *growth bound* of $\{T_A(t)\}_{t \geq 0}$ is defined by

$$\omega_0(A) := \lim_{t \to +\infty} \frac{\ln(\|T_A(t)\|)}{t} \in [-\infty, +\infty)$$

and the *essential growth bound* of $\{T_A(t)\}_{t \geq 0}$ is defined by

$$\omega_{0,\text{ess}}(A) := \lim_{t \to +\infty} \frac{\ln(\|T_A(t)\|_{\text{ess}})}{t} \in [-\infty, +\infty).$$

Lemma 4.4.2. *Let $\{T_A(t)\}_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then we have the following properties*

(a) $\omega_0(A) = \lim_{t \to +\infty} \frac{\ln(\|T_A(t)\|)}{t} \in [-\infty, +\infty)$ *exists;*
(b) $\omega_{0,\text{ess}}(A) = \lim_{t \to +\infty} \frac{\ln(\|T_A(t)\|_{\text{ess}})}{t} \in [-\infty, +\infty)$ *exists;*
(c) $\omega_{0,\text{ess}}(A) \leq \omega_0(A)$;
(d) $r(T_A(t)) = e^{\omega_0(A)t}$, $\forall t \geq 0$;
(e) $r_{\text{ess}}(T_A(t)) = e^{\omega_{0,\text{ess}}(A)t}$, $\forall t \geq 0$.

Proof. (a) We have for $t \in [n, n+1]$ that

$$\begin{aligned} \frac{\ln(\|T_A(t)\|)}{t} &= \frac{\ln(\|T_A(t-n)T_A(n)\|)}{t} \\ &\leq \frac{n}{t} \frac{\ln(\|T_A(t)\|) + \ln(\|T_A(n)\|)}{n}. \end{aligned}$$

Thus, for $t \in [n, n+1]$ we have

$$\begin{aligned} \frac{\ln(\|T_A(n+1)\|)}{n+1} &= \frac{\ln(\|T_A(n+1-t)\|) + \ln(\|T_A(t)\|)}{n+1} \\ &\leq \frac{t}{n+1} \frac{\ln(\|T_A(n+1-t)\|) + \ln(\|T_A(t)\|)}{t}. \end{aligned}$$

Hence

$$\lim_{n \to +\infty} \ln\left(\|T_A(n)\|^{1/n}\right) \leq \limsup_{t \to +\infty} \frac{\ln(\|T_A(t)\|)}{t} \leq \lim_{n \to +\infty} \ln\left(\|T_A(n)\|^{1/n}\right).$$

So when $t \to +\infty$, we obtain

$$\lim_{n \to +\infty} \ln\left(\|T_A(n)\|^{1/n}\right) \leq \liminf_{t \to +\infty} \frac{\ln(\|T_A(t)\|)}{t},$$

and (a) follows. The proof of (b) is similar.

(c) It is sufficient to note that $\|T_A(t)\|_{\text{ess}} \leq \|T_A(t)\|$, $\forall t \geq 0$. We also remark that

$$\|T_A(tn)\|^{1/n} = e^{t \frac{\ln(\|T_A(nt)\|)}{nt}}.$$

So when n goes to $+\infty$, we obtain (c). The proof of (d) is similar. $\square$

The following theorem, which was proved by Webb [362][363], gives relationship between the spectrum of a semigroup and the spectrum of its infinitesimal generator.

Theorem 4.4.3 (Webb). *Let $\{T_A(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then we have the following statements:*

(i) If $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T_A(t))$ for $t \geq 0$; if $\mu \in \sigma_p(T_A(t))$ for some $t > 0, \mu \neq 0$, then there exists $\lambda \in \sigma_p(A)$ such that $e^{\lambda t} = \mu$ and $\mathscr{N}(e^{\lambda t}I - T_A(t))$ is the closed linear extension of the linear independent subspaces $\mathscr{N}(\lambda_k I - A)$, where $\lambda_k \in \sigma_p(A)$ and $e^{\lambda_k t} = \mu$;

(ii) If $\lambda \in \sigma(A)$, then $e^{\lambda t} \in \sigma(T_A(t))$ for $t \geq 0$, and $\mathscr{N}(\lambda I - A) \subset \mathscr{N}(e^{\lambda t}I - T_A(t))$ for $t \geq 0$;

(iii) If $\lambda \in \sigma_{\text{ess}}(A)$, then $e^{\lambda t} \in \sigma_{\text{ess}}(T_A(t))$, $t > 0$;

(iv) $\sup\limits_{\lambda\in\sigma(A)} \operatorname{Re}\lambda \leq \omega_0(A)$ and $\sup\limits_{\lambda\in\sigma_{\text{ess}}(A)} \operatorname{Re}\lambda \leq \omega_{0,\text{ess}}(A)$;

(iv) $\omega_0(A) = \max\left\{\omega_{0,\text{ess}}(A), \sup\limits_{\lambda\in\sigma(A)\setminus\sigma_{\text{ess}}(A)} \operatorname{Re}(\lambda)\right\}$.

Proof. (i) The results were proved in Hille and Phillips [187, Theorem 16.7.2, p. 467].

(ii) The first part was Theorem 16.7.1 of Hille and Phillips [187]. To prove the second part, let $x \in \mathscr{N}(\lambda I - A)$. Note that $e^{\lambda t}x$ satisfies the initial value problem $\frac{du}{dt} = Au(t),\ t \geq 0,\ \ u(0) = x$, and $T_A(t)x$ is the unique solution of the initial value problem, we have $T_A(t)x = e^{\lambda t}x$ for all $t \geq 0$. Hence, $\mathscr{N}(\lambda I - A) \subset \mathscr{N}(e^{\lambda t}I - T_A(t))$ for $t \geq 0$. Let k be a positive integer and assume for induction that $\mathscr{N}((\lambda I - A)^k) \subset \mathscr{N}((e^{\lambda t}I - T_A(t))^k)$ for $t \geq 0$. Let $x \in \mathscr{N}((\lambda I - A)^{k+1})$. Then $(\lambda I - A)^k x \in \mathscr{N}(\lambda I - A)$, which implies that $(\lambda I - A)^k x \in \mathscr{N}(e^{\lambda t}I - T_A(t))$. Thus,

$$0 = (e^{\lambda t}I - T_A(t))(\lambda I - A)^k x = (\lambda I - A)^k (e^{\lambda t}I - T_A(t))x,$$

which by induction implies that $(e^{\lambda t}I - T_A(t))x \in \mathscr{N}((e^{\lambda t}I - T_A(t))^k)$. Therefore, $\mathscr{N}((\lambda I - A)^{k+1}) \subset \mathscr{N}((e^{\lambda t}I - T_A(t)^{k+1}))$ for all $t \geq 0$, $k = 1, 2, \ldots.$

(iii) Let $\lambda \in \sigma_{\text{ess}}(A)$ and let $\tau > 0$. By (ii) we have $e^{\lambda\tau} \in \sigma(T_A(t))$. Assume that $e^{\lambda\tau} \in \sigma(T_A(t)) \setminus \sigma_{\text{ess}}(T_A(t))$. Then $e^{\lambda\tau}$ is isolated in $\sigma(T_A(r))$, $\mathscr{N}(e^{\lambda\tau}I - T_A(\tau))$ is finite dimensional and $\mathscr{R}(e^{\lambda\tau}I - T_A(\tau))$ is closed. By (ii), $\mathscr{N}(\lambda I - A) \subset \mathscr{N}(e^{\lambda\tau}I - T_A(\tau))$, so $\mathscr{N}(\lambda I - A)$ is finite dimensional. To show that λ is isolated in $\sigma(A)$, suppose that there exists a sequence $\{z_k\} \subset \sigma(A)$, $z_k \neq \lambda$ for any k, such that $z_k \to \lambda$. Once again by (ii) one has $e^{z_k\tau} \in \sigma(T_A(t))$ for all k. Moreover, $e^{z_k\tau} \neq e^{\lambda\tau}$ for all k sufficiently large since $e^{z_k\tau} = e^{\lambda\tau}$ if and only if $\operatorname{Re} z_k = \operatorname{Re}\lambda$ and $\operatorname{Im} z_k = \operatorname{Im}\lambda + 2j\pi$ for some integer j. Since $e^{z_k\tau} \to e^{\lambda\tau}$, $e^{\lambda\tau}$ is not isolated in $\sigma(T_A(\tau))$. Thus, λ is isolated in $\sigma(A)$.

Next we show that $\mathscr{R}(\lambda I - A)$ is closed. Since $\mathscr{N}(e^{\lambda\tau}I - T_A(\tau))$ is finite dimensional, there exists a positive integer m such that

$$\mathscr{N}(e^{\lambda\tau}I - T_A(\tau)) = \mathscr{N}((e^{\lambda\tau}I - T_A(\tau))^m).$$

So $e^{\lambda\tau}$ is a pole of $(\mu I - T_A(\tau))^{-1}$, $e^{\lambda\tau} \in \sigma_p(T_A(\tau))$, and $X = M_1 \oplus M_2$, where

$$M_1 = \mathscr{N}((e^{\lambda\tau}I - T_A(\tau))^m),\ M_2 = \mathscr{R}((e^{\lambda\tau}I - T_A(\tau))^m)$$

are closed subspaces. Let Π_1, Π_2 be the projections induced by this direct sum; that is,

$$\Pi_i x = x_i,\ \ x = x_1 + x_2,\ \ x_i \in M_i,\ \ i = 1,2.$$

For $t \geq 0, T_A(t)$ commutes with $(e^{\lambda\tau}I - T_A(\tau))^m$, so $T_A(t)$ also commutes with Π_i for each $t \geq 0$. In fact, for each $t \geq 0, T_A(t)$ is completely reduced by M_i, thus A is completely reduced by M_i. Let $\{T_{A_i}(t)\}_{t\geq 0}$ and $A_i (i = 1,2)$ be the restriction of $\{T_A(t)\}_{t\geq 0}$ and A to $M_i (i = 1,2)$. Observe that A_i is the infinitesimal generator of $\{T_{A_i}(t)\}_{t\geq 0}$ on $M_i (i = 1,2)$.

We first claim that $e^{\lambda\tau}I - T_{A_2}(\tau)$ is one-to-one on M_2. Suppose that $\mathscr{N}(e^{\lambda\tau}I - T_{A_2}(\tau)) \neq \{0\}$. By (i) it follows that there exists $\lambda_k \in \sigma_p(A_2)$ such that $e^{\lambda\tau} = e^{\lambda_k\tau}$. Thus, there exists $x \in M_2, x \neq 0$, such that $\lambda_k x = A_2 x$. However, (i) implies that

$$x \in \mathscr{N}(\lambda_k I - A_2) \subset \mathscr{N}(e^{\lambda\tau}I - T_{A_2}(\tau)) \subset \mathscr{N}(e^{\lambda\tau}I - T_A(\tau)) \subset M_1.$$

Hence, $x \in M_1 \cap M_2 = \{0\}$, a contradictoion.

We then claim that $\mathscr{R}(e^{\lambda\tau}I - T_{A_2}(\tau))$ is closed in M_2. Let $\{y_k\} \subset \mathscr{R}(e^{\lambda\tau}I - T_{A_2}(\tau))$ such that $y_k \to y_0$ in M_2. Since $\mathscr{R}(e^{\lambda\tau}I - T_{A_2}(\tau))$ is closed in X, there exists $x_0 \in X$ such that

$$y_0 = (e^{\lambda\tau}I - T_A(\tau))x_0 = (e^{\lambda\tau}I - T_A(\tau))(\Pi_1 x_0 + \Pi_2 x_0).$$

By the uniqueness of the direct sum representation, $(e^{\lambda\tau}I - T_A(\tau))\Pi_1 x_0 = 0$. Thus, $(e^{\lambda\tau}I - T_A(\tau))\Pi_2 x_0 = y_0$.

We next claim that there is a constant c_1 such that $\|(e^{\lambda\tau}I - T_A(\tau))x\| \leq c_1\|(\lambda I - A)x\|$ for all $x \in D(A)$. Let $x \in D(A)$ and define

$$\begin{aligned} u(t) &= (e^{\lambda\tau}I - T_A(\tau))x,\ t \geq 0, \\ v(t) &= \int_0^t e^{\lambda(t-s)} T_A(s)(\lambda I - A)x ds,\ t \geq 0. \end{aligned}$$

Note that $u(t)$ and $v(t)$ are both solutions of the initial value problem

$$\frac{dw}{dt} = \lambda w(t) + (\lambda I - A)T_A(t)x,\ t \geq 0;\ \ w(0) = 0.$$

The uniqueness of solutions to the problem implies that $u(t) = v(t)$ for $t \geq 0$. By the uniform boundedness of $T_A(t) : [0,\tau] \to \mathscr{L}(X)$, there exists a constant c_1 such that for all $x \in D(A)$,

$$\|(e^{\lambda\tau}I - T_A(\tau))x\| = \|\int_0^\tau e^{\lambda(\tau-s)} T_A(s)(\lambda I - A)x ds\| \leq c_1\|(\lambda I - A)x\|. \quad (4.4.1)$$

Now using the fact that the necessary and sufficient condition for the range $\mathscr{R}(T)$ of a one-to-one closed linear operator T in the Banach space X to be closed is $\|x\| \le c\|Tx\|$ for all $x \in X$ for some constant c (Schechter [311]), we obtain c_2 such that for all $x \in M_2$, $\|x\| \le c_2(e^{\lambda\tau}I - T_A(\tau))x\|$. Thus, from (4.4.1) we have for $x \in M_2$ that

$$\|x\| \le c_1c_2\|(\lambda I - A_2)x\|.$$

So $\mathscr{R}(\lambda I - A_2)$ is closed in M_2.

We finally argue that $\mathscr{R}(\lambda I - A_2)$ is closed in X. Let $\{y_k\} \subset \mathscr{R}(\lambda I - A)$ such that $y_k \to y_0$. For $k = 1, 2, \ldots$, there exists $x_k \in D(A)$ such that $y_k = (\lambda I - A)x_k$. Then

$$\Pi_i y_k = (\lambda I - A)\Pi_i x_k = (\lambda I - A_i)\Pi_i x_k \to \Pi_i y_0,\ i = 1, 2.$$

Since $\mathscr{R}(\lambda I - A_2)$ is closed in M_2, there exists $x_{0,2} \in M_2$ such that $(\lambda I - A_2)x_{0,2} = \Pi_2 y_0$. Since M_1 is finite dimensional, $\mathscr{R}(\lambda I - A_1)$ is closed in M_1 and there exists $x_{0,1} \in M_1$ such that $(\lambda I - A_1)x_{0,1} = \Pi_1 y_0$. Thus,

$$(\lambda I - A)(x_{0,1} + x_{0,2}) = \Pi_1 y_0 + \Pi_2 y_0 = y_0.$$

Hence, $\mathscr{R}(\lambda I - A)$ is closed, which means that $\lambda \notin \sigma_{\text{ess}}(A)$, a contradiction.

(iv) Follow from (i), (ii), Lemma 4.4.2 (d) and (e).

(v) Define

$$\omega_2(A) = \max\{\omega_{0,\text{ess}}(A), \sup_{\lambda\in\sigma(A)\setminus\sigma_{\text{ess}}(A)} \operatorname{Re}(\lambda)\}.$$

Lemma 4.4.2(c) and (iv) imply that $\omega_2(A) \le \omega_0(A)$. To prove $\omega_0(A) \le \omega_2(A)$, by Lemma 4.4.2(d) it suffices to show that for some $t \ge 0$, $r(T_A(t)) \le e^{\omega_2(A)t}$. Let $t > 0$ and let $\mu \in \sigma(T_A(t)) \setminus \sigma_{\text{ess}}(T_A(t))$, then Lemma 4.3.22 implies that $\mu \in \sigma_p(T_A(t))$ and by (i), there exists $\lambda \in \sigma_p(A)$ such that $e^{\lambda t} = \mu$. By (iii), $\lambda \in \sigma(A) \setminus \sigma_{\text{ess}}(A)$. Therefore, $|\mu| \le e^{\operatorname{Re}\lambda t} \le e^{\omega_2(A)t}$, which implies that

$$r(T_A(t)) \le e^{\omega_2(A)t}.$$

This completes the proof. $\square$

4.5 Spectral Decomposition of the State Space

The goal of this section is to investigate the spectral properties of the linear operator A. Indeed, since A_0 is the infinitesimal generator of a linear C^0-semigroup of X_0, we can apply the standard theory to the linear operator A_0. We first investigate the properties of projectors which commute with the resolvents of A_0 and A. Then we will turn to the spectral decomposition of the state spaces X_0 and X. Assume that $A : D(A) \subset X \to X$ is a linear operator on a complex Banach X. We start with some basic facts.

Lemma 4.5.1. *We have the following:*

(i) If Y is invariant by A, then $A\mid_Y = A_Y$ (i.e. $D(A_Y) = D(A) \cap Y$);
(ii) If $(\lambda I - A)^{-1} Y \subset Y$ for some $\lambda \in \rho(A)$, then

$$D(A_Y) = (\lambda I - A)^{-1} Y,\ \lambda \in \rho(A_Y),\ \text{and}\ (\lambda I_Y - A_Y)^{-1} = (\lambda I - A)^{-1}\mid_Y .$$

Proof. (i) Assume that Y is invariant by A, we have

$$D(A_Y) = \{x \in D(A) \cap Y : Ax \in Y\} = D(A) \cap Y = D(A\mid_Y),$$

so $A\mid_Y = A_Y$.

(ii) Assume that $(\lambda I - A)^{-1} Y \subset Y$ for some $\lambda \in \rho(A)$. Then we have

$$\begin{aligned} D(A_Y) &= \{x \in D(A) \cap Y : Ax \in Y\} = \{x \in D(A) \cap Y : (\lambda I - A)x \in Y\} \\ &= (\lambda I - A)^{-1} Y, \end{aligned}$$

and the result follows. $\square$

Let $\Pi : X \to X$ be a bounded linear projector on a Banach space X and let Y be a subspace (closed or not) of X. Then we have the following equivalence

$$\Pi(Y) \subset Y \Leftrightarrow \Pi(Y) = Y \cap \Pi(X). \tag{4.5.1}$$

Lemma 4.5.2. *Let $(X, \|.\|)$ be a Banach space. Let $A : D(A) \subset X \to X$ be a linear operator and let $\Pi : X \to X$ be a bounded linear projector. Assume that*

$$\Pi(\lambda I - A)^{-1} = (\lambda I - A)^{-1} \Pi$$

for some $\lambda \in \rho(A)$. Then we have the following

(i) $\Pi(D(A)) = D(A) \cap \Pi(X)$ and $\Pi\left(\overline{D(A)}\right) = \overline{D(A)} \cap \Pi(X)$;
(ii) $A\Pi x = \Pi A x, \forall x \in D(A)$;
(iii) $A_{\Pi(X)} = A\mid_{\Pi(X)}$;
(iv) For $\lambda \in \rho\left(A_{\Pi(X)}\right)$, one has $D(A_{\Pi(X)}) = (\lambda I - A)^{-1} \Pi(X)$ and $\left(\lambda I - A_{\Pi(X)}\right)^{-1} = (\lambda I - A)^{-1}\mid_{\Pi(X)}$;
(v) $\left(A\mid_{\Pi(X)}\right)_{\overline{D(A\mid_{\Pi(X)})}} = \left(A_{\overline{D(A)}}\right)\mid_{\Pi\left(\overline{D(A)}\right)}$.

Proof. We have

$$\Pi(D(A)) = \Pi(\lambda I - A)^{-1}(X) = (\lambda I - A)^{-1} \Pi(X) \subset D(A).$$

Thus, $\Pi(D(A)) \subset D(A)$. Since Π is bounded, we have $\Pi\left(\overline{D(A)}\right) \subset \overline{D(A)}$. So by using (4.5.1), we obtain $\Pi(D(A)) = D(A) \cap \Pi(X)$ and $\Pi\left(\overline{D(A)}\right) = \overline{D(A)} \cap \Pi(X)$. This proves (i).

Let $x \in D(A)$ be fixed. Set $y = (\lambda I - A)x$. Then

$$\Pi Ax = \Pi A(\lambda I - A)^{-1} y = A(\lambda I - A)^{-1} \Pi y = A\Pi x,$$

which gives (ii). Hence, $\Pi(X)$ is invariant by A, and by using Lemma 4.5.1, we obtain (iii). Moreover, we have

$$(\lambda I - A)^{-1} \Pi(X) = \Pi(\lambda I - A)^{-1} X \subset \Pi(X).$$

So Lemma 4.5.1 implies (iv). Finally, we have

$$\begin{aligned} D\left((A\mid_{\Pi(X)})_{\overline{D(A|_{\Pi(X)})}} \right) &= \left\{ x \in D(A\mid_{\Pi(X)}) : Ax \in \overline{D(A\mid_{\Pi(X)})} \right\} \\ &= \left\{ x \in \Pi(X) \cap D(A) : Ax \in \overline{D(A)} \cap \Pi(X) \right\} \\ &= \left\{ x \in \Pi\left(\overline{D(A)}\right) \cap D(A) : Ax \in \Pi\left(\overline{D(A)}\right) \right\} \\ &= D\left(\left(A_{\overline{D(A)}}\right) \mid_{\Pi\left(\overline{D(A)}\right)} \right). \end{aligned}$$

This shows that (v) holds. □

Lemma 4.5.3. *Let the assumptions of Lemma 4.5.2 be satisfied. Assume in addition that Π has a finite rank. Then $\Pi(D(A))$ is closed, $\Pi\left(\overline{D(A)}\right) = \Pi(D(A)) \subset D(A)$, and $A\mid_{\Pi(X)}$ is a bounded linear operator from $\Pi(D(A))$ into $\Pi(X)$.*

Proof. By using Lemma 4.5.2, we have $\Pi(D(A)) = D(A) \cap \Pi(X)$, so $\Pi(D(A))$ is a finite dimensional subspace of X. It follows that $\Pi(D(A))$ is closed and $A\mid_{\Pi(X)}$ is bounded. Now since Π is bounded, we have $\Pi\left(\overline{D(A)}\right) \subset \overline{\Pi(D(A))} = \Pi(D(A))$, and the result follows. □

Lemma 4.5.4. *Let Assumption 3.4.1 be satisfied. Let $\Pi_0 : X_0 \to X_0$ be a bounded linear projector. Then*

$$\Pi_0 T_{A_0}(t) = T_{A_0}(t)\Pi_0, \ \forall t \geq 0 \tag{4.5.2}$$

if and only if

$$\Pi_0(\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1} \Pi_0, \forall \lambda > \omega. \tag{4.5.3}$$

If we assume in addition that (4.5.2) is satisfied, then we have the following:

(i) $\Pi_0(D(A_0)) = D(A_0) \cap \Pi_0(X_0)$ *and* $A_0\Pi_0 x = \Pi_0 A_0 x, \forall x \in D(A_0)$;
(ii) $A_0\mid_{\Pi_0(X_0)} = (A_0)_{\Pi_0(X_0)}$;
(iii) $T_{A_0|_{\Pi_0(X_0)}}(t) = T_{A_0}(t)\mid_{\Pi_0(X_0)}, \forall t \geq 0$;
(iv) *If Π_0 has a finite rank, then $\Pi_0(X_0) = \Pi_0(D(A_0)) \subset D(A_0)$, $A_0\mid_{\Pi_0(X_0)}$ is a bounded linear operator from $\Pi_0(X_0)$ into itself, and*

$$T_{A_0|_{\Pi_0(X_0)}}(t) = e^{A_0|_{\Pi_0(X_0)} t}, \ \forall t \geq 0.$$

Proof. (4.5.2)⇒(4.5.3) follows from the following formula

$$(\lambda I - A_0)^{-1} x = \int_0^{+\infty} e^{-\lambda s} T_{A_0}(s) x ds, \ \forall \lambda > \omega, \ \forall x \in Y.$$

(4.5.3)⇒(4.5.2) follows from the exponential formula (see Pazy [281, Theorem 8.3, p.33])

$$T_{A_0}(t)x = \lim_{n \to +\infty} \left(I - \frac{t}{n} A_0 \right)^{-n} x, \ \ \forall x \in X_0.$$

By applying Lemmas 4.5.2 and 4.5.3 to A_0, we obtain (i)-(iv). □

The following result will be crutial to construct a center mainfold in Chapter 6.

Proposition 4.5.5. *Let Assumption 3.4.1 be satisfied. Let $\Pi_0 : X_0 \to X_0$ be a bounded linear projector satisfying the following properties*

$$\Pi_0 (\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1} \Pi_0, \ \ \forall \lambda > \omega$$

and

$$\Pi_0 (X_0) \subset D(A_0) \text{ and } A_0 \mid_{\Pi_0(X_0)} \text{ is bounded.}$$

Then there exists a unique bounded linear projector Π on X satisfying the following properties:

(i) $\Pi \mid_{X_0} = \Pi_0$;
(ii) $\Pi (X) \subset X_0$;
(iii) $\Pi (\lambda I - A)^{-1} = (\lambda I - A)^{-1} \Pi, \ \ \forall \lambda > \omega$.

Moreover, for each $x \in X$ we have the following approximation formula

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda (\lambda I - A)^{-1} x = \lim_{h \to 0^+} \frac{1}{h} \Pi_0 S_A(h) x.$$

Proof. Assume first that there exists a bounded linear projector Π on X satisfying (i)-(iii). Let $x \in X$ be fixed. Then from (ii) we have $\Pi x \in X_0$, so

$$\Pi x = \lim_{\lambda \to +\infty} \lambda (\lambda I - A)^{-1} \Pi x.$$

Using (i) and (iii), we deduce that

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda (\lambda I - A)^{-1} x.$$

Thus, there exists at most one bounded linear projector Π satisfying (i)-(iii).

It remains to prove the existence of such an operator Π. To simplify the notation, set $B = A_0 \mid_{\Pi_0(X_0)}$. Then by assumption, B is a bounded linear operator from $\Pi_0 (X_0)$ into itself, and

$$T_{A_0}(t) \Pi_0 x = e^{Bt} \Pi_0 x, \ \ \forall t \geq 0, \forall x \in X_0.$$

Let $x \in X$ be fixed. Since $S_A(t)x \in X_0$ for each $t \geq 0$, we have for each $h > 0$ and each $\lambda > \omega$ that

$$(\lambda I - A_0)^{-1} S_A(h)x = S_A(h)(\lambda I - A)^{-1} x = \int_0^h T_{A_0}(h-s)(\lambda I - A)^{-1} x ds$$

and

$$\begin{aligned}\Pi_0 (\lambda I - A_0)^{-1} S_A(h)x &= (\lambda I - A_0)^{-1} \Pi_0 S_A(h)x \\ &= \int_0^h \Pi_0 T_{A_0}(h-s)(\lambda I - A)^{-1} x ds \\ &= \int_0^h e^{B(h-s)} \Pi_0 (\lambda I - A)^{-1} x ds.\end{aligned}$$

Since B is a bounded linear operator, $t \to e^{Bt}$ is operator norm continuous and

$$\frac{1}{h}\int_0^h e^{B(h-s)} ds = I_{\Pi_0(X_0)} + \frac{1}{h}\int_0^h \left[e^{B(h-s)} - I_{\Pi_0(X_0)}\right] ds.$$

Thus, there exists $h_0 > 0$, such that for each $h \in [0, h_0]$,

$$\left\| \frac{1}{h}\int_0^h \left[e^{B(h-s)} - I_{\Pi_0(X_0)}\right] ds \right\|_{\mathscr{L}(\Pi_0(X_0))} < 1.$$

It follows that for each $h \in [0, h_0]$, the linear operator $\frac{1}{h}\int_0^h e^{B(h-s)} ds$ is invertible from $\Pi_0(X_0)$ into itself and

$$\begin{aligned}\left(\frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^{-1} &= \left(I_{\Pi_0(X_0)} - \left(I_{\Pi_0(X_0)} - \frac{1}{h}\int_0^h e^{B(h-s)} ds\right)\right)^{-1} \\ &= \sum_{k=0}^{\infty} \left(I_{\Pi_0(X_0)} - \frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^k.\end{aligned}$$

We have for each $\lambda > \omega$ and each $h \in (0, h_0]$ that

$$\left(\frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^{-1} (\lambda I - A_0)^{-1} \Pi_0 \frac{1}{h} S_A(h)x = \Pi_0 (\lambda I - A)^{-1} x.$$

Since for each $t \geq 0$, $e^{Bt}\Pi_0 = T_{A_0}(t)\Pi_0$ commutes with $(\lambda I - A_0)^{-1}$, it follows that for each $h \in [0, h_0]$, $\left(\frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^{-1} \Pi_0$ commutes with $(\lambda I - A_0)^{-1}$. Therefore, we obtain for each $\lambda > \omega$ and each $h \in (0, h_0]$ that

$$\lambda (\lambda I - A_0)^{-1} \left(\frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^{-1} \Pi_0 \frac{1}{h} S_A(h)x = \Pi_0 \lambda (\lambda I - A)^{-1} x. \tag{4.5.4}$$

Now it is clear that the left hand side of (4.5.4) converges as $\lambda \to +\infty$. So we can define $\Pi : X \to X$ for each $x \in X$ by

$$\Pi x = \lim_{\lambda \to +\infty} \Pi_0 \lambda \left(\lambda I - A\right)^{-1} x. \tag{4.5.5}$$

Moreover, for each $h \in (0, h_0]$ and each $x \in X$,

$$\Pi x = \left(\frac{1}{h}\int_0^h e^{B(h-s)} ds\right)^{-1} \Pi_0 \frac{1}{h} S_A(h) x. \tag{4.5.6}$$

It follows from (4.5.6) that $\Pi : X \to X$ is a bounded linear operator and $\Pi(X) \subset X_0$. Furthermore, by using (4.5.5), we know that $\Pi \mid_{X_0} = \Pi_0$ and Π commutes with the resolvent of A. Also notice that for each $h \in (0, h_0]$,

$$\frac{1}{h} \Pi_0 S_A(h) x = \frac{1}{h}\int_0^h e^{B(h-s)} \Pi x ds.$$

So

$$\Pi x = \lim_{h \searrow 0} \frac{1}{h} \Pi_0 S_A(h) x.$$

Finally, for each $x \in X$,

$$\begin{aligned} \Pi \Pi x &= \lim_{\lambda \to +\infty} \Pi \Pi_0 \lambda \left(\lambda I - A\right)^{-1} x = \lim_{\lambda \to +\infty} \Pi_0^2 \lambda \left(\lambda I - A\right)^{-1} x \\ &= \lim_{\lambda \to +\infty} \Pi_0 \lambda \left(\lambda I - A\right)^{-1} x = \Pi x. \end{aligned}$$

This implies that Π is a projector. $\square$

Note that if the linear operator Π_0 has a finite rank, then $A_0 \mid_{\Pi_0(X_0)}$ is bounded. So we can apply the above proposition. By Proposition 3.4.3, Lemmas 4.5.2 and 4.5.4, we obtain the following results.

Lemma 4.5.6. *Let Assumption 3.4.1 be satisfied. Let $\Pi : X \to X$ be a bounded linear projector. Assume that*

$$\Pi \left(\lambda I - A\right)^{-1} = \left(\lambda I - A\right)^{-1} \Pi, \ \forall \lambda \in (\omega, +\infty).$$

Then $A \mid_{\Pi(X)} = A_{\Pi(X)}$ satisfies Assumption 3.4.1 on $\Pi(X)$. Moreover,

(i) $\left(A \mid_{\Pi(X)}\right)_{\overline{D\left(A\mid_{\Pi(X)}\right)}} = \left(A_{\overline{D(A)}}\right) \mid_{\Pi\left(\overline{D(A)}\right)} = A_0 \mid_{\Pi(X_0)}$;
(ii) $S_A(t)\Pi = \Pi S_A(t), \forall t \geq 0$;
(iii) $S_{A\mid_{\Pi(X)}}(t) = S_A(t) \mid_{\Pi(X)}, \forall t \geq 0$.

From the above results, we obtain the second main result of this section.

Proposition 4.5.7. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. Let $\Pi : X \to X$ be a bounded linear projector. Assume that*

$$\Pi \left(\lambda I - A\right)^{-1} = \left(\lambda I - A\right)^{-1} \Pi, \ \forall \lambda \in (\omega, +\infty).$$

Then the linear operator $A\mid_{\Pi(X)} = A_{\Pi(X)}$ satisfies Assumptions 3.4.1 and 3.5.2 in $\Pi(X)$. Moreover, for each $\tau > 0$, each $f \in C([0,\tau], X)$, and each $x \in X_0$, if we set for each $t \in [0,\tau]$ that

$$u(t) = T_{A_0}(t)x + \frac{d}{dt}(S_A * f)(t),$$

then

$$\Pi u(t) = T_{A_0|_{\Pi(X_0)}}(t)\Pi x + \frac{d}{dt}\left(S_{A|_{\Pi(X)}} * \Pi f\right)(t),$$

$$\Pi u(t) = \Pi x + A\mid_{\Pi(X)} \int_0^t \Pi u(s)ds + \int_0^t \Pi f(s)ds,$$

and

$$\|\Pi u(t)\| \leq M e^{\omega t} \|\Pi x\| + \delta(t) \sup_{s\in[0,t]} \|\Pi f(s)\|, \ \forall t \in [0,\tau].$$

Furthermore, if Π has a finite rank and $\Pi(X) \subset X_0$, then $\Pi(X) = \Pi(X_0) \subset \Pi(D(A_0)) \subset D(A_0)$, $A\mid_{\Pi(X)}$ is a bounded linear operator from $\Pi(X_0)$ into itself. In particular, $A\mid_{\Pi(X)} = A_0\mid_{\Pi(X_0)}$ and the map $t \to \Pi u(t)$ is a solution of the following ordinary differential equation in $\Pi(X_0)$:

$$\frac{d\Pi u(t)}{dt} = A_0\mid_{\Pi(X_0)} \Pi u(t) + \Pi f(t), \ \forall t \in [0,\tau]; \ \Pi u(0) = \Pi x.$$

By combining Lemma 4.4.2 and Theorem 4.4.3 and by applying Theorem 4.3.27 to $T_A(t)$ for some $t > 0$, we obtain the following theorem, which is one of the main results in this chapter. This theorem can also be obtain by combining Theorem 4.4.3, Webb [362, Proposition 4.11, p. 166], and Engel and Nagel [126, Corollary 2.11, p. 258].

Theorem 4.5.8. *Let $(X, \|.\|)$ be a complex Banach space and let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 3.4.1. Assume that $\omega_0(A_0) > \omega_{0,\text{ess}}(A_0)$. Then for each $\eta > \omega_{0,\text{ess}}(A_0)$ such that*

$$\Sigma_\eta := \sigma(A_0) \cap \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \geq \eta\}$$

is nonempty and finite, each $\lambda_0 \in \Sigma_\eta$ is a pole of $(\lambda I - A_0)^{-1}$ and $B^0_{\lambda_0,-1}$ has a finite rank. Moreover, if we set

$$\Pi = \sum_{\lambda_0 \in \Sigma_\eta} B^0_{\lambda_0,-1},$$

then

$$\Pi_{\lambda_0}(\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1} \Pi_{\lambda_0}, \ \forall \lambda \in \rho(A),$$

$$\omega_0(A_0) = \omega_0\left(A_0\mid_{\Pi_{\lambda_0}(X)}\right) = \sup_{\lambda\in\Sigma_\eta} \text{Re}(\lambda),$$

and

$$\omega_0\left(A_0\mid_{\left(I-\Pi_{\lambda_0}\right)(X)}\right)\leq\eta.$$

Remark 4.5.9. In order to apply the above theorem, we need to check that $\omega_0(A_0)>\omega_{0,\text{ess}}(A_0)$. This property can be verified by using perturbation techniques and by applying the results of Thieme [331] in the Hille-Yosida case, or the results in Ducrot et al. [110] in the present context.

4.6 Asynchronous Exponential Growth of Linear Operators.

Definition 4.6.1. Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. We say that $\{T(t)\}_{t\geq 0}$ has *asynchronous exponential growth* with intrinsic growth constant $\lambda_0\in\mathbb{R}$ if there exists a nonzero finite rank operator $P_0\in X$ such that

$$\lim_{t\to+\infty}e^{-\lambda_0 t}T(t)=P_0.$$

Webb [363] gave necessary and sufficient conditions for $\{T(t)\}_{t\geq 0}$ to have asynchronous exponential growth.

Theorem 4.6.2 (Webb). *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then $\{T(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant $\lambda_0\in\mathbb{R}$ if and only if*

(i) $\omega_{0,\text{ess}}(A)<\lambda_0$;
(ii) $\lambda_0=\sup\{\text{Re}\lambda:\lambda\in\sigma(A)\}$;
(iii) λ_0 *is a simple pole of* $(\lambda I-A)^{-1}$.

Proof. (Necessity) Suppose $\{T(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant λ_0.

(i) We can see that P_0 is a projection and

$$T(t)P_0=P_0T(t)=e^{\lambda_0 t}P_0,\ t\geq 0.$$

Thus,

$$AP_0x=\lim_{t\to 0}\frac{T(t)P_0x-P_0x}{t}=\lambda_0P_0x,\ x\in X,$$

so that $\lambda_0\in\sigma_p(A)$. Since P_0 is a projection, there exists a direct sum decomposition $X=P_0X\oplus(I-P_0)X$. Let $\hat{X}=(I-P_0)X,\ \hat{T}(t)=e^{-\lambda_0 t}T(t)(I-P_0)$, and observe that $\hat{X}$ is invariant under $\hat{T}(t)$. Consider the semigroup $\{\hat{T}(t)\}_{t\geq 0}$ in the Banach space $\hat{X}$ and let $\hat{A}$ be its infinitesimal generator. Notice that

$$\lim_{t\to+\infty}\|\hat{T}(t)\|=\lim_{t\to+\infty}[\|e^{-\lambda_0 t}T(t)-P_0\|+\|(e^{-\lambda_0 t}T(t)-I)P_0\|]=0.$$

It follows from Lemma 4.4.2 (d) that $e^{\omega_0(\hat{A})t} = r(\hat{T}(t)) \leq \|\hat{T}(t)\|$, which means that $\omega_0(\hat{A}) < 0$. Hence, there exists $\gamma < 0$ and $M_\gamma \geq 1$ such that $\|\hat{T}(t)\| \leq M_\gamma e^{\gamma t}, t \geq 0$. Since P_0 has finite rank, $T(t)P_0$ is compact and so

$$\kappa(T(t)) \leq \kappa(T(t)P_0) + \kappa(T(t)(I - P_0)) \leq M_\gamma e^{(\lambda_0+\gamma)t}.$$

Thus, $\omega_{0,\text{ess}}(A) \leq \lambda_0 + \gamma < \lambda_0$.

(ii) Suppose that there exists $\lambda_1 \in \sigma(A)$ such that $\text{Re}\lambda_1 \geq \lambda_0$. By Theorem 4.4.3 (iv), $\lambda_1 \notin \sigma_{\text{ess}}(A)$, it follows from Theorem 4.4.3 (v) that $\lambda_1 \in \sigma_p(A)$. There exists $z \neq 0$ such that $T(t)z = e^{\lambda_1 t}z$. Thus,

$$\begin{aligned}
\text{Re}T(t)z &= e^{\text{Re}\lambda_1 t}[(\cos \text{Im}\lambda_1 t)\text{Re}z - (\sin \text{Im}\lambda_1 t)\text{Im}z],\\
\text{Im}T(t)z &= e^{\text{Re}\lambda_1 t}[(\cos \text{Im}\lambda_1 t)\text{Im}z + (\sin \text{Im}\lambda_1 t)\text{Re}z].
\end{aligned}$$

Since $e^{-\lambda_0 t}\text{Re}T(t)z$ and $e^{-\lambda_0 t}\text{Im}T(t)z$ converge, $\text{Re}\lambda_1 = \lambda_0$ and $\text{Im}\lambda_1 = 0$. Thus, $\lambda_0 = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}$.

(iii) Assume that λ_0 is not a simple pole of $(\lambda I - A)^{-1}$. From (i) we know that $\lambda_0 \in \sigma_p(A)$ and λ_0 is isolated, therefore by Theorem 4.1.3 there exists the Laurent expansion (4.2.2) with B_k given by (4.2.3) and satisfying (4.2.4). Choose x such that $B_{-k}x \neq 0$ and let $y = (A - \lambda_0 I)^{k-2}B_{-1}x$. We have $AB_{-k}x = \lambda_0 B_{-k}x$ and $Ay = B_{-k}x + \lambda_0 y$. Since

$$\begin{aligned}
\frac{d}{dt}\left(e^{\lambda_0 t}(tB_{-k}x + y)\right) &= \lambda_0 e^{\lambda_0 t}(tB_{-k}x + y) + e^{\lambda_0 t}B_{-k}x\\
&= A\left(e^{\lambda_0 t}(tB_{-k}x + y)\right)
\end{aligned}$$

and the solution of the initial value problem

$$\frac{d}{dt}T(t)y = AT(t)y,\ t \geq 0;\ \ T(0)y = y$$

is unique, $T(t)y = e^{\lambda_0 t}(tB_{-k}x + y)$. But $e^{-\lambda_0 t}T(t)y$ does not converge. Therefore, λ_0 is a simple pole.

(Sufficiency) Suppose that $\omega_{0,\text{ess}}(A) < \lambda_0$, $\lambda_0 = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}$, and λ_0 is a simple pole of $(\lambda I - A)^{-1}$. By Theorem 4.4.3 (iv) and Proposition 4.11 in Webb [362], $\lambda_0 \in \sigma_p(A)$ and $\mathscr{N}_{\lambda_0}(A)$ is finite dimensional. Let $\omega_{0,\text{ess}}(A) < \gamma < \lambda_0$ and assume that there exists an infinite sequence $\{\lambda_k\} \subseteq \sigma(A)$ such that $\text{Re}\lambda_k \geq \gamma$. Then, $\lambda_k \in \sigma_p(A)$ and it follows from Theorem 4.4.3 (i) that $e^{\lambda_k t} \in \sigma_p(T(t))$. Fix $t > 0$. If $\{e^{\lambda_k t}\}$ is infinite, then $\sigma(T(t))$ has an accumulation point. Thus,

$$r_{\text{ess}}(T(t)) \geq e^{\gamma t} \geq e^{\omega_{0,\text{ess}}(A)t},$$

which contradicts Lemma 4.4.2 (e). If $\{e^{\lambda_k t}\}$ is finite, then $e^{\lambda_k t} = \mu$ for infinitely many k. By Theorem 4.4.3 (i), $\mathscr{N}_\mu(T(t))$ is infinite dimensional, since it must contain all linearly independent subspaces $\mathscr{N}(\lambda_k I - A)$ whenever $e^{\lambda_k t} = \mu$. Thus,

$$r_{\text{ess}}(T(t)) \geq \text{Re}\mu \geq e^{\gamma t} \geq e^{\omega_{0,\text{ess}}(A)t},$$

which again contradicts Lemma 4.4.2 (e). There must exist $\gamma > \omega_{0,\text{ess}}(A)$ such that

$$\sup_{\lambda \in \sigma(A) \setminus \sigma_{\text{ess}}(A), \lambda \neq \lambda_0} \text{Re}\lambda < \gamma < \lambda_0.$$

Since λ_0 is a simple pole, $\mathcal{N}_{\lambda_0}(A) = \mathcal{N}(\lambda_0 I - A)$ and $T(t)P_0 = e^{\lambda_0 t}P_0$, $t \geq 0$. Thus, $P_0 \neq 0, P_0$ has finite rank, and

$$\lim_{t\to+\infty} \|e^{-\lambda_0 t}T(t) - P_0\| = \lim_{t\to+\infty} \|e^{-\lambda_0 t}T(t)(I-P_0)\| \leq \lim_{t\to+\infty} M_1 e^{(\gamma-\lambda_0)t}\|(I-P_0)\| = 0.$$

This completes the proof. □

Theorem 4.6.2 (i) requires $\omega_{0,\text{ess}}(A) < \lambda_0$. Now we give a means to estimate $\omega_{0,\text{ess}}(A)$.

Proposition 4.6.3. *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X. Suppose that $T(t) = U(t) + V(t)$ for sufficiently large t, where $\|U(t)\} \leq Ce^{\gamma t}$ (C and γ are independent of t) and $V(t)$ is compact. Then*

$$\omega_{0,\text{ess}}(A) \leq \gamma.$$

Proof. By the properties of $\kappa(.)$ (Lemma 4.3.2) we have

$$\kappa(T(t)) \leq \kappa(U(t)) + \kappa(V(t)) = \kappa(U(t)) \leq Ce^{\gamma t}$$

for t sufficiently large. The conclusion follows from Lemma 4.4.2 (b). □

In applications, it is more convenient to consider the semigroups in Banach lattices (Schaefer [310]).

Definition 4.6.4. Let $(X, \|.\|)$ be a Banach space. We say that X_+ is a *positive cone* if it is a closed convex subset of X satisfying the following properties

(i) $\lambda x \in X_+, \forall \lambda \geq 0, x \in X_+$;
(ii) $X_+ \cap (-X_+) = \{0\}$.

We say that $(X, \leq)$ is an *ordered Banach space* if we can find a positive cone X_+ such that

$$x \geq 0 \Leftrightarrow x \in X_+.$$

An ordered Banach space $(X, \leq)$ is called a *Banach lattice* if it satisfies the following additional properties

(i) Any two elements $x, y \in X$ have a supremum $x \vee y = \sup\{x, y\}$ and an infimum $x \wedge y = \inf\{x, y\}$ in X;
(ii) $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in X$, where the *modulus of* x defined by $|x| = x \vee (-x)$.

Recall that a bounded linear operator L is *positive* if and only if

$$LX_+ \subset X_+.$$

The following theorem addresses the asymptotic behavior of strongly continuous semigroups in a Banach lattice which combines the results and ideas from Greiner [152], Greiner and Nagel [153], Greiner et al. [154], and Webb [363]. Detailed proofs can be found in Webb [363] and are omitted here.

Proposition 4.6.5. *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of positive bounded linear operators on a Banach lattice X with infinitesimal generator A. Let $\lambda_0 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ and assume that $\omega_{0,\mathrm{ess}}(A) < \lambda_0$. Then*

(i) $\lambda_0 > \mathrm{Re}\lambda$ *for all* $\lambda \in \sigma(A) \setminus \{\lambda_0\}$;
(ii) *There exists* $x_0 \in X_+, x_0 \neq 0$, *such that* $Ax_0 = \lambda_0 x_0$;
(iii) *If there exists a strictly positive functional* $f \in X^*$ *and* $\lambda_1 \in \mathbb{R}$ *such that for all* $x \in X_+ \cap \mathscr{N}(\lambda_0 I - A)$, $\langle f, e^{-\lambda_1 t} T(t)x\rangle$ *is bounded in* t, *then* $\lambda_0 \leq \lambda_1$;
(iv) *If there exists a strictly positive functional* $f \in X^*$ *and* $\lambda_1 \in \mathbb{R}$ *such that for all* $x \in X_+ \cap \mathscr{N}(\lambda_0 I - A)$, $\lim_{t\to+\infty}\langle f, e^{-\lambda_1 t} T(t)x\rangle$ *exists and is positive, then* $\lambda_1 = \lambda_0$;
(v) *If there exists a strictly positive functional* $f \in X^*$ *such that for all* $x \in \mathscr{N}_{\lambda_0}(A)$, $\langle f, e^{-\lambda_0 t} T(t)x\rangle$ *is bounded in* t, *then* λ_0 *is a simple pole of* $(\lambda I - A)^{-1}$ *(*f *is strictly positive means* $\langle f, x\rangle > 0$ *for all* $x \in X_+, x \neq 0$*).*

By Theorem 4.6.2 and Proposition 4.6.5, we have the following result on asynchronous exponential growth which is more applicable in practice.

Corollary 4.6.6. *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of positive bounded linear operators on a Banach lattice X with infinitesimal generator A. Assume that*

(i) $\omega_{0,\mathrm{ess}}(A) < \lambda_0 := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$;
(ii) *There exists a strictly positive functional* $f \in X^*$ *such that for all* $x \in \mathscr{N}_{\lambda_0}(A)$, $\langle f, e^{-\lambda_0 t} T(t)x\rangle$ *is bounded in* t.

Then $\{T(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant λ_0 in a Banach lattice.

Definition 4.6.7. Denote the dual of the positive cone X_+ by $X_+^* := \{f \in X^* : \langle f, x\rangle \geq 0,\ \forall x \geq 0\}$. A strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ is *irreducible* if for $x \in X_+ \setminus \{0\},\ f \in X_+^* \setminus \{0\}$, there exists $t > 0$ such that $\langle f, T(t)x\rangle > 0$.

By Theorem 4.6.2, Proposition 4.6.5, and Theorem 1.3 of Greiner [152], we have the following result which gives another sufficient condition for asynchronous exponential growth.

Corollary 4.6.8. *Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of positive bounded linear operators on a Banach lattice X with infinitesimal generator A. Assume that*

(i) $\omega_{0,\mathrm{ess}}(A) < \lambda_0 := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$;
(ii) $\{T(t)\}_{t\geq 0}$ *is irreducible.*

Then $\{T(t)\}_{t\geq 0}$ *has asynchronous exponential growth with intrinsic growth constant* λ_0 *in a Banach lattice.*

4.7 Remarks and Notes

The spectral theory of linear operators has been well-developed, we refer to the classical references on this topic: Brezis [48], Dunford and Schwartz [123], Hille and Phillips [187], Kato [205], Schechter [311], Schaefer [310], van Neerven [346], Webb [362], Yagi [376], and Yosida [381]. See also a survey by Arino [24]. The part on the non-essential spectrum of bounded linear operators is based on the paper of Nussbaum [280] where the essential spectral radius was first introduced and the essential spectrum was first investigated. In fact, the work of Nussbaum was based on the early work of Gohberg and Kreĭn [149] concerning Fredholm's index for linear operators. Here we gave a direct proof of Nussbaum's results about the non-essential spectrum of linear operators. The results on the relationship between the spectrum of a semigroup and the spectrum of its infinitesimal generator were given in Webb [362, 363]. The estimates of growth bound and essential growth bound were taken from Webb [362] and Engel and Nagel [126]. The presentation of this chapter was mainly from Magal and Ruan [248].

(a) Essential Growth and Bounded Linear Perturbation. It is important to find a method to evaluate the essential growth bound of linear operators. The first result on this aspect is due to Webb [358, Proposition 3.3]. The following version is a consequence of Theorem 3.2 in Magal and Thieme [251].

Theorem 4.7.1. *Let* $\{T_A(t)\}_{t\geq 0}$ *be a strongly continuous semigroup of bounded linear operators on a Banach space X and with infinitesimal generator* $A : D(A) \subset X \to X$*. Let* $L \in \mathscr{L}(X)$ *be a bounded linear operator. Assume that* $LT_A(t)$ *is compact for each* $t > 0$*. Then*

$$\omega_{0,\mathrm{ess}}(A+L) \leq \omega_{0,\mathrm{ess}}(A).$$

Such a result has been extended first by Thieme [331, Theorem 3] when A is a non-densely defined Hille-Yosida operator and $L : \overline{D(A)} \to X$ is a bounded linear operator. When A is not a Hille-Yosida operator we make the following assumption.

Assumption 4.7.2. Let $A : D(A) \subset X \to X$ be a linear operator satisfying Assumption 3.4.1. Let $p \in [1, +\infty)$ be fixed. Assume that there exist two constants, $\widehat{M} > 0$ and $\widehat{\omega} \in \mathbb{R}$, such that for each $\tau > 0$ and each $f \in L^p((0,\tau), X)$, there exists an integrated solution $u_f \in C([0,\tau], X)$ of the Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t),\ t \in [0,\tau_0];\ \ u(0) = 0$$

satisfying

$$\left\| u_f(t) \right\| \leq \widehat{M} \left\| e^{\widehat{\omega}(t-.)} f(.) \right\|_{L^p((0,t),X)}, \ \forall t \in [0,\tau].$$

The following theorem was proved by Ducrot et al. [110, Theorem 1.2].

Theorem 4.7.3. *Let Assumption 4.7.2 be satisfied. Let $L : \overline{D(A)} \to X$ be a bounded linear operator. Assume that*

$$LT_{A_0}(t) \text{ is compact for each } t > 0.$$

Then we have the following inequality

$$\omega_{0,\text{ess}}\left((A+L)_0\right) \leq \omega_{0,\text{ess}}\left(A_0\right).$$

Theorem 4.7.3 will be frequently used in the rest of the monograph to estimate the essential growth bounds of linear operators.

(b) Asynchronous Exponential Growth. The property of asynchronous (or balanced) exponential growth is one of the most important phenomena in population dynamics since it is observed in many reproducing populations before the effects of crowding and resource limitation take hold. The property means that the population density $u(x,t)$ with respect to a structure variable x is asymptotic to $e^{\lambda_0 t} u_0(x)$ as time t approaches infinity. The constant λ_0 is intrinsic to the population in its environment. The characteristic distribution $u_0(x)$ depends only on the initial state. An important outcome of this property is that the proportion of the population with structure variable x between two given values tends to a constant as time becomes large.

Sharpe and Lotka [315] were the first to study asynchronous exponential growth in age-structured populations. Feller [139] was the first to give a rigorous proof of asynchronous exponential growth in age-structured population dynamics. In the 1980s, it was recognized that the idea of asynchronous exponential growth can be described in the framework of strongly continuous semigroups of bounded linear operators in Banach spaces, see for example, Diekmann et al. [102], Greiner [152], Greiner and Nagel [153], Greiner et al. [154], Webb [363], and the references cited therein. Webb [361] indeed provided a new proof of Sharpe and Lotka Theorem using the theory of semigroups of operators in Banach spaces. Since then, many researchers have studied asynchronous exponential growth in various structured biological models, see for example, Arino et al. [26, 31], Bai and Cui [35], Banasiak et al. [37], Dyson et al. [124], Farkas [132], Piazzera and Tonetto [288], Webb and Grabosch [366], Yan et al. [377], and so on. Thieme [333] characterized strong and uniform approach to asynchronous exponential growth and derived applicable sufficient conditions. Thieme [334] derived conditions for the positively perturbed semigroups to have asynchronous exponential growth and applied the results to age-structured population models.

The presentation in Section 4.6 was mainly taken from Webb [363] which deals with asynchronous exponential growth of semigroups of linear operators. Gyllenberg and Webb [165] considered the following abstract nonlinear differential equation

$$\frac{dz}{dt} = Az(t) + F(z(t)),\ t \geq 0;\ \ z(0) = x \in X, \tag{4.7.1}$$

where A is the infinitesimal generator of a semigroup of linear operators in the Banach space X and F is a nonlinear operator in X. They showed that if the linear semigroup generated by A has asynchronous exponential growth and F satisfies $\|F\| \leq f(\|x\|)\|x\|$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\int^\infty f(r)/r dr < \infty$, then the nonlinear semigroup associated with the abstract Cauchy problem (4.7.1) also has asynchronous exponential growth.

Chapter 5
Semilinear Cauchy Problems with Non-dense Domain

The main purpose of this chapter is to present a comprehensive semilinear theory that will allow us to study the properties of solutions of the non-densely defined Cauchy problems, such as existence and uniqueness of a maximal semiflow, positivity, Lipschitz perturbation, differentiability with respect to the state variable, time differentiability, classical solutions, stability of equilibria, etc.

5.1 Introduction

Consider the Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(t, u(t)), \ \ t \geq 0; \ \ u(0) = x \in \overline{D(A)}, \tag{5.1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator in a Banach space X and $F : [0, +\infty) \times \overline{D(A)} \to X$ is a continuous map.

When A is a Hille-Yosida operator and is densely defined, the problem has been extensively studied (see Segal [313], Weissler [372], Martin [258], Pazy [281], Cazenave and Haraux [58], Hirsch and Smith [189]). When A is a Hille-Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [85] investigated the existence of several types of solutions for (5.1.1). Thieme [328] investigated the semilinear Cauchy problem with a Lipschitz perturbation of the closed linear operator A which is non-densely defined but is a Hille-Yosida operator. See also Thieme [329, 335]. We are interested in studying the problem when $\overline{D(A)}$ is not dense in X and A is not a Hille-Yosida operator.

Since the domain is not dense the integrated solution of (5.1.1) will belong to the smaller subspace

$$X_0 := \overline{D(A)}.$$

Since the linear operator A is not a Hille-Yosida operator, we first assume that the resolvent set $\rho(A)$ of A is non-empty and that A_0, the part of A in $\overline{D(A)}$, is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$. This is equivalent to make the following assumption.

Assumption 5.1.1. Assume that $A : D(A) \subset X \to X$ is a linear operator on a Banach space $(X, \|.\|)$ satisfying the following properties:

(a) There exist two constants, $\omega_A \in \mathbb{R}$ and $M_A \geq 1$, such that $(\omega_A, +\infty) \subset \rho(A)$ and

$$\left\|(\lambda I - A)^{-k}\right\|_{\mathscr{L}(X_0)} \leq \frac{M_A}{(\lambda - \omega_A)^k}, \ \forall \lambda > \omega_A, \ \forall k \geq 1;$$

(b) $\lim_{\lambda \to +\infty} (\lambda I - A)^{-1} x = 0, \ \forall x \in X$.

Then A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X defined for all $t \geq 0$ by

$$S_A(t) = (\lambda I - A_0) \int_0^t T_{A_0}(l) dl \, (\lambda I - A)^{-1}$$

for each $\lambda \in \rho(A)$.

As we already explained in Chapter 3, we need to impose some extra conditions to assure the existence of integrated (or mild) solutions of the nonhomogeneous Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = 0. \tag{5.1.2}$$

We will only require that for each $f \in C([0, \tau], X)$ the Cauchy problem (5.1.2) has an integrated solution $u_f(t)$, and there exists a map $\delta : [0, +\infty) \to [0, +\infty)$ (independent of f) such that for each $t \in [0, \tau]$,

$$\left\|u_f(t)\right\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \tag{5.1.3}$$

where

$$\delta(t) \to 0 \text{ as } t \to 0.$$

This is also equivalent to the following assumption.

Assumption 5.1.2. Let $\tau_0 > 0$ be fixed. Assume that $\{S_A(t)\}_{t\geq 0}$ has a bounded semi-variation on $[0, \tau_0]$ (that is,

$$V^\infty(S_A, 0, \tau_0) := \sup\left\{\left\|\sum_{i=1}^n \left(S_A(t_i) - S_A(t_{i-1})\right) x_i\right\|\right\} < +\infty,$$

where the supremum is taken over all partitions $0 = t_0 < .. < t_n = \tau_0$ and over any $(x_1, ..., x_n) \in X^n$ with $\|x_i\|_X \leq 1, \ \forall i = 1, .., n$) and for any $t \in [0, \tau_0]$,

$$\lim_{t(>0) \to 0} V^\infty(S_A, 0, t) = 0.$$

Remark 5.1.3. We always have by the definition of semi-variation that

$$V^{\infty}(S_A, 0, t) \leq \delta(t), \forall t \in [0, \tau]. \tag{5.1.4}$$

Then for each $f \in C([0, \tau], X)$ the map $t \to (S_A * f)(t)$ is continuously differentiable and (5.1.2) has a unique integrated solution $u_f(t)$ which is given by

$$u_f(t) = \frac{d}{dt}(S_A * f)(t), \ \forall t \in [0, \tau].$$

Remark 5.1.4. When the domain of A is dense in X or when $f \in C\left([0, \tau], \overline{D(A)}\right)$, we have

$$\frac{d}{dt}(S_A * f)(t) = \int_0^t T_{A_0}(t-s) f(s) ds.$$

The first main ingredient to derive a semilinear theory is the approximation formula proved in Proposition 3.4.8; that is,

$$\frac{d}{dt}(S_A * f)(t) = \lim_{\mu \to +\infty} \int_0^t T_{A_0}(t-l) \mu (\mu I - A)^{-1} f(l) dl \tag{5.1.5}$$

whenever $t \in [0, \tau]$ and $f \in C([0, \widehat{\tau}], X)$. From this approximation formula, we deduced the following formula in Corollary 3.4.9

$$\frac{d}{dt}(S_A * f)(t) = T_{A_0}(t-s) \frac{d}{dt}(S_A * f)(s) + \frac{d}{dt}(S_A * f(.+s))(t-s) \tag{5.1.6}$$

whenever $t, s \in [0, \tau]$ with $s \leq t$ and $f \in C([0, \tau], X)$. The second ingredient is the estimation (5.1.3) which is equivalent to

$$\left\| \frac{d}{dt}(S_A * f)(t) \right\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|$$

whenever $t \in [0, \tau]$.

As a consequence we are in a position to use Proposition 3.5.3. Let $s \geq 0$ be fixed. The nonautonomous Cauchy problem (5.1.1) will generate a nonautonomous semiflow which will be an *integrated solution* of (5.1.1); that is, the map $t \to U(t, s)x$ satisfies

$$U(t, s)x = x + A \int_s^t U(l, s) x dl + \int_s^t F(l, U(l, s)x) dl, \ \forall t \geq s \tag{5.1.7}$$

or equivalently $U(t, s)$ satisfies the following variation of constants formula

$$U(t, s)x = T_{A_0}(t-s)x + \frac{d}{dt}(S_A * F(.+s, U(.+s, s)x)(t-s), \ \forall t \geq s. \tag{5.1.8}$$

As mentioned above it is important to observe that the problem is similar to the densely defined semilinear Cauchy problem. Indeed when the domain of A is dense in X or $F(X) \subset \overline{D(A)}$, the integrated solution satisfies the following variation of constants formula

$$U(t,s)x = T_{A_0}(t-s)x + \int_0^{t-s} T_{A_0}(t-s-l)F(l+s,U(l+s,s)x)dl$$

or equivalently after a change of variable

$$U(t,s)x = T_{A_0}(t-s)x + \int_s^t T_{A_0}(t-r)F(r,U(r,s)x)dr$$

whenever $t \geq s$.

The main difficulty in this chapter is to extend the known results on semilinear Cauchy problem with dense domain for which one has L^1 estimation, that is,

$$\| \int_0^t T_{A_0}(t-s)f(s)ds\| \leq M_A \int_0^t e^{\omega_A(t-s)} \|f(s)\| ds,$$

to the L^∞ estimation in (5.1.5).

5.2 Existence and Uniqueness of a Maximal Semiflow: the Blowup Condition

We start by making the following assumption.

Assumption 5.2.1. Assume that $F : [0,+\infty) \times \overline{D(A)} \to X$ is a continuous map such that for each $\tau_0 > 0$ and each $\xi > 0$, there exists $K(\tau_0, \xi) > 0$ such that

$$\|F(t,x) - F(t,y)\| \leq K(\tau_0, \xi) \|x-y\|$$

whenever $t \in [0, \tau_0]$, $y, x \in X_0$, and $\|x\| \leq \xi$, $\|y\| \leq \xi$.

First note that without loss of generality we can assume that $\delta(t)$ is non-decreasing. Moreover, by using the Bounded Perturbation Theorem 3.5.1, for each $\alpha \in R$ replacing τ_0 by some $\tau_\alpha \in (0, \tau_0)$ such that $\delta(\tau_\alpha)|\alpha| < 1$, we know that $A + \alpha I$ satisfies Assumptions 5.1.1 and 5.1.2. Replacing A by $A - \omega I$ and $F(t,.)$ by $F(t,.) + \omega I$, we can assume that $\omega = 0$. From now on we assume that $\delta(t)$ is non-decreasing and $\omega = 0$.

In the following, we will use the norm $|.|$ on X_0 defined by

$$|x| = \sup_{t \geq 0} \left\| T_{A_0}(t)x \right\|, \ \forall x \in X_0.$$

Then we have

$$\|x\| \leq |x| \leq M \|x\| \text{ and } \left| T_{A_0}(t)x \right| \leq |x|, \ \forall x \in X_0, \ \forall t \geq 0. \tag{5.2.1}$$

Notice that by the assumption, for each $f \in C([0,\tau_0],X)$, $\frac{d}{dt}(S_A * f)(t)$ is well-defined $\forall t \in [0,\tau_0]$. Let $f \in C^1([0,2\tau_0],X)$. Then, for $t \in [\tau_0, 2\tau_0]$,

$$\begin{aligned}\frac{d}{dt}(S_A * f)(t) &= \lim_{\mu\to+\infty}\int_0^t T_{A_0}(t-s)\mu(\mu I - A)^{-1} f(s)ds \\ &= \frac{d}{dt}(S_A * f(.+\tau_0))(t-\tau_0) + T_{A_0}(t-\tau_0)\frac{d}{dt}(S_A * f(.))(\tau_0),\end{aligned}$$

so

$$\left\|\frac{d}{dt}(S_A * f)(t)\right\| \leq \delta(t-\tau_0)\sup_{l\in[\tau_0,t-\tau_0]}\|f(l)\| + \delta(t-\tau_0)\sup_{l\in[0,\tau_0]}\|f(l)\|.$$

Thus, Assumption 5.1.2 is satisfied with $Z = C([0,2\tau_0],X)$, we deduce that $\frac{d}{dt}(S_A * f)(t)$ is well-defined for all $t \in [0,2\tau_0]$ and satisfies the conclusions of Theorem 3.4.7. By induction, we obtain that for each $\tau_0 > 0$ and each $f \in C([0,\tau_0],X)$, $t \to (S_A * f)(t)$ is continuously differentiable on $[0,\tau_0]$, $(S_A * f)(t) \in D(A), \forall t \in [0,\tau_0]$, and if we denote $u(t) = \frac{d}{dt}(S_A * f)(t)$, then

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s)ds,\ \forall t \in [0,\tau_0].$$

In the following definition τ is the blow-up time of the maximal solution of (5.1.1).

Definition 5.2.2. Consider two maps $\tau : [0,+\infty)\times X_0 \to (0,+\infty]$ and $U : D_\tau \to X_0$, where

$$D_\tau = \left\{(t,s,x) \in [0,+\infty)^2 \times X_0 : s \leq t < s + \tau(s,x)\right\}.$$

We say that U is *a maximal nonautonomous semiflow on* X_0 if U satisfies the following properties:

(i) $\tau(r,U(r,s)x) + r = \tau(s,x) + s, \forall s \geq 0, \forall x \in X_0, \forall r \in [s, s+\tau(s,x))$;
(ii) $U(s,s)x = x, \forall s \geq 0, \forall x \in X_0$;
(iii) $U(t,r)U(r,s)x = U(t,s)x, \forall s \geq 0, \forall x \in X_0, \forall t,r \in [s,s+\tau(s,x))$ with $t \geq r$;
(iv) If $\tau(s,x) < +\infty$, then

$$\lim_{t\to(s+\tau(s,x))^-} |U(t,s)x| = +\infty.$$

Set

$$D = \left\{(t,s,x) \in [0,+\infty)^2 \times X_0 : t \geq s\right\}.$$

In order to present the main result of this section, we introduce some lemmas.

Lemma 5.2.3 (Uniqueness). *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then for each $x \in X_0$, each $s \geq 0$, and each $\tau > 0$, equation (5.1.1) has at most one integrated solution $U(.,s)x \in C([s,\tau+s],X_0)$.*

Proof. Assume that there exist two solutions of equation (5.1.1), $u,v \in C([s,\tau+s],X_0)$ with $u(s) = v(s)$. Define

$$t_0 = \sup\{t \geq s : u(l) = v(l), \forall l \in [s,t]\}.$$

Then, for each $t \geq t_0$, we have

$$u(t) - v(t) = A\int_{t_0}^{t}[u(l) - v(l)]dl + \int_{t_0}^{t}[F(l,u(l)) - F(l,v(l))]dl.$$

It follows that

$$\begin{aligned}(u-v)(t-t_0+t_0) = A\int_0^{t-t_0}(u-v)(l+t_0)dl \\ + \int_0^{t-t_0}[F(l+t_0,u(l+t_0)) - F(l+t_0,v(l+t_0))]dl.\end{aligned}$$

Thus,

$$u(t) - v(t) = \frac{d}{dt}(S_A * (F(.+t_0,u(.+t_0)) - F(.+t_0,v(.+t_0))))(t-t_0).$$

Let $\xi = \max\left(\|u\|_{\infty,[s,\tau+s]}, \|v\|_{\infty,[s,\tau+s]}\right)$. Thus, we have for each $t \in [t_0, t_0+\tau_0]$ that

$$\|u(t) - v(t)\| \leq \delta(t)K(\tau+s,\xi) \sup_{l\in[t_0,t_0+t]} \|u(l) - v(l)\|.$$

Let $\varepsilon > 0$ be fixed such that $\delta(\varepsilon)K(\tau+s,\xi) < 1$. We obtain that

$$\sup_{l\in[t_0,t_0+\varepsilon]} \|u(l) - v(l)\| \leq \delta(\varepsilon)K(\tau+s,\xi) \sup_{l\in[t_0,t_0+\varepsilon]} \|u(l) - v(l)\|.$$

So

$$u(t) = v(t),\ \forall t \in [t_0, t_0+\varepsilon],$$

which gives a contradiction with the definition of t_0. $\square$

Lemma 5.2.4 (Local Existence). *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then for each $\tau > 0$, each $\beta > 0$, and each $\xi > 0$, there exists $\gamma(\tau,\beta,\xi) \in (0,\tau_0]$ such that for each $s \in [0,\tau]$ and each $x \in X_0$ with $|x| \leq \xi$, equation (5.1.1) has a unique integrated solution $U(.,s)x \in C([s,s+\delta(\gamma(\tau,\beta,\xi))],X_0)$ which satisfies*

$$|U(t,s)x| \leq (1+\beta)\xi,\ \forall t \in [s,s+\delta(\gamma(\tau,\beta,\xi))].$$

Proof. Let $s \in [0,\tau]$ and $x \in X_0$ with $\|x\| \leq \xi$ be fixed. Let $\gamma(\tau,\beta,\xi) \in (0,\tau_0]$ such that

$$\delta(\gamma(\tau,\beta,\xi))M\left[\widehat{\xi}_{\tau+\tau_0} + (1+\beta)\xi K(\tau+\tau_0,(1+\beta)\xi)\right] \leq \beta\xi$$

with $\widehat{\xi}_\alpha = \sup_{s\in[0,\alpha]}\|F(s,0)\|, \forall \alpha \geq 0$. Set

$$E = \{u \in C([s,s+\delta(\gamma(\tau,\beta,\xi))],X_0) : |u(t)| \leq (1+\beta)\xi, \forall t \in [s,s+\delta(\gamma(\tau,\beta,\xi))]\}.$$

Consider the map $\Phi_{x,s} : C([s,s+\delta(\gamma(\tau,\beta,\xi))],X_0) \to C([s,s+\delta(\gamma(\tau,\beta,\xi))],X_0)$ defined for each $t \in [s,s+\delta(\gamma(\tau,p,C))]$ by

$$\Phi_{x,s}(u)(t) = T_{A_0}(t-s)x + \frac{d}{dt}(S_A * F(.+s,u(.+s)))(t-s).$$

We have $\forall u \in E$ that (using (5.2.1) repeatedly)

$$\begin{aligned}
|\Phi_{x,s}(u)(t)| &\leq \xi + M\left\| \frac{d}{dt}(S_A * F(.+s,u(.+s)))(t-s) \right\| \\
&\leq \xi + M\delta(\gamma(\tau,\beta,\xi)) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} \|F(t,u(t))\| \\
&\leq \xi + M\delta(\gamma(\tau,\beta,\xi)) \left[\widehat{\xi}_\alpha + K(\tau+\tau_0,(1+\beta)\xi) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)| \right] \\
&\leq (1+\beta)\xi.
\end{aligned}$$

Hence, $\Phi_{x,s}(E) \subset E$. Moreover, for all $u,v \in E$, we have (again using (5.2.1))

$$\begin{aligned}
&|\Phi_{x,s}(u)(t) - \Phi_{x,s}(v)(t)| \\
&\quad \leq M\delta(\gamma(\tau,\beta,\xi))K(\tau+\tau_0,(1+\beta)\xi) \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)| \\
&\quad \leq \frac{K(\tau+\tau_0,(1+\beta)\xi)\beta\xi}{1+\widehat{\xi}_\alpha + K(\tau+\tau_0,(1+\beta)\xi)(1+\beta)\xi} \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)| \\
&\quad \leq \frac{\beta}{1+\beta} \sup_{t\in[s,s+\delta(\gamma(\tau,\beta,\xi))]} |u(t)-v(t)|.
\end{aligned}$$

Therefore, $\Phi_{x,s}$ is a $\left(\frac{\beta}{1+\beta}\right)$-contraction on E and the result follows. $\square$

For each $s \geq 0$ and each $x \in X_0$, define

$$\tau(s,x) = \sup\{t \geq 0 : \exists U(.,s)x \in C([s,s+t],X_0) \text{ an integrated solution of (5.1.1)}\}.$$

By Lemma 5.2.4 we already knew that

$$\tau(s,x) > 0,\ \forall s \geq 0,\ \forall x \in X_0.$$

Moreover, we have the following lemma.

Lemma 5.2.5. *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then $U : D_\tau \to X_0$ is a maximal nonautonomous semiflow on X_0.*

Proof. Let $s \geq 0$ and $x \in X_0$ be fixed. We first prove assertions (i)-(iii) of Definition 5.2.2. Let $r \in [s,s+\tau(s,x))$ be fixed. Then, for all $t \in [r,s+\tau(s,x))$,

$$U(t,s)x = x + A\int_s^t U(l,s)xdl + \int_s^t F(l,U(l,s)x)dl$$

$$= U(r,s)x + A\int_s^t U(l,s)xdl + \int_s^t F(l,U(l,s)x)dl.$$

By Lemma 5.2.3, we obtain that

$$U(t,s)x = U(t,r)U(r,s)x,\ \forall t \in [r, s+\tau(s,x)).$$

So $\tau(r,U(r,s)x) + r \geq \tau(s,x) + s$. Moreover, if we set

$$v(t) = \begin{cases} U(t,r)U(r,s)x & \forall t \in [r, r+\tau(r,U(r,s)x)), \\ U(t,s)x & \forall t \in [s,r], \end{cases}$$

then

$$v(t) = x + A\int_s^t v(l)dl + \int_s^t F(l,v(l))dl,\ \forall t \in [s, r+\tau(r,U(r,s)x)].$$

Thus, by the definition of $\tau(s,x)$ we have $s+\tau(s,x) \geq r+\tau(r,U(r,s)x)$ and the result follows.

It remains to prove assertion (iv) of Definition 5.2.2. Assume that $\tau(s,x) < +\infty$ and that $\|U(t,s)x\| \nrightarrow +\infty$ as $t \nearrow s+\tau(s,x)$. Then we can find a constant $\xi > 0$ and a sequence $\{t_n\}_{n\geq 0} \subset [s, s+\tau(s,x))$, such that $t_n \to s+\tau(s,x)$ as $n \to +\infty$ and

$$|U(t_n,s)x| \leq \xi,\ \forall n \geq 0.$$

Using Lemma 5.2.4 with $\tau = s+\tau(s,x)$ and $\beta = 2$, we know that there exists $\gamma(\tau,\beta,\xi) \in (0,\tau_0]$ for each $n \geq 0$, $t_n + \tau(t_n,x) \geq t_n + \gamma(\tau,\beta,\xi)$. From the first part of the proof we have

$$s+\tau(s,x) \geq t_n + \gamma(\tau,\beta,\xi)$$

and, when $n \to +\infty$, we obtain

$$s+\tau(s,x) \geq s+\tau(s,x)+\gamma(\tau,\beta,\xi),$$

which is impossible since $\gamma(\tau,\beta,\xi) > 0$. □

Lemma 5.2.6. *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then the following properties are satisfied*

(i) The map $(s,x) \to \tau(s,x)$ is lower semi-continuous on $[0,+\infty) \times X_0$;
(ii) For each $(s,x) \in [0,+\infty) \times X_0$, each $\tau \in (0,\tau(s,x))$, and each sequence $\{(s_n,x_n)\}_{n\geq 0} \subset [0,+\infty) \times X_0$ such that $(s_n,x_n) \to (s,x)$ as $n \to +\infty$, one has

$$\sup_{l\in[0,\tau]} |U(l+s_n,s_n)x_n - U(l+s,s)x| \to 0 \text{ as } n \to +\infty;$$

(iii) D_τ is open in D;
(iv) The map $(t,s,x) \to U(t,s)x$ is continuous from D_τ into X_0.

Proof. Let $(s,x) \in [0,+\infty) \times X_0$ be fixed. Consider a sequence $\{(s_n,x_n)\}_{n\geq 0} \subset [0,+\infty) \times X_0$ satisfying $(s_n,x_n) \to (s,x)$ as $n \to +\infty$. Let $\widehat{\tau} \in (0,\tau(s,x))$ be fixed. Define

$$\xi = 2 \sup_{t\in[s,s+\widehat{\tau}]} |U(t,s)x| + 1 > 0$$

and

$$\widehat{\tau}_n = \sup\{t \in (0,\tau(s_n,x_n)) : |U(l+s_n,s_n)x_n| \leq 2\xi, \forall l \in [0,t]\}.$$

Let $\varepsilon \in (0,\tau_0]$ be given such that

$$\xi_1 := \delta(\varepsilon) MK(\widehat{\tau}+\widehat{s},2\xi) < 1,\ \widehat{s} = \sup_{n\geq 0} s_n.$$

Set

$$\xi_2^n = \delta(\varepsilon) M \sup_{l\in[0,\widehat{\tau}]} \|F(l+s_n,U(l+s,s)x) - F(l+s,U(l+s,s)x)\| \to 0 \text{ as } n \to +\infty.$$

Then, we have for each $l \in [0,\min(\widehat{\tau}_n,\widehat{\tau})]$ and each $r \in [0,l]$ with $l-r \leq \varepsilon$ that

$$\begin{aligned} U(l+s,s)x &= U(l+s,r+s)U(r+s,s)x \\ &= T_{A_0}(l-r)U(r+s,s)x + \frac{d}{dt}(S_A * F(.+r+s,U(.+r+s,s)x)(l-r). \end{aligned}$$

Hence,

$$\begin{aligned} &|U(l+s_n,s_n)x_n - U(l+s,s)x| \\ &\quad = |U(l+s_n,r+s_n)U(r+s_n,s_n)x_n - U(l+s,r+s)U(r+s,s)x| \\ &\quad \leq \left|T_{A_0}(l-r)\left[U(r+s_n,s_n)x_n - U(r+s,s)x\right]\right| \\ &\qquad + M\delta(\varepsilon) \sup_{h\in[r,l]} \|F(h+s_n,U(h+s_n,s_n)x_n) - F(h+s,U(h+s,s)x)\| \\ &\quad \leq |U(r+s_n,s_n)x_n - U(r+s,s)x| \\ &\qquad + \xi_1 \sup_{h\in[r,l]} |U(h+s_n,s_n)x_n - U(h+s,s)x| + \xi_2^n. \end{aligned}$$

Therefore, for each $l \in [0,\min(\widehat{\tau}_n,\widehat{\tau})]$ and each $r \in [0,l]$ with $l-r \leq \varepsilon$,

$$\sup_{h\in[r,l]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \leq \frac{1}{1-\xi_1}\left[|U(r+s_n,s_n)x_n - U(r+s,s)x| + \xi_2^n\right].$$

From this we deduce for $r = 0$ that

$$\sup_{h\in[0,\min(\varepsilon,\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \to 0 \text{ as } n \to +\infty.$$

Thus, we have proved (ii) on a subinterval $[0,\min(\varepsilon,\widehat{\tau}_n,\widehat{\tau})]$. By induction on the number of subintervals, we have that

$$\sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| \to 0 \text{ as } n\to+\infty. \tag{5.2.2}$$

It follows that

$$\sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n| \leq \sup_{h\in[0,\min(\widehat{\tau}_n,\widehat{\tau})]} |U(h+s_n,s_n)x_n - U(h+s,s)x| + \xi.$$

Since $\xi>0$, there exists $n_0\geq 0$ such that $\widehat{\tau}_n > \widehat{\tau}, \forall n\geq n_0$, and the result follows by using (5.2.2).

Now (iii) follows from (i). Moreover, if $(t_n,s_n,x_n)\to(t,s,x)$, then we have

$$\begin{aligned}|U(t_n,s_n)x_n - U(t,s)x| \leq{}& |U((t_n-s_n)+s_n,s_n)x_n - U((t_n-s_n)+s,s)x| \\ &+|U((t_n-s_n)+s,s)x - U((t-s)+s,s)x|\end{aligned}$$

and by using (ii),

$$|U(t_n,s_n)x_n - U(t,s)x| \to 0 \text{ as } n\to+\infty.$$

This proves (iv). □

Summarizing the above lemmas, we now state the main result of this section which is a generalization of Theorem 4.3.4 in Cazenave and Haraux [58].

Theorem 5.2.7. *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then there exist a map* $\tau:[0,+\infty)\times X_0\to(0,+\infty]$ *and a maximal nonautonomous semiflow* $U: D_\tau\to X_0$, *such that for each* $x\in X_0$ *and each* $s\geq 0$, $U(.,s)x\in C([s,s+\tau(s,x)),X_0)$ *is a unique maximal integrated solution of (5.1.1) (i.e. satisfies (5.1.7) or (5.1.8)). Moreover,* D_τ *is open in* D *and the map* $(t,s,x)\to U(t,s)x$ *is continuous from* D_τ *into* X_0.

5.3 Positivity

We are now interested in the positivity of the solutions of equation (5.1.1). Let $X_+\subset X$ be a positive cone of X. It is clear that

$$X_{0+} := X_0\cap X_+$$

is also a positive cone of X_0.

We need the following assumption to prove the positivity of solutions of equation (5.1.1).

Assumption 5.3.1. Assume that there exists a linear operator $B\in\mathscr{L}(X_0,X)$ such that

(a) For each $\gamma\geq 0$, the linear operator $A-\gamma B$ is resolvent positive; that is,

$$(\lambda I-(A-\gamma B))^{-1}X_+ \subset X_+$$

for all $\lambda > \omega_A$ large enough;

(b) For each $\xi > 0$ and each $\sigma > 0$, there exists $\gamma = \gamma(\xi, \sigma) > 0$, such that

$$F(t,x)+\gamma Bx \in X_+$$

whenever $x \in X_{0+}$, $\|x\| \leq \xi$ and $t \in [0,\sigma]$.

Proposition 5.3.2. *Let Assumptions 5.1.1, 5.1.2, 5.2.1 and 5.3.1 be satisfied. Then for each $x \in X_{0+}$ and each $s \geq 0$, we have*

$$U(t,s)x \in X_{0+}, \ \ \forall t \in [s, s+\chi(s,x)).$$

Proof. Without loss of generality we can assume that $s=0$ and $x \in X_{0+}$. Moreover, using the semiflow property, it is sufficient to prove that there exists $\varepsilon \in (0, \chi(0,x))$ such that $U(t,0)x \in X_{0+}, \forall t \in [0,\varepsilon]$. Let $x \in X_{0+}$ be fixed. Set $\xi := 2(\|x\|+1)$. Let $\gamma > 0$ be given such that

$$F(t,x)+\gamma Bx \in X_+$$

when $x \in X_{0+}$, $\|x\| \leq \xi$ and $t \in [0,1]$. Fix $\tau_\gamma > 0$ such that $\gamma \|B\|_{\mathscr{L}(X_0,X)} V^\infty(S_A, 0, \tau_\gamma) < 1$. For each $\sigma \in (0, \tau_\gamma)$, define

$$E^\sigma = \{\varphi \in C([0,\sigma], X_{0+}) : \|\varphi(t)\| \leq \xi, \ \ \forall t \in [0,\sigma]\}.$$

Then it is sufficient to consider the fixed point problem

$$u(t) = T_{(A-\gamma B)_0}(t)x + (S_{A-\gamma B} \diamond [F(.,u(.))+\gamma Bu(.)])(t) =: \Psi(u)(t), \forall t \in [0,\sigma].$$

Since $A-\gamma B$ is resolvent positive, we have $T_{(A-\gamma B)_0}(t)X_{0+} \subset X_{0+}, \forall t \geq 0$. Using the approximation formula (5.1.5), we have for each $\tau > 0$ that

$$\left(S_{A-\gamma B} \diamond \varphi\right)(t) \in X_{0+}, \ \ \forall t \in [0,\tau], \ \ \forall \varphi \in C([0,\tau], X_+).$$

Moreover, by using Theorem 3.5.1, for each $\varphi \in E^\sigma$ and each $t \in [0,\sigma]$, we deduce that

$$\begin{aligned}
\|\Psi(\varphi)(t)\| &= \left\|T_{(A-\gamma B)_0}(t)x + \left(S_{A-\gamma B} \diamond [F(.,\varphi(.))+\gamma B\varphi(.)]\right)(t)\right\| \\
&\leq \left\|T_{(A-\gamma B)_0}(t)x\right\| \\
&\quad + \frac{V^\infty(S_A,0,t)}{1-\gamma\|B\|_{\mathscr{L}(X_0,X)} V^\infty(S_A,0,\tau_\gamma)} \sup_{s\in[0,t]} \|F(s,\varphi(s))+\gamma B\varphi(s)\| \\
&\leq \sup_{t\in[0,\sigma]} \left\|T_{(A-\gamma B)_0}(t)x\right\| \\
&\quad + \frac{V^\infty(S_A,0,\sigma)}{1-\gamma\|B\|_{\mathscr{L}(X_0,X)} V^\infty(S_A,0,\tau_\gamma)} \left[\sup_{s\in[0,\sigma]} \|F(s,0)\| + \left[K(1,\xi)+\gamma\|B\|_{\mathscr{L}(X_0,X)}\right]\xi\right].
\end{aligned}$$

Hence, there exists $\sigma_1 \in (0,1)$ such that

$$\Psi(E^{\sigma}) \subset E^{\sigma}, \ \forall \sigma \in (0,\sigma_1].$$

Therefore, for each $\sigma \in (0,\sigma_1]$ and each pair $\varphi,\psi \in E^{\sigma}$, we have for $t \in [0,\sigma]$ that

$$\begin{aligned}
&\|\Psi(\varphi)(t) - \Psi(\psi)(t)\| \\
&\quad = \left\| \left(S_{A-\gamma B} \diamond [F(.,\varphi(.)) - F(.,\psi(.)) + \gamma B(\varphi-\psi)(.)] \right)(t) \right\| \\
&\quad \leq \frac{V^{\infty}(S_A,0,\sigma)}{1-\gamma\|B\|_{\mathscr{L}(X_0,X)} V^{\infty}(S_A,0,\tau_\gamma)} \left[K(1,\xi) + \gamma \|B\|_{\mathscr{L}(X_0,X)} \right] \sup_{s\in[0,\sigma]} \|(\varphi-\psi)(s)\|.
\end{aligned}$$

Thus, there exists $\sigma_2 \in (0,\sigma_1]$ such that $\Psi(E^{\sigma_2}) \subset E^{\sigma_2}$ and Ψ is a contraction strict on E^{σ_2}. The result then follows. $\square$

Example 5.3.3. Usually Proposition 5.3.2 is applied with $B = I$. But the case $B \neq I$ can also be useful. Consider the following functional differential equation:

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x_t), \ \forall t \geq 0, \\ x_0 = \varphi \in C([-\tau,0],\mathbb{R}^n), \end{cases} \tag{5.3.1}$$

where $f : C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$ is Lipschitz continuous on bounded sets of $C([-\tau,0],\mathbb{R}^n)$. In order to obtain the positivity of solutions, it is sufficient to assume that for each $M \geq 0$ there exists $\gamma = \gamma(M) > 0$ such that

$$f(\varphi) + \gamma\varphi(0) \geq 0$$

whenever $\|\varphi\|_\infty \leq M$ and $\varphi \in C\left([-\tau,0],\mathbb{R}^n_+\right)$. It is well known that this condition is sufficient to ensure the positivity of solutions (see Martin and Smith [259, 260]). In order to prove this, one may also apply Proposition 5.3.2. By identifying x_t with $v(t) = \begin{pmatrix} 0 \\ x_t \end{pmatrix}$, system (5.3.1) can be rewritten as a non-densely defined Cauchy problem (see Liu et al. [233] for more details)

$$\frac{dv(t)}{dt} = Av(t) + F(v(t)), \ \forall t \geq 0, \text{ and } v(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

with $X = \mathbb{R}^n \times C([-\tau,0],\mathbb{R}^n), D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-\tau,0],\mathbb{R}^n)$, where $A : D(A) \subset X \to X$ is defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) \\ \varphi' \end{pmatrix}$$

and $F : \overline{D(A)} \to X$ by

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} f(\varphi) \\ 0_C \end{pmatrix}.$$

Then Proposition 5.3.2 applies with

$$B\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}\varphi(0)\\ 0_C\end{pmatrix}.$$

Recall that a cone X_+ of a Banach space $(X,\|.\|)$ is *normal* if there exists a norm $\|.\|_1$ equivalent to $\|.\|$, which is monotone; that is,

$$\forall x,y\in X_+,\ 0\le x\le y\Rightarrow \|x\|_1\le \|y\|_1 .$$

Corollary 5.3.4. *Let Assumptions 5.1.1, 5.1.2, 5.2.1 and 5.3.1 be satisfied. Assume in addition that*

(a) X_+ is a normal cone of $(X,\|.\|)$;
(b) There exist a continuous map $G:[0,+\infty)\times X_{0+}\to X_+$ and two real numbers $k_1\ge 0$ and $k_2\ge 0$ such that for each $t\ge 0$ and each $x\in X_{0+}$,

$$F(t,x)\le G(t,x) \text{ and } \|G(t,x)\|\le k_1\|x\|+k_2.$$

Then

$$\chi(s,x)=+\infty,\ \ \forall s\ge 0,\ \ \forall x\in X_{0+}.$$

Moreover, for each $\gamma>0$ large enough, there exist $C_1>0$ and $C_2>0$ such that we have the following estimate

$$\|U(t,s)x\|\le e^{\gamma(t-s)}\left[C_1\|x\|+C_2\right].$$

Proof. Without loss of generality, we can assume that $s=0$ and the norm $\|.\|$ is monotone. Let $\varepsilon\in(0,\frac{1}{2k_1})$ and $\tau_\varepsilon>0$ be given such that $M_A V^\infty(S_A,0,\tau_\varepsilon)\le\varepsilon$. Let $x\in X_{0+}$ be fixed. Then by Proposition 5.3.2, we have for each $t\in[0,\chi(0,x))$ that

$$\begin{aligned}0\le U(t,0)x&=T_{A_0}(t)x+(S_A\diamond F(.,U(.,0)x))(t)\\&\le T_{A_0}(t)x+(S_A\diamond G(.,U(.,0)x))(t).\end{aligned}$$

Hence, for each $\gamma>\max(\omega_A,0)$, we have for each $t\in[0,\chi(0,x))$ that

$$\begin{aligned}e^{-\gamma t}\|U(t,0)x\|&\le e^{-\gamma t}\left\|T_{A_0}(t)x\right\|+e^{-\gamma t}\|(S_A\diamond G(.,U(.,0)x))(t)\|\\&\le M_A e^{(-\gamma+\omega_A)t}\|x\|+C(\varepsilon,\gamma)\sup_{s\in[0,t]}e^{-\gamma s}\|G(s,U(s,0)x)\|\\&\le M_A\|x\|+C(\varepsilon,\gamma)\sup_{s\in[0,t]}e^{-\gamma s}\left[k_1\|U(s,0)x\|+k_2\right]\\&\le M_A\|x\|+k_2C(\varepsilon,\gamma)+k_1C(\varepsilon,\gamma)\sup_{s\in[0,t]}e^{-\gamma s}\|U(s,0)x\|.\end{aligned}$$

Since $2k_1\varepsilon<1$, for $\gamma>\max(\omega_A,0)$ sufficiently large, we obtain $k_1C(\varepsilon,\gamma)=\frac{2k_1\varepsilon}{1-e^{(\omega_A-\gamma)\tau_\varepsilon}}<1$ and the result follows. $\square$

5.4 Lipschitz Perturbation

Let E be a subset of Y and $G : Y \to Z$ be a map from a Banach space $(Y, \|.\|_Y)$ into a Banach space $(Z, \|.\|_Z)$. Define

$$\|G\|_{\mathrm{Lip}(E,Z)} := \sup_{x,y\in E: x\neq y} \frac{\|G(x)-G(y)\|_Z}{\|x-y\|_Y}.$$

Proposition 5.4.1. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $F : [0,+\infty) \times \overline{D(A)} \to X$ be a continuous map and $\sigma \in (0,+\infty]$ be a fixed constant. Assume that*

$$\Gamma_F(\sigma) := \sup_{t\in[0,\sigma)} \|F(t,.)\|_{\mathrm{Lip}(X_0,X)} < +\infty.$$

Then for each $x \in X_0$ and each $s \in [0,\sigma)$, there exists a unique solution $U(.,s)x \in C([s,\sigma), X_0)$ of

$$U(t,s)x = x + A\int_s^t U(l,s)x dl + \int_s^t F(l, U(l,s)x) dl, \ \forall t \in [s,\sigma).$$

Moreover, there exists $\gamma_0 > \max(0,\omega_A)$ such that for each $\gamma \geq \gamma_0$, each pair $t,s \in [0,\sigma)$ with $t \geq s$, and each pair $x,y \in X_0$, we have

$$\|U(t,s)x\| \leq e^{\gamma(t-s)} \left[2M_A \|x\| + \sup_{l\in[s,\sigma)} e^{-\gamma(l-s)} \|F(l,0)\| \right]$$

and

$$\|U(t,s)x - U(t,s)y\| \leq 2M_A e^{\gamma(t-s)} \|x-y\|.$$

Proof. Fix $s,t \in [0,\sigma)$ with $s < t$. Let $\varepsilon > 0$ such that

$$\varepsilon \max(\Gamma_F(\sigma), 1) < 1/8.$$

Let $\tau_\varepsilon > 0$ be given such that $M_A V^\infty(S_A, 0, \tau_\varepsilon) \leq \varepsilon$. Then by Lemma 3.5.5 we have for each $\gamma > \omega_A$ that

$$\|\mathscr{L}_s(\varphi)\|_{\mathscr{L}(BC^\gamma([s,+\infty),X), BC^\gamma([s,+\infty),X_0))} \leq C(\gamma,\varepsilon) = \frac{2\varepsilon \max(1, e^{-\gamma\tau_\varepsilon})}{1-e^{(\omega_A-\gamma)\tau_\varepsilon}}.$$

Let $\gamma_0 \geq \max(0,\omega_A)$ be fixed such that

$$\frac{1}{1-e^{(\omega_A-\gamma)\tau_\varepsilon}} < 2, \ \forall \gamma \geq \gamma_0.$$

To prove the proposition it is sufficient to prove that the following fixed point problem

$$U(.,s)x = T_{A_0}(.-s)x + \mathscr{L}_s \circ \Psi(U(.,s)x) \tag{5.4.1}$$

admits a solution $U(.,s)x \in BC^{\gamma}([s,\sigma),X_0)$, where $\Psi : BC^{\gamma}([s,\sigma),X_0) \to BC^{\gamma}([s,\sigma),X)$ is a nonlinear operator defined by

$$\Psi(\varphi)(l) = F(l,\varphi(l)),\ \ \forall l \in [s,t],\ \forall \varphi \in BC^{\gamma}([s,\sigma),X_0).$$

We have

$$\left\| T_{A_0}(.-s) \right\|_{\mathscr{L}(X_0,BC^{\gamma}([s,\sigma),X_0))} \leq M_A,$$

$$\|\mathscr{L}_s\|_{\mathscr{L}(BC^{\gamma}([s,\sigma),X),BC^{\gamma}([s,\sigma),X_0))} \leq 4\varepsilon,$$

and

$$\|\Psi\|_{\mathrm{Lip}(BC^{\gamma}([s,\sigma),X_0),BC^{\gamma}([s,\sigma),X))} \leq \Gamma_F(\sigma).$$

From this we deduce that

$$\|\mathscr{L}_s \circ \Psi\|_{\mathrm{Lip}(BC^{\gamma}([s,\sigma),X_0),BC^{\gamma}([s,\sigma),X_0))} \leq 4\varepsilon \Gamma_F(\sigma) \leq 1/2.$$

Thus, the fixed point problem (5.4.1) has a unique solution. Moreover, for each $x \in X_0$, there exists a unique solution in $BC^{\gamma}([s,\sigma),X_0)$.

$$\begin{aligned}
&\|U(.,s)x\|_{BC^{\gamma}([s,t],X_0)} \\
&\quad \leq M_A \|x\| + \|\mathscr{L}_s(\Psi(0))\| + \|\mathscr{L}_s(\Psi(U(.,s)x) - \Psi(0))\| \\
&\quad \leq M_A \|x\| + 4\varepsilon \|\Psi(0)\|_{BC^{\gamma}([s,\sigma),X)} + \frac{1}{2} \|U(.,s)x\|_{BC^{\gamma}([s,\sigma),X_0)},
\end{aligned}$$

which implies that

$$\|U(.,s)x\|_{BC^{\gamma}([s,\sigma),X_0)} \leq 2M_A \|x\| + 8\varepsilon \|\Psi(0)\|_{BC^{\gamma}([s,\sigma),X)}.$$

Since by construction $\varepsilon \leq \frac{1}{8}$, we have

$$\sup_{l\in[s,\sigma)} e^{-\gamma(l-s)} \|U(.,s)x\| \leq 2M_A \|x\| + \sup_{l\in[s,\sigma)} e^{-\gamma(l-s)} \|F(l,0)\|.$$

Similarly, we have for each pair $x, y \in X_0$ that

$$U(.,s)x - U(.,s)y = T_{A_0}(.-s)(x-y) + \mathscr{L}_s[\Psi(U(.,s)x) - \Psi(U(.,s)y)].$$

Therefore,

$$\begin{aligned}
&\|U(.,s)x - U(.,s)y\|_{BC^{\gamma}([s,\sigma),X_0)} \\
&\quad \leq M_A \|x-y\| + \frac{1}{2} \|U(.,s)x - U(.,s)y\|_{BC^{\gamma}([s,\sigma),X_0)}.
\end{aligned}$$

This completes the proof. $\square$

5.5 Differentiability with Respect to the State Variable

In this section we investigate the differentiability of solutions with respect to the state variable. Let Y be a Banach space. Define open and closed balls as follows

$$\begin{aligned} B_Y(x,r) &:= \{y \in Y : \|x-y\|_Y < r\}, \\ \overline{B}_Y(x,r) &:= \{y \in Y : \|x-y\|_Y \leq r\} \end{aligned}$$

whenever $x \in Y$ and $r > 0$.

Proposition 5.5.1. *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Assume in addition that*

(a) For each $t \geq 0$ the map $x \to F(t,x)$ is continuously differentiable from X_0 into X;
(b) The map $(t,x) \to D_x F(t,x)$ is continuous from $[0,+\infty) \times X_0$ into $\mathscr{L}(X_0, X)$.

Let $x_0 \in X_0$, $s \geq 0$, $\tau \in [0, \chi(s,x_0))$, and $\gamma \in (0, \chi(s,x_0) - \tau)$. Let $\eta > 0$ (there exists such a constant since D_χ is open in D) such that

$$\chi(s,y) > \tau + \gamma,\ \forall y \in B_{X_0}(x_0, \eta).$$

Then for each $t \in [s, s+\tau+\gamma]$, the map $x \to U(t,s)x$ is defined from $B_{X_0}(x,\eta)$ into X_0 and is differentiable at x_0. Moreover, if we set

$$V(t,s)y = D_x U(t,s)(x)(y),\ \forall y \in X_0,$$

then $t \to V(t,s)y$ is an integrated solution of the Cauchy problem

$$\begin{aligned} \frac{dV(t,s)y}{dt} &= AV(t,s)y + D_x F(t, U(t,s)x_0)V(t,s)y,\ t \in [s, s+\chi(s,x_0)), \\ V(s,s)y &= y \end{aligned}$$

or equivalently, $\forall t \in [s, s+\chi(s,x_0))$, $t \to V(t,s)y$ is a solution of

$$V(t,s)y = T_{A_0}(t-s)y + (S_A \diamond D_x F(.,U(.,s)x_0)V(.,s)y)(t-s).$$

Proof. First by using the result in the Section 5.4 about the Lipschitz case, it is clear that $V(t,s)$ is well defined. Set

$$R(t)(y) = U(t,s)(x_0+y) - U(t,s)(x_0) - V(t,s)y.$$

Then

$$\begin{aligned} R(t)(y) = {} & (S_A \diamond [F(.,U(.,s)(x_0+y)) - F(.,U(.,s)(x_0)) \\ & -D_x F(.,U(.,s)x_0)V(.,s)y])(t-s). \end{aligned}$$

But

$$F(t,U(t,s)(x_0+y)) - F(t,U(t,s)(x_0))$$

$$
\begin{aligned}
&= \int_0^1 D_x F(t, rU(t,s)(x_0+y) + (1-r)U(t,s)(x_0))(U(t,s)(x_0+y) \\
&\quad - U(t,s)(x_0))dr \\
&= \int_0^1 \Psi_1(t,r,y)\,(U(t,s)(x_0+y) - U(t,s)(x_0))\,dr \\
&\quad + D_x F(t, U(t,s)(x_0))\,(U(t,s)(x_0+y) - U(t,s)(x_0)),
\end{aligned}
$$

where

$$\Psi_1(t,r,y) = D_x F(t, rU(t,s)(x_0+y) + (1-r)U(t,s)(x_0)) - D_x F(t, U(t,s)(x_0)).$$

Thus,

$$R(t)y = (S_A \diamond [K(.) + D_x F(., U(.,s)x_0)R(.)y])\,(t-s),$$

where

$$K(t) = \int_0^1 \Psi_2(t,r,y)\,(U(t,s)(x_0+y) - U(t,s)(x_0))\,dr$$

and

$$\Psi_2(t,r,y) = D_x F(t, rU(t,s)(x_0+y) + (1-r)U(t,s)(x_0)) - D_x F(t, U(t,s)(x_0)).$$

The result follows from Proposition 3.5.3 and the continuity of $(t,x) \to U(t,s)x$. $\square$

5.6 Time Differentiability and Classical Solutions

In this section, we study the time differentiability of the solutions. Consider a solution $u \in C([0,\tau], \overline{D(A)})$ of

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t F(s,u(s))ds, \ t \in [0,\tau].$$

Assume that $x \in D(A)$ and $F : [0,\tau] \times \overline{D(A)} \to X$ is a C^1 map. When the domain of A is dense, it is well known (see Pazy [281], Theorem 6.1.5, p. 187) that for each $x \in D(A)$, the map $t \to u(t)$ is a classical solution; that is, the map $t \to u(t)$ is continuously differentiable, $u(t) \in D(A)$ for all $t \in [0,\tau]$, and satisfies

$$u'(t) = Au(t) + f(t,u(t)), \ \forall t \in [0,\tau], \ \ u(0) = x.$$

Now we consider the same problem but in the context of non-densely defined Cauchy problems. When A satisfies the Hille-Yosida condition, this problem has been studied by Thieme [328] and Magal [242]. So the goal is to extend these results to the non-Hille-Yosida case. For each $\tau > 0$, set

$$C^{1,+}([0,\tau],X)=\left\{f\in C([0,\tau],X):\frac{d^+f}{dt}\in C([0,\tau),X),\lim_{t\nearrow\tau}\frac{d^+f}{dt}(t)<\infty\right\}.$$

The following lemma is a variant of a result due to Da Prato and Sinestrari [85].

Lemma 5.6.1 (Da Prato-Sinestrari). *Let $A:D(A)\subset X\to X$ be a closed linear operator. Let $\tau>0$, $f\in C([0,\tau],X)$, and $x\in X_0$ be fixed. Assume that $u\in C([0,\tau],X)$ is a solution of*

$$u(t)=x+A\int_0^t u(s)ds+\int_0^t f(s)ds,\ \forall t\in[0,\tau].$$

Assume in addition that u belongs to $C^{1,+}([0,\tau],X)$ or $C([0,\tau],D(A))$. Then

$$u\in C^1([0,\tau],X)\cap C([0,\tau],D(A))$$

and

$$u'(t)=Au(t)+f(t),\ \forall t\in[0,\tau].$$

Proof. If $u\in C([0,T],D(A))$, since A is closed, we have

$$u(t)=x+\int_0^t Au(s)ds+\int_0^t f(s)ds,\ \ \forall t\in[0,\tau].$$

So $u\in C^1([0,\tau],X)$ and $u'(t)=Au(t)+f(t),\forall t\in[0,\tau]$.

If $u\in C^{1,+}([0,\tau],X)$, then we have for each $t\in[0,\tau)$ and $h>0$ that

$$\frac{u(t+h)-u(t)}{h}-\frac{\int_t^{t+h}f(s)ds}{h}=A\frac{\int_t^{t+h}u(s)ds}{h}.$$

Since A is closed, we deduce that $u(t)\in D(A)$ and $Au(t)=\frac{d^+u}{dt}(t)-f(t),\forall t\in[0,\tau)$. Since $u\in C^{1,+}([0,\tau],X)$, we then have that $u\in C([0,\tau],D(A))$ and complete the proof. □

Lemma 5.6.2. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Assume that $g\in C^1([0,T],X)$ and $g(0)\in\overline{D(A)}$, then $t\to(S_A\diamond g)(t)$ is continuously differentiable and*

$$\frac{d}{dt}(S_A\diamond g)(t)=T_{A_0}(t)g(0)+\left(S_A\diamond g'\right)(t),\ \ \forall t\in[0,T].$$

Proof. Since g is continuously differentiable, the map $t\to(S_A*g)(t)$ is continuously differentiable,

$$\frac{d}{dt}(S_A*g)(t)=S_A(t)g(0)+\left(S_A*g'\right)(t),\ \forall t\in[0,T],$$

Since $g(0)\in\overline{D(A)}$, we have $S_A(t)g(0)=\int_0^t T_{A_0}(l)g(0)dl$, $\forall t\in[0,T]$, and the result follows. □

The following theorem is due to Vanderbauwhede [343, Theorem 3.5], a generalized version is given and proved in Chapter 6 (see Lemma 6.1.13 and its proof) when it is used to prove the smoothness of center manifolds.

Theorem 5.6.3 (Fibre Contraction Theorem). *Let M_1 and M_2 be two complete metric spaces and $\Psi : M_1 \times M_2 \to M_1 \times M_2$ a mapping of the form*

$$\Psi(x,y) = (\Psi_1(x), \Psi_2(x,y)), \forall (x,y) \in M_1 \times M_2$$

satisfying the following properties:

(i) *Ψ_1 has a fixed point $\bar{x} \in M_1$ such that for each $x \in M_1$,*

$$\Psi_1^n(x) \to \bar{x} \text{ as } n \to +\infty;$$

(ii) *There exists $k \in [0,1)$ such that for each $x \in M_1$ the map $y \to \Psi_2(x,y)$ is k-Lipschitz continuous;*

(iii) *The map $x \to \Psi_2(x,\bar{y})$ is continuous, where $\bar{y} \in M_2$ is a fixed point of the map $y \to \Psi_2(\bar{x},y)$.*

Then for each $(x,y) \in M_1 \times M_2$,

$$\Psi^n(x,y) \to (\bar{x},\bar{y}) \text{ as } n \to +\infty.$$

The key result of this section is the following lemma.

Lemma 5.6.4. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $\tau > 0$ be fixed and $F : [0,\tau] \times \overline{D(A)} \to X$ be continuously differentiable. Assume that there exists an integrated solution $u \in C([0,\tau], X)$ of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + F(t,u(t)),\ t \in [0,\tau],\ \ u(0) = x \in X_0.$$

Assume in addition that

$$x \in D(A_0) \text{ and } F(0,x) \in \overline{D(A)}.$$

Then there exists $\varepsilon > 0$ such that $u \in C^1([0,\varepsilon],X) \cap C([0,\varepsilon],D(A))$ and

$$u'(t) = Au(t) + F(t,u(t)),\ \ \forall t \in [0,\varepsilon].$$

Proof. We apply the Fibre Contraction Theorem (Theorem 5.6.3) to prove the lemma.

(i) Construction of the map Ψ. Since F is continuously differentiable, there exist $\varepsilon_0 > 0$, $K_1 > 0$, and $K_2 > 0$ such that

$$\|\partial_t F(t,y)\| \le K_1 \text{ and } \|\partial_x F(t,y)\|_{\mathscr{L}(X_0,X)} \le K_2$$

whenever $\|x - y\| \le \varepsilon_0$ and $0 \le t \le \varepsilon_0$. For each $\varepsilon \in (0,\varepsilon_0]$, set

$$\begin{aligned}
M_1^\varepsilon &= \{\varphi \in C([0,\varepsilon],X_0) : \varphi(0) = x,\ \|\varphi(t) - x\| \le \varepsilon_0, \forall t \in [0,\varepsilon]\},\\
M_2^\varepsilon &= \{\varphi \in C([0,\varepsilon],X_0) : \varphi(0) = A_0x + F(0,x),\\
&\qquad \|\varphi(t) - A_0x + F(0,x)\| \le \varepsilon_0,\ \forall t \in [0,\varepsilon]\}.
\end{aligned}$$

From now on, we assume that for each $i = 1,2$, M_i^ε is endowed with the metric $d(\varphi,\widehat{\varphi}) = \|\varphi - \widehat{\varphi}\|_{\infty,[0,\varepsilon]}$ and $M_1^\varepsilon \times M_2^\varepsilon$ is endowed with the usual product distance $d((\varphi,\psi),(\widehat{\varphi},\widehat{\psi})) = d(\varphi,\widehat{\varphi}) + d(\psi,\widehat{\psi})$.

For each $\varepsilon \in (0,\varepsilon_0]$, set

$$E^\varepsilon = \left\{ (\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon : \varphi_1(t) = x + \int_0^t \varphi_2(s)ds, \forall t \in [0,\varepsilon] \right\}.$$

Then it is clear that E^ε is a closed subset of $M_1^\varepsilon \times M_2^\varepsilon$.

Consider a map $\Psi : M_1^\varepsilon \times M_2^\varepsilon \to C([0,\varepsilon],X_0) \times C([0,\varepsilon],X_0)$ defined by

$$\Psi(\varphi_1,\varphi_2) = (\Psi_1(\varphi_1), \Psi_2(\varphi_1,\varphi_2)),\ \forall (\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon,$$

where for each $t \in [0,\varepsilon]$,

$$\begin{aligned}
\Psi_1(\varphi_1)(t) &= T_{A_0}(t)x + (S_A \diamond F(.,\varphi_1(.)))(t),\\
\Psi_2(\varphi_1,\varphi_2)(t) &= T_{A_0}(t)[A_0x + F(0,x)]\\
&\quad + (S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.))(t).
\end{aligned}$$

(ii) $\Psi(M_1^\varepsilon \times M_2^\varepsilon) \subset M_1^\varepsilon \times M_2^\varepsilon$. One can easily check that Ψ is a continuous map. We now prove that for some $\varepsilon > 0$ small enough, $\Psi(M_1^\varepsilon \times M_2^\varepsilon) \subset M_1^\varepsilon \times M_2^\varepsilon$, and

$$\Psi_1(\varphi_1)(0) = x,\ \Psi_2(\varphi_1,\varphi_2)(0) = [A_0x + F(0,x)].$$

For each $\varepsilon \in (0,\varepsilon_0]$, each $t \in [0,\varepsilon]$, and each $\varphi \in M_1^\varepsilon$, we have

$$\begin{aligned}
&\|\Psi_1(\varphi)(t) - x\|\\
&\quad \le \left\|T_{A_0}(t)x - x\right\| + \|(S_A \diamond F(.,\varphi(.)))(t)\|\\
&\quad \le \left\|T_{A_0}(t)x - x\right\| + V^\infty(S_A,0,t) \sup_{s\in[0,t]} \|F(s,\varphi(s))\|\\
&\quad \le \left\|T_{A_0}(t)x - x\right\|\\
&\qquad + V^\infty(S_A,0,\varepsilon)\left(\sup_{s\in[0,t]} \|F(s,x)\| + K_2 \sup_{s\in[0,t]} \|\varphi(s) - x\| \right)\\
&\quad \le \sup_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)x - x\right\| + V^\infty(S_A,0,\varepsilon)\left(\sup_{s\in[0,\varepsilon]} \|F(s,x)\| + K_2\varepsilon_0 \right).
\end{aligned}$$

Thus, there exists $\varepsilon_1 \in (0,\varepsilon_0]$ such that for each $\varepsilon \in (0,\varepsilon_1]$, $\Psi_1(M_1^\varepsilon) \subset M_1^\varepsilon$.

Moreover, for each $\varepsilon \in (0,\varepsilon_1]$, each $t \in [0,\varepsilon]$, and each $(\varphi_1,\varphi_2) \in M_1^\varepsilon \times M_2^\varepsilon$, we have

$$\begin{aligned}
&\|\Psi_2(\varphi_1,\varphi_2)(t) - [A_0x + F(0,x)]\| \\
&\quad\le \left\|T_{A_0}(t)[A_0x + F(0,x)] - [A_0x + F(0,x)]\right\| \\
&\qquad + \|(S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.))(t)\| \\
&\quad\le \sup\nolimits_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)[A_0x + F(0,x)] - [A_0x + F(0,x)]\right\| \\
&\qquad + V^\infty(S_A,0,\varepsilon) \sup\nolimits_{s\in[0,\varepsilon]} \|\partial_t F(s,\varphi_1(s))\| \\
&\qquad + V^\infty(S_A,0,\varepsilon) \sup\nolimits_{s\in[0,\varepsilon]} \|\partial_x F(s,\varphi_1(s))\| \, \|\varphi_2(.)\| \\
&\quad\le \sup\nolimits_{t\in[0,\varepsilon]} \left\|T_{A_0}(t)[A_0x + F(0,x)] - [A_0x + F(0,x)]\right\| \\
&\qquad + V^\infty(S_A,0,\varepsilon)\{K_1 + K_2[\|A_0x + F(0,x)\| + \varepsilon_0]\}.
\end{aligned} \tag{5.6.1}$$

Therefore, there exists $\varepsilon_2 \in (0,\varepsilon_1]$ such that for each $\varepsilon \in (0,\varepsilon_2]$,

$$\Psi_2(M_1^\varepsilon \times M_2^\varepsilon) \subset M_2^\varepsilon.$$

Similarly, for each $\varepsilon \in (0,\varepsilon_2]$, $\Psi(M_1^\varepsilon \times M_2^\varepsilon) \subset M_1^\varepsilon \times M_2^\varepsilon$.

(iii) $\Psi(E^\varepsilon) \subset E^\varepsilon$. Let $(\varphi_1,\varphi_2) \in E^\varepsilon$. Then $\varphi_1 \subset C^1([0,\varepsilon],X_0)$ and $\varphi_1'(t) = \varphi_2(t), \forall t \in [0,\varepsilon]$. Notice that

$$\Psi_1(\varphi_1)(t) = T_{A_0}(t)x + (S_A \diamond F(.,\varphi_1(.)))(t),$$

using Lemma 5.6.2 and the fact that $x \in D(A_0)$ and $F(0,x) \in \overline{D(A)}$, we have

$$\begin{aligned}
\frac{d\Psi_1(\varphi_1)(t)}{dt} &= A_0 T_{A_0}(t)x + T_{A_0}(t)F(0,x) + \left(S_A \diamond \frac{d}{dt}F(.,\varphi_1(.))\right)(t) \\
&= T_{A_0}(t)[A_0x + F(0,x)] \\
&\qquad + (S_A \diamond \partial_t F(.,\varphi_1(.)) + \partial_x F(.,\varphi_1(.))\varphi_2(.))(t).
\end{aligned}$$

Thus,

$$\frac{d\Psi_1(\varphi_1)(t)}{dt} = \Psi_2(\varphi_1,\varphi_2)(t)$$

and

$$\Psi(E^\varepsilon) \subset E^\varepsilon.$$

(iv) Convergence of Ψ^n. To apply Lemma 5.6.3, it remains to verify conditions (i) and (ii) for some $\varepsilon \in (0,\varepsilon_2]$ small enough. Let $(\varphi_1,\varphi_2), (\widehat{\varphi}_1,\widehat{\varphi}_2) \in M_1^\varepsilon \times M_2^\varepsilon$ be fixed. We have for each $\varepsilon \in (0,\varepsilon_2]$ that

$$\begin{aligned}
\|\Psi_1(\varphi_1)(t) - \Psi_1(\widehat{\varphi}_1)(t)\| &= \|(S_A \diamond F(.,\varphi_1(.)) - F(.,\widehat{\varphi}_1(.)))(t)\| \\
&\le V^\infty(S_A,0,\varepsilon)\|F(s,\varphi_1(s)) - F(s,\widehat{\varphi}_1(s))\| \\
&\le V^\infty(S_A,0,\varepsilon)K_2 \sup_{s\in[0,\varepsilon]} \|\varphi_1(s) - \widehat{\varphi}_1(s)\|.
\end{aligned}$$

So there exists $\varepsilon_3 \in (0,\varepsilon_2]$ such that $\delta_1 := V^\infty(S_A,0,\varepsilon_3)K_2 \in (0,1)$, we have for each $\varepsilon \in (0,\varepsilon_3]$ that

$$\|\Psi_1(\varphi_1) - \Psi_1(\widehat{\varphi}_1)\|_{\infty,[0,\varepsilon]} \le \delta_1 \|\varphi_1 - \widehat{\varphi}_1\|_{\infty,[0,\varepsilon]}.$$

Moreover,

$$\begin{aligned}\|\Psi_2(\varphi_1,\varphi_2)(t)-\Psi_2(\varphi_1,\widehat{\varphi}_2)(t)\| &= \|(S_A\diamond\partial_x F(.,\varphi_1(.))(\varphi_2(.)-\widehat{\varphi}_2))(t)\| \\ &\le V^\infty(S_A,0,\varepsilon)K_2\sup_{s\in[0,\varepsilon]}\|\varphi_2(s)-\widehat{\varphi}_2(s)\| \\ &\le \delta_1\sup_{s\in[0,\varepsilon]}\|\varphi_2(s)-\widehat{\varphi}_2(s)\|,\end{aligned}$$

which implies that

$$\|\Psi_2(\varphi_1,\varphi_2)(t)-\Psi_2(\varphi_1,\widehat{\varphi}_2)\|_{\infty,[0,\varepsilon]}\le\delta_1\|\varphi_2-\widehat{\varphi}_2\|_{\infty,[0,\varepsilon]}.$$

Hence, for $\varepsilon=\varepsilon_3$ we have $\Psi(M_1^\varepsilon\times M_2^\varepsilon)\subset M_1^\varepsilon\times M_2^\varepsilon$, $\Psi(E^\varepsilon)\subset E^\varepsilon$ and Ψ satisfies the assumptions of Lemma 5.6.3. We deduce that there exists $(u,v)\in M_1^\varepsilon\times M_2^\varepsilon$ such that for each $(\varphi_1,\varphi_2)\in M_1^\varepsilon\times M_2^\varepsilon$,

$$\Psi^n(\varphi_1,\varphi_2)\to(u,v)\text{ as }n\to+\infty.$$

Since $\Psi(E^\varepsilon)\subset E^\varepsilon$ and E^ε is closed, we deduce that $(u,v)\in E^\varepsilon$. In particular, $u\in C^1([0,\varepsilon],X)$, and the result follows. □

In next lemma, we show that the conclusions of Lemma 5.6.4 hold for $t\in[0,\tau]$.

Lemma 5.6.5. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $\tau>0$ be fixed and $F:[0,\tau]\times\overline{D(A)}\to X$ be continuously differentiable. Assume that there exists an integrated solution $u\in C([0,\tau],X)$ of the Cauchy problem*

$$\frac{du(t)}{dt}=Au(t)+F(t,u(t)),\ t\in[0,\tau],\ \ u(0)=x\in X_0.$$

Assume in addition that

$$x\in D(A_0)\text{ and }F(0,x)\in\overline{D(A)}.$$

Then $u\in C^1([0,\tau],X)\cap C([0,\tau],D(A))$ and

$$u'(t)=Au(t)+F(t,u(t)),\ \ \forall t\in[0,\tau].$$

Proof. Let $w\in C([0,\tau],\overline{D(A)})$ be a solution of the equation

$$\begin{aligned}w(t) &= Ax+F(0,x)+A\int_0^t w(s)ds \\ &\quad+\int_0^t\frac{\partial}{\partial t}F(s,u(s))+D_xF(s,u(s))w(s)ds,\forall t\in[0,\tau].\end{aligned}$$

From Section 5.4 concerning global Lipschitz perturbation, it is clear that the solution $w(t)$ exists and is uniquely determined. Since $u(t)$ exists on $[0,\tau]$, let $t\in[0,\tau)$ be fixed. We have for each $h\in(0,\tau-t)$ that

$$
\begin{aligned}
&\frac{u(t+h)-u(t)}{h} \\
&= \frac{1}{h} A \left[\int_0^{t+h} u(s)ds - \int_0^t u(s)ds \right] \\
&\quad + \frac{1}{h} \left[\int_0^{t+h} F(s,u(s))ds - \int_0^t F(s,u(s))ds \right] \\
&= A \left[\int_0^t \frac{u(s+h)-u(s)}{h} ds \right] + \frac{1}{h} A \int_0^h u(s)ds \\
&\quad + \int_0^t \frac{F(s+h,u(s+h))-F(s,u(s))}{h} ds + \frac{1}{h} \int_0^h F(s,u(s))ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{u(t+h)-u(t)}{h} - w(t) \\
&= A \int_0^t \left[\frac{u(s+h)-u(s)}{h} - w(s) \right] ds \\
&\quad + \frac{1}{h} A \int_0^h u(s)ds + \frac{1}{h} \int_0^h F(s,u(s))ds - Ax - F(0,x) \\
&\quad + \int_0^t \left[\frac{F(s+h,u(s+h))-F(s+h,u(s))}{h} - D_x F(s,u(s)) w(s) \right] ds \\
&\quad + \int_0^t \left[\frac{F(s+h,u(s))-F(s,u(s))}{h} - \frac{\partial}{\partial t} F(s,u(s)) \right] ds.
\end{aligned}
$$

Denote

$$
v_h(t) := \frac{u(t+h)-u(t)}{h} - w(t)
$$

and

$$
x_h := \frac{1}{h} A \int_0^h u(s)ds + \frac{1}{h} \int_0^h F(s,u(s))ds - Ax - F(0,x).
$$

We have

$$
\begin{aligned}
v_h(t) &= x_h + A \int_0^t v_h(s)ds \\
&\quad + \int_0^t \int_0^1 D_x F\left(l\left(u(s+h)-u(s)\right)+u(s)\right) \\
&\qquad\qquad \left(\frac{u(s+h)-u(s)}{h} - w(s) \right) dl ds \\
&\quad + \int_0^t \int_0^1 \left[D_x F\left(l\left(u(s+h)-u(s)\right)+u(s)\right) - D_x F\left(u(s)\right) \right] w(s) dl ds \\
&\quad + \int_0^t \left[\frac{F(s+h,u(s))-F(s,u(s))}{h} - \frac{\partial}{\partial t} F(s,u(s)) \right] ds.
\end{aligned}
$$

Set

$$K = \sup_{l\in[0,1],s\in[0,\tau],h\in[0,\tau-s]} \|D_x F\left(l\left(u(s+h)-u(s)\right)+u(s)\right)\|_{\mathscr{L}(X_0,X)} < +\infty.$$

Let $\widehat{\tau} > 0$ be given such that

$$M_A V^\infty(S_A,0,t) \le \frac{1}{8(K+1)},\ \forall t\in[0,\widehat{\tau}].$$

Choose $\gamma > \max(0,\omega_A)$ so that

$$\frac{1}{4\left(1-e^{(\omega_A-\gamma)\widehat{\tau}}\right)} < \frac{1}{2}.$$

Then by Proposition 3.5.3, we have for all $\gamma > \max(0,\omega_A)$ that

$$\begin{aligned}
&e^{-\gamma t}\|v_h(t)\| \\
&\quad\le M_A\|x_h\| + \frac{1}{2}\sup_{s\in[0,\tau]} e^{-\gamma s}\|v_h(s)\| \\
&\qquad + \sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[D_xF(l(u(s+h)-u(s))+u(s))-D_xF(u(s))\right]w(s)dl\right\| \\
&\qquad + \sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[\frac{F(s+h,u(s))-F(s,u(s))}{h}-\frac{\partial}{\partial t}F(s,u(s))\right]dl\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&e^{-\gamma t}\|v_h(t)\| \\
&\quad\le 2M_A\|x_h\| \\
&\qquad +2\sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[D_xF(l(u(s+h)-u(s))+u(s))-D_xF(u(s))\right]w(s)dl\right\| \\
&\qquad +2\sup_{s\in[0,\tau]} e^{-\gamma s}\left\|\int_0^1 \left[\frac{F(s+h,u(s))-F(s,u(s))}{h}-\frac{\partial}{\partial t}F(s,u(s))\right]dl\right\|.
\end{aligned}$$

We now claim that

$$\lim_{h\searrow 0} x_h = 0.$$

Indeed, we have

$$\frac{u(h)-u(0)}{h} = \frac{1}{h}A\int_0^h u(s)ds + \frac{1}{h}\int_0^h F(s,u(s))ds$$

and by Lemma 5.6.4, we have

$$\lim_{h\to 0^+}\frac{u(h)-u(0)}{h}=Ax+F(0,x),$$

so

$$\lim_{h\searrow 0}x_h=0.$$

We conclude that for each $t\in[0,\tau)$, we will have

$$\lim_{h\to 0^+}\frac{u(t+h)-u(t)}{h}=w(t).$$

Since $w\in C([0,\tau],X)$, we deduce that $u\in C^{1,+}([0,\tau],X)$. By using Da Prato-Sinestrari Lemma 5.6.1, we obtain the result. $\square$

To extend the differentiability result to the case where $F(0,x)\notin\overline{D(A)}$, we notice that since $u(t)\in\overline{D(A)}$ for all $t\in[0,T]$, a necessary condition for differentiability is

$$Ax+F(0,x)\in\overline{D(A)}.$$

In fact, this condition is also sufficient. Indeed, taking any bounded linear operator $B\in\mathscr{L}(X)$, if u satisfies

$$u(t)=x+A\int_0^t u(s)ds+\int_0^t F(s,u(s))ds,\ \forall t\in[0,\tau],$$

then we have

$$u(t)=x+(A+B)\int_0^t u(s)ds+\int_0^t (F(s,u(s))-Bu(s))\,ds,\ \ t\in[0,\tau].$$

So to prove the differentiability of $u(t)$ it is sufficient to find B such that $(A+B)x\in\overline{D(A)}$. Take $B(\varphi)=-x^*(\varphi)Ax$, where $x^*\in X^*$ is a continuous linear form with $x^*(x)=1$ if $x\neq 0$, which is possible by the Hahn-Banach theorem. We then have

$$x\in D(A)=D(A+B)\text{ and }(A+B)x\in\overline{D(A)}=\overline{D(A+B)}.$$

Moreover, assuming that $Ax+F(0,x)\in\overline{D(A)}$, we obtain $F(0,x)-Bx\in\overline{D(A)}$. By using Theorem 3.5.1, we deduce that $A+B$ satisfies Assumptions 5.1.1 and 5.1.2. Therefore we obtain the following theorem.

Theorem 5.6.6. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $\tau>0$ be fixed and $F:[0,\tau]\times\overline{D(A)}\to X$ be continuously differentiable. Assume that there exists an integrated solution $u\in C([0,\tau],X)$ of the Cauchy problem*

$$\frac{du(t)}{dt}=Au(t)+F(t,u(t)),\ t\in[0,\tau],\ \ u(0)=x\in X_0.$$

Assume in addition that

$$x\in D(A)\textit{ and }Ax+F(0,x)\in\overline{D(A)}.$$

Then $u \in C^1([0,\tau],X) \cap C([0,\tau],D(A))$ *and*

$$u'(t) = Au(t) + F(t,u(t)), \ \forall t \in [0,\tau].$$

We now consider the nonlinear generator

$$A_N \varphi = A\varphi + F(0,\varphi), \ \varphi \in D(A_N) = D(A).$$

As in the linear case, one may define $A_{N,0}$ (the part A_N in $\overline{D(A)}$) as follows

$$A_{N,0} = A_N \text{ on } D(A_{N,0}) = \left\{ y \in D(A) : A_N y \in \overline{D(A)} \right\}.$$

Of course, one may ask about the density of the domain $D(A_{N,0})$ in $\overline{D(A)}$.

Lemma 5.6.7. *Let Assumptions 5.1.1, 5.1.2 and 5.2.1 be satisfied. Then the domain* $D(A_{N,0})$ *is dense in* $X_0 = \overline{D(A)}$. *Assume in addition that* X *has a positive cone* X_+ *and that Assumption 5.3.1 is satisfied. Then* $D(A_{N,0}) \cap X_{0+}$ *is dense in* X_{0+}.

Proof. Let $y \in \overline{D(A)}$ be fixed. Consider the following fixed point problem: $x_\lambda \in D(A)$ satisfies

$$(\lambda I - A - F)x_\lambda = \lambda y \Leftrightarrow x_\lambda = \lambda(\lambda I - A)^{-1} y + (\lambda I - A)^{-1} F(0, x_\lambda).$$

Denote

$$\Phi_\lambda(x) = \lambda(\lambda I - A)^{-1} y + (\lambda I - A)^{-1} F(0,x), \ \forall x \in X_0.$$

Fix $r > 0$. Since $y \in \overline{D(A)}$, by Lemma 3.5.4, $\lim_{\lambda \to +\infty} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} = 0$, we deduce that there exists $\lambda_0 > \omega_A$ such that

$$\Phi_\lambda(B_{X_0}(y,r)) \subset B_{X_0}(y,r), \ \forall \lambda \geq \lambda_0,$$

where $B_{X_0}(y,r)$ denotes the ball centered at y with radius r in X_0. Moreover, there exists $\lambda_1 \geq \lambda_0$, such that for each $\lambda \geq \lambda_1$, Φ_λ is a strict contraction on $B_{X_0}(y,r)$. Hence, $\forall \lambda \geq \lambda_1$, there exists $x_\lambda \in B_{X_0}(y,r)$ such that $\Phi_\lambda(x_\lambda) = x_\lambda$. Finally, using the fact that $y \in \overline{D(A)}$, we have $\lim\limits_{\lambda \to +\infty} \lambda(\lambda I - A)^{-1} y = y$, so

$$\lim_{\lambda \to +\infty} x_\lambda = y.$$

The proof of the positive case is similar. □

5.7 Stability of Equilibria

In this section we first investigate the local stability of an equilibrium.

Proposition 5.7.1. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $F:\overline{D(A)}\to X$ be a continuous map. Assume that*

(a) There exists $\overline{x}\in D(A)$ such that $A\overline{x}+F(\overline{x})=0$;
(b) There exist $\widehat{M}\geq 1$, $\widehat{\omega}<0$, and $L\in\mathscr{L}(X_0,X)$ such that

$$\left\|T_{(A+L)_0}(t)\right\|_{\mathscr{L}(X_0)}\leq \widehat{M}e^{\widehat{\omega}t},\ \ \forall t\geq 0;$$

(c) $\|F-L\|_{\mathrm{Lip}\left(\overline{B}_{X_0}(\overline{x},r),X\right)}\to 0$ as $r\to 0$.

Then for each $\gamma\in(\widehat{\omega},0)$ there exists $\varepsilon>0$, such that for each $x\in\overline{B}_{X_0}(\overline{x},\varepsilon)$, there exists a unique solution $U(.)x\in C([0,+\infty),X_0)$ of

$$U(t)x=x+A\int_0^t U(s)xds+\int_0^t F(U(s)x)\,ds,\ \ \forall t\geq 0$$

which satisfies

$$\|U(t)x-\overline{x}\|\leq 2\widehat{M}e^{\gamma t}\|x-\overline{x}\|,\ \ \forall t\geq 0,\forall x\in X_0.$$

Proof. Without loss of generality we can assume that $\overline{x}=0$, $L=0$, $\omega_A<0$, and

$$\|F\|_{\mathrm{Lip}\left(\overline{B}_{X_0}(\overline{x},\eta),X\right)}\to 0 \text{ as } \eta\to 0.$$

Choose $\eta_0>0$ such that

$$\|F\|_{\mathrm{Lip}\left(\overline{B}_{X_0}(\overline{x},\eta_0),X\right)}<+\infty.$$

Let $\phi:(-\infty,+\infty)\to[0,+\infty)$ be a Lipschitz continuous map such that

$$\phi(\alpha)\begin{cases}=0 & \text{if } 2\leq|\alpha|\\ \in[0,1] & \text{if } 1\leq|\alpha|\leq 2\\ =1 & \text{if } |\alpha|\leq 1.\end{cases}$$

Set

$$F_r(x)=\phi(r\|x\|)F(x),\ \forall x\in X_0,\ \forall r>0.$$

Then

$$F_r(x)=\begin{cases}0 & \text{if } \frac{2}{r}\leq\|x\|,\\ F(x) & \text{if } \|x\|\leq\frac{1}{r}.\end{cases}$$

Choose $\eta\in(0,\eta_0]$ and fix $r=\dfrac{2}{\eta}$. Let $x,y\in X_0$. Define $\varphi:[0,1]\to\mathbb{R}$ by

$$\varphi(t)=\|F_r(t(x-y)+y)-F_r(y)\|,\ \forall t\in[0,1].$$

Since $\|F\|_{\mathrm{Lip}\left(\overline{B}_{X_0}(\overline{x},\eta),X\right)}<+\infty$, the map φ is Lipschitz continuous, we have for each pair $t,s\in[0,1]$ that

$$|\varphi(t)-\varphi(s)| \leq \|F\|_{\text{Lip}\left(\overline{B}_{X_0}(\overline{x},\eta),X\right)} \left(2\|\phi\|_{\text{Lip}}+1\right) \|x-y\| \, |t-s| \, .$$

In particular, for $t=1$ and $s=0$, we deduce that

$$\|F_r(x)-F_r(y)\| \leq \|F\|_{\text{Lip}\left(\overline{B}_{X_0}\left(\overline{x},\frac{2}{r}\right),X\right)} \left(2\|\phi\|_{\text{Lip}}+1\right) \|(x-y)\| \, .$$

So for all $r \geq \dfrac{2}{\eta_0}$, $F_r \in \text{Lip}(X_0, X)$ and

$$\|F_r\|_{\text{Lip}(X_0,X)} \leq \|F\|_{\text{Lip}\left(\overline{B}_{X_0}\left(\overline{x},\frac{2}{r}\right),X\right)} \left(2\|\phi\|_{\text{Lip}}+1\right) \to 0 \text{ as } r \to +\infty. \tag{5.7.1}$$

For each $r \geq \dfrac{2}{\eta_0}$, we consider the nonlinear semigroup $\{U_r(t)\}_{t\geq 0}$ which is a solution of

$$U_r(t)x = x + A\int_0^t U_r(s)x ds + \int_0^t F_r\left(U_r(s)x\right) ds, \ \ \forall t \geq 0.$$

Let $\gamma \in (\widehat{\omega}, 0)$ be fixed. By Proposition 5.4.1 and (5.7.1), there exists $r_0 = r_0(\gamma) \geq \dfrac{2}{\eta_0}$ such that

$$\left\|U_{r_0}(t)x\right\| \leq 2Me^{\gamma t}\|x\|, \ \ \forall t \geq 0, \ \forall x \in X_0.$$

Let $\varepsilon \in \left(0, \frac{1}{2r_0}\frac{1}{2M}\right)$. Then for each $x \in B_{X_0}(0,\varepsilon)$,

$$\left\|U_{r_0}(t)x\right\| \leq 2Me^{\gamma t}\|x\| \leq \frac{1}{2r_0}.$$

On the other hand, since $F = F_r$ on $B_{X_0}\left(0, \frac{1}{2r_0}\right)$, we deduce that for each $x \in B_{X_0}(0,\varepsilon)$, $U_{r_0}(.)x$ is a solution of

$$U_r(t)x = x + A\int_0^t U_r(s)x ds + \int_0^t F\left(U_r(s)x\right) ds, \ \ \forall t \geq 0.$$

The uniqueness of the solution with initial value x in $B_{X_0}(0,\varepsilon)$ follows from the fact that F is locally Lipschitz continuous around 0 and by using the argument of Lemma 5.2.3. □

Remark 5.7.2. (1) If F is continuously differentiable in $B_{X_0}(\overline{x}, r_0)$, set $L = DF(\overline{x})$. Then by the formula

$$F(x)-F(y) = \int_0^1 DF(s(x-y)+y)(x-y)ds, \ \ \forall x,y \in B_{X_0}(\overline{x},\varepsilon),$$

it is clear that

$$\|F - DF(\overline{x})\|_{\text{Lip}\left(\overline{B}_{X_0}(\overline{x},r),X\right)} \to 0 \text{ as } r \to 0.$$

So if $\overline{x}$ is an equilibrium (i.e., assertion (a) is satisfied) and

$$\left\| T_{(A+DF(\overline{x}))_0}(t) \right\|_{\mathscr{L}(X_0)} \leq \widehat{M} e^{\widehat{\omega} t}, \ \forall t \geq 0$$

for some $\widehat{M} \geq 1$ and $\widehat{\omega} < 0$, the conclusion of the proposition holds.

(2) In order to see an example where the condition (c) is more appropriate than the usual differentiability condition, consider the following case. Assume that F is quasi-linear; that is, $F(x) = L(x)x$, where $L : X_0 \to \mathscr{L}(X_0, X)$ is a Lipschitz continuous map (but not necessarily differentiable in a neighborhood of 0). Then

$$\begin{aligned}
\|(F - L(0))\,x - (F - L(0))\,y\| &= \|(L(x) - L(0))\,x - (L(y) - L(0))\,y\| \\
&\leq \|(L(x) - L(0))\,x - (L(y) - L(0))\,x\| \\
&\quad + \|(L(y) - L(0))\,x - (L(y) - L(0))\,y\| \\
&\leq [\|x\| + \|y\|]\, \|L\|_{\mathrm{Lip}} \|x - y\|.
\end{aligned}$$

So

$$\|(F - L(0))\|_{\mathrm{Lip}\left(B_{X_{0+}}(0,\varepsilon), X\right)} \leq 2\varepsilon \|L\|_{\mathrm{Lip}} \to 0 \text{ as } \varepsilon \to 0.$$

Thus, in this case we can apply the condition (c), but F is not differentiable.

We now investigate the global asymptotic stability of an equilibrium.

Proposition 5.7.3. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $F : \overline{D(A)} \to X$ be a Lipschitz continuous map. Assume that:*

(a) There exists $\overline{x} \in D(A)$ such that $A\overline{x} + F(\overline{x}) = 0$;
(b) There exist $\widehat{M} > 0$, $\widehat{\omega} < 0$, and $L \in \mathscr{L}(X_0, X)$, such that

$$\left\| T_{(A+L)_0}(t) \right\|_{\mathscr{L}(X_0)} \leq \widehat{M} e^{\widehat{\omega} t}, \ \forall t \geq 0.$$

Consider a C_0-semigroup of nonlinear operators $\{U(t)\}_{t \geq 0}$ on X_0 which is a solution of

$$U(t)x = x + A \int_0^t U(s)x ds + \int_0^t F(U(s)x)\, ds, \ \forall t \geq 0.$$

Then for each $\gamma \in (\widehat{\omega}, 0)$, there exists $\delta_0 = \delta_0(\gamma) > 0$, such that

$$\|F - L\|_{\mathrm{Lip}(X_0,X)} \leq \delta_0 \Rightarrow \|U(t)x - \overline{x}\| \leq 2\widehat{M} e^{\gamma t} \|x - \overline{x}\|, \ \forall t \geq 0, \ \forall x \in X_0.$$

So $\overline{x}$ is a globally exponentially stable equilibrium of $\{U(t)\}_{t \geq 0}$.

Proof. Replacing $U(t)x$ by $V(t)x = U(t)(x + \overline{x}) - \overline{x}$ and $F(.)$ by $G(.) = F(. + \overline{x}) - F(\overline{x})$, respectively. Without loss of generality we can assume that $\overline{x} = 0$. Moreover, using Theorem 3.5.1 and replacing M by $\widehat{M}$, ω_A by $\widehat{\omega}$, A by $A + L$, and F by $F - L$, respectively. We can further assume that $L = 0$ and $\omega_A < 0$.

Fix $\tau > 0$ and set $\varepsilon := M\delta(\tau)$. Let $\gamma \in (\omega_A, 0)$ be fixed. Choose $\delta_0 = \delta_0(\gamma) > 0$ such that

$$\delta_0 \frac{2\varepsilon e^{-\gamma \tau_\varepsilon}}{1 - e^{(\omega_A - \gamma)\tau_\varepsilon}} \leq \frac{1}{2}.$$

Then by Lemma 3.5.5 we have

$$\|\mathscr{L}_0(\varphi)\|_{\mathscr{L}(BC^\gamma([0,+\infty),X),BC^\gamma([0,+\infty),X_0))} \leq \frac{2\varepsilon e^{-\gamma\tau_\varepsilon}}{1-e^{(\omega_A-\gamma)\tau_\varepsilon}} \leq \frac{1}{2\delta_0}.$$

It is sufficient to consider the problem $U(.)x \in BC^\gamma([0,+\infty),X_0)$,

$$U(t)x = T_{A_0}(t)x + \mathscr{L}_0(\Psi(U(.)x))(t),\ \forall t \in [0,+\infty),$$

where $\Psi : BC^\gamma([0,+\infty),X_0) \to BC^\gamma([0,+\infty),X)$ is defined by

$$\Psi(\varphi)(t) = F(\varphi(t)),\ \forall t \in [0,+\infty).$$

If $\|F\|_{\mathrm{Lip}(X_0,X)} \leq \delta_0$, we have $\|\mathscr{L}_0 \circ \Psi\|_{\mathrm{Lip}(BC^\gamma([0,+\infty),X_0),BC^\gamma([0,+\infty),X_0))} \leq 1/2$, so for each $t \geq 0$

$$\|U(.)x\|_{BC^\gamma([0,+\infty),X_0)} \leq M\|x\| + \frac{1}{2}\|U(.)x\|_{BC^\gamma([0,+\infty),X_0)}$$

and the result follows. □

As a consequence of Theorem 2.2 in Desch and Schappacher [94] and Proposition 5.5.1, we have the following result on the instability of an equilibrium.

Proposition 5.7.4. *Let Assumptions 5.1.1 and 5.1.2 be satisfied. Let $F : \overline{D(A)} \to X$ be a Lipschitz continuous map. Assume that there exists $\overline{x} \in D(A)$ such that $A\overline{x} + F(\overline{x}) = 0$. Assume in addition that*

$$\omega_{0,\mathrm{ess}}((A+DF(\overline{x}))_0) := \lim_{t\to+\infty} \frac{\ln\left(\left\|T_{(A+L)_0}(t)\right\|_{\mathrm{ess}}\right)}{t} < 0$$

and there exists $\lambda \in \sigma_p((A+DF(\overline{x}))_0)$ with $\mathrm{Re}(\lambda) > 0$. Then $\overline{x}$ is an unstable equilibrium in the following sense: There exist a constant $\varepsilon > 0$ and a sequence $\{x_n\}(\subset X_0) \to \overline{x}$ as $t_n \to +\infty$, such that

$$\|U(t_n)x_n - \overline{x}\| \geq \varepsilon \text{ for all } n \geq 0.$$

5.8 Remarks and Notes

For densely defined Cauchy problems we refer to Segal [313], Weissler [372], Martin [258], Pazy [281], Cazenave and Haraux [58], Hirsch and Smith [189]. When A is a Hille-Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [85] investigated the existence of several types of solutions for the semilinear Cauchy problem. Thieme [328] investigated the semilinear Cauchy problem with a Lipschitz perturbation of the closed linear operator A which is non-densely defined but is a Hille-Yosida operator. Integrated semigroup theory was used to obtain a variation of constants formula which allows to transform the integrated solutions of the evolution equation to solutions of an abstract semilinear Volterra integral equation, which in turn was used to find integrated solutions to the Cauchy problem. Moreover, sufficient and necessary conditions for the invariance of closed convex sets under the solution flow were found. Conditions for the regularity of the solution flow in time and initial state were derived. The steady states of the solution flow were characterized and sufficient conditions for local stability and instability were given. See also Thieme [329, 335]. This chapter is taken from Magal and Ruan [245, 247] which generalized the results of Thieme [328, 329, 335] to non-densely defined semilinear Cauchy problems where the linear operator is not a Hille-Yosida operator.

We also refer to Friedman [146], Pazy [281], Henry [183], and Lunardi [240] for more results about Cauchy problems for abstract parabolic equations and to Barbu [38], Goldstein [150], Webb [362], and Pavel [282] for a nonlinear semigroup approach.

Chapter 6
Center Manifolds, Hopf Bifurcation and Normal Forms

The purpose of this chapter is to develop the center manifold theory, Hopf bifurcation theorem, and normal form theory for abstract semilinear Cauchy problems with nondense domain.

6.1 Center Manifold Theory

In this section, we investigate the existence and smoothness of the center manifold for a nonlinear semiflow $\{U(t)\}_{t\geq 0}$ on X_0, generated by integrated solutions of the semilinear Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(u(t)),\ t \geq 0;\ \ u(0) = x \in X_0, \tag{6.1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator satisfying Assumptions 3.4.1 and 3.5.2, and $F : X_0 \to X$ is Lipschitz continuous. So $t \to U(t)x$ is a solution of

$$U(t)x = x + A\int_0^t U(s)xds + \int_0^t F(U(s)x)ds,\ \forall t \geq 0, \tag{6.1.2}$$

or equivalently

$$U(t)x = T_{A_0}(t)x + (S_A \diamond F(U(.)x))(t),\ \forall t \geq 0. \tag{6.1.3}$$

We know that for each $x \in X_0$, (6.1.2) has a unique integrated solution $t \to U(t)x$ from $[0,+\infty)$ into X_0. Moreover, the family $\{U(t)\}_{t\geq 0}$ defines a continuous semiflow; that is,

(i) $U(0) = I$ and $U(t)U(s) = U(t+s), \forall t,s \geq 0$;
(ii) The map $(t,x) \to U(t)x$ is continuous from $[0,+\infty) \times X_0$ into X_0.

Furthermore, there exists $\gamma > 0$ such that

$$\|U(t)x - U(t)y\| \leq Me^{\gamma t} \|x - y\|, \ \forall t \geq 0, \forall x, y \in X_0.$$

Assume that $\bar{x} \in X_0$ is an equilibrium of $\{U(t)\}_{t\geq 0}$ (i.e. $U(t)\bar{x} = \bar{x}, \forall t \geq 0$, or equivalently $\bar{x} \in D(A)$ and $A\bar{x} + F(\bar{x}) = 0$). Then by using (6.1.2) and by replacing $U(t)x$ by $V(t)x = U(t)x - \bar{x}$, and $F(x)$ by $F(x+\bar{x}) - F(\bar{x})$, without loss of generality we can assume that $\bar{x} = 0$. Moreover, assume that F is differentiable at 0 and denote by $DF(0)$ its differential at 0. Then by using Theorem 3.5.1 and by replacing A by $A + DF(0)$, and F by $F - DF(0)$, without loss of generality we can also assume that $DF(0) = 0$. So in the following, we assume that the space X_0 can be decomposed into X_{0s}, X_{0c}, and X_{0u}, the stable, center, and unstable linear manifold, respectively, corresponding to the spectral decomposition of A_0.

Assumption 6.1.1. Assume that Assumption 3.4.1 and 3.5.2 are satisfied and there exist two bounded linear projectors with finite rank, $\Pi_{0c} \in \mathscr{L}(X_0) \setminus \{0\}$ and $\Pi_{0u} \in \mathscr{L}(X_0)$, such that

$$\Pi_{0c}\Pi_{0u} = \Pi_{0u}\Pi_{0c} = 0$$

and

$$\Pi_{0k}T_{A_0}(t) = T_{A_0}(t)\Pi_{0k}, \ \forall t \geq 0, \ \forall k = \{c, u\}.$$

Assume in addition that

(a) If $\Pi_{0u} \neq 0$, then $\omega_0\left(-A_0 \mid_{\Pi_{0u}(X_0)}\right) < 0$;
(b) $\sigma\left(A_0 \mid_{\Pi_{0c}(X_0)}\right) \subset i\mathbb{R}$;
(c) If $\Pi_{0s} := I - (\Pi_{0c} + \Pi_{0u}) \neq 0$, then $\omega_0\left(A_0 \mid_{\Pi_{0s}(X_0)}\right) < 0$.

Remark 6.1.2. By Theorem 4.5.8, Assumption 6.1.1 is satisfied if and only if

(a) $\omega_{0,\mathrm{ess}}(A_0) < 0$;
(b) $\sigma(A_0) \cap i\mathbb{R} \neq \emptyset$.

For each $k = \{c, u\}$, denote by $\Pi_k : X \to X$ the unique extension of Π_{0k} satisfying (i)-(iii) in Proposition 4.5.5. Denote

$$\Pi_s = I - (\Pi_c + \Pi_u) \text{ and } \Pi_h = I - \Pi_c.$$

Then we have for each $k \in \{c, h, s, u\}$ that

$$\Pi_k (\lambda I - A)^{-1} = (\lambda I - A)^{-1} \Pi_k, \forall \lambda > \omega,$$

$$\Pi_k (X_0) \subset X_0,$$

and for each $k \in \{c, u\}$ that

$$\Pi_k (X) \subset X_0.$$

For each $k \in \{c, h, s, u\}$, set

$$X_{0k} = \Pi_k (X_0), \ X_k = \Pi_k (X), \ A_k = A \mid_{X_k}, \text{ and } A_{0k} = A_0 \mid_{X_{0k}}.$$

So for each $k \in \{c, u\}$,

$$X_k = X_{0k}.$$

Thus, by using Lemma 4.5.6(i) and (4.5.1) we have for each $k \in \{c,h,s,u\}$ that

$$(A_k)_{\overline{D(A_k)}} = A_0 \mid_{X_{0k}} \text{ and } X_{0k} = X_k \cap X_0.$$

In other words, A_{0k} is the part of A_k in $X_{0k} = \overline{D(A_k)}$. Moreover, we have

$$X = X_s \oplus X_c \oplus X_u \text{ and } X_h = X_s \oplus X_u.$$

Lemma 6.1.3. *Fix $\beta \in (0, \min(-\omega_0(A_{0s}), -\omega_0(-A_{0u})))$. Then we have*

$$\left\| T_{A_{0s}}(t) \right\|_{\mathscr{L}(X_{0s})} \leq M_s e^{-\beta t},\ \forall t \geq 0, \tag{6.1.4}$$

$$\left\| e^{-A_{0u}t} \right\|_{\mathscr{L}(X_{0u})} \leq M_u e^{-\beta t},\ \forall t \geq 0 \tag{6.1.5}$$

with

$$\begin{aligned} M_s &= \sup_{t\geq 0} \left\| T_{A_{0s}}(t) \right\|_{\mathscr{L}(X_{0s})} e^{\beta t} < +\infty, \\ M_u &= \sup_{t\geq 0} \left\| e^{-A_{0u}t} \right\|_{\mathscr{L}(X_{0u})} e^{\beta t} < +\infty. \end{aligned}$$

Moreover, for each $\eta \in (0, \beta)$, we have

$$\left\| e^{A_{0c}t} \right\|_{\mathscr{L}(X_{0c})} \leq e^{\eta |t|} M_{c,\eta},\ \ \forall t \in \mathbb{R}, \tag{6.1.6}$$

with

$$M_{c,\eta} = \sup_{t\in\mathbb{R}} \left\| e^{A_{0c}t} \right\|_{\mathscr{L}(X_{0c})} e^{-\eta |t|} < +\infty.$$

Let $(Y, \|.\|_Y)$ be a Banach space. Let $\eta \in \mathbb{R}$ be a constant and $I \subset \mathbb{R}$ be an interval. Define

$$BC^\eta(I,Y) = \left\{ f \in C(I,Y) : \sup_{t\in I} e^{-\eta|t|} \|f(t)\|_Y < +\infty \right\}.$$

It is well known that $BC^\eta(I,Y)$ is a Banach space when it is endowed with the norm

$$\|f\|_{BC^\eta(I,Y)} = \sup_{t\in I} e^{-\eta|t|} \|f(t)\|_Y.$$

Moreover, the family $\{(BC^\eta(I,Y), \|.\|_{BC^\eta(I,Y)})\}_{\eta>0}$ forms a *scale of Banach spaces*; that is, if $0 < \zeta < \eta$ then $BC^\zeta(I,Y) \subset BC^\eta(I,Y)$ and the embedding is continuous. More precisely, we have

$$\|f\|_{BC^\eta(I,Y)} \leq \|f\|_{BC^\zeta(I,Y)},\ \ \forall f \in BC^\zeta(I,Y).$$

Let $(Z, \|.\|_Z)$ be a Banach spaces. From now on, we denote by Lip(Y,Z) (resp. Lip$_B(Y,Z)$) the space of Lipschitz (resp. Lipschitz and bounded) maps from Y into

Z. Set

$$\|F\|_{\mathrm{Lip}(Y,Z)} := \sup_{x,y\in Y:x\neq y} \frac{\|F(x)-F(y)\|_Z}{\|x-y\|_Y}.$$

We shall study the existence and smoothness of center manifolds in the following two sections.

6.1.1 Existence of Center Manifolds

In this subsection, we investigate the existence of center manifolds. From now on we fix $\beta \in (0, \min(-\omega_0(A_{0s}), -\omega_0(-A_{0u})))$. Recall that $u \in C(\mathbb{R}, X_0)$ is a *complete orbit* of $\{U(t)\}_{t\geq 0}$ if

$$u(t) = U(t-s)u(s), \ \forall t,s \in \mathbb{R} \text{ with } t \geq s, \tag{6.1.7}$$

where $\{U(t)\}_{t\geq 0}$ is a continuous semiflow generated by (6.1.2).

Note that equation (6.1.7) is also equivalent to

$$u(t) = u(s) + A\int_0^{t-s} u(s+r)dr + \int_0^{t-s} F(u(s+r))\,dr$$

for all $t,s \in \mathbb{R}$ with $t \geq s$, or to

$$u(t) = T_{A_0}(t-s)u(s) + (S_A \diamond F(u(s+.)))(t-s) \tag{6.1.8}$$

for each $t,s \in \mathbb{R}$ with $t \geq s$.

Definition 6.1.4. Let $\eta \in (0,\beta)$. The η*-center manifold* of (6.1.1), denoted by V_η, is the set of all points $x \in X_0$, so that there exists $u \in BC^\eta(\mathbb{R}, X_0)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0) = x$.

Let $u \in BC^\eta(\mathbb{R}, X_0)$ be given. For all $\tau \in \mathbb{R}$, we have

$$e^{-\eta|\tau|}\|u\|_{BC^\eta(\mathbb{R},X_0)} \leq \|u(.+\tau)\|_{BC^\eta(\mathbb{R},X_0)} \leq e^{\eta|\tau|}\|u\|_{BC^\eta(\mathbb{R},X_0)}.$$

So for each $\eta > 0$, V_η is invariant under the semiflow $\{U(t)\}_{t\geq 0}$; that is,

$$U(t)V_\eta = V_\eta, \ \forall t \geq 0.$$

Moreover, we say that $\{U(t)\}_{t\geq 0}$ is *reduced on* V_η if there exists a map $\Psi^\eta : X_{0c} \to X_{0h}$ such that

$$V_\eta = \mathrm{Graph}(\Psi) = \{x_c + \Psi(x_c) : x_c \in X_{0c}\}.$$

Before proving the main results of this chapter, we need some preliminary lemmas.

Lemma 6.1.5. *Let Assumption 6.1.1 be satisfied. Let $\tau > 0$ be fixed. Then for each $f \in C([0,\tau], X)$ and each $t \in [0,\tau]$, we have*

$$\Pi_{0s}(S_A \diamond f)(t) = (S_A \diamond \Pi_s f)(t) = (S_{A_s} \diamond \Pi_s f)(t), \tag{6.1.9}$$

and for each $k \in \{c,u\}$,

$$\Pi_{0k}(S_A \diamond f)(t) = (S_A \diamond \Pi_k f)(t) = \int_0^t e^{A_{0k}(t-r)} \Pi_k f(r) dr, \ \forall t \in [0,\tau]. \tag{6.1.10}$$

Furthermore, for each $\gamma > -\beta$, *there exists* $\widehat{C}_{s,\gamma} > 0$, *such that for each* $f \in C([0,\tau],X)$ *and each* $t \in [0,\tau]$, *we have*

$$e^{-\gamma t} \|\Pi_{0s}(S_A \diamond f)(t)\| \le \widehat{C}_{s,\gamma} \sup_{s\in[0,t]} e^{-\gamma s} \|f(s)\| ds. \tag{6.1.11}$$

Proof. The first part (i.e. equations (6.1.9) and (6.1.10)) of the lemma is a consequence of Proposition 4.5.7. Moreover, applying Proposition 3.5.3 to $(S_{A_s} \diamond \Pi_s f)(t)$ and using (6.1.4), we obtain (6.1.11). □

Lemma 6.1.6. *Let Assumption 6.1.1 be satisfied. Then we have the following:*

(i) *For each* $\eta \in [0,\beta)$, *each* $f \in BC^\eta(\mathbb{R},X)$, *and each* $t \in \mathbb{R}$,

$$K_s(f)(t) := \lim_{r\to-\infty} \Pi_{0s}(S_A \diamond f(r+.))(t-r) \textit{ exists;}$$

(ii) *For each* $\eta \in [0,\beta)$, K_s *is a bounded linear operator from* $BC^\eta(\mathbb{R},X)$ *into* $BC^\eta(\mathbb{R},X_{0s})$. *More precisely, for each* $\nu \in (-\beta,0)$, *we have*

$$\|K_s\|_{\mathscr{L}(BC^\eta(\mathbb{R},X),BC^\eta(\mathbb{R},X_{0s}))} \le \widehat{C}_{s,\nu}, \forall \eta \in [0,-\nu],$$

where $\widehat{C}_{s,\nu} > 0$ *is the constant introduced in (6.1.11);*

(iii) *For each* $\eta \in [0,\beta)$, *each* $f \in BC^\eta(\mathbb{R},X)$, *and each* $t,s \in \mathbb{R}$ *with* $t \ge s$,

$$K_s(f)(t) - T_{A_{0s}}(t-s)K_s(f)(s) = \Pi_{0s}(S_A \diamond f(s+.))(t-s).$$

Proof. (i) and (iii) Let $\eta \in (0,\beta)$ be fixed. By using (3.4.12), we have for each $t,s,r \in \mathbb{R}$ with $r \le s \le t$, and each $f \in BC^\eta(\mathbb{R},X)$ that

$$(S_A \diamond f(r+.))(t-r) = T_{A_0}(t-s)(S_A \diamond f(r+.))(s-r) + (S_A \diamond f(s+.))(t-s).$$

By projecting this equation on X_{0s}, we obtain for all $t,s,r \in \mathbb{R}$ with $r \le s \le t$ that

$$\begin{aligned}\Pi_{0s}(S_A \diamond f(r+.))(t-r) &= T_{A_{0s}}(t-s)\Pi_{0s}(S_A \diamond f(r+.))(s-r) \\ &\quad + \Pi_{0s}(S_A \diamond f(s+.))(t-s).\end{aligned} \tag{6.1.12}$$

Let $\nu \in (-\beta,-\eta)$ be fixed. Then by using (6.1.4) and (6.1.11), we have for all $t,s,r \in \mathbb{R}$ with $r \le s \le t$ that

$$\begin{aligned}&\|\Pi_{0s}(S_A \diamond f(r+.))(t-r) - \Pi_{0s}(S_A \diamond f(s+.))(t-s)\| \\ &\quad = \left\|T_{A_{0s}}(t-s)\Pi_{0s}(S_A \diamond f(r+.))(s-r)\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq M_s e^{-\beta(t-s)} \widehat{C}_{s,\nu} e^{\nu(s-r)} \sup_{l\in[0,s-r]} e^{-\nu l} \|f(r+l)\| \\
&= M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu(s-r)} \sup_{\sigma\in[r,s]} e^{-\nu(\sigma-r)} \|f(\sigma)\| \\
&= M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu s} \sup_{l\in[r,s]} e^{-\nu\sigma} e^{\eta|\sigma|} e^{-\eta|\sigma|} \|f(\sigma)\| \\
&\leq \|f\|_{BC^{\eta}(\mathbb{R},X)} M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu s} \sup_{\sigma\in[r,s]} e^{-\nu\sigma} e^{\eta|\sigma|}.
\end{aligned}
$$

Hence, for all $s, r \in \mathbb{R}_-$ with $s \geq r$, we obtain

$$
\begin{aligned}
&\|\Pi_{0s}(S_A \diamond f(r+.))(t-r) - \Pi_{0s}(S_A \diamond f(s+.))(t-s)\| \\
&\quad \leq \|f\|_{BC^{\eta}(\mathbb{R},X)} M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu s} \sup_{\sigma\in[r,s]} e^{-(\nu+\eta)\sigma}.
\end{aligned}
$$

Because $-(\nu+\eta) > 0$, we have

$$
\begin{aligned}
&\|\Pi_{0s}(S_A \diamond f(r+.))(t-r) - \Pi_{0s}(S_A \diamond f(s+.))(t-s)\| \\
&\quad \leq \|f\|_{BC^{\eta}(\mathbb{R},X)} M_s \widehat{C}_{s,\nu} e^{-\beta(t-s)} e^{\nu s} e^{-(\nu+\eta)s} \\
&\quad = \|f\|_{BC^{\eta}(\mathbb{R},X)} M_s \widehat{C}_{s,\nu} e^{-\beta t} e^{(\beta-\eta)s}.
\end{aligned}
$$

Since $\beta - \eta > 0$, by using Cauchy sequences, we deduce that

$$
K_s(f)(t) = \lim_{s\to-\infty} \Pi_{0s}(S_A \diamond f(s+.))(t-s) \text{ exists.}
$$

Taking the limit as r goes to $-\infty$ in (6.1.12), we obtain (iii).

(ii) Let $\nu \in (-\beta, 0)$ and $\eta \in [0, -\nu]$ be fixed. For each $f \in BC^{\eta}(\mathbb{R}, X)$ and each $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\|K_s(f)(t)\| &= \lim_{r\to-\infty} \|\Pi_{0s}(S_A \diamond f(r+.))(t-r)\| \\
&\leq \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu(t-r)} \sup_{l\in[0,t-r]} e^{-\nu l} \|f(r+l)\| \\
&= \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu(t-r)} \sup_{\sigma\in[r,t]} e^{-\nu(\sigma-r)} \|f(\sigma)\| \\
&= \widehat{C}_{s,\nu} \limsup_{r\to-\infty} e^{\nu t} \sup_{\sigma\in[r,t]} e^{-\nu\sigma} e^{\eta|\sigma|} e^{-\eta|\sigma|} \|f(\sigma)\| \\
&= \widehat{C}_{s,\nu} e^{\nu t} \|f\|_{\eta} \sup_{\sigma\in(-\infty,t]} e^{-\nu\sigma} e^{\eta|\sigma|}.
\end{aligned}
$$

Since $\nu + \eta \leq 0$, we deduce that if $t \leq 0$,

$$
\begin{aligned}
e^{-\eta|t|} \|K_s(f)(t)\| &\leq \widehat{C}_{s,\nu} e^{(\nu+\eta)t} \|f\|_{\eta} \sup_{\sigma\in(-\infty,t]} e^{-(\nu+\eta)\sigma} \\
&= \widehat{C}_{s,\nu} e^{(\nu+\eta)t} \|f\|_{\eta} e^{-(\nu+\eta)t}
\end{aligned}
$$

$$= \widehat{C}_{s,\nu} \|f\|_{\eta}$$

and since $\eta - \nu > 0$, it follows that if $t \geq 0$,

$$\begin{aligned} e^{-\eta|t|} \|K_s(f)(t)\| &\leq \widehat{C}_{s,\nu} e^{(\nu-\eta)t} \|f\|_{\eta} \sup_{\sigma \in (-\infty,t]} e^{-\nu\sigma} e^{\eta|\sigma|} \\ &\leq \widehat{C}_{s,\nu} \|f\|_{\eta} e^{(\nu-\eta)t} \max\Big(\sup_{\sigma \in (-\infty,0]} e^{-(\nu+\eta)\sigma}, \sup_{\sigma \in [0,t]} e^{(\eta-\nu)\sigma} \Big) \\ &= \widehat{C}_{s,\nu} \|f\|_{\eta} e^{(\nu-\eta)t} e^{(\eta-\nu)t} = \widehat{C}_{s,\nu} \|f\|_{\eta}. \end{aligned}$$

This completes the proof. □

Lemma 6.1.7. *Let Assumption 6.1.1 be satisfied. Let $\eta \in [0, \beta)$ be fixed. Then we have the following:*

(i) For each $f \in BC^{\eta}(\mathbb{R}, X)$ and each $t \in \mathbb{R}$,

$$K_u(f)(t) := -\int_t^{+\infty} e^{-A_{0u}(l-t)} \Pi_u f(l) dl := -\lim_{r \to +\infty} \int_t^r e^{-A_{0u}(l-t)} \Pi_u f(l) dl$$

exists;

(ii) K_u is a bounded linear operator from $BC^{\eta}(\mathbb{R}, X)$ into $BC^{\eta}(\mathbb{R}, X_{0u})$ and

$$\|K_u\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))} \leq \frac{M_u \|\Pi_u\|_{\mathscr{L}(X)}}{\beta - \eta};$$

(iii) For each $f \in BC^{\eta}(\mathbb{R}, X)$ and each $t, s \in \mathbb{R}$ with $t \geq s$,

$$K_u(f)(t) - e^{A_{0u}(t-s)} K_u(f)(s) = \Pi_{0u} (S_A \diamond f(s+.)) (t-s).$$

Proof. By using (6.1.5) and the same argument as in the proof of Lemma 6.1.6, we obtain (i) and (ii). Moreover, for each $s, t, r \in \mathbb{R}$ with $s \leq t \leq r$, we have

$$\begin{aligned} \int_s^r e^{A_{0u}(s-l)} \Pi_u f(l) dl &= \int_s^t e^{A_{0u}(s-l)} \Pi_u f(l) dl + \int_t^r e^{A_{0u}(s-l)} \Pi_u f(l) dl \\ &= \int_s^t e^{A_{0u}(s-l)} \Pi_u f(l) dl + e^{A_{0u}(s-t)} \int_t^r e^{A_{0u}(t-l)} \Pi_u f(l) dl. \end{aligned}$$

It follows that

$$e^{A_{0u}(t-s)} \int_s^r e^{A_{0u}(s-l)} \Pi_u f(l) dl = \int_s^t e^{A_{0u}(t-l)} \Pi_u f(l) dl + \int_t^r e^{A_{0u}(t-l)} \Pi_u f(l) dl.$$

When $r \to +\infty$, we obtain for all $s, t \in \mathbb{R}$ with $s \leq t$ that

$$\begin{aligned} -e^{A_{0u}(t-s)} K_{u,\eta}(f)(s) &= \int_0^{t-s} e^{A_{0u}(t-s-r)} \Pi_u f(s+r) dr - K_{u,\eta}(f)(t) \\ &= \Pi_u (S_A \diamond f(s+.)) (t-s) - K_{u,\eta}(f)(t). \end{aligned}$$

This gives (iii). □

Lemma 6.1.8. *Let Assumption 6.1.1 be satisfied. Let $\eta \in (0,\beta)$ be fixed. For each $x_c \in X_{0c}$, each $f \in BC^{\eta}(\mathbb{R},X)$, and each $t \in \mathbb{R}$, denote*

$$K_1(x_c)(t) := e^{A_{0c}t}x_c, \quad K_c(f)(t) := \int_0^t e^{A_{0c}(t-s)}\Pi_c f(s)ds.$$

Then K_1 (respectively K_c) is a bounded linear operator from X_{0c} into $BC^{\eta}(\mathbb{R},X_{0c})$ (respectively from $BC^{\eta}(\mathbb{R},X)$ into $BC^{\eta}(\mathbb{R},X_{0c})$, and

$$\|K_1\|_{\mathscr{L}(X_c,BC^{\eta}(\mathbb{R},X))} \leq \max\left(\sup_{t\geq 0}\left\|e^{(A_c-\eta I)t}\right\|, \sup_{t\geq 0}\left\|e^{-(A_c+\eta I)t}\right\|\right),$$
$$\|K_c\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))} \leq \|\Pi_c\|_{\mathscr{L}(X)}\max\left(\int_0^{+\infty}\left\|e^{(A_c-\eta I)l}\right\|dl, \int_0^{+\infty}\left\|e^{-(A_c+\eta I)l}\right\|dl\right).$$

Proof. The proof is straightforward. □

Lemma 6.1.9. *Let Assumption 6.1.1 be satisfied. Let $\eta \in (0,\beta)$ and $u \in BC^{\eta}(\mathbb{R},X_0)$ be fixed. Then u is a complete orbit of $\{U(t)\}_{t\geq 0}$ if and only if for each $t \in \mathbb{R}$,*

$$\begin{aligned}u(t) = {} & K_1(\Pi_{0c}u(0))(t) + K_c(F(u(.)))(t)\\ & + K_u(F(u(.)))(t) + K_s(F(u(.)))(t).\end{aligned} \tag{6.1.13}$$

Proof. Let $u \in BC^{\eta}(\mathbb{R},X_0)$ be fixed. Assume first that u is a complete orbit of $\{U(t)\}_{t\geq 0}$. Then for $k \in \{c,u\}$ we have for all $l,r \in \mathbb{R}$ with $r \leq l$ that

$$\Pi_{0k}u(l) = e^{A_{0k}(l-r)}\Pi_{0k}u(r) + \int_r^l e^{A_{0k}(l-s)}\Pi_k F(u(s))ds.$$

Thus,

$$\frac{d\Pi_{0k}u(t)}{dt} = A_{0k}\Pi_{0k}u(t) + \Pi_k F(u(t)), \ \forall t \in \mathbb{R}.$$

From this ordinary differential equation, we first deduce that

$$\Pi_{0c}u(t) = e^{A_{0c}t}\Pi_{0c}u(0) + \int_0^t e^{A_{0c}(t-s)}\Pi_c F(u(s))ds, \ \forall t \in \mathbb{R}. \tag{6.1.14}$$

Hence, for each $t,l \in \mathbb{R}$,

$$\Pi_{0u}u(t) = e^{A_{0u}(t-l)}\Pi_{0u}u(l) + \int_l^t e^{A_{0u}(t-s)}\Pi_u F(u(s))ds, \ \forall t,l \in \mathbb{R}.$$

It follows that for each $l \geq 0$,

$$\left\|e^{-A_{0u}(l-t)}\Pi_{0u}u(l)\right\| \leq M_u\|\Pi_u\|_{\mathscr{L}(X)}e^{-\beta(l-t)}e^{\eta l}\|u\|_{BC^{\eta}(\mathbb{R},X_0)}.$$

So when l goes to $+\infty$, we obtain

$$\Pi_{0u}u(t) = -\int_t^{+\infty} e^{A_{0u}(t-s)}\Pi_u F(u(s))ds, \ \forall t \in \mathbb{R}. \tag{6.1.15}$$

Furthermore, we have for all $t, l \in \mathbb{R}$ with $t \geq l$ that

$$\Pi_{0s}u(t) = T_{A_{0s}}(t-l)\Pi_{0s}u(l) + \Pi_{0s}\left(S_A \diamond F(u(l+.))\right)(t-l)$$

and for each $l \leq 0$ that

$$\left\| T_{A_{0s}}(t-l)\Pi_{0s}u(l) \right\| \leq e^{-\beta t} M_s \|u\|_\eta \, e^{(\beta-\eta)l}.$$

Taking $l \to -\infty$, we obtain

$$\Pi_{0s}u(t) = K_{s,\eta}\left(F(u(.))\right)(t), \ \forall t \in \mathbb{R}. \tag{6.1.16}$$

Finally, summing up (6.1.14), (6.1.15), and (6.1.16), we obtain (6.1.13).

Conversely, assume that u is a solution of (6.1.13). Then

$$\Pi_{0c}u(t) = e^{A_{0c}t}\Pi_{0c}u(0) + \int_0^t e^{A_{0c}(t-s)}\Pi_c F(u(s))ds, \ \forall t \in \mathbb{R}.$$

It follows that

$$\frac{d\Pi_{0c}u(t)}{dt} = A_{0c}\Pi_{0c}u(t) + \Pi_c F(u(t)), \ \forall t \in \mathbb{R}.$$

Thus, for $l, r \in \mathbb{R}_-$ with $r \leq l$,

$$\Pi_{0c}u(l) = T_{A_0}(t-s)\Pi_{0c}u(r) + \Pi_{0c}\left(S_A \diamond F(u(s+.))\right)(t-s).$$

Moreover, using Lemma 6.1.6 (iii) and Lemma 6.1.7 (iii), we deduce that for all $t, s \in \mathbb{R}$ with $t \geq s$

$$\begin{aligned}
\Pi_{0s}u(t) - T_{A_0}(t-s)\Pi_{0s}u(s) &= \Pi_{0s}\left(S_A \diamond F(u(s+.))\right)(t-s), \\
\Pi_{0u}u(t) - T_{A_0}(t-s)\Pi_{0u}u(s) &= \Pi_{0u}\left(S_A \diamond F(u(s+.))\right)(t-s).
\end{aligned}$$

Therefore, u satisfies (6.1.8) and is a complete orbit of $\{U(t)\}_{t\geq 0}$. $\square$

Let $\eta \in (0, \beta)$ be fixed. We rewrite equation (6.1.13) as the following fixed point problem: To find $u \in BC^\eta(\mathbb{R}, X)$ such that

$$u = K_1(\Pi_{0c}u(0)) + K_2\Phi_F(u), \tag{6.1.17}$$

where the nonlinear operator $\Phi_F \in \mathrm{Lip}\left(BC^\eta(\mathbb{R}, X_0), BC^\eta(\mathbb{R}, X)\right)$ is defined by

$$\Phi_F(u)(t) = F(u(t)), \ \forall t \in \mathbb{R}$$

and $K_2 \in \mathscr{L}\left(BC^\eta(\mathbb{R}, X), BC^\eta(\mathbb{R}, X_0)\right)$ is the linear operator defined by

$$K_2 = K_c + K_s + K_u.$$

Moreover, we have the following estimates

$$\|K_1\|_{\mathscr{L}(X_c,BC^{\eta}(\mathbb{R},X))} \leq \max(\sup_{t\geq 0}\left\|e^{(A_c-\eta Id)t}\right\|, \sup_{t\geq 0}\left\|e^{-(A_c+\eta Id)t}\right\|),$$

$$\|\Phi_F\|_{\mathrm{Lip}} \leq \|F\|_{\mathrm{Lip}},$$

and for each $\nu \in (-\beta, 0)$, we have

$$\|K_2\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))} \leq \gamma(\nu,\eta), \ \forall \eta \in (0,-\nu],$$

where

$$\begin{aligned}\gamma(\nu,\eta) := {} & \widehat{C}_{s,\nu} + \frac{M_u \|\Pi_u\|_{\mathscr{L}(X)}}{(\beta-\eta)} \\ & + \|\Pi_c\|_{\mathscr{L}(X)} \max\left(\int_0^{+\infty}\left\|e^{(A_c-\eta Id)l}\right\| dl, \int_0^{+\infty}\left\|e^{-(A_c+\eta)l}\right\| dl\right).\end{aligned} \tag{6.1.18}$$

Furthermore, by Lemma 6.1.9, the η-center manifold is given by

$$V_\eta = \{x \in X_0 : \exists u \in BC^{\eta}(\mathbb{R},X_0) \text{ a solution of (6.1.17) and } u(0) = x\}. \tag{6.1.19}$$

We are now in the position to prove the existence of center manifolds for semilinear equations with non-dense domain, which is a generalization of Vanderbauwhede and Iooss [345, Theorem 1, p.129].

Theorem 6.1.10. *Let Assumption 6.1.1 be satisfied. Let $\eta \in (0,\beta)$ be fixed and let $\delta_0 = \delta_0(\eta) > 0$ be such that*

$$\delta_0 \|K_2\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))} < 1.$$

Then for each $F \in \mathrm{Lip}(X_0,X)$ with $\|F\|_{\mathrm{Lip}(X_0,X)} \leq \delta_0$, there exists a Lipschitz continuous map $\Psi : X_{0c} \to X_{0h}$ such that

$$V_\eta = \{x_c + \Psi(x_c) : x_c \in X_{0c}\}.$$

Moreover, we have the following properties:

(i) $\sup_{x_c\in X_c}\|\Psi(x_c)\| \leq \|K_s + K_u\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))} \sup_{x\in X_0}\|\Pi_h F(x)\|$;

(ii) *We have*

$$\|\Psi\|_{\mathrm{Lip}(X_{0c},X_{0h})} \leq \frac{\|K_s+K_u\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))}\|F\|_{\mathrm{Lip}(X_0,X)}\|K_1\|_{\mathscr{L}(X_c,BC^{\eta}(\mathbb{R},X_0))}}{1-\|K_2\|_{\mathscr{L}(BC^{\eta}(\mathbb{R},X))}\|F\|_{\mathrm{Lip}(X_0,X)}}; \tag{6.1.20}$$

(iii) *For each $u \in C(\mathbb{R},X_0)$, the following statements are equivalent:*

(1) $u \in BC^{\eta}(\mathbb{R},X_0)$ *is a complete orbit of* $\{U(t)\}_{t\geq 0}$;

(2) $\Pi_{0h}u(t) = \Psi(\Pi_{0c}u(t)), \forall t \in \mathbb{R}$, *and* $\Pi_{0c}u(.) : \mathbb{R} \to X_{0c}$ *is a solution of the ordinary differential equation*

$$\frac{dx_c(t)}{dt} = A_{0c}x_c(t) + \Pi_c F\left[x_c(t) + \Psi\left(x_c(t)\right)\right]. \tag{6.1.21}$$

Proof. (i) Since $\|F\|_{\text{Lip}} \|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R},X))} < 1$, the map $(Id - K_2\Phi_F)$ is invertible, $(Id - K_2\Phi_F)^{-1}$ is Lipschitz continuous, and

$$\left\|(Id - K_2\Phi_F)^{-1}\right\|_{\text{Lip}(BC^\eta(\mathbb{R},X_0))} \leq \frac{1}{1-\|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R},X))}\|F\|_{\text{Lip}(X_0,X)}}. \tag{6.1.22}$$

Let $x \in X_0$ be fixed. By Lemma 6.1.9, we know that $x \in V_\eta$ if and only if there exists $u_{\Pi_{0c}x} \in BC^\eta(\mathbb{R},X)$, such that $u_{\Pi_{0c}x}(0) = x$ and

$$u_{\Pi_{0c}x} = K_1(\Pi_{0c}x) + K_2\Phi_F\left(u_{\Pi_{0c}x}\right).$$

So

$$V_\eta = \left\{(Id - K_2\Phi_F)^{-1}K_1(x_c)(0) : x_c \in X_{0c}\right\}.$$

We define $\Psi : X_{0c} \to X_{0h}$ by

$$\Psi(x_c) = \Pi_{0h}(Id - K_2\Phi_F)^{-1}K_1(x_c)(0),\ \forall x_c \in X_{0c}. \tag{6.1.23}$$

Then

$$V_\eta = \{x_c + \Psi(x_c) : x_c \in X_{0c}\}.$$

For each $x_c \in X_{0c}$, set

$$u_{x_c} = (Id - K_2\Phi_F)^{-1}K_1(x_c).$$

We have

$$u_{x_c} = K_1(x_c) + K_2\Phi_F\left(u_{x_c}\right).$$

By projecting on X_{0h}, we obtain

$$\Pi_{0h}u_{x_c} = [K_s + K_u]\,\Phi_F\left(u_{x_c}\right),$$

so

$$\Psi(x_c) = [K_s + K_u]\,\Phi_F\left(u_{x_c}\right)(0) \tag{6.1.24}$$

and (i) follows.

(ii) It follows from (6.1.22) and (6.1.24).

(iii) Assume first that $u \in BC^\eta(\mathbb{R},X_0)$ is a complete orbit of $\{U(t)\}_{t\geq 0}$. Then by the definition of V_η, we have $u(t) \in V_\eta, \forall t \in \mathbb{R}$. Hence,

$$\Pi_{0h}u(t) = \Psi(\Pi_{0c}u(t)),\ \ \forall t \in \mathbb{R}.$$

Moreover, by projecting (6.1.8) on X_{0c}, we have for each $t, s \in \mathbb{R}$ with $t \geq s$ that

$$\Pi_{0c}u(t) = e^{A_{0c}(t-s)}\Pi_{0c}u(s) + \int_0^{t-s} e^{A_{0c}(t-s-l)}\Pi_c F\left(u(s+l)\right)dl.$$

Thus, $t \to \Pi_{0c}u(t)$ is a solution of (6.1.21).

Conversely assume that $u \in C(\mathbb{R},X_0)$ satisfies (iii)(2). Then

$$\Pi_{0h}u(t) = \Psi(\Pi_{0c}u(t)), \ \forall t \in \mathbb{R},$$

and $\Pi_{0c}u(.) : \mathbb{R} \to X_{0c}$ is a solution of (6.1.21). Set $x = u(0)$. We know that $x \in V_\eta$, and by the definition of V_η, there exists $v \in BC^\eta(\mathbb{R}, X_0)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $v(0) = x$. But since V_η is invariant under the semiflow, we deduce that

$$\Pi_{0h}v(t) = \Psi(\Pi_{0c}v(t)), \ \forall t \in \mathbb{R},$$

and $\Pi_{0c}v(.) : \mathbb{R} \to X_{0c}$ is a solution of (6.1.21). Finally, since $\Pi_{0c}v(0) = \Pi_{0c}u(0)$, and since F and Ψ are Lipschitz continuous, we deduce that (6.1.21) has at most one solution. It follows that

$$\Pi_{0c}v(t) = \Pi_{0c}u(t), \ \forall t \in \mathbb{R},$$

and by construction

$$\Pi_{0h}v(t) = \Psi(\Pi_{0c}v(t)) = \Psi(\Pi_{0c}u(t)) = \Pi_{0h}u(t), \ \forall t \in \mathbb{R}.$$

Thus,

$$u(t) = v(t), \ \forall t \in \mathbb{R}.$$

Therefore, $u \in BC^\eta(\mathbb{R}, X_0)$ is a complete orbit of $\{U(t)\}_{t\geq 0}$. $\square$

Proposition 6.1.11. *Let Assumption 6.1.1 be satisfied. Assume in addition that* $F \in \mathrm{Lip}_B(X_0, X)$ *(i.e. F is Lipschitz and bounded). Then*

$$V_\eta = V_\xi, \ \forall \eta, \xi \in (0, \beta).$$

Proof. Let $\eta, \xi \in (0, \beta)$ be given such that $\xi < \eta$. Let $x \in V_\xi$. By the definition of V_ξ there exists $u \in BC^\xi(\mathbb{R}, X_0)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0) = x$. But $BC^\xi(\mathbb{R}, X_0) \subset BC^\eta(\mathbb{R}, X_0)$, so $u \in BC^\eta(\mathbb{R}, X_0)$, and we deduce that $x \in V_\eta$.

Conversely, let $x \in V_\eta$ be fixed. By the definition of V_η there exists $u \in BC^\eta(\mathbb{R}, X_0)$, a complete orbit of $\{U(t)\}_{t\geq 0}$, such that $u(0) = x$. By Lemma 6.1.9 we deduce that u is a solution of

$$u = K_1(\Pi_{0c}u(0)) + K_2\Phi_F(u).$$

But $K_1(\Pi_{0c}u(0)) \in BC^\xi(\mathbb{R}, X_0)$ and F is bounded, so we have $\Phi_F(u) \in BC^0(\mathbb{R}, X_0) \subset BC^\xi(\mathbb{R}, X_0)$ and

$$K_2\Phi_F(u) \in BC^\xi(\mathbb{R}, X_0).$$

Hence, $u \in BC^\xi(\mathbb{R}, X_0)$ and

$$u = K_1(\Pi_{0c}u(0)) + K_2\Phi_F(u).$$

Using Lemma 6.1.9 once more, we obtain that $x \in V_\xi$. $\square$

6.1.2 Smoothness of Center Manifolds

We introduce the following notation. Let $k \geq 1$ be an integer, let $Y_1, Y_2, .., Y_k$, Y and Z be Banach spaces, and let V be an open subset of Y. Denote $\mathscr{L}^{(k)}(Y_1, Y_2, .., Y_k, Z)$ (resp. $\mathscr{L}^{(k)}(Y, Z)$) the space of bounded k-linear maps from $Y_1 \times ... \times Y_k$ into Z (resp. from Y^k into Z). Let $W \in C^k(V, Z)$ be fixed. We choose the convention that if $l = 1, ..., k-1$ and $u, \widehat{u} \in V$ with $u \neq \widehat{u}$, the quantity

$$\sup_{u_1, ..., u_l \in B_Y(0,1)} \frac{\left\| \left[D^l W(u) - D^l W(\widehat{u}) \right](u_1, ..., u_l) - D^{l+1} W(\widehat{u})(u - \widehat{u}, u_1, ..., u_l) \right\|}{\| u - \widehat{u} \|}$$

goes to 0 as $\| u - \widehat{u} \| \to 0$. Set

$$C_b^k(V, Z) := \left\{ W \in C^k(V, Z) : |W|_{j,V} := \sup_{x \in V} \left\| D^j W(x) \right\| < +\infty, \ 0 \leq j \leq k \right\}.$$

For each $\eta \in [0, \beta)$, consider $K_h : BC^\eta(\mathbb{R}, X) \to BC^\eta(\mathbb{R}, X_{0h})$, the bounded linear operator defined by

$$K_h = K_s + K_u,$$

where K_s and K_u are the bounded linear operators defined, respectively, in Lemma 6.1.6 and Lemma 6.1.7. For each $\rho > 0$ and each $\eta \geq 0$, set

$$V_\rho := \{ x \in X_0 : \| \Pi_h x \| < \rho \}, \quad \overline{V}_\rho := \{ x \in X_0 : \| \Pi_h x \| \leq \rho \},$$

and

$$\overline{V}_\rho^\eta := \left\{ u \in BC^\eta(\mathbb{R}, X_0) : u(t) \in \overline{V}_\rho, \forall t \in \mathbb{R} \right\}.$$

Note that since $\overline{V}_\rho$ is a closed and convex subset of X_0, so is $\overline{V}_\rho^\eta$ for each $\eta \geq 0$.

Definition 6.1.12. Let X be a metric space and $H : X \to X$ be a map. A fixed point $\overline{x} \in X$ of H is said to be *attracting* if

$$\lim_{n \to +\infty} H^n(x) = \overline{x} \quad \text{for each } x \in X.$$

The following lemma is an extension of the Fibre Contraction Theorem (Theorem 5.6.3 which corresponds to the case $k = 1$). This result is taken from Vanderbauwhede [343, Corollary 3.6].

Lemma 6.1.13. *Let $k \geq 1$ be an integer and let $(M_0, d_0), (M_1, d_1), ..., (M_k, d_k)$ be complete metric spaces. Let $H : M_0 \times M_1 \times ... \times M_k \to M_0 \times M_1 \times ... \times M_k$ be a mapping of the form*

$$H(x_0, x_1, ..., x_k) = (H_0(x_0), H_1(x_0, x_1), ..., H_k(x_0, x_1, ..., x_k)),$$

where for each $l = 0, ..., k$, $H_l : M_0 \times M_1 \times ... \times M_l \to M_l$ is a uniform contraction; that is, H_0 is a contraction, and for each $l = 1, .., k$, there exists $\tau_l \in [0, 1)$ such that for each $(x_0, x_1, ..., x_{l-1}) \in M_0 \times M_1 \times ... \times M_{l-1}$ and each $x_l, \widehat{x}_l \in M_l$,

$$d_l\left(H_l\left(x_0,x_1,...,x_{l-1},x_l\right),H_l\left(x_0,x_1,...,x_{l-1},\widehat{x}_l\right)\right) \leq \tau_l d\left(x_l,\widehat{x}_l\right).$$

Then H has a unique fixed point $(\overline{x}_0,\overline{x}_1,...,\overline{x}_k)$. If, moreover, for each $l = 1,...,k$,

$$H_l\left(.,\overline{x}_l\right) : M_0 \times M_1 \times ... \times M_{l-1} \to M_l$$

is continuous, then $(\overline{x}_0,\overline{x}_1,...,\overline{x}_k)$ is an attracting fixed point of H.

Proof. We prove the lemma for $k = 1$. The proof for any integer $k \geq 1$ can be easily derived from this case. It is clear that $(\overline{x}_0,\overline{x}_1)$ is the unique fixed point of H, so we only need to prove its attractivity.

Let $(x_0,x_1) \in M_0 \times M_1$. Consider the sequence $(x_0(n),x_1(n))$ defined by

$$(x_0(0),x_1(0)) := (x_0,x_1)$$

and

$$(x_0(n+1),x_1(n+1)) := (H_0(x_0(n)),H_1(x_0(n),x_1(n)))\,,\ \forall n \geq 0.$$

Since H_0 is a contraction, it is clear that $\lim_{n\to+\infty} x_0(n) = \overline{x}_0$. It remains to show that $\lim_{n\to+\infty} x_1(n) = \overline{x}_1$. We observe first that

$$\begin{aligned} d(x_1(n+1),\overline{x}_1) &= d(H_1(x_0(n),x_1(n)),H_1(\overline{x}_0,\overline{x}_1)) \\ &\leq d(H_1(x_0(n),x_1(n)),H_1(x_0(n),\overline{x}_1)) + d(H_1(x_0(n),\overline{x}_1),H_1(x_0(n),\overline{x}_1) \\ &\leq \tau_1 d(x_1(n),\overline{x}_1) + \alpha_n, \end{aligned}$$

where $\alpha_n := d(H_1(x_0(n),\overline{x}_1),H_1(x_0(n),\overline{x}_1) \to 0$ as $n \to +\infty$.

Setting $\delta_n := d(x_1(n),\overline{x}_1)$, we obtain

$$\delta_{n+1} \leq \tau_1 \delta_n + \alpha_n,\ \forall n \geq 0.$$

Since $\tau_1 < 1$, it is not difficult to prove that $\{\delta_n\}$ is bounded sequence and

$$\limsup_{n\to+\infty} \delta_n \leq \tau_1 \limsup_{n\to+\infty} \delta_n.$$

Hence, $\limsup_{n\to+\infty} \delta_n = 0.$ $\square$

We recall that the map $\Psi : X_{0c} \to X_{0h}$ is defined by

$$\Psi\left(x_c\right) = \Pi_h\left(I - K_2\Phi_F\right)^{-1}\left(K_1 x_c\right)(0),\ \ \forall x_c \in X_{0c}.$$

We define the map $\Gamma_0 : BC^\eta\left(\mathbb{R},X_{0c}\right) \to BC^\eta\left(\mathbb{R},X_0\right)$ by

$$\Gamma_0\left(u\right) = \left(I - K_2\Phi_F\right)^{-1}\left(u\right),\ \ \forall u \in BC^\eta\left(\mathbb{R},X_{0c}\right).$$

For each $\delta \geq 0$, the bounded linear operator $L : BC^\delta\left(\mathbb{R},X_0\right) \to X_{0h}$ is defined by

$$L(u) = \Pi_h u(0),\ \ \forall u \in BC^\delta\left(\mathbb{R},X_{0c}\right).$$

Then we have

$$\Psi(x_c) = L\Gamma_0(K_1 x_c), \ \forall x_c \in X_{0c}$$

and

$$\Gamma_0(u) = u + K_2 \Phi_F(\Gamma_0(u)), \ \forall u \in BC^\eta(\mathbb{R}, X_{0c}).$$

So we obtain

$$\Gamma_0 = J + K_2 \circ \Phi_F \circ (\Gamma_0), \tag{6.1.25}$$

where J is the continuous imbedding from $BC^\eta(\mathbb{R}, X_{0c})$ into $BC^\eta(\mathbb{R}, X_0)$.

From (6.1.25), we deduce that for each $u \in BC^\eta(\mathbb{R}, X_{0c})$,

$$\|\Gamma_0(u) - u\|_{BC^\eta(\mathbb{R},X_0)} \leq \|K_2\|_{\mathscr{L}(BC^\eta(\mathbb{R},X), BC^\eta(\mathbb{R},X_0))} |F|_{0,X_0},$$

$$\|\Pi_h \Gamma_0(u)(t)\|_{BC^0(\mathbb{R},X)} \leq \|K_h\|_{\mathscr{L}\left(BC^0(\mathbb{R},X)\right)} \|\Pi_h F\|_{0,X_0} = \rho_0, \ \forall t \in \mathbb{R}.$$

For each subset $E \subset BC^\eta(\mathbb{R}, X_{0c})$, denote

$$M_{0,E} = \left\{ \Theta \in C\left(E, \overline{V}^0_{\rho_0}\right) : \sup_{u \in E} \|\Theta(u) - u\|_{BC^\eta(\mathbb{R},X_0)} < +\infty \right\}$$

and set

$$M_0 = M_{0, BC^\eta(\mathbb{R}, X_{0c})}.$$

From the above remarks, it follows that Γ_0 (respectively $\Gamma_0 \mid_E$) must be an element of M_0 (respectively $M_{0,E}$). Since $\overline{V}^0_{\rho_0}$ is a closed subset of $BC^\eta(\mathbb{R}, X_0)$, we know that for each subset $E \subset BC^\eta(\mathbb{R}, X_{0c})$, $M_{0,E}$ is a complete metric space endowed with the metric

$$d_{0,E}\left(\Theta, \widetilde{\Theta}\right) = \sup_{u \in E} \left\| \Theta(u) - \widetilde{\Theta}(u) \right\|_{BC^\eta(\mathbb{R},X_0)}.$$

Set

$$d_0 = d_{0, BC^\eta(\mathbb{R}, X_{0c})}.$$

Lemma 6.1.14. *Let E be a Banach space and $W \in C^1_b\left(V_\rho, E\right)$. Let $\xi \geq \delta \geq 0$ be fixed. Define $\Phi_W : V^\eta_\rho \to BC^\xi(\mathbb{R}, E)$, $\Phi_{DW} : V^\eta_\rho \to BC^\xi(\mathbb{R}, \mathscr{L}(X_0, E))$, and $\Phi^{(1)}_W : V^\eta_\rho \to \mathscr{L}(BC^\delta(\mathbb{R}, X_0), BC^\xi(\mathbb{R}, E))$ for each $t \in \mathbb{R}$, each $u \in V^\eta_\rho$, and each $v \in BC^\delta(\mathbb{R}, X_0)$ by*

$$\begin{aligned}
&\Phi_W(u)(t) := W(u(t)), \\
&\Phi_{DW}(u)(t) := DW(u(t)), \\
&\left(\Phi^{(1)}_W(u)(v)\right)(t) := DW(u(t))(v(t)),
\end{aligned}$$

respectively. Then we have the following:

(a) If $\xi > 0$, then Φ_W and Φ_{DW} are continuous;

(b) For each $u, v \in V^\eta_\rho$, $\Phi^{(1)}_W(u) \in \mathscr{L}(BC^\delta(\mathbb{R}, X_0), BC^\xi(\mathbb{R}, E))$,

$$\left\| \Phi_W^{(1)}(u) - \Phi_W^{(1)}(v) \right\|_{\mathscr{L}\left(BC^{\delta}(\mathbb{R},X_0),BC^{\xi}(\mathbb{R},E)\right)} \leq \left\| \Phi_{DW}(u) - \Phi_{DW}(v) \right\|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))}$$

and

$$\left\| \Phi_W^{(1)}(u) \right\|_{\mathscr{L}\left(BC^{\delta}(\mathbb{R},X_0),BC^{\xi}(\mathbb{R},E)\right)} \leq \left\| \Phi_{DW}(u) \right\|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))} \leq |W|_{1,V_\rho};$$

(c) *If $\xi > \delta$, then $\Phi_W^{(1)}$ is continuous;*
(d) *If $\xi \geq \delta \geq \eta$, we have for each $u, \widehat{u} \in V_\rho^\eta$ that*

$$\left\| \Phi_W(u) - \Phi_W(\widehat{u}) - \Phi_W^{(1)}(\widehat{u})(u-\widehat{u}) \right\|_{BC^{\xi}(\mathbb{R},E)} \leq \|u-\widehat{u}\|_{BC^{\delta}(\mathbb{R},X_0)} \varkappa_{\xi-\delta}(u,\widehat{u}),$$

where

$$\varkappa_{\xi-\delta}(u,\widehat{u}) = \sup_{s\in[0,1]} \left\| \Phi_{DW}(su+(1-s)\widehat{u}) - \Phi_{DW}(\widehat{u}) \right\|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))},$$

and if $\xi > \delta \geq \eta$, we have (by continuity of Φ_{DW})

$$\varkappa_{\xi-\delta}(u,\widehat{u}) \to 0 \quad as \ \|u-\widehat{u}\|_{BC^{\eta}(\mathbb{R},X_0)} \to 0.$$

Proof. We first prove that $\Phi_W \in C_b^0(V_\rho^\eta, BC^{\xi}(\mathbb{R},E))$. For each $u, \widehat{u} \in V_\rho^\eta$ and each $R > 0$, we have

$$\begin{aligned} \|\Phi_W(u) - \Phi_W(\widehat{u})\|_{BC^{\xi}(\mathbb{R},E)} &= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \|W(u(t)) - W(\widehat{u}(t))\| \\ &= \max\left(\sup_{|t|\leq R} e^{-\xi|t|} \|W(u(t)) - W(\widehat{u}(t))\|, 2\|W\|_0 e^{-\xi R} \right). \end{aligned} \tag{6.1.26}$$

Fix some arbitrary $\varepsilon > 0$. Let $R > 0$ be given such that $2\|W\|_0 e^{-\xi R} < \varepsilon$ and denote $\Omega = \{\widehat{u}(t) : |t| \leq R\}$. Since W is continuous and Ω is compact, we can find $\delta_1 > 0$ such that

$$\|W(x) - W(\widehat{x})\| \leq \varepsilon \text{ if } \widehat{x} \in \Omega, \text{ and } \|x-\widehat{x}\| \leq \delta_1.$$

Let $\delta = e^{-\eta R}\delta_1$. If $\|u-\widehat{u}\|_{BC^{\eta}(\mathbb{R},X_0)} \leq \delta$, then $\|u(t)-\widehat{u}(t)\| \leq \delta_1, \forall t \in [-R,R]$, and (6.1.26) implies $\|\Phi_W(u) - \Phi_W(\widehat{u})\|_{BC^{\xi}(\mathbb{R},E)} \leq \varepsilon$.

We now prove that $\Phi_W^{(1)} \in C(V_\rho^\eta, \mathscr{L}(BC^{\delta}(\mathbb{R},X_0), BC^{\xi}(\mathbb{R},E)))$. From the first part of the proof, since E is an arbitrary Banach space, we deduce that Φ_{DW} is continuous. Moreover, for each $u, \widehat{u} \in V_\rho^\eta$ and each $v \in BC^{\delta}(\mathbb{R},X_0)$,

$$\begin{aligned} \left\| \left(\Phi_W^{(1)}(u)(v)\right) \right\|_{BC^{\xi}(\mathbb{R},E)} &= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \|DW(u(t))(v(t))\| \\ &\leq \|\Phi_{DW}(u)\|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))} \|v\|_{BC^{\delta}(\mathbb{R},X_0)} \end{aligned}$$

and

$$\left\| \left(\left[\Phi_W^{(1)}(u) - \Phi_W^{(1)}(\widehat{u}) \right](v) \right) \right\|_{BC^{\xi}(\mathbb{R},E)}$$
$$\leq \| \Phi_{DW}(u) - \Phi_{DW}(\widehat{u}) \|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))} \|v\|_{BC^{\delta}(\mathbb{R},X_0)}.$$

Thus, if $\xi \geq \delta$, we have for each $u \in V_\rho^\eta$ that

$$\Phi_W^{(1)}(u) \in \mathscr{L}\left(BC^{\delta}(\mathbb{R},X_0), BC^{\xi}(\mathbb{R},E)\right), \ \forall u \in V_\rho^\eta$$

and if $\xi > \delta$,

$$\Phi_W^{(1)} \in C\left(V_\rho^\eta, \mathscr{L}\left(BC^{\delta}(\mathbb{R},X_0), BC^{\xi}(\mathbb{R},E)\right)\right), \ \forall \mu > 0.$$

Since V_ρ is an open and convex subset of X_0, we have the following classical formula

$$W(x) - W(y) = \int_0^1 DW(sx + (1-s)y)(x-y)\, ds, \ \forall x, y \in V_\rho.$$

Therefore, for each $u, \widehat{u} \in V_\rho^\eta$,

$$\begin{aligned}
&\left\| \Phi_W(u) - \Phi_W(\widehat{u}) - \Phi_W^{(1)}(\widehat{u})(u - \widehat{u}) \right\|_{BC^{\xi}(\mathbb{R},E)} \\
&= \sup_{t\in\mathbb{R}} e^{-\xi|t|} \| W(u(t)) - W(\widehat{u}(t)) - DW(\widehat{u}(t))(u(t) - \widehat{u}(t)) \| \\
&\leq \sup_{t\in\mathbb{R}} \sup_{s\in[0,1]} e^{-\xi|t|} \| [DW(su(t) + (1-s)\widehat{u}(t)) - DW(\widehat{u}(t))](u(t) - \widehat{u}(t)) \| \\
&\leq \|u - \widehat{u}\|_{BC^{\delta}(\mathbb{R},X_0)} \sup_{s\in[0,1]} \| \Phi_{DW}(su + (1-s)\widehat{u}) - \Phi_{DW}(\widehat{u}) \|_{BC^{\xi-\delta}(\mathbb{R},\mathscr{L}(X_0,E))}.
\end{aligned}$$

The proof is complete. □

The following lemma is taken from Vanderbauwhede and Iooss [345, Lemma 3].

Lemma 6.1.15. *Let E be a Banach space and $W \in C_b^1(V_\rho, E)$. Let Φ_W and $\Phi_W^{(1)}$ be defined as in Lemma 6.1.14. Let $\Theta \in C\left(BC^{\eta}(\mathbb{R},X_{0c}), V_\rho^\eta\right)$ be such that*

(a) Θ is of class C^1 from $BC^{\eta}(\mathbb{R},X_{0c})$ into $BC^{\eta+\mu}(\mathbb{R},X_0)$ for each $\mu > 0$;
(b) Its derivative takes the form

$$D\Theta(u)(v) = \Theta^{(1)}(u)(v), \ \forall u, v \in BC^{\eta}(\mathbb{R},X_{0c}),$$

for some globally bounded $\Theta^{(1)} : BC^{\eta}(\mathbb{R},X_{0c}) \to \mathscr{L}(BC^{\eta}(\mathbb{R},X_{0c}), BC^{\eta}(\mathbb{R},X_0))$.

Then $\Phi_W \circ \Theta \in C_b^0(BC^{\eta}(\mathbb{R},X_{0c}), BC^{\eta}(\mathbb{R},E)) \cap C^1(BC^{\eta}(\mathbb{R},X_{0c}), BC^{\eta+\mu}(\mathbb{R},E))$ for each $\mu > 0$ and

$$D(\Phi_W \circ \Theta)(u)(v) = \Phi_W^{(1)}(\Theta(u))\Theta^{(1)}(u)(v), \ \forall u, v \in BC^{\eta}(\mathbb{R},X_{0c}).$$

Proof. By using Lemma 6.1.14, it follows that

$$\Phi_W \circ \Theta \in C_b^0\left(BC^\eta\left(\mathbb{R}, X_{0c}\right), BC^\eta\left(\mathbb{R}, E\right)\right)$$

and

$$\Phi_W^{(1)}\left(\Theta\left(.\right)\right)\Theta^{(1)}(.) \in C\left(BC^\eta\left(\mathbb{R}, X_{0c}\right), \mathscr{L}\left(BC^\eta\left(\mathbb{R}, X_{0c}\right), BC^{\eta+\mu}\left(\mathbb{R}, E\right)\right)\right).$$

Let $u, \widehat{u} \in BC^\eta\left(\mathbb{R}, X_{0c}\right)$. By Lemma 6.1.14, we also have

$$\begin{aligned}
&\left\|\Phi_W\left(\Theta\left(u\right)\right) - \Phi_W\left(\Theta\left(\widehat{u}\right)\right) - \Phi_W^{(1)}\left(\Theta\left(\widehat{u}\right)\right)\Theta^{(1)}(\widehat{u})\left(u - \widehat{u}\right)\right\|_{BC^{\eta+\mu}(\mathbb{R},E)} \\
&\quad\leq \left\|\Phi_W\left(\Theta\left(u\right)\right) - \Phi_W\left(\Theta\left(\widehat{u}\right)\right) - \Phi_W^{(1)}\left(\Theta\left(\widehat{u}\right)\right)\left(\Theta\left(u\right) - \Theta\left(\widehat{u}\right)\right)\right\|_{BC^{\eta+\mu}(\mathbb{R},E)} \\
&\qquad + \left\|\Phi_W^{(1)}\left(\Theta\left(\widehat{u}\right)\right)\left[\Theta\left(u\right) - \Theta\left(\widehat{u}\right) - \Theta^{(1)}(\widehat{u})\left(u - \widehat{u}\right)\right]\right\|_{BC^{\eta+\mu}(\mathbb{R},E)} \\
&\quad\leq \left\|\Theta\left(u\right) - \Theta\left(\widehat{u}\right)\right\|_{BC^{\eta+\mu/2}(\mathbb{R},X_0)} \varkappa_{\mu/2}\left(\Theta\left(u\right), \Theta\left(\widehat{u}\right)\right) \\
&\qquad + \left\|\Phi_{DW}\left(\Theta\left(\widehat{u}\right)\right)\right\|_{BC^{\mu/2}(\mathbb{R},\mathscr{L}(X_0,E))}\left\|\Theta\left(u\right) - \Theta\left(\widehat{u}\right) - \Theta^{(1)}(\widehat{u})\left(u - \widehat{u}\right)\right\|_{BC^{\eta+\mu/2}(\mathbb{R},X_0)}
\end{aligned}$$

and the result follows. □

One may extend the previous lemma to any order $k > 1$.

Lemma 6.1.16. *Let E be a Banach space and let $W \in C_b^k\left(V_\rho, E\right)$ (for some integer $k \geq 1$). Let $l \in \{1, ..., k\}$ be an integer. Suppose $\xi \geq 0, \mu \geq 0$ are two real numbers and $\delta_1, \delta_2, ..., \delta_l \geq 0$ such that $\xi = \mu + \delta_1 + \delta_2 + ... + \delta_l$. Define*

$$\Phi_{D^{(l)}W}\left(u\right)\left(t\right) := D^{(l)}W\left(u\left(t\right)\right),\ \forall t \in \mathbb{R}, \forall u \in V_\rho^\eta,$$

$$\Phi_W^{(l)}\left(u\right)\left(u_1, u_2, ..., u_l\right)\left(t\right) := D^{(l)}W\left(u\left(t\right)\right)\left(u_1\left(t\right), u_2\left(t\right), ..., u_l\left(t\right)\right),$$
$$\forall t \in \mathbb{R}, \forall u \in V_\rho^\eta, \forall u_i \in BC^{\delta_i}\left(\mathbb{R}, X_0\right),\ \text{for } i = 1, ..., l.$$

Then we have the following:

(a) If $\xi > 0$, then $\Phi_{D^{(l)}W} : V_\rho^\eta \to BC^\xi\left(\mathbb{R}, \mathscr{L}^{(l)}\left(X_0, E\right)\right)$ is continuous;

(b) For each $u, v \in V_\rho^\eta$, $\Phi_W^{(l)}\left(u\right) \in \mathscr{L}^{(l)}\left(BC^{\delta_1}\left(\mathbb{R}, X_0\right), ..., BC^{\delta_l}\left(\mathbb{R}, X_0\right); BC^\xi\left(\mathbb{R}, E\right)\right)$,

$$\begin{aligned}
&\left\|\Phi_W^{(l)}\left(u\right) - \Phi_W^{(l)}\left(v\right)\right\|_{\mathscr{L}^{(l)}\left(BC^{\delta_1}(\mathbb{R},X_0),...,BC^{\delta_l}(\mathbb{R},X_0);BC^\xi(\mathbb{R},E)\right)} \\
&\quad\leq \left\|\Phi_{D^{(l)}W}\left(u\right) - \Phi_{D^{(l)}W}\left(v\right)\right\|_{BC^\mu\left(\mathbb{R},\mathscr{L}^{(l)}(X_0,E)\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Phi_W^{(l)}\left(u\right)\right\|_{\mathscr{L}^{(l)}\left(BC^{\delta_1}(\mathbb{R},X_0),...,BC^{\delta_l}(\mathbb{R},X_0);BC^\xi(\mathbb{R},E)\right)} \\
&\quad\leq \left\|\Phi_{D^{(l)}W}\left(u\right)\right\|_{BC^\mu\left(\mathbb{R},\mathscr{L}^{(l)}(X_0,E)\right)} \leq |W|_{l,V_\rho};
\end{aligned}$$

(c) If $\mu > 0$, then $\Phi_W^{(l)}$ is continuous;

(d) *If $\delta_1 \geq \eta$, we have for each $u, \widehat{u} \in V_\rho^\eta$ that*

$$\left\| \Phi_W^{(l-1)}(u) - \Phi_W^{(l-1)}(\widehat{u}) - \Phi_W^{(l)}(\widehat{u})(u-\widehat{u}) \right\|_{\mathscr{L}^{(l-1)}\left(BC^{\delta_2}(\mathbb{R},X_0),\ldots,BC^{\delta_l}(\mathbb{R},X_0);BC^{\xi}(\mathbb{R},E)\right)}$$
$$\leq \|u-\widehat{u}\|_{BC^{\delta_1}(\mathbb{R},X_0)} \varkappa_\mu^{(l)}(u,\widehat{u}),$$

where

$$\varkappa_\mu^{(l)}(u,\widehat{u}) = \sup_{s\in[0,1]} \left\| \Phi_{D^{(l)}W}(su+(1-s)\widehat{u}) - \Phi_{D^{(l)}W}(\widehat{u}) \right\|_{BC^\mu\left(\mathbb{R},\mathscr{L}^{(l)}(X_0,E)\right)},$$

and if $\mu > 0$, we have by continuity of $\Phi_{D^{(l)}W}$ that

$$\varkappa_\mu^{(l)}(u,\widehat{u}) \to 0 \text{ as } \|u-\widehat{u}\|_{BC^\eta(\mathbb{R},X_0)} \to 0.$$

Proof. The proof is similar to that of Lemma 6.1.14. □

In the following lemma we use a formula for the k^{th}-derivative of the composed map. This formula is taken from Avez [34, p. 38] which also corrects the one used in Vanderbauwhede [343, Proof of Lemma 3.11].

Lemma 6.1.17. *Let E be a Banach space and let $W \in C_b^k\left(V_\rho, E\right)$. Let Φ_W and $W^{(k)}$ be defined as above. Let $\Theta \in C\left(BC^\eta(\mathbb{R},X_{0c}), V_\rho^\eta\right)$ be such that*

(a) *Θ is of class C^k from $BC^\eta(\mathbb{R},X_{0c})$ into $BC^{k\eta+\mu}(\mathbb{R},X_0)$ for each $\mu > 0$;*
(b) *For each $l = 1,\ldots,k$, its derivative takes the form*

$$D^l\Theta(u)(v_1,v_2,\ldots,v_l) = \Theta^{(l)}(u)(v_1,v_2,\ldots,v_l), \forall u,v_1,v_2,\ldots,v_l \in BC^\eta(\mathbb{R},X_{0c}),$$

for some globally bounded $\Theta^{(l)} : BC^\eta(\mathbb{R},X_{0c}) \to \mathscr{L}^{(l)}(BC^\eta(\mathbb{R},X_{0c}); BC^\eta(\mathbb{R},X_0))$.

Then $\Phi_W \circ \Theta \in C_b^0(BC^\eta(\mathbb{R},X_{0c}), BC^\eta(\mathbb{R},E)) \cap C^k\left(BC^\eta(\mathbb{R},X_{0c}), BC^{k\eta+\mu}(\mathbb{R},E)\right)$ for each $\mu > 0$. Moreover, for each $l = 1,\ldots,k$ and each $u,v_1,v_2,\ldots,v_l \in BC^\eta(\mathbb{R},X_{0c})$,

$$D^l(\Phi_W \circ \Theta)(u)(v) = (\Phi_W \circ \Theta)^{(l)}(u)(v_1,v_2,\ldots,v_l)$$

for some globally bounded $(\Phi_W \circ \Theta)^{(l)} : BC^\eta(\mathbb{R},X_{0c}) \to \mathscr{L}^{(l)}(BC^\eta(\mathbb{R},X_{0c}); BC^\eta(\mathbb{R},E))$. More precisely, we have for $j = 1,\ldots,k$ that

(i) *$(\Phi_W \circ \Theta)^{(j)}(u) = \Phi_W^{(1)}(\Theta(u)) D^{(j)}\Theta(u) + \widetilde{\Phi}_{W,j}(u)$;*
(ii) *$\widetilde{\Phi}_{W,1}(u) = 0$;*
(iii) *For $j > 1$, the map $\widetilde{\Phi}_{W,j}(u)$ is a finite sum $\sum_{\lambda\in\Lambda_j} \widetilde{\Phi}_{W,\lambda,j}(u)$, where for each $\lambda \in \Lambda_j$ the map $\widetilde{\Phi}_{W,\lambda,j}(u) : BC^\eta(\mathbb{R},X_{0c}) \to \mathscr{L}^{(j)}(BC^\eta(\mathbb{R},X_{0c}), BC^\eta(\mathbb{R},E))$ has the following form*

$$\widetilde{\Phi}_{W,\lambda,j}(u)(u_1,u_2,\ldots,u_j) = \Phi_W^{(l)}(\Theta(u)) \begin{pmatrix} D^{(r_1)}\Theta(u)\left(u_{i_1^{r_1}}, u_{i_2^{r_1}}, \ldots, u_{i_{r_1}^{r_1}}\right), \ldots, \\ D^{(r_l)}\Theta(u)\left(u_{i_1^{r_l}}, \ldots, u_{i_{r_l}^{r_l}}\right) \end{pmatrix}$$

with $2 \leq l \leq j$, $1 \leq r_i \leq j-1$ *for* $1 \leq i \leq l$, $r_1 + r_2 + ... + r_l = j$,

$$\left\{ i_1^{rm}, ..., i_{r_m}^{r_m} \right\} \subset \{1, ..., j\}, \; \forall m = 1, ..., l$$

$$\left\{ i_1^{rm}, ..., i_{r_m}^{r_m} \right\} \cap \left\{ i_1^{rn}, ..., i_{r_n}^{r_n} \right\} = \emptyset, \; \text{if } m \neq n,$$
$$i_1^{rm} \leq i_2^{rm} \leq ... \leq i_{r_m}^{r_m}, \; \forall m = 1, ..., l,$$

and each $\lambda \in \Lambda_j$ *corresponds to each such a particular choice.*

Proof. The proof is similar to that of Lemma 6.1.15. □

We make the following assumption.

Assumption 6.1.18. Let $k \geq 1$ be an integer and let $\eta, \widehat{\eta} \in (0, \frac{\beta}{k})$ such that $k\eta < \widehat{\eta} < \beta$. Assume that

a) $F \in \text{Lip}(X_0, X) \cap C_b^k \left(V_\rho, X \right)$;
b) $\rho_0 := \|K_h\|_{\mathscr{L}\left(BC^0(\mathbb{R},X)\right)} \|\Pi_h F\|_{0,X_0} < \rho$;
c) $\sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathscr{L}\left(BC^\theta(\mathbb{R},X)\right)} \|F\|_{\text{Lip}(X_0,X)} < 1$.

Note that by using (6.1.18) we deduce that

$$\sup_{\theta \in [\eta, \widehat{\eta}]} \|K_2\|_{\mathscr{L}\left(BC^\theta(\mathbb{R},X)\right)} < +\infty.$$

Thus, Assumption 6.1.18 makes sense.

Following the approach of Vanderbauwhede [343, Corollary 3.6] and Vanderbauwhede and Iooss [345, Theorem 2], we obtain the following result on the smoothness of center manifolds.

Theorem 6.1.19. *Let Assumptions 6.1.1 and 6.1.18 be satisfied. Then the map* Ψ *given by Theorem 6.1.10 belongs to the space* $C_b^k(X_c, X_h)$.

Proof. **Step 1. Existence of a fixed point.** Let k, η, and $\widehat{\eta}$ be the numbers introduced in Assumption 6.1.18. Let $\mu > 0$ be given such that $k\eta + (2k-1)\mu = \widehat{\eta}$. We first apply Lemma 6.1.13. For each $j = 1, ..., k$ and each subset $E \subset BC^\eta(\mathbb{R}, X_{0c})$, define $M_{j,E}$ as the Banach space of all continuous maps

$$\Theta_j : E \to \mathscr{L}^{(j)} \left(BC^\eta(\mathbb{R}, X_{0c}), BC^{j\eta + (2j-1)\mu}(\mathbb{R}, X_0) \right)$$

such that

$$\left| \Theta_j \right|_j = \sup_{u \in E} \left\| \Theta_j(u) \right\|_{\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_{0c}), BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)} < +\infty.$$

For $j = 0, ..., k$, define the map $H_{j,E} : M_{0,E} \times M_{1,E} \times ... \times M_{j,E} \to M_{j,E}$ as follows: If $j = 0$, set for each $u \in E$ that

$$H_{0,E}(\Theta_0)(u) = u + K_2 \circ \Phi_F \circ \Theta_0(u).$$

If $j=1$, set for each $u\in E$ that

$$H_{1,E}\left(\Theta_0,\Theta_1\right)(u)(.)=J^1+K_2\circ\Phi_F^{(1)}\left(\Theta_0\left(u\right)\right)\circ\Theta_1\left(u\right), \tag{6.1.27}$$

where J^1 is the continuous imbedding from $BC^\eta\left(\mathbb{R},X_{0c}\right)$ into $BC^{\eta+\mu}\left(\mathbb{R},X_0\right)$.
If $k\geq 2$, set for each $j=2,...,k$ and each $u\in E$ that

$$\begin{aligned}&H_{j,E}\left(\Theta_0,\Theta_1,...,\Theta_j\right)(u)\\&\quad=K_2\circ\Phi_F^{(1)}\left(\Theta_0\left(u\right)\right)\circ\Theta_j\left(u\right)+\widehat{H}_{j,E}\left(\Theta_0,\Theta_1,...,\Theta_{j-1}\right)(u),\end{aligned} \tag{6.1.28}$$

where

$$\widehat{H}_{j,E}\left(\Theta_0,\Theta_1,...,\Theta_{j-1}\right)(u)=\sum_{\lambda\in\Lambda_j}\widehat{H}_{\lambda,j,E}\left(\Theta_0,\Theta_1,...,\Theta_{j-1}\right)(u)$$

and

$$\begin{aligned}&\widehat{H}_{\lambda,j,E}\left(\Theta_0,\Theta_1,...,\Theta_{j-1}\right)(u)\left(u_0,u_1,...,u_j\right)\\&\quad=K_2\circ\Phi_F^{(l)}\left(\Theta_0(u)\right)\left(\Theta_{r_1}\left(u\right)\left(u_{i_1^{r_1}},u_{i_2^{r_1}},...,u_{i_{r_1}^{r_1}}\right),...,\Theta_{r_l}\left(u\right)\left(u_{i_1^{r_l}},...,u_{i_{r_l}^{r_l}}\right)\right)\end{aligned}$$

with the same constraints as in Lemma 6.1.17 for λ, r_j, l, and $i_k^{r_j}$.

Define

$$H_j=H_{j,BC^\eta(\mathbb{R},X_{0c})}\quad\text{for each } j=0,...,k.$$

In the above definition one has to consider K_2 as a linear operator from $BC^{j\eta+(2j-1)\mu}\left(\mathbb{R},X\right)$ into $BC^{j\eta+(2j-1)\mu}\left(\mathbb{R},X_0\right)$, and $\Phi_F^{(l)}\left(\Theta_0(u)\right)$ as an element of

$$\mathscr{L}^{(j)}\left(BC^{r_1\eta+(2r_1-1)\mu}\left(\mathbb{R},X_0\right),...,BC^{r_l\eta+(2r_l-1)\mu}\left(\mathbb{R},X_0\right);BC^{j\eta+(2j-1)\mu}\left(\mathbb{R},X\right)\right).$$

Notice that

$$j\eta+\left(2j-1\right)\mu>\sum_{k=1}^{l}r_k\eta+(2r_k-1)\mu$$

since $2\leq l\leq j$ and $r_1+r_2+...+r_l=j$. Finally, define $H:M_0\times M_1\times...\times M_k\to M_0\times M_1\times...\times M_k$ by

$$H\left(\Theta_0,\Theta_1,...,\Theta_k\right)=\left(H_0\left(\Theta_0\right),H_1\left(\Theta_0,\Theta_1\right),...,H_k\left(\Theta_0,\Theta_1,...,\Theta_k\right)\right).$$

We now check that the conditions of Lemma 6.1.13 are satisfied. We have already shown that H_0 is a contraction on X_0. It follows from (6.1.27) and (6.1.28) that H_j $(1\leq j\leq k)$ is a contraction on X_j. More precisely, from Assumption 6.1.18 c), we have for each $j=1,...,k$ that

$$\begin{aligned}
&\sup_{u\in V_\rho^\eta}\left\|K_2\circ\Phi_F^{(1)}(u)\right\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)}\\
&\quad\le\|K_2\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)}\sup_{u\in V_\rho^\eta}\left\|\Phi_F^{(1)}(u)\right\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)}\\
&\quad\le\sup_{\theta\in[\eta,\widehat{\eta}]}\|K_2\|_{\mathscr{L}\left(BC^{\theta}(\mathbb{R},X)\right)}|F|_{1,V_\rho}\\
&\quad\le\sup_{\theta\in[\eta,\widehat{\eta}]}\|K_2\|_{\mathscr{L}\left(BC^{\theta}(\mathbb{R},X)\right)}\|F\|_{\mathrm{Lip}(X_0,X)}<1.
\end{aligned}$$

Thus, each H_j is a contraction. The fixed point of H_0 is Γ_0, and we denote by $\Gamma=(\Gamma_0,\Gamma_1,...,\Gamma_k)$ the fixed point of H. Moreover, for $\mu=0$, each H_j is still a contraction so we have for each $j=1,...,k$ that

$$\sup_{u\in BC^\eta(\mathbb{R},X_{0c})}\left\|\Gamma_j(u)\right\|_{\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_0),BC^{j\eta}(\mathbb{R},X_0)\right)}<+\infty.$$

Step 2. Attractivity of the fixed point. In this part we apply Lemma 6.1.13 to prove that for each compact subset C of $BC^\eta(\mathbb{R},X_{0c})$ and each $\Theta\in M_0\times M_1\times...\times M_k$,

$$\lim_{m\to+\infty}H_C^m(\Theta\mid_C)=\Gamma\mid_C. \tag{6.1.29}$$

Let C be a compact subset of $BC^\eta(\mathbb{R},X_{0c})$. From the definition of H_C, it is clear that

$$\Gamma\mid_C=H_C(\Gamma\mid_C)$$

and from the Step 1, it is not difficult to see that for each $j=0,...,k$, $H_{j,C}$ is a contraction. In order to apply Lemma 6.1.13, it remains to prove that for each $j=1,...,k$, $H_{j,C}\left(\Theta_{0,C},\Theta_{1,C},...,\Theta_{j-1,C},\Gamma_j\mid_C\right)\in M_j$ dependents continuously on $\left(\Theta_{0,C},\Theta_{1,C},...,\Theta_{j-1,C}\right)\in M_{0,C}\times M_{1,C}\times...\times M_{j-1,C}$.

We have

$$\begin{aligned}
&H_j\left(\Theta_{0,C},\Theta_{1,C},...,\Theta_{j-1,C},\Gamma^{(j)}\mid_C\right)(u)\\
&\quad=K_2\circ\Phi_F^{(1)}(\Theta_{0,C}(u))\circ\Gamma^{(j)}(u)+\widehat{H}_j\left(\Theta_{0,C},\Theta_{1,C},...,\Theta_{j-1,C}\right)(u).
\end{aligned}$$

Since $\Gamma^{(j)}(u)\in\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_0),BC^{j\eta}(\mathbb{R},X_0)\right)$ and $\Phi(u)\in V_\rho^\eta$, we can consider $\Phi_F^{(1)}$ as a map from V_ρ^η into $\mathscr{L}(BC^{j\eta}(\mathbb{R},X_0),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0))$, and by Lemma 6.1.14 this map is continuous.

Indeed, let $\Theta_0,\widehat{\Theta}_0\in M_0$ be two maps. Then we have

$$\begin{aligned}
&\sup_{u\in C}\left\|K_2\circ\left[\Phi_F^{(1)}(\Theta_0(u))-\Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right]\circ\Gamma^{(j)}(u)\right\|_{\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_{0c}),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)}\\
&\quad\le\|K_2\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)}\\
&\qquad\cdot\sup_{u\in C}\left\|\left[\Phi_F^{(1)}(\Theta_0(u))-\Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right)\right]\circ\Gamma^{(j)}(u)\right\|_{\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_{0c}),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)}\\
&\quad\le\|K_2\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)}\sup_{u\in C}\left\|\Gamma^{(j)}(u)\right\|_{\mathscr{L}^{(j)}\left(BC^\eta(\mathbb{R},X_{0c}),BC^{j\eta}(\mathbb{R},X_0)\right)}
\end{aligned}$$

$$\cdot \sup_{u\in C} \left\| \Phi_F^{(1)}\left(\Theta_0(u)\right) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathscr{L}^{(j)}\left(BC^{j\eta}(\mathbb{R},X_0),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)}$$

and by Lemma 6.1.14 we have

$$\begin{aligned}
&\sup_{u\in C} \left\| \Phi_F^{(1)}\left(\Theta_0(u)\right) - \Phi_F^{(1)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathscr{L}^{(j)}\left(BC^{j\eta}(\mathbb{R},X_0),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)} \\
&\quad \leq \sup_{u\in C} \left\| \Phi_{DF}\left(\Theta_0(u)\right) - \Phi_{DF}\left(\widehat{\Theta}_0(u)\right) \right\|_{BC^{(2j-1)\mu}(\mathbb{R},\mathscr{L}(X_0,X))} \\
&\quad \leq \max \begin{pmatrix} \sup_{|t|\geq R} e^{-(2j-1)\mu|t|} \left\| DF\left(\Theta_0(u)(t)\right) - DF\left(\widehat{\Theta}_0(u)(t)\right) \right\|_{\mathscr{L}(X_0,X)}, \\ \sup_{|t|\leq R} e^{-(2j-1)\mu|t|} \left\| DF\left(\Theta_0(u)(t)\right) - DF\left(\widehat{\Theta}_0(u)(t)\right) \right\|_{\mathscr{L}(X_0,X)} \end{pmatrix}.
\end{aligned}$$

Since $\widehat{\Theta}_0$ is continuous, C is compact, it follows that $\widehat{\Theta}_0(C)$ is compact, and by Ascoli's theorem (see for example Lang [224]), the set $\widehat{C} = \overline{\bigcup_{|t|\leq R, u\in C} \left\{ \widehat{\Theta}_0(u)(t) \right\}}$ is compact. But since $DF(.)$ is continuous, we have that for each $\varepsilon > 0$, there exists $\eta > 0$, such that for each $x, y \in X_0$,

$$d\left(x,\widehat{C}\right) \leq \eta, \ d\left(y,\widehat{C}\right) \leq \eta, \ \text{ and } \|x-y\| \leq \eta \Rightarrow \|DF(x) - DF(y)\| \leq \varepsilon.$$

Hence, the map $\Theta_{0,C} \to K_2 \circ \Phi_F^{(1)}\left(\Theta_{0,C}(.)\right) \circ \Gamma^{(j)}(.)$ is continuous.

It remains to consider $1 \leq r_i \leq j-1$, $r_1 + r_2 + ... + r_l = j$. We have

$$\begin{aligned}
&\left\| K_2 \circ \left[\Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right] \left(\widetilde{\Theta}_{r_1}(u), ..., \widetilde{\Theta}_{r_l}(u) \right) \right\|_{\mathscr{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_{0c}),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)} \\
&\quad \leq \|K_2\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)} \\
&\qquad \cdot \sup\nolimits_{u\in C} \left\| \left[\Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right] \right. \\
&\qquad\qquad \left. \cdot \left(\widetilde{\Theta}_{r_1}(u), ..., \widetilde{\Theta}_{r_l}(u) \right) \right\|_{\mathscr{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_{0c}),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)} \\
&\quad \leq \|K_2\|_{\mathscr{L}\left(BC^{j\eta+(2j-1)\mu}(\mathbb{R},X),BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)\right)} \\
&\qquad \cdot \left\| \Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathscr{L}^{(l)}\left(\prod_{p=1,...,l} BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},X_0);BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)} \\
&\qquad \cdot \prod_{p=1,...,l} \left\| \widetilde{\Theta}_{r_p}(u) \right\|_{\mathscr{L}^{(j)}\left(BC^{\eta}(\mathbb{R},X_{0c}),BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},X_0)\right)}
\end{aligned}$$

and by Lemma 6.1.16 we have

$$\begin{aligned}
&\sup\nolimits_{u\in C} \left\| \Phi_F^{(l)}\left(\Theta_0(u)\right) - \Phi_F^{(l)}\left(\widehat{\Theta}_0(u)\right) \right\|_{\mathscr{L}^{(l)}\left(\prod_{p=1,...,l} BC^{r_p\eta+(2r_p-1)\mu}(\mathbb{R},X_0);BC^{j\eta+(2j-1)\mu}(\mathbb{R},X)\right)} \\
&\quad \leq \sup\nolimits_{u\in C} \left\| \Phi_{D^{(l)}F}\left(\Theta_0(u)\right) - \Phi_{D^{(l)}F}\left(\widehat{\Theta}_0(u)\right) \right\|_{BC^{\delta}\left(\mathbb{R},\mathscr{L}^{(l)}(X_0,X)\right)}
\end{aligned}$$

with $\delta = (j\eta + (2j-1)\mu) - \sum_{k=1}^{l} r_k\eta + (2r_k - 1)\mu > 0$. By using the same compactness argument as previously, we deduce that

$$\sup_{u\in C}\left\|\Phi_{D^{(l)}F}\left(\Theta_0(u)\right) - \Phi_{D^{(l)}F}\left(\widehat{\Theta}_0(u)\right)\right\|_{BC^{\delta}\left(\mathbb{R},\mathscr{L}^{(l)}(X_0,X)\right)} \to 0$$

as $d_{0,C}(\Theta_0, \widehat{\Theta}_0) \to 0$. We conclude that the continuity condition of Lemma 6.1.13 is satisfied for each $H_{j,C}$ and (6.1.29) follows.

Step 3. It now remains to prove that for each $u, v \in BC^{\eta}\left(\mathbb{R}, X_{0c}\right), \forall j = 1, ..., k,$

$$\Gamma_{j-1}(u) - \Gamma_{j-1}(v) = \int_0^1 \Gamma_j(s(u-v)+v)\,(u-v)\,ds, \tag{6.1.30}$$

where the last integral is a Riemann integral. As assumed that (6.1.30) is satisfied, we deduce that $\Gamma_0 : BC^{\eta}\left(\mathbb{R}, X_{0c}\right) \to BC^{k\eta+(2k-1)\mu}\left(\mathbb{R}, X_0\right)$ is k-time continuously differentiable, and since

$$\Psi(x_c) = L \circ \Gamma_0 \circ K_1\left(x_c\right)$$

and L is a bounded linear operator from $BC^{k\eta+(2k-1)\mu}\left(\mathbb{R}, X_0\right)$ into X_{0h}, we know that $\Psi : X_{0c} \to X_{0h}$ is k-time continuously differentiable.

We now prove (6.1.30). Set

$$\Theta^0 = \left(\Theta_0^0, \Theta_1^0, ..., \Theta_k^0\right)$$

with

$$\Theta_0^0\left(u\right) = u, \Theta_1^0\left(u\right) = J, \text{ and } \Theta_j^0 = 0, \ \forall j = 2, ..., k$$

and set

$$\Theta^m = (\Theta_0^m, \Theta_1^m, ..., \Theta_k^m) = H^m\left(\Theta^0\right), \ \forall m \geq 1.$$

Then from Lemma 6.1.17, we know that $\Theta_0^m : BC^{\eta}\left(\mathbb{R}, X_{0c}\right) \to BC^{k\eta+(2k-1)\mu}\left(\mathbb{R}, X_0\right)$ is a C^k-map and

$$D^j\Theta_0^m(u) = \Theta_j^m(u), \ \ \forall j = 1, ..., k, \ \ \forall u \in BC^{\eta}\left(\mathbb{R}, X_{0c}\right).$$

For each $u, v \in BC^{\eta}\left(\mathbb{R}, X_{0c}\right)$ and each $\forall j = 1, ..., k, \forall m \geq 1,$

$$\Theta_{j-1}^m(u) - \Theta_{j-1}^m(v) = \int_0^1 \Theta_j^m(s(u-v)+v)\,(u-v)\,ds.$$

Let $u, v \in BC^{\eta}\left(\mathbb{R}, X_{0c}\right)$ be fixed. Denote

$$C = \left\{s(u-v)+v : s \in [0,1]\right\}.$$

Then clearly C is a compact set, and from Step 2, we have for each $j = 0, ..., k$ that

$$\sup_{w\in C}\left\|\Theta_j^m(w) - \Gamma_j(w)\right\|_{BC^{j\eta+(2j-1)\mu}(\mathbb{R},X_0)} \to 0 \ \text{ as } m \to +\infty.$$

Thus, (6.1.30) follows. $\square$

It follows from the foregoing treatment that we can obtain the derivatives of $\Gamma_0(u)$ at $u = 0$. Assume that $F(0) = 0$ and $DF(0) = 0$, we have

$$
\begin{aligned}
&D\Gamma_0(0) = J,\\
&D^{(2)}\Gamma_0(0)(u_1,u_2) = K_2 \circ \Phi_F^{(2)}(0)\left(D\Gamma_0(0)(u_1), D\Gamma_0(0)(u_2)\right),\\
&D^{(3)}\Gamma_0(0)(u_1,u_2,u_3) = K_2 \circ \Phi_F^{(2)}(0)\left(D^{(2)}\Gamma_0(0)(u_1,u_3), D\Gamma_0(0)(u_2)\right)\\
&\qquad +K_2 \circ \Phi_F^{(2)}(0)\left(D\Gamma_0(0)(u_1), D^{(2)}\Gamma_0(0)(u_2,u_3)\right)\\
&\qquad +K_2 \circ \Phi_F^{(3)}(0)\left(D\Gamma_0(0)(u_1), D\Gamma_0(0)(u_2), D\Gamma_0(0)(u_3)\right),\\
&\vdots\\
&D^{(l)}\Gamma_0(0) = \sum_{\lambda\in\Lambda_j} K_2 \circ \Phi_F^{(l)}(0)\left(D^{(r_1)}\Gamma(0), \ldots, D\Gamma^{(r_l)}(0)\right).
\end{aligned}
\tag{6.1.31}
$$

We have the following Lemma.

Lemma 6.1.20. *Let Assumptions 6.1.1 and 6.1.18 be satisfied. Assume also that $F(0) = 0$ and $DF(0) = 0$. Then*

$$\Psi(0) = 0, \quad D\Psi(0) = 0,$$

and if $k > 1$,

$$D^j\Psi(0)(x_1,\ldots,x_n) = \Pi_h D^{(l)}\Gamma_0(0)(K_1x_1,\ldots,K_1x_n)(0),$$

where $D^{(l)}\Gamma_0(0)$ is given by (6.1.31). In particular, if $k > 1$ and

$$\Pi_h D^j F(0)\mid_{X_{0c}\times\ldots\times X_{0c}} = 0 \text{ for } 2 \le j \le k,$$

then

$$D^j\Psi(0) = 0 \ \text{ for } 1 \le j \le k.$$

In the context of Hopf bifurcation, we need an explicit formula for $D^2\Psi(0)$. Since $D\Gamma_0(0) = J$, we obtain from the above formula that $\forall x_1, x_2 \in X_{0c}$,

$$D^2\Psi(0)(x_1,x_2) = \Pi_h K_h\left[D^{(2)}F(0)(K_1x_1, K_1x_2)\right](0),$$

where

$$K_h = K_s + K_u, \ \ K_1(x_c)(t) := e^{A_{0c}t}x_c,$$

$$K_u(f)(t) := -\int_t^{+\infty} e^{-A_{0u}(l-t)}\Pi_u f(l)\,dl,$$

and

$$K_s(f)(t) := \lim_{r\to-\infty} \Pi_{0s}\left(S_A \diamond f(r+.)\right)(t-r).$$

Hence,

$$\begin{aligned}
&D^2\Psi(0)(x_1,x_2) \\
&\quad = -\int_0^{+\infty} e^{-A_{0u}l}\Pi_u D^{(2)}F(0)\left(e^{A_{0c}l}x_1, e^{A_{0c}l}x_2\right)dl \\
&\qquad + \lim_{r\to-\infty} \Pi_{0s}\left(S_A \diamond D^{(2)}F(0)\left(e^{A_{0c}(r+.)}x_1, e^{A_{0c}(r+.)}x_2\right)\right)(-r).
\end{aligned}$$

In order to express the terms in the above formula, we note that

$$\begin{aligned}
&(\lambda I - A)^{-1} \lim_{r\to-\infty} \Pi_{0s}\left(S_A \diamond D^{(2)}F(0)\left(e^{A_{0c}(r+.)}x_1, e^{A_{0c}(r+.)}x_2\right)\right)(-r) \\
&\quad = \lim_{r\to-\infty} \Pi_{0s}\int_0^{-r} T_{A_0}(-r-s)(\lambda I - A)^{-1} D^{(2)}F(0)\left(e^{A_{0c}(r+s)}x_1, e^{A_{0c}(r+s)}x_2\right)ds \\
&\quad = \lim_{r\to-\infty}\int_0^{-r} T_{A_0}(l)(\lambda I - A)^{-1} D^{(2)}F(0)\left(e^{-A_{0c}l}x_1, e^{-A_{0c}l}x_2\right)dl \\
&\quad = \int_0^{+\infty} T_{A_0}(l)\Pi_{0s}(\lambda I - A)^{-1} D^{(2)}F(0)\left(e^{-A_{0c}l}x_1, e^{-A_{0c}l}x_2\right)dl.
\end{aligned}$$

Therefore, we obtain the following formula

$$\begin{aligned}
&D^2\Psi(0)(x_1,x_2) \\
&\quad = -\int_0^{+\infty} e^{-A_{0u}l}\Pi_u D^{(2)}F(0)\left(e^{A_{0c}l}x_1, e^{A_{0c}l}x_2\right)dl \\
&\qquad + \lim_{\lambda\to+\infty}\int_0^{+\infty} T_{A_0}(l)\Pi_{0s}\lambda(\lambda I - A)^{-1} D^{(2)}F(0)\left(e^{-A_{0c}l}x_1, e^{-A_{0c}l}x_2\right)dl.
\end{aligned}$$

Assume that X is a complex Banach space and F is twice continuously differentiable in X considered as a $\mathbb{C}$-Banach space. We assume in addition that A_{0c} is diagonalizable, and denote by $\{v_1,...,v_n\}$ a basis of X_c such that for each $i=1,...,n$, $A_{0c}v_i = \lambda_i v_i$. Then by Assumption 6.1.1, we must have $\lambda_i \in i\mathbb{R}$, $\forall i = 1,...,n$. Moreover, for each $i,j=1,...,n$, we have

$$\begin{aligned}
&D^2\Psi(0)(v_i,v_j) \\
&\quad = -\int_0^{+\infty} e^{(\lambda_i+\lambda_j)l} e^{-A_{0u}l}\Pi_u D^{(2)}F(0)(v_i,v_j)\,dl \\
&\qquad + \lim_{\lambda\to+\infty}\int_0^{+\infty} T_{A_0}(l)\Pi_{0s}\lambda(\lambda I - A)^{-1} D^{(2)}F(0)\left(e^{-\lambda_i l}v_i, e^{-\lambda_j l}v_j\right)dl \\
&\quad = -\left(-(\lambda_i+\lambda_j)I - (-A_{0u})\right)^{-1}\Pi_u D^{(2)}F(0)(v_i,v_j) \\
&\qquad + \lim_{\lambda\to+\infty}\int_0^{+\infty} e^{-(\lambda_i+\lambda_j)l} T_{A_{0,s}}(l)\Pi_{0s}\lambda(\lambda I - A)^{-1} D^{(2)}F(0)(v_i,v_j)\,dl \\
&\quad = -\left(-(\lambda_i+\lambda_j)I - (-A_{0u})\right)^{-1}\Pi_u D^{(2)}F(0)(v_i,v_j) \\
&\qquad + \lim_{\lambda\to+\infty}\lambda(\lambda I - A)^{-1}\left((\lambda_i+\lambda_j)I - A_s\right)^{-1}\Pi_s D^{(2)}F(0)(v_i,v_j).
\end{aligned}$$

Thus,

$$D^2\Psi(0)(v_i,v_j) = ((\lambda_i+\lambda_j)I - A_{0u})^{-1}\Pi_u D^{(2)}F(0)(v_i,v_j) + ((\lambda_i+\lambda_j)I - A_s)^{-1}\Pi_s D^{(2)}F(0)(v_i,v_j).$$

Note that by Assumption 6.1.1 $i\mathbb{R}\subset\rho(A_s)$, so the above formula is well defined.

As in Vanderbauwhede and Iooss [345, Theorem 3], we have the following theorem about the existence of the local center manifold.

Theorem 6.1.21 (Local Center Manifold). *Let Assumption 6.1.1 be satisfied. Let $F : B_{X_0}(0,\varepsilon)\to X$ be a map. Assume that there exists an integer $k\geq 1$ such that F is k-time continuously differentiable in some neighborhood of 0 with $F(0)=0$ and $DF(0)=0$. Then there exist a neighborhood Ω of the origin in X_0 and a map $\Psi\in C_b^k(X_{0c},X_{0h})$ with $\Psi(0)=0$ and $D\Psi(0)=0$, such that the following properties hold:*

(i) If I is an interval of $\mathbb{R}$ and $x_c : I\to X_{0c}$ is a solution of

$$\frac{dx_c(t)}{dt} = A_{0c}x_c(t) + \Pi_c F[x_c(t)+\Psi(x_c(t))] \tag{6.1.32}$$

such that

$$u(t) := x_c(t)+\Psi(x_c(t))\in\Omega,\ \forall t\in I,$$

then for each $t,s\in I$ with $t\geq s$,

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))\,dl.$$

(ii) If $u:\mathbb{R}\to X_0$ is a map such that for each $t,s\in\mathbb{R}$ with $t\geq s$,

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))\,dl$$

and $u(t)\in\Omega$, $\forall t\in\mathbb{R}$, then

$$\Pi_h u(t) = \Psi(\Pi_c u(t)), \forall t\in\mathbb{R},$$

and $\Pi_c u:\mathbb{R}\to X_{0c}$ is a solution of (6.1.32).

(iii) If $k\geq 2$, then for each $x_1,x_2\in X_{0c}$,

$$\begin{aligned} &D^2\Psi(0)(x_1,x_2)\\ &\quad = -\int_0^{+\infty} e^{-A_{0u}l}\Pi_u D^{(2)}F(0)\left(e^{A_{0c}l}x_1, e^{A_{0c}l}x_2\right)dl\\ &\qquad + \lim_{r\to-\infty}\Pi_{0s}\left(S_A\diamond D^{(2)}F(0)\left(e^{A_{0c}(r+.)}x_1, e^{A_{0c}(r+.)}x_2\right)\right)(-r). \end{aligned}$$

Moreover, X is a $\mathbb{C}$-Banach space, and if $\{v_1,\dots,v_n\}$ is a basis of X_c such that for each $i=1,\dots,n$, $A_{0c}v_i=\lambda_i v_i$, with $\lambda_i\in i\mathbb{R}$, then for each $i,j=1,\dots,n$,

$$D^2\Psi(0)(v_i,v_j) = ((\lambda_i+\lambda_j)I - A_{0u})^{-1}\Pi_u D^{(2)}F(0)(v_i,v_j)$$

$$+((\lambda_i+\lambda_j)I-A_s)^{-1}\Pi_s D^{(2)}F(0)(v_i,v_j).$$

Proof. Set for each $r>0$ that

$$F_r(x)=F(x)\chi_c\left(r^{-1}\Pi_{0c}(x)\right)\chi_h\left(r^{-1}\|\Pi_{0h}(x)\|\right),\forall x\in X_0,$$

where $\chi_c : X_{0c}\to[0,+\infty)$ is a C^∞ map with $\chi_{0c}(x)=1$ if $\|x\|\le 1$, $\chi_{0c}(x)=0$ if $\|x\|\ge 2$, and $\chi_h:[0,+\infty)\to[0,+\infty)$ is a C^∞ map with $\chi_h(x)=1$ if $|x|\le 1$, $\chi_h(x)=0$ if $|x|\ge 2$. Then by using the same argument as in the proof of Theorem 3 in [345], we deduce that there exists $r_0>0$, such that for each $r\in(0,r_0]$, F_r satisfies Assumption 6.1.18. By applying Theorem 6.1.19 to

$$\frac{du(t)}{dt}=Au(t)+F_r(u(t)),\ \ t\ge 0,\text{ and } u(0)=x\in\overline{D(A)}$$

for $r>0$ small enough, the result follows. □

In order to investigate the existence of Hopf bifurcation we also need the following result.

Proposition 6.1.22. *Let the assumptions of Theorem 6.1.21 be satisfied. Assume that $\overline{x}\in X_0$ is an equilibrium of $\{U(t)\}_{t\ge 0}$ (i.e. $\overline{x}\in D(A)$ and $A\overline{x}+F(\overline{x})=0$) such that*

$$\overline{x}\in\Omega.$$

Then

$$\Pi_{0h}\overline{x}=\Psi(\Pi_{0c}\overline{x})$$

and $\Pi_{0c}\overline{x}$ is an equilibrium of the reduced equation (6.1.32). Moreover, if we consider the linearized equation of (6.1.32) at $\Pi_{0c}\overline{x}$:

$$\frac{dy_c(t)}{dt}=L(\overline{x})y_c(t)$$

with

$$L(\overline{x})=[A_{0c}+\Pi_c DF(\overline{x})[I+D\Psi(\Pi_{0c}\overline{x})]],$$

then we have the following spectral properties

$$\sigma(L(\overline{x}))=\sigma((A+DF(\overline{x}))_0)\cap\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)\in[-\eta,\eta]\}.$$

Proof. Let $\overline{x}\in X_0$ be an equilibrium of $\{U(t)\}_{t\ge 0}$ such that $\overline{x}\in\Omega$. We set

$$\overline{x}_c=\Pi_c\overline{x}\ \text{ and } \overline{u}(t)=\overline{x},\ \forall t\in\mathbb{R}.$$

Then the linearized equation at $\overline{x}$ is given by

$$\frac{dw(t)}{dt}=(A+DF(\overline{x}))w(t)\text{ for } t\ge 0\text{ and } w(0)=w_0\in X_0. \tag{6.1.33}$$

So

$$w(t) = T_{(A+DF(\bar{x}))_0}(t)w_0,\ \forall t \geq 0.$$

Moreover, we have

$$D\Psi(x_c)\, y_c = \Pi_h \left[\Gamma_0^1(\overline{u})\,(K_1 y_c)\right]$$

and

$$\Gamma_0^1(\overline{u})(v) = v + K_2\Phi_{DF(\bar{x})}\left(\Gamma_0^1(u)(v)\right),\ \forall v \in BC^\eta(\mathbb{R}, X_{0c}).$$

It follows that

$$\Gamma_0^1(\overline{u}) = \left(I - K_2\Phi_{DF(\bar{x})}\right)^{-1} v.$$

Thus,

$$D\Psi(\bar{x}_c)\, y_c = \Pi_h \left[\left(I - K_2\Phi_{DF(\bar{x})}\right)^{-1}(K_1 y_c)\right].$$

This is exactly the formula for the center manifold of equation (6.1.32) (see (6.1.23) in the proof of Theorem 6.1.10). By applying Theorem 6.1.10 to equation (6.1.33), we deduce that

$$W_\eta = \{y_c + D\Psi(\bar{x}_c)\, y_c : y_c \in X_{0c}\}$$

is invariant by $\left\{T_{(A+DF(\bar{x}))_0}(t)\right\}_{t\geq 0}$. Moreover, for each $w \in C(\mathbb{R}, X_0)$ the following statements are equivalent:

(1) $w \in BC^\eta(\mathbb{R}, X_0)$ is a complete orbit of $\left\{T_{(A+DF(\bar{x}))_0}(t)\right\}_{t\geq 0}$.

(2) $\Pi_{0h}w(t) = D\Psi(\bar{x}_c)\,(\Pi_{0c}w(t)), \forall t \in \mathbb{R}$, and $\Pi_{0c}w(.) : \mathbb{R} \to X_{0c}$ is a solution of the ordinary differential equation

$$\frac{dw_c(t)}{dt} = A_{0c}w_c(t) + \Pi_c DF(\bar{x})\left[w_c(t) + D\Psi(\bar{x}_c)\,(w_c(t))\right].$$

The result follows from the above equivalence. $\square$

6.2 Hopf Bifurcation

The main purpose of this section is to present a general Hopf bifurcation theory for the non-densely defined abstract Cauchy problem:

$$\frac{du(t)}{dt} = Au(t) + F(\mu, u(t)), \forall t \geq 0,\ u(0) = x \in \overline{D(A)}, \tag{6.2.1}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space X, $F : \mathbb{R} \times \overline{D(A)} \to X$ is a C^k-map with $k \geq 2$, and $\mu \in \mathbb{R}$ is the bifurcation parameter. Here, we study the Cauchy problem (6.2.1) when $D(A)$ is not dense in X and A is not a Hille-Yosida operator. Also the solutions must be understood as integrated solutions of (6.2.1). We apply the Center Manifold Theorem in Section 6.1 to prove a Hopf bifurcation theorem for the abstract non-densely defined Cauchy problem (6.2.1).

Assume that 0 is an equilibrium of (6.2.1) for each $\mu \in \mathbb{R}$ small enough; that is,

$$F(\mu,0)=0,\forall\mu\in\mathbb{R}.$$

Moreover, replacing A by $A+\partial_x F(0,0)$ and F by $G(\mu,u(t))=F(\mu,x)-\partial_x F(0,0)x$, the problem is unchanged (since Theorem 3.5.1 implies that $A+\partial_x F(0,0)$ satisfies Assumptions 3.4.1 and 3.5.2). So without loss of generality, we can assume that

$$\partial_x F(0,0)=0.$$

We make the following assumption.

Assumption 6.2.1. Let $\varepsilon>0$ and $F\in C^k\left((-\varepsilon,\varepsilon)\times B_{X_0}(0,\varepsilon);X\right)$ for some $k\geq 4$. Assume that the following conditions are satisfied:

(a) $F(\mu,0)=0,\forall\mu\in(-\varepsilon,\varepsilon)$, and $\partial_x F(0,0)=0$;
(b) **(Transversality condition)** For each $\mu\in(-\varepsilon,\varepsilon)$, there exists a pair of conjugated simple eigenvalues of $(A+\partial_x F(\mu,0))_0$, denoted by $\lambda(\mu)$ and $\overline{\lambda(\mu)}$, such that
$$\lambda(\mu)=\alpha(\mu)+i\omega(\mu),$$
the map $\mu\to\lambda(\mu)$ is continuously differentiable,
$$\omega(0)>0,\ \alpha(0)=0,\ \frac{d\alpha(0)}{d\mu}\neq 0,$$
and
$$\sigma(A_0)\cap i\mathbb{R}=\left\{\lambda(0),\overline{\lambda(0)}\right\}; \tag{6.2.2}$$
(c) The essential growth rate of $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ is strictly negative; that is,
$$\omega_{0,\mathrm{ess}}(A_0)<0.$$

The above conditions are closely related to the usual conditions for the finite dimensional case. The only difference with respect to the finite dimensional case is assumption (c) which is necessary to deal with spectral theory of the semigroup generated by A_0.

In order to apply the reduction technics and results in Theorem 6.1.21 and Proposition 6.1.22, we first incorporate the parameter into the state variable by considering the following system

$$\begin{cases}\dfrac{d\mu(t)}{dt}=0\\ \dfrac{du(t)}{dt}=Au(t)+F(\mu(t),u(t))\\ (\mu(0),u(0))=(\mu_0,u_0)\in(-\varepsilon,\varepsilon)\times\overline{D(A)}.\end{cases} \tag{6.2.3}$$

Note that F is only defined in a neighborhood of $(0,0)\in\mathbb{R}\times X$. In order to rewrite (6.2.3) as an abstract Cauchy problem, consider the Banach space $\mathbb{R}\times X$ endowed with the usual product norm

$$\left\| \begin{pmatrix} \mu \\ x \end{pmatrix} \right\| = \max\left(|\mu|, \|x\|\right),$$

and the linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{R}\times X \to \mathbb{R}\times X$ defined by

$$\mathscr{A} \begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ Ax + \partial_\mu F(0,0)\,\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \partial_\mu F(0,0) & A \end{pmatrix} \begin{pmatrix} \mu \\ x \end{pmatrix}$$

with

$$D(\mathscr{A}) = \mathbb{R}\times D(A).$$

Observe that by Assumption 6.2.1 (a) we have $\partial_x F(0,0) = 0$, and the linear operator $\mathscr{A}$ is the generator of the linearized equation of system (6.2.3) at $(0,0)$. Consider the function $\mathscr{F} : (-\varepsilon,\varepsilon) \times B_{X_0}(0,\varepsilon) \to \mathbb{R}\times X$ defined by

$$\mathscr{F} \begin{pmatrix} \mu \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ F(\mu,x) - \partial_\mu F(0,0)\,\mu \end{pmatrix}.$$

Using the variable $v(t) = \begin{pmatrix} \mu(t) \\ u(t) \end{pmatrix}$, we can rewrite system (6.2.3) as the following abstract Cauchy problem

$$\frac{dv(t)}{dt} = \mathscr{A} v(t) + \mathscr{F}(v(t)),\ t \geq 0,\ \ v(0) = v_0 \in \overline{D(\mathscr{A})}. \tag{6.2.4}$$

We first observe that $\mathscr{F}$ is defined on $B_{\mathbb{R}\times X}(0,\varepsilon)$ and is 4-time continuously differentiable. Moreover, by using Assumption 6.2.1 (a), we have

$$\mathscr{F}(0) = 0 \text{ and } D\mathscr{F}(0) = 0.$$

In order to apply Theorem 6.1.21 and Proposition 6.1.22 to system (6.3.7) we need to verify Assumption 3.5.2.

6.2.1 State Space Decomposition

In order to apply the Center Manifold Theorem, we need to study the spectral properties of the linear operator $\mathscr{A}$. From Assumption 6.2.1 (b) and (c), we know that

$$\sigma(A_0) \cap i\mathbb{R} = \left\{ \lambda(0), \overline{\lambda(0)} \right\} \text{ and } \omega_{0,\text{ess}}(A_0) < 0.$$

For each $\lambda_0 \in \sigma(A_0)$ with $\operatorname{Re}(\lambda_0) > \omega_{0,\text{ess}}(A_0)$, λ_0 is a pole of the resolvent of A_0. That is, there exists an integer $\widehat{k} \geq 1$ such that

$$(\lambda I - A_0)^{-1} = \sum_{k=-\widehat{k}}^{\infty} (\lambda - \lambda_0)^k B_{k,\lambda_0}^{A_0},$$

where

$$B_{k,\lambda_0}^{A_0} := \frac{1}{2\pi i}\int_{S(\lambda_0,\varepsilon)^+} (\lambda-\lambda_0)^{-(k+1)} (\lambda I - A_0)^{-1} d\lambda$$

for $\varepsilon > 0$ small enough. The bounded linear operator $B_{-1,\lambda_0}^{A_0}$ is the projector on the generalized eigenspace of A_0 associated to λ_0.

Set

$$\Pi_{0c}^{A_0} = B_{-1,\lambda(0)}^{A_0} + B_{-1,\overline{\lambda(0)}}^{A_0}$$

and

$$\Pi_{0u}^{A_0} = \sum_{\lambda\in\sigma(A_0):\mathrm{Re}(\lambda)>0} B_{-1,\lambda}^{A_0}.$$

Since $\lambda(0)$ and $\overline{\lambda(0)}$ are simple eigenvalues of A_0, we have

$$B_{-1,\gamma}^{A_0} = \lim_{\lambda\to\gamma} (\lambda-\gamma)(\lambda I - A_0)^{-1} \text{ for } \gamma=\lambda(0) \text{ or } \gamma=\overline{\lambda(0)}.$$

Lemma 6.2.2. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. Then*

$$\sigma(\mathscr{A}) = \sigma(\mathscr{A}_0) = \sigma(A_0)\cup\{0\} = \sigma(A)\cup\{0\},$$

where $\mathscr{A}_0$ is the part of $\mathscr{A}$ in $D(\mathscr{A})$, and for each $\lambda\in\rho(\mathscr{A})$,

$$(\lambda I-\mathscr{A})^{-1}\begin{pmatrix}\mu\\ x\end{pmatrix} = \begin{pmatrix}\lambda^{-1}\mu\\ (\lambda I - A)^{-1}\left[x+\partial_\mu F(0,0)\lambda^{-1}\mu\right]\end{pmatrix}.$$

Proof. Let $\lambda\in\mathbb{C}\diagdown(\sigma(A)\cup\{0\})$. Then

$$\begin{aligned}(\lambda I-\mathscr{A})\begin{pmatrix}\mu\\ x\end{pmatrix} &= \begin{pmatrix}\widehat{\mu}\\ \widehat{x}\end{pmatrix}\\ \Leftrightarrow \begin{pmatrix}\lambda\mu\\ \lambda x - Ax - \partial_\mu F(0,0)\mu\end{pmatrix} &= \begin{pmatrix}\widehat{\mu}\\ \widehat{x}\end{pmatrix}\\ \Leftrightarrow &\begin{cases}\mu=\lambda^{-1}\widehat{\mu},\\ x=(\lambda I-A)^{-1}\left[\widehat{x}+\partial_\mu F(0,0)\lambda^{-1}\widehat{\mu}\right].\end{cases}\end{aligned}$$

It follows that

$$(\lambda I-\mathscr{A})^{-1}\begin{pmatrix}\widehat{\mu}\\ \widehat{x}\end{pmatrix} = \begin{pmatrix}\lambda^{-1}\widehat{\mu}\\ (\lambda I - A)^{-1}\left[\widehat{x}+\partial_\mu F(0,0)\lambda^{-1}\widehat{\mu}\right]\end{pmatrix},$$

so

$$\rho(\mathscr{A}) \supset \mathbb{C}\diagdown\sigma(A)\cup\{0\}.$$

It is clear that $0\in\sigma(\mathscr{A})$ because

$$\mathscr{A}\begin{pmatrix}\mu\\ (-A)^{-1}\partial_\mu F(0,0)\mu\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Moreover, if $\lambda \in \sigma(A)$, we have

$$(\lambda I - \mathscr{A}) \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{x} \end{pmatrix} \Leftrightarrow (\lambda I - A)x = \widehat{x}.$$

So $\lambda \in \sigma(\mathscr{A})$. $\square$

Lemma 6.2.3. *Let Assumptions 3.4.1 and 3.5.2 be satisfied. Then the linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{R}\times X \to \mathbb{R}\times X$ satisfies Assumptions 3.4.1 and 3.5.2. Moreover, we have*

$$T_{\mathscr{A}_0}(t) \begin{pmatrix} \mu \\ x \end{pmatrix} := \begin{pmatrix} \mu \\ T_{A_0}(t)x + S_A(t)\partial_\mu F(0,0)\mu \end{pmatrix} \tag{6.2.5}$$

and

$$S_{\mathscr{A}}(t) \begin{pmatrix} \mu \\ x \end{pmatrix} := \begin{pmatrix} t\mu \\ S_A(t)x + \int_0^t S_A(l)\partial_\mu F(0,0)\mu dl \end{pmatrix}. \tag{6.2.6}$$

Furthermore

$$\omega_{0,\mathrm{ess}}(\mathscr{A}_0) = \omega_{0,\mathrm{ess}}(A_0).$$

Proof. To prove that $\mathscr{A}$ satisfies Assumptions 3.4.1 and 3.5.2, it is sufficient to apply Theorem 3.5.1. Recall that

$$(\lambda I - A_0)^{-1} x = \int_0^{+\infty} e^{-\lambda t} T_{A_0}(t)xdt$$

and

$$(\lambda I - A)^{-1} x = \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t)xdt.$$

Thus, for each $\lambda > 0$ large enough,

$$\begin{aligned} &\int_0^{+\infty} e^{-\lambda t} \begin{pmatrix} \mu \\ T_{A_0}(t)x + S_A(t)\partial_\mu F(0,0)\mu \end{pmatrix} dt \\ &\quad = \begin{pmatrix} \lambda^{-1}\mu \\ (\lambda I - A_0)^{-1} x + \lambda^{-1} (\lambda I - A)^{-1} \partial_\mu F(0,0)\mu \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\lambda \int_0^{+\infty} e^{-\lambda t} \begin{pmatrix} t\mu \\ S_A(t)x + \int_0^t S_A(l)\partial_\mu F(0,0)\mu dl \end{pmatrix} dt \\ &\quad = \begin{pmatrix} \lambda^{-1}\mu \\ (\lambda I - A)^{-1} x + \lambda^{-1} (\lambda I - A)^{-1} \partial_\mu F(0,0)\mu \end{pmatrix}. \end{aligned}$$

It follows that $T_{\mathscr{A}_0}(t)$ and $S_{\mathscr{A}}(t)$ are defined respectively by (6.2.5) and (6.2.6).

By using formula (6.2.5) we deduce that

$$\left\| T_{\mathscr{A}_0}(t) \right\|_{\mathrm{ess}} = \left\| T_{A_0}(t) \right\|_{\mathrm{ess}}, \ \forall t \geq 0,$$

(since $\mu \in \mathbb{R}$) and it follows that

$$\omega_{0,\mathrm{ess}}(\mathscr{A}_0) = \lim_{t\to+\infty} \frac{\ln\left(\left\|T_{\mathscr{A}_0}(t)\right\|_{\mathrm{ess}}\right)}{t} = \lim_{t\to+\infty} \frac{\ln\left(\left\|T_{A_0}(t)\right\|_{\mathrm{ess}}\right)}{t} = \omega_{0,\mathrm{ess}}(A_0).$$

This completes the proof. $\square$

Next we compute the projectors on the generalized eigenspace associated to some eigenvalue of $\mathscr{A}$. Consider $\lambda_0 \in \{\lambda \in \sigma(\mathscr{A}) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(A_0)\} \setminus \{0\}$. Since

$$\omega_{0,\mathrm{ess}}(\mathscr{A}_0) = \omega_{0,\mathrm{ess}}(A_0) \text{ and } \sigma(\mathscr{A}) = \sigma(\mathscr{A}_0) = \sigma(A_0) \cup \{0\} = \sigma(A) \cup \{0\},$$

it follows that λ_0 is a pole of order k_0 of the resolvent of $\mathscr{A}_0$. Since $\mathrm{Re}(\lambda_0) > \omega_{0,\mathrm{ess}}(\mathscr{A}_0)$, by Lemma 4.2.13, we deduce that λ_0 is a pole of order k_0 of the resolvent of $\mathscr{A}$. Moreover, λ_0 is a pole of order k_1 of the resolvent of A. We have

$$(\lambda I - \mathscr{A})^{-1} = \sum_{k=-k_0}^{\infty} (\lambda - \lambda_0)^k B_{k,\lambda_0}^{\mathscr{A}}$$

and

$$(\lambda I - A)^{-1} = \sum_{k=-k_1}^{\infty} (\lambda - \lambda_0)^k B_{k,\lambda_0}^{A}$$

for $|\lambda - \lambda_0|$ small enough. The projector on the generalized eigenspace of A (respectively $\mathscr{A}$) associated to λ_0 is B_{-1,λ_0}^{A} (respectively $B_{-1,\lambda_0}^{\mathscr{A}}$).

We have $k_1 = k_0$. Indeed, we have

$$(\lambda I - \mathscr{A})^{-1}\begin{pmatrix}\mu \\ x\end{pmatrix} = \begin{pmatrix}\lambda^{-1}\mu \\ (\lambda I - A)^{-1}\left[x + \partial_\mu F(0,0)\lambda^{-1}\mu\right]\end{pmatrix},$$

so

$$\begin{aligned}
&\lim_{\lambda(\neq\lambda_0)\to\lambda_0} (\lambda-\lambda_0)^{k_1}(\lambda I - \mathscr{A})^{-1}\begin{pmatrix}\mu \\ x\end{pmatrix} \\
&= \lim_{\lambda(\neq\lambda_0)\to\lambda_0} \begin{pmatrix}(\lambda-\lambda_0)^{k_1}\lambda^{-1}\mu \\ (\lambda-\lambda_0)^{k_1}(\lambda I - A)^{-1}\left[x + \partial_\mu F(0,0)\lambda^{-1}\mu\right]\end{pmatrix} \\
&= \begin{pmatrix}0 \\ B_{-k_1,\lambda_0}^{A}\left[x + \partial_\mu F(0,0)\lambda_0^{-1}\mu\right]\end{pmatrix}.
\end{aligned}$$

Since the above limit exists it follows that $k_0 \leq k_1$, and since $B_{-k_1,\lambda_0}^{A} \neq 0$ it follows that $k_0 = k_1$. So we obtain the following lemma.

Lemma 6.2.4. *Let $\lambda_0 \in \{\lambda \in \sigma(\mathscr{A}) : \mathrm{Re}(\lambda) > \omega_{0,\mathrm{ess}}(A_0)\} \setminus \{0\}$. Then λ_0 is a pole of order k_0 of the resolvent of A if and only if λ_0 is a pole of order k_0 of the resolvent of $\mathscr{A}$.*

Now we compute

$$B_{k,\lambda_0}^{\mathscr{A}} := \frac{1}{2\pi i}\int_{S(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-(k+1)}(\lambda I - \mathscr{A})^{-1} d\lambda.$$

Set

$$\begin{pmatrix} \widehat{\mu} \\ \widehat{x} \end{pmatrix} := B^{\mathscr{A}}_{k,\lambda_0} \begin{pmatrix} \mu \\ x \end{pmatrix}.$$

Since

$$\begin{aligned}
(\lambda I - \mathscr{A})^{-1} \begin{pmatrix} \mu \\ x \end{pmatrix} &= \begin{pmatrix} \lambda^{-1}\mu \\ (\lambda I - A)^{-1} \left[x + \partial_\mu F(0,0)\, \lambda^{-1}\mu \right] \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sum_{k=-k_0} (\lambda - \lambda_0)^k B^A_{k,\lambda_0} x \end{pmatrix} \\
&\quad + \begin{pmatrix} \lambda^{-1}\mu \\ \sum_{k=-k_0} (\lambda - \lambda_0)^k B^A_{k,\lambda_0} \partial_\mu F(0,0)\, \lambda^{-1}\mu \end{pmatrix},
\end{aligned}$$

it follows that

$$\widehat{\mu} = \frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-k-1} \lambda^{-1} \mu d\lambda,$$

$$\begin{aligned}
\widehat{x} = &\sum_{j=-k_0} \frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-k-1} (\lambda - \lambda_0)^j B^A_{j,\lambda_0} x \, d\lambda \\
&+ \sum_{j=-k_0} \frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-k-1} (\lambda - \lambda_0)^j \lambda^{-1} B^A_{j,\lambda_0} \partial_\mu F(0,0)\, \mu d\lambda
\end{aligned}$$

and

$$\lambda^{-1} = \sum_{l=0}^{+\infty} (\lambda - \lambda_0)^l \frac{(-1)^l}{\lambda_0^{l+1}}.$$

Since

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-k-1} (\lambda - \lambda_0)^j \lambda^{-1} d\lambda \\
&= \frac{1}{2\pi i} \sum_{l=0}^{+\infty} \frac{(-1)^l}{\lambda_0^{l+1}} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{[j+l-(k+1)]} d\lambda
\end{aligned}$$

and

$$\begin{aligned}
\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{[j+l-(k+1)]} d\lambda &= \int_0^{2\pi} \left(\rho e^{i\theta} \right)^{[j+l-(k+1)]} i\rho e^{i\theta} d\theta \\
&= i\rho^{[j+l-k]} \int_0^{2\pi} \left(e^{i\theta} \right)^{[j+l-k]} d\theta \\
&= \begin{cases} 2\pi i & \text{if } l = k - j \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

it implies that

$$\frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{-k-1}(\lambda-\lambda_0)^j\lambda^{-1}d\lambda=\frac{(-1)^{k-j}}{\lambda_0^{k-j+1}}.$$

For $l=0$, it yields

$$\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{[j-(k+1)]}d\lambda=\begin{cases}2\pi i & \text{if } j=k\\ 0 & \text{otherwise}\end{cases}$$

and

$$\sum_{j=-k_0}\frac{1}{2\pi i}\int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+}(\lambda-\lambda_0)^{-k-1}(\lambda-\lambda_0)^j B^A_{j,\lambda_0}x\,d\lambda=B^A_{k,\lambda_0}x.$$

Therefore, we obtain

$$\widehat{\mu}=\begin{cases}\frac{(-1)^k\mu}{\lambda_0^{k+1}} & k\geq 0\\ 0 & k<0\end{cases}$$

and

$$\widehat{x}=B^A_{k,\lambda_0}x+\sum_{j=-k_0}^{k}\frac{(-1)^{k-j}}{\lambda_0^{k-j+1}}B^A_{j,\lambda_0}\partial_\mu F(0,0)\mu.$$

From the above computation we obtain the following lemma.

Lemma 6.2.5. *We have the following:*

(i) The projector on the generalized eigenspace of $\mathscr{A}$ associated to

$$\lambda_0\in\{\lambda\in\sigma(\mathscr{A}):\operatorname{Re}(\lambda)>0\},$$

a pole of order k_0 of the resolvent of $\mathscr{A}$ is given by

$$B^{\mathscr{A}}_{-1,\lambda_0}\begin{pmatrix}\mu\\ x\end{pmatrix}=\begin{pmatrix}0\\ B^A_{-1,\lambda_0}x+\sum_{j=-k_0}^{-1}\frac{(-1)^{-1-j}}{\lambda_0^{-j}}B^A_{j,\lambda_0}\partial_\mu F(0,0)\mu\end{pmatrix}.$$

(ii) $\lambda(0)$ and $\overline{\lambda(0)}$ are simple eigenvalues of $\mathscr{A}$ and the projectors on the generalized eigenspace of $\mathscr{A}$ associated to $\lambda(0)$ and $\overline{\lambda(0)}$ are given by

$$B^{\mathscr{A}}_{-1,\gamma}\begin{pmatrix}\mu\\ x\end{pmatrix}=\begin{pmatrix}0\\ B^A_{-1,\gamma}\left[x+\gamma^{-1}\partial_\mu F(0,0)\mu\right]\end{pmatrix} \text{ for } \gamma=\lambda(0) \text{ or } \gamma=\overline{\lambda(0)}.$$

The projector on the generalized eigenspace of $\mathscr{A}$ associated to 0 is given in the following lemma.

Lemma 6.2.6. 0 *is a simple eigenvalue of $\mathscr{A}$ and the projector on the generalized eigenspace of $\mathscr{A}$ associated to* 0 *is given by*

$$B^{\mathscr{A}}_{-1,0}\begin{pmatrix}\mu\\ x\end{pmatrix}=\begin{pmatrix}\mu\\ (-A)^{-1}\partial_\mu F(0,0)\mu\end{pmatrix}.$$

Proof. Since $0 \in \rho(A)$ it follows that

$$\lim_{\lambda \to 0} \lambda (\lambda I - \mathscr{A})^{-1} \begin{pmatrix} \mu \\ x \end{pmatrix} = \lim_{\lambda \to 0} \begin{pmatrix} \mu \\ \lambda (\lambda I - A)^{-1} x + (\lambda I - A)^{-1} \partial_\mu F(0,0) \mu \end{pmatrix}$$
$$= \begin{pmatrix} \mu \\ (-A)^{-1} \partial_\mu F(0,0) \mu \end{pmatrix} =: \Pi_0 \begin{pmatrix} \mu \\ x \end{pmatrix}.$$

This completes the proof. □

From the above results we obtain a state space decomposition with respect to the spectral properties of the linear operator $\mathscr{A}$. More precisely, the projector on the linear unstable manifold is given by

$$\Pi_u^{\mathscr{A}} = \sum_{\lambda \in \sigma(A): \mathrm{Re}(\lambda) > 0} B_{-1,\lambda}^{\mathscr{A}}$$

and the projector on the linear center manifold is defined by

$$\Pi_c^{\mathscr{A}} = B_{-1,0}^{\mathscr{A}} + B_{-1,\lambda(0)}^{\mathscr{A}} + B_{-1,\overline{\lambda(0)}}^{\mathscr{A}}.$$

Set

$$\Pi_s^{\mathscr{A}} := I - \left(\Pi_c^{\mathscr{A}} + \Pi_u^{\mathscr{A}} \right).$$

6.2.2 Hopf Bifurcation Theorem

The main result of this section is the following theorem.

Theorem 6.2.7 (Hopf Bifurcation). *Let Assumptions 3.4.1, 3.5.2 and 6.2.1 be satisfied. Then there exist a constant $\varepsilon^* > 0$ and three C^{k-1} maps, $\varepsilon \to \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to x_\varepsilon$ from $(0, \varepsilon^*)$ into $\overline{D(A)}$, and $\varepsilon \to T(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $T(\varepsilon)$-periodic function $u_\varepsilon \in C^k(\mathbb{R}, X_0)$, which is an integrated solution of (6.2.1) with the parameter value $\mu = \mu(\varepsilon)$ and the initial value $x = x_\varepsilon$. So for each $t \geq 0$, $u_\varepsilon(t)$ satisfies*

$$u_\varepsilon(t) = x_\varepsilon + A \int_0^t u_\varepsilon(l) dl + \int_0^t F(\mu(\varepsilon), u_\varepsilon(l)) dl.$$

Moreover, we have the following properties

(i) There exist a neighborhood N of 0 in X_0 and an open interval I in $\mathbb{R}$ containing 0, such that for $\widehat{\mu} \in I$ and any periodic solution $\widehat{u}(t)$ in N with minimal period $\widehat{T}$ close to $\frac{2\pi}{\omega(0)}$ of (6.2.1) for the parameter value $\widehat{\mu}$, there exists $\varepsilon \in (0, \varepsilon^*)$ such that $\widehat{u}(t) = u_\varepsilon(t + \theta)$ (for some $\theta \in [0, \gamma(\varepsilon))$), $\mu(\varepsilon) = \widehat{\mu}$, and $T(\varepsilon) = \widehat{T}$;
(ii) The map $\varepsilon \to \mu(\varepsilon)$ is a C^{k-1} function and we have the Taylor expansion

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0,\varepsilon^*),$$

where $[\frac{k-2}{2}]$ *is the integer part of* $\frac{k-2}{2}$*;*

(iii) The period $\gamma(\varepsilon)$ *of* $t \to u_\varepsilon(t)$ *is a* C^{k-1} *function and*

$$T(\varepsilon) = \frac{2\pi}{\omega(0)}[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \tau_{2n}\varepsilon^{2n}] + O(\varepsilon^{k-1}), \forall \varepsilon \in (0,\varepsilon^*),$$

where $\omega(0)$ *is the imaginary part of* $\lambda(0)$ *defined in Assumption 6.2.1;*

(iv) The Floquet exponent $\beta(\varepsilon)$ *is a* C^{k-1} *function satisfying* $\beta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and having the Taylor expansion*

$$\beta(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \beta_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0,\varepsilon^*).$$

The periodic solution $x_\varepsilon(t)$ *is orbitally asymptotically stable with asymptotic phase if* $\beta(\varepsilon) < 0$ *and unstable if* $\beta(\varepsilon) > 0$.

Proof. By using the results of Section 6.2.1, we deduce that $\mathscr{A}$ satisfies Assumption 3.5.2 and we can apply Theorem 6.1.21 to the system

$$\frac{dv(t)}{dt} = \mathscr{A}v(t) + \mathscr{F}(v(t)),\ t \geq 0,\ \ v(0) = v_0 \in \overline{D(\mathscr{A})}. \tag{6.2.7}$$

Set

$$\mathscr{X}_{0c} = \Pi_c^{\mathscr{A}}\left(\mathbb{R}\times\overline{D(A)}\right)$$

and

$$\mathscr{X}_{0h} = \left(I - \Pi_c^{\mathscr{A}}\right)\left(\mathbb{R}\times\overline{D(A)}\right).$$

By using Theorem 6.1.21, we can find $\Psi \in C_b^k(\mathscr{X}_{0c}, \mathscr{X}_{0h})$ such that the manifold

$$M = \{x_c + \Psi(x_c) : x_c \in \mathscr{X}_{0c}\}$$

is locally invariant by the semiflow generated by (6.2.7).

By applying $\Pi_c^{\mathscr{A}}$ to both sides of (6.2.7), we obtain the reduced system in $\mathscr{X}_{0c} = \Pi_c^{\mathscr{A}}(\mathbb{R}\times X)$:

$$\frac{d}{dt}\begin{pmatrix}\mu(t)\\ x_c(t)\end{pmatrix} = \mathscr{A}_{0c}\begin{pmatrix}\mu(t)\\ x_c(t)\end{pmatrix} + \Pi_c^{\mathscr{A}}\mathscr{F}\left(\begin{pmatrix}\mu(t)\\ x_c(t)\end{pmatrix} + \Psi\begin{pmatrix}\mu(t)\\ x_c(t)\end{pmatrix}\right), \tag{6.2.8}$$

where

$$\begin{pmatrix}\mu(t)\\ x_c(t)\end{pmatrix} = \Pi_c^{\mathscr{A}}\begin{pmatrix}\mu(t)\\ u(t)\end{pmatrix}.$$

Now since $\begin{pmatrix} \mu \\ 0 \end{pmatrix}$ is a branch of the equilibrium of (6.2.7), it corresponds to a branch of the equilibrium $\begin{pmatrix} \mu \\ \overline{x}_c(\mu) \end{pmatrix} = \Pi_c^{\mathscr{A}} \begin{pmatrix} \mu \\ 0 \end{pmatrix}$ of system (6.2.8). Applying Proposition 6.1.22 to system (6.2.7) and using Assumption 6.2.1, we deduce that the spectrum of the linearized equation of (6.2.8) around $\begin{pmatrix} \mu \\ \overline{x}_c(\mu) \end{pmatrix}$ consists of

$$\left\{0, \lambda(\mu), \overline{\lambda}(\mu)\right\}.$$

It follows that we can apply the Hopf bifurcation theorem in the book by Hassard et al. [181] to system (6.2.8). The proof is complete. □

Remark 6.2.8. In Assumption 6.2.1, if we only assume that $k \geq 2$ and the condition (6.2.2) is replaced by

$$\sigma(A_0) \cap i\omega(0)\mathbb{Z} = \left\{\lambda(0), \overline{\lambda(0)}\right\}$$

(i.e. the spectrum of A_0 does not contain a multiple of $\lambda(0)$). Then by the Hopf bifurcation theorem of Crandall and Rabinowitz [76], we deduce that the assertion (i) of Theorem 6.2.7 holds.

6.3 Normal Form Theory

6.3.1 Nonresonant Type Results

Let $m \geq 1$ be a given integer. Let Y be a closed subspace of X. Let $\mathscr{L}_s\left(X_0^m, Y\right)$ be the space of bounded m-linear symmetric maps from $X_0^m = X_0 \times X_0 \times ... \times X_0$ into Y and $\mathscr{L}_s(X_c^m, D(A))$ be the space of bounded $m-$linear symmetric maps from $X_c^m = X_c \times X_c \times ... \times X_c$ into $D(A)$; that is, for each $L \in \mathscr{L}_s(X_c^m, D(A))$,

$$L(x_1, ..., x_m) \in D(A), \ \forall (x_1, ..., x_m) \in X_c^m,$$

and the maps $(x_1, .., x_m) \to L(x_1, ..., x_m)$ and $(x_1, .., x_m) \to AL(x_1, ..., x_m)$ are $m-$linear bounded from X_c^m into X. Let $\mathscr{L}_s(X_c^m, X_h \cap D(A))$ be the space of bounded $m-$linear symmetric maps from $X_c^m = X_c \times X_c \times ... \times X_c$ into $D(A_h) = X_h \cap D(A)$ which belongs to $\mathscr{L}_s(X_c^m, D(A))$.

Let $k = \dim(X_c)$ and Y be a subspace of X. We define $V^m(X_c, Y)$ the linear space of homogeneous polynomials of degree m. More precisely, given a basis $\left\{b_j\right\}_{j=1,...,k}$ of X_c, $V^m(X_c, Y)$ is the space of finite linear combinations of maps of the form

$$x_c = \sum_{j=1}^{k} x_j b_j \in X_c \to x_1^{n_1} x_2^{n_2} ... x_k^{n_k} V$$

with

$$n_1 + n_2 + ... + n_k = m \text{ and } V \in Y.$$

Define a map $\mathscr{G} : \mathscr{L}_s(X_c^m, Y) \to V^m(X_c, Y)$ by

$$\mathscr{G}(L)(x_c) = L(x_c, \cdots, x_c), \ \forall L \in \mathscr{L}_s(X_c^m, Y).$$

Let $G \in V^m(X_c, Y)$ be given. We have $G(x_c) = \frac{1}{m!}D^m G(0)(x_c, \cdots, x_c)$. So

$$\mathscr{G}^{-1}(G) = \frac{1}{m!}D^m G(0).$$

In other words, we have

$$L = \frac{1}{m!}D^m G(0) \Leftrightarrow G(x_c) = L(x_c, ..., x_c), \ \forall x_c \in X_c.$$

It follows that $\mathscr{G}$ is a bijection from $\mathscr{L}_s(X_c^m, Y)$ into $V^m(X_c, Y)$. So we can also define $V^m(X_c, D(A))$ as

$$V^m(X_c, D(A)) := \mathscr{G}(\mathscr{L}_s(X_c^m, D(A))).$$

In order to use the usual formalism in the context of normal form theory, we now define the Lie bracket (Guckenheimer and Holmes [155, p.141]). Recall that

$$X_c = X_{0c} \subset D(A_0) \subset D(A),$$

so the following definition makes sense.

Definition 6.3.1. Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. Then for each $G \in V^m(X_c, D(A))$, we define the *Lie bracket*

$$[A, G](x_c) := DG(x_c)(Ax_c) - AG(x_c), \ \forall x_c \in X_c. \tag{6.3.1}$$

Recall that $A_c \in \mathscr{L}(X_c)$ is the part of A in X_c, we obtain

$$[A, G](x_c) = DG(x_c)(A_c x_c) - AG(x_c), \forall x_c \in X_c.$$

Setting $L := \frac{1}{m!}D^m G(0) \in \mathscr{L}_s(X_c^m, D(A) \cap X_h)$. We also have

$$DG(x_c)(y) = mL(y, x_c, ..., x_c), \ \ DG(x_c)A_c x_c = mL(A_c x_c, x_c, ..., x_c),$$

and

$$[A, G](x_c) = \frac{d}{dt}\left[L(e^{A_c t}x_c, \ldots, e^{A_c t}x_c)\right](0) - AL(x_c, \ldots, x_c). \tag{6.3.2}$$

We consider two cases when G belongs to different subspaces, namely, $G \in V^m(X_c, D(A) \cap X_h)$ and $G \in V^m(X_c, D(A))$, respectively.

(i) $G \in V^m(X_c, D(A) \cap X_h)$. We consider the change of variables

$$v := u - G(\Pi_c u) \Leftrightarrow \begin{cases} \Pi_c v = \Pi_c u \\ \Pi_h v = \Pi_h u - G(\Pi_c u) \end{cases} \Leftrightarrow u = v + G(\Pi_c v). \tag{6.3.3}$$

Then

$$G(x_c) := L(x_c, x_c, ..., x_c), \forall x_c \in X_c.$$

The map $x_c \to AG(x_c)$ is differentiable and

$$D(AG)(x_c)(y) = ADG(x_c)(y) = mAL(y, x_c, ..., x_c).$$

Define a map $\xi : X \to X$ by

$$\xi(x) := x + G(\Pi_c x), \forall x \in X.$$

Since the range of G is included in X_h, we obtain the following equivalence

$$y = \xi(x) \Leftrightarrow x = \xi^{-1}(y),$$

where

$$\xi^{-1}(y) := y - G(\Pi_c y), \forall y \in X,$$

and

$$\Pi_c \xi^{-1}(x) = \Pi_c x, \forall x \in X.$$

Finally, since $G(x) \in D(A)$, we have

$$\xi\left(\overline{D(A)}\right) \subset \overline{D(A)} \text{ and } \xi^{-1}\left(\overline{D(A)}\right) \subset \overline{D(A)}.$$

The following result justifies the change of variables (6.3.3).

Lemma 6.3.2. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. Let $L \in \mathscr{L}_s(X_c^m, X_h \cap D(A))$. Assume that $u \in C([0,\tau], X)$ is an integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + F(u(t)), \ t \in [0,\tau], \ u(0) = x \in \overline{D(A)}. \tag{6.3.4}$$

Then $v(t) = \xi^{-1}(u(t))$ is an integrated solution of the system

$$\frac{dv(t)}{dt} = Av(t) + H(v(t)), \ t \in [0,\tau], \ v(0) = \xi^{-1}(x) \in \overline{D(A)}, \tag{6.3.5}$$

where $H : \overline{D(A)} \to X$ is the map defined by

$$H(\xi(x)) = F(\xi(x)) - [A, G](\Pi_c x) - DG(\Pi_c x)[\Pi_c F(\xi(x))].$$

Conversely, if $v \in C([0,\tau], X)$ is an integrated solution of (6.3.5), then $u(t) = \xi(v(t))$ is an integrated solution of (6.3.4).

Proof. Assume that $u \in C([0,\tau], X)$ is an integrated solution of the system (6.3.4); that is,

$$\int_0^t u(l)dl \in D(A), \forall t \in [0,\tau],$$

and

$$u(t) = x + A\int_0^t u(l)dl + \int_0^t F(u(l))dl, \forall t \in [0,\tau].$$

Set

$$v(t) = \xi^{-1}(u(t)), \forall t \in [0,\tau].$$

We have

$$\begin{aligned} A\int_0^t v(l)dl &= A\int_0^t u(l)dl - \int_0^t AG(\Pi_c u(l))\,dl \\ &= u(t) - x - \int_0^t F(u(l))dl - \int_0^t AG(\Pi_c u(l))\,dl \\ &= u(t) - G(\Pi_c u(t)) - (x - G(\Pi_c x)) \\ &\quad + (G(\Pi_c u(t)) - G(\Pi_c x)) \\ &\quad - \int_0^t F(u(l))dl - \int_0^t AG(\Pi_c u(l))\,dl \\ &= v(t) - \xi^{-1}(x) + (G(\Pi_c u(t)) - G(\Pi_c x)) \\ &\quad - \int_0^t F(u(l))dl - \int_0^t AG(\Pi_c u(l))\,dl. \end{aligned}$$

Since $\dim(X_c) < +\infty$, $t \to \Pi_c u(t)$ satisfies the following ordinary differential equations

$$\frac{d\Pi_c u(t)}{dt} = A_{0c}\Pi_c u(t) + \Pi_c F(u(t)).$$

By integrating both sides of the above ordinary differential equations, we obtain

$$\begin{aligned} G(\Pi_c u(t)) - G(\Pi_c x) &= \int_0^t DG(\Pi_c u(l))\left(\frac{d\Pi_c u(l)}{dl}\right)dl \\ &= \int_0^t DG(\Pi_c u(l))(A_{0c}\Pi_c u(l)) + DG(\Pi_c u(l))(\Pi_c F(u(l)))\,dl. \end{aligned}$$

It follows that

$$\begin{aligned} A\int_0^t v(l)dl &= v(t) - \xi(x) \\ &\quad + \int_0^t DG(\Pi_c u(l))\left[A_{0c}\Pi_c u(l) + \Pi_c F(u(l))\right]dl \\ &\quad - \int_0^t F(u(l))dl - \int_0^t AG(\Pi_c u(l))\,dl. \end{aligned}$$

Thus

$$v(t) = \xi(x) + A\int_0^t v(l)dl + \int_0^t H(v(l))dl,$$

in which

$$\begin{aligned} H(\xi(x)) = {} & F(\xi(x)) + AG(\Pi_c \xi(x)) \\ & - DG(\Pi_c \xi(x))\left[A_c \Pi_c \xi(x) + \Pi_c F(\xi(x))\right]. \end{aligned}$$

Since $\Pi_c \xi = \Pi_c$, the first implication follows. The converse follows from the first implication by replacing F by H and ξ by ξ^{-1}. $\square$

Set for each $\eta > 0$,

$$BC^{\eta}(\mathbb{R}, X) := \left\{ f \in C(\mathbb{R}, X) : \sup_{t \in \mathbb{R}} e^{-\eta |t|} \|f(t)\| < +\infty \right\}.$$

We have the following lemma.

Lemma 6.3.3. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. If*

$$f(t) = t^k e^{\lambda t} x$$

for some $k \in \mathbb{N}$, $\lambda \in i\mathbb{R}$, and $x \in X$, then

$$(K_u + K_s)(\Pi_h f)(0) = (-1)^k k! (\lambda I - A_h)^{-(k+1)} \Pi_h x \in D(A_h) \subset D(A).$$

Proof. We have

$$\begin{aligned}
K_u(f)(0) &= -\int_0^{+\infty} e^{\lambda l} l^k e^{-A_{0u} l} \Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k} \int_0^{+\infty} e^{\lambda l} e^{-A_{0u} l} \Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k} (-\lambda I + A_{0u})^{-1} \Pi_u x \\
&= \frac{d^k}{d\lambda^k} (\lambda I - A_{0u})^{-1} \Pi_u x \\
&= (-1)^k k! (\lambda I - A_{0u})^{-(k+1)} \Pi_u x.
\end{aligned}$$

Similarly, we have for $\mu > \omega_A$ that

$$\begin{aligned}
(\mu I - A)^{-1} K_s(f)(0) &= \lim_{\tau \to -\infty} (\mu I - A)^{-1} \Pi_s (S_A \diamond f(\tau + .))(-\tau) \\
&= \lim_{\tau \to -\infty} \int_0^{-\tau} T_{A_{0s}}(-\tau - s)(\mu I - A)^{-1} \Pi_s f(s + \tau) ds \\
&= \lim_{r \to +\infty} \int_0^r T_{A_{0s}}(r - s)(\mu I - A)^{-1} \Pi_s f(s - r) ds \\
&= \int_0^{+\infty} T_{A_{0s}}(l)(\mu I - A)^{-1} \Pi_s f(-l) dl.
\end{aligned}$$

So we obtain that

$$\begin{aligned}
(\mu I - A)^{-1} K_s(f)(0) &= \int_0^{+\infty} (-l)^k e^{-\lambda l} T_{A_0}(l)(\mu I - A)^{-1} \Pi_s x dl \\
&= \frac{d^k}{d\lambda^k} (\lambda I - A_0)^{-1} (\mu I - A)^{-1} \Pi_s x
\end{aligned}$$

$$= (-1)^k k! \, (\lambda I - A_0)^{-(k+1)} \, (\mu I - A)^{-1} \, \Pi_s x$$
$$= (\mu I - A)^{-1} \, (-1)^k k! \, (\lambda I - A_s)^{-(k+1)} \, \Pi_s x.$$

Since $(\mu I - A)^{-1}$ is one-to-one, we deduce that

$$K_s(f)(t) = (-1)^k k! \, (\lambda I - A_s)^{-(k+1)} \, \Pi_s x$$

and the result follows. $\square$

The first result of this section is the following proposition which is related to nonresonant normal forms for ordinary differential equations (see Guckenheimer and Holmes [155], Chow and Hale [62], and Chow et al. [63]).

Proposition 6.3.4. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. For each $R \in V^m(X_c, X_h)$, there exists a unique map $G \in V^m(X_c, X_h \cap D(A))$ such that*

$$[A, G](x_c) = R(x_c), \forall x_c \in X_c. \tag{6.3.6}$$

Moreover, (6.3.6) is equivalent to

$$G(x_c) = (K_u + K_s)\,(R(e^{A_c \cdot} x_c))(0),$$

or

$$L(x_1, \ldots, x_m) = (K_u + K_s)\,(H\left(e^{A_c \cdot} x_1, \ldots, e^{A_c \cdot} x_m\right))(0),$$

with $L := \frac{1}{m!} D^m G(0)$ and $H := \frac{1}{m!} D^m R(0)$.

Proof. Assume first that $G \in V^m(X_c, X_h \cap D(A))$ satisfies (6.3.6). Then $L = \frac{1}{m!} D^m G(0) \in \mathscr{L}_s(X_c^m, X_h \cap D(A))$ satisfies

$$\frac{d}{dt}\left[L(e^{A_c t} x_1, ..., e^{A_c t} x_m)\right](0) = A_h L(x_1, ..., x_m) + H(x_1, ..., x_m),$$

where $H = \frac{1}{m!} D^m R(0) \in \mathscr{L}_s(X_c^m, X_h)$. Then (6.3.6) is satisfied if and only if for each $(x_1, ..., x_m) \in X_c^m$ and each $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}\left[L(e^{A_c t} x_1, ..., e^{A_c t} x_m)\right](t) = A_h L(e^{A_c t} x_1, ..., e^{A_c t} x_m) \\ + H(e^{A_c t} x_1, ..., e^{A_c t} x_m). \end{aligned} \tag{6.3.7}$$

Set

$$v(t) := L(e^{A_c t} x_1, ..., e^{A_c t} x_m), \ \forall t \in \mathbb{R}$$

and

$$w(t) := H(e^{A_c t} x_1, ..., e^{A_c t} x_m), \ \forall t \in \mathbb{R}.$$

The Cauchy problem (6.3.7) can be rewritten as

$$\frac{dv(t)}{dt} = A_h v(t) + w(t), \ \ \forall t \in \mathbb{R}. \tag{6.3.8}$$

Since L and H are bounded multilinear maps and $\sigma(A_{0c}) \subset i\mathbb{R}$, it follows that for each $\eta > 0$,

$$v \in BC^{\eta}(\mathbb{R}, X) \text{ and } w \in BC^{\eta}(\mathbb{R}, X).$$

Let $\eta \in \left(0, \min\left(-\omega_0(A_{0s}), \inf_{\lambda \in \sigma(A_{0u})} \mathrm{Re}(\lambda)\right)\right)$. By projecting (6.3.8) on X_u, we have

$$\frac{d\Pi_u v(t)}{dt} = A_u \Pi_u v(t) + \Pi_u w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_u v(t) = e^{A_u(t-s)} \Pi_u v(s) + \int_s^t e^{A_u(t-l)} \Pi_u w(l) dl,$$
$$\Pi_u v(s) = e^{-A_u(t-s)} \Pi_u v(t) - \int_s^t e^{-A_u(l-s)} \Pi_u w(l) dl.$$

By using the fact that $v \in BC^{\eta}(\mathbb{R}, X)$, we obtain when t goes to $+\infty$ that

$$\Pi_u v(s) = K_u(\Pi_u w)(s), \ \forall s \in \mathbb{R}.$$

Thus, for $s = 0$ we have

$$\Pi_u L(x_1, ..., x_m) = K_u(\Pi_u H(e^{A_c \cdot} x_1, ..., e^{A_c \cdot} x_m))(0). \tag{6.3.9}$$

By projecting (6.3.8) on X_s, we obtain

$$\frac{d\Pi_s v(t)}{dt} = A_s \Pi_s v(t) + \Pi_s w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_s v(t) = T_{A_s}(t-s) \Pi_s v(s) + (S_{A_s} \diamond \Pi_s w(.+s))(t-s).$$

By using the fact that $v \in BC^{\eta}(\mathbb{R}, X)$, we have when s goes to $-\infty$ that

$$\Pi_s v(t) = K_s(\Pi_s w)(t), \ \forall t \in \mathbb{R}.$$

Thus, for $t = 0$ it follows that

$$\Pi_s L(x_1, ..., x_m) = K_s(\Pi_s H(e^{A_c \cdot} x_1, ..., e^{A_c \cdot} x_m))(0). \tag{6.3.10}$$

Summing up (6.3.9) and (6.3.10), we deduce that

$$L(x_1, ..., x_m) = (K_u + K_s)(H(e^{A_c \cdot} x_1, ..., e^{A_c \cdot} x_m))(0). \tag{6.3.11}$$

Conversely, assume that $L(x_1, ..., x_m)$ is defined by (6.3.11) and set

$$v(t) := (K_u + K_s)(H\left(e^{A_c(t+.)} x_1, ..., e^{A_c(t+.)} x_m\right))(0), \ \forall t \in \mathbb{R}.$$

Then we have
$$v(t) = L(e^{A_c t}x_1, ..., e^{A_c t}x_m), \ \forall t \in \mathbb{R}.$$
Moreover, using Lemma 6.1.6-(iii) and Lemma 6.1.7-(iii), we deduce that for each $t, s \in \mathbb{R}$ with $t \geq s$,
$$v(t) = T_{A_0}(t-s)v(s) + (S_A \diamond w(.+s))(t-s),$$
or equivalently,
$$v(t) = v(s) + A\int_s^t v(l)dl + \int_s^t w(l)dl.$$
Since $t \to v(t)$ is continuously differentiable and A is closed, we deduce that
$$v(t) \in D(A), \ \forall t \in \mathbb{R},$$
and
$$\frac{dv(t)}{dt} = Av(t) + w(t), \ \forall t \in \mathbb{R}.$$
The result follows. $\square$

Remark 6.3.5. (An explicit formula for L) Since $n := \dim(X_c) < +\infty$, we can find a basis $\{e_1, ..., e_n\}$ of X_c such that the matrix of A_c (with respect to this basis) is reduced to the Jordan's form. Then for each $x_c \in X_c$, $e^{A_c t}x_c$ is a linear combination of elements of the form
$$t^k e^{\lambda t} x_j$$
for some $k \in \{1, ..., n\}$, some $\lambda \in \sigma(A_c) \subset i\mathbb{R}$, and some $x_j \in \{e_1, ..., e_n\}$. Let $\lambda_1, ..., \lambda_m \in \sigma(A_c) \subset i\mathbb{R}$, $x_1, ..., x_m \in \{e_1, ..., e_n\}$, $k_1, ..., k_m \in \{1, ..., n\}$. Define
$$f(t) := H\left(t^{k_1} e^{\lambda_1 t \cdot} x_1, ..., t^{k_m} e^{\lambda_m t \cdot} x_m\right), \forall t \in \mathbb{R}.$$
Since H is m-linear, we obtain
$$f(t) = t^k e^{\lambda} y$$
with
$$k = k_1 + k_2 + ... + k_m, \ \ \lambda = \lambda_1 + ... + \lambda_m,$$
and
$$y = H(x_1, ..., x_m).$$
Now by using Lemma 6.3.3, we obtain the explicit formula
$$(K_u + K_s)(H\left((.)^{k_1} e^{\lambda_1 \cdot} x_1, ..., (.)^{k_m} e^{\lambda_m \cdot} x_m\right))(0) = (-1)^k k! (\lambda I - A_h)^{-(k+1)} \Pi_h y \in D(A).$$

(ii) $G \in V^m(X_c, D(A))$. From (6.3.2), for each $H \in V^m(X_c, X)$, to find $G \in V^m(X_c, D(A))$ satisfying
$$[A, G] = H, \tag{6.3.12}$$

is equivalent to find $L \in \mathscr{L}_s(X_c^m, D(A))$ satisfying

$$\frac{d}{dt}\left[L(e^{A_c t}x_1, \ldots, e^{A_c t}x_m)\right]_{t=0} = AL(x_1, \ldots, x_m) + \widehat{H}(x_1, \ldots, x_m) \tag{6.3.13}$$

for each $(x_1, \ldots, x_m) \in X_c^m$ with

$$\mathscr{G}(\widehat{H}) = H.$$

Define $\Theta_m^c : V^m(X_c, X_c) \to V^m(X_c, X_c)$ by

$$\Theta_m^c(G_c) := [A_c, G_c], \forall G_c \in V^m(X_c, X_c) \tag{6.3.14}$$

and $\Theta_m^h : V^m(X_c, X_h \cap D(A)) \to V^m(X_c, X_h)$ by

$$\Theta_m^h(G_h) := [A, G_h], \forall G_h \in V^m(X_c, X_h \cap D(A)).$$

We decompose $V^m(X_c, X_c)$ into the direct sum

$$V^m(X_c, X_c) = \mathscr{R}_m^c \oplus \mathscr{C}_m^c, \tag{6.3.15}$$

where

$$\mathscr{R}_m^c := R(\Theta_m^c)$$

is the range of Θ_m^c, and $\mathscr{C}_m^c$ is some complementary space of $\mathscr{R}_m^c$ into $V^m(X_c, X_c)$.

The range of the linear operator Θ_m^c can be characterized by using the so called non-resonance theorem. The second result of this section is the following theorem.

Proposition 6.3.6. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. Let $H \in R_m^c \oplus V^m(X_c, X_h)$. Then there exists $G \in V^m(X_c, D(A))$ (non-unique in general) satisfying*

$$[A, G] = H. \tag{6.3.16}$$

Furthermore, if $N(\Theta_m^c) = \{0\}$ (the null space of Θ_m^c), then G is uniquely determined.

Proof. By projecting on X_c and X_h and using the fact that $X_c \subset D(A)$, it follows that solving system (6.3.12) is equivalent to find $G_c \in V^m(X_c, X_c)$ and $G_h \in V^m(X_c, X_h \cap D(A))$ satisfying

$$[A_c, G_c] = \Pi_c H \tag{6.3.17}$$

and

$$[A, G_h] = \Pi_h H. \tag{6.3.18}$$

Now it is clear that we can solve (6.3.17). Moreover, by using the equivalence between (6.3.12) and (6.3.13), we can apply Proposition 6.3.4 and deduce that (6.3.18) can be solved. □

Remark 6.3.7. In practice, we often have

$$N(\Theta_m^c) \cap R(\Theta_m^c) = \{0\},$$

In this case, a natural splitting of $V^m(X_c, X_c)$ will be

$$V^m(X_c, X_c) = R(\Theta_m^c) \oplus N(\Theta_m^c).$$

Define $P_m : V^m(X_c, X) \to V^m(X_c, X)$ the bounded linear projector satisfying

$$\mathscr{P}_m (V^m(X_c, X)) = \mathscr{R}_m^c \oplus V^m(X_c, X_h), \text{ and } (I - \mathscr{P}_m)(V^m(X_c, X)) = \mathscr{C}_m^c.$$

Again consider the Cauchy problem (6.3.3). Assume that $DF(0) = 0$. Without loss of generality we also assume that for some $m \in \{2, ..., k\}$,

$$\Pi_h D^j F(0) \mid_{X_c \times X_c \times ... \times X_c} = 0, \; \mathscr{G}\left(\Pi_c D^j F(0) \mid_{X_c \times X_c \times ... \times X_c}\right) \in \mathscr{C}_j^c, \qquad (C_{m-1})$$

for each $j = 1, ..., m-1$.

Consider the change of variables

$$u(t) = w(t) + G(\Pi_c w(t)) \tag{6.3.19}$$

and the map $I + \frac{1}{m!} G \circ \Pi_c : \overline{D(A)} \to \overline{D(A)}$ is locally invertible around 0. We will show that we can find $G \in V^m(X_c, D(A))$ such that after the change of variables (6.3.19) we can rewrite the system (6.3.3) as

$$\frac{dw(t)}{dt} = Aw(t) + H(w(t)) \text{ for } t \geq 0, \text{ and } w(0) = (I + G \circ \Pi_c)x \in \overline{D(A)}, \tag{6.3.20}$$

where H satisfies the condition (C_m). This will provide a normal form method which is analogous to the one proposed by Faria and Magalhães [136].

Lemma 6.3.8. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. Let $G \in V^m(X_c, D(A))$. Assume that $u \in C([0, \tau], X)$ is an integrated solution of the Cauchy problem (6.3.3). Then $w(t) = (I + G \circ \Pi_c)^{-1}(u(t))$ is an integrated solution of the system (6.3.20), where $H : \overline{D(A)} \to X$ is the map defined by*

$$H(w(t)) = F(w(t)) - [A, G](\Pi_c w(t)) + O(\|w(t)\|^{m+1}).$$

Conversely, if $w \in C([0, \tau], X)$ is an integrated solution of (6.3.20), then $u(t) = (I + G \circ \Pi_c) w(t)$ is an integrated solution of (6.3.3).

Lemma 6.3.8 can be proved similarly as Lemma 6.3.2, here we omit it.

Proposition 6.3.9. *Let Assumptions 3.4.1, 3.5.2 and 6.1.1 be satisfied. Let $r > 0$ and let $F : B_{X_0}(0, r) \to X$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{X_0}(0, r)$ with $F(0) = 0$ and $DF(0) = 0$. Let $m \in \{2, ..., k\}$ be such that F satisfies the condition (C_{m-1}). Then there exists a map $G \in V^m(X_c, D(A))$ such that after the change of variables*

$$u(t) = w(t) + G(\Pi_c w(t)),$$

we can rewrite system (6.3.3) as (6.3.20) and H satisfies the condition (C_m), where

$$H(w(t)) = F(w(t)) - [A,G](\Pi_c w(t)) + O(\|w(t)\|^{m+1}).$$

Proof. Let $x_c \in X_c$ be fixed. We have

$$H(x_c) = F(x_c) - [A,G](\Pi_c x_c) + O(\|x_c\|^{m+1}).$$

It follows that

$$\begin{aligned} H(x_c) = & \frac{1}{2!}D^2F(0)(x_c,x_c) + ... + \frac{1}{(m-1)!}D^{m-1}F(0)(x_c,\dots,x_c) \\ & + \mathscr{P}_m\left[\frac{1}{m!}D^mF(0)(x_c,\dots,x_c)\right] + (I-\mathscr{P}_m)\left[\frac{1}{m!}D^mF(0)(x_c,\dots,x_c)\right] \\ & - [A,G](x_c) + O(\|x_c\|^{m+1}) \end{aligned}$$

since $DF(0) = 0$. Moreover, by using Proposition 6.3.6 we obtain that there exists a map $G \in V^m(X_c, D(A))$ such that

$$[A,G](x_c) = \mathscr{P}_m\left[\frac{1}{m!}D^mF(0)(x_c,\dots,x_c)\right].$$

Hence,

$$\begin{aligned} H(x_c) = & \frac{1}{2!}D^2F(0)(x_c,x_c) + ... + \frac{1}{(m-1)!}D^{m-1}F(0)(x_c,\dots,x_c) \\ & + (I-\mathscr{P}_m)\left[\tfrac{1}{m!}D^mF(0)(x_c,\dots,x_c)\right] + O(\|x_c\|^{m+1}). \end{aligned} \tag{6.3.21}$$

By the assumption, we have for all $j = 1,...,m-1$ that

$$\Pi_h D^j H(0) \mid_{X_c\times X_c\times\ldots\times X_c} = \Pi_h D^j F(0) \mid_{X_c\times X_c\times\ldots\times X_c} = 0$$

and

$$\mathscr{G}\left(\Pi_c D^j H(0) \mid_{X_c\times X_c\times\ldots\times X_c}\right) = \mathscr{G}\left(\Pi_c D^j F(0) \mid_{X_c\times X_c\times\ldots\times X_c}\right) \in \mathscr{C}_j^c.$$

Now by using (6.3.21), we have

$$\frac{1}{m!}\Pi_h D^m H(0) \mid_{X_c\times X_c\times\ldots\times X_c} = \Pi_h \mathscr{G}^{-1}\left[(I-\mathscr{P}_m)\left(\frac{1}{m!}D^mF(0)(x_c,\dots,x_c)\right)\right] = 0$$

and

$$\mathscr{G}(\Pi_c D^m H(0) \mid_{X_c\times X_c\times\ldots\times X_c}) = \mathscr{G}\left\{\Pi_c \mathscr{G}^{-1}\left[(I-\mathscr{P}_m)(D^mF(0)(x_c,\dots,x_c))\right]\right\} \in \mathscr{C}_m^c.$$

The result follows. $\square$

6.3.2 Normal Form Computation

In this subsection we provide the method to compute the Taylor's expansion at any order and normal form of the reduced system of a system topologically equivalent to the original system:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(u(t)), \ t \geq 0, \\ u(0) = x \in \overline{D(A)}. \end{cases} \tag{6.3.22}$$

Assumption 6.3.10. Assume that $F \in C^k\left(\overline{D(A)}, X\right)$ for some integer $k \geq 2$ with

$$F(0) = 0 \text{ and } DF(0) = 0.$$

Set

$$F_1 := F.$$

Once again we consider two cases; namely, $G \in V^m(X_c, D(A) \cap X_h)$ and $G \in V^m(X_c, D(A))$, respectively.

(i) $G \in V^m(X_c, D(A) \cap X_h)$. For $j = 2, \ldots, k$, we apply Proposition 6.3.4. Then there exists a unique function $G_j \in V^j(X_c, X_h \cap D(A))$ satisfying

$$[A, G_j](x_c) = \frac{1}{j!} \Pi_h D^j F_{j-1}(0)(x_c, \ldots, x_c), \forall x_c \in X_c. \tag{6.3.23}$$

Define $\xi_j : X \to X$ and $\xi_j^{-1} : X \to X$ by

$$\xi_j(x) := x + G_j(\Pi_c x) \text{ and } \xi_j^{-1}(x) := x - G_j(\Pi_c x), \forall x \in X.$$

Then

$$F_j(x) := F_{j-1}(\xi_j(x)) - [A, G_j](\Pi_c x) - DG_j(\Pi_c x)\left[\Pi_c F_{j-1}(\xi_j(x))\right].$$

Moreover, we have for $x \in X_0$ that

$$\Pi_c F_j(x) = \Pi_c F_{j-1}(\xi_j(x)) = \Pi_c F_{j-1}(x + G_j(\Pi_c x)).$$

Since the range of G_j is included in X_h, by induction we have

$$\Pi_c F_j(x) = \Pi_c F(x + G_2(\Pi_c x) + G_3(\Pi_c x) + \ldots + G_j(\Pi_c x)).$$

Now, we obtain

$$\Pi_h D^j F_k(0)\,|_{X_c \times X_c \times \ldots \times X_c} = 0 \text{ for all } j = 1, \ldots, k.$$

Setting

$$u_k(t) = \xi_k^{-1} \circ \xi_{k-1}^{-1} \circ \ldots \xi_2^{-1}(u(t)) = u(t) - G_2(\Pi_c u(t)) - G_3(\Pi_c u(t)) - \ldots - G_k(\Pi_c u(t)),$$

we deduce that $u_k(t)$ is an integrated solution of the system

$$\begin{cases} \dfrac{du_k(t)}{dt} = Au_k(t) + F_k(u_k(t)), \ t \geq 0, \\ u_k(0) = x_k \in \overline{D(A)}. \end{cases} \tag{6.3.24}$$

Applying Lemma 6.1.20 and Theorem 6.1.21 to system (6.3.24), we obtain the following result which is one of the main results of this paper.

Theorem 6.3.11. *Let Assumptions 3.4.1, 3.5.2, 6.1.1, and 6.3.10 be satisfied. Then by using the change of variables*

$$\begin{cases} u_k(t) = u(t) - G_2(\Pi_c u(t)) - G_3(\Pi_c u(t)) - \ldots - G_k(\Pi_c u(t)) \\ \Leftrightarrow \\ u(t) = u_k(t) + G_2(\Pi_c u_k(t)) + G_3(\Pi_c u_k(t)) + \ldots + G_k(\Pi_c u_k(t)), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (6.3.22) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (6.3.24). Moreover, the reduced system of Cauchy problem (6.3.24) is given by the ordinary differential equations on X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F \begin{bmatrix} x_c(t) + G_2(x_c(t)) + \\ G_3(x_c(t)) + \ldots + G_k(x_c(t)) \end{bmatrix} + R_c(x_c(t)), \tag{6.3.25}$$

where the remainder term $R_c \in C^k(X_c, X_c)$ satisfies

$$D^j R_c(0) = 0 \text{ for each } j = 1, \ldots, k,$$

or in other words $R_c(x_c(t))$ is a remainder term of order k.

If we assume in addition that $F \in C^{k+2}\left(\overline{D(A)}, X\right)$, then the map $R_c \in C^{k+2}(X_c, X_c)$ and $R_c(x_c(t))$ is a remainder term of order $k+2$; that is

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c), \tag{6.3.26}$$

where $O(x_c)$ is a function of x_c which remains bounded when x_c goes to 0, or equivalently,

$$D^j R_c(0) = 0 \text{ for each } j = 1, \ldots, k+1.$$

Proof. By Theorem 6.1.21 and Lemma 6.1.20, there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (6.3.24) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F[x_c(t) + G_2(x_c(t)) + G_3(x_c(t)) + \ldots + G_k(x_c(t)) + \Psi_k(x_c(t))]$$

and

$$D^j \Psi_k(0) = 0 \text{ for } j = 1, \ldots, k.$$

By setting

$$\begin{aligned} R_c(x_c) = & \Pi_c F\left[x_c + G_2(x_c) + G_3(x_c) + \ldots + G_k(x_c) + \Psi_k(x_c)\right] \\ & - \Pi_c F\left[x_c + G_2(x_c) + G_3(x_c) + \ldots + G_k(x_c)\right], \end{aligned}$$

we obtain the first part of the theorem. If we assume in addition that $F \in C^{k+2}\left(\overline{D(A)}, X\right)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus,

$$R_c \in C^{k+2}(X_c, X_c).$$

Set

$$h(x_c) := x_c + G_2(x_c) + G_3(x_c) + \ldots + G_k(x_c).$$

We have

$$\begin{aligned} R_c(x_c) &= \Pi_c \left\{ F\left[h(x_c) + \Psi_k(x_c)\right] - F\left[h(x_c)\right] \right\} \\ &= \Pi_c \int_0^1 DF\left(h(x_c) + s\Psi_k(x_c)\right)\left(\Psi_k(x_c)\right) ds. \end{aligned}$$

Define

$$\widehat{h}(x_c) := h(x_c) + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$\begin{aligned} DF\left(\widehat{h}(x_c)\right)\left(\Psi_k(x_c)\right) &= DF(0)\left(\Psi_k(x_c)\right) + \int_0^1 D^2F\left(l\widehat{h}(x_c)\right)\left(\widehat{h}(x_c), \Psi_k(x_c)\right) dl \\ &= \int_0^1 D^2F\left(l\widehat{h}(x_c)\right)\left(\widehat{h}(x_c), \Psi_k(x_c)\right) dl. \end{aligned}$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2F\left(l\left(h(x_c) + s\Psi_k(x_c)\right)\right)\left(h(x_c) + s\Psi_k(x_c), \Psi_k(x_c)\right) dl ds$$

and $h(x_c)$ is a term of order 1, $\Psi_k(x_c)$ is a term of order $k+1$, it follows that (6.3.26) holds. This completes the proof. □

Remark 6.3.12. In order to apply the above approach, we first need to compute Π_c and A_c, then $\Pi_h := I - \Pi_c$ can be derived. The point to apply the above procedure is to solve system (6.3.23). To do this, one may compute

$$(\lambda I - A_h)^{-k} \frac{1}{j!} \Pi_h D^j F(0) \tag{6.3.27}$$

for each $\lambda \in i\mathbb{R}$ and each $k \geq 1$ by using Remark 6.3.5, or one may directly solve system (6.3.23) by computing $\Pi_h \frac{1}{j!} D^j F_{j-1}$. This last approach will involve the computation of (6.3.27) for some specific values of $\lambda \in i\mathbb{R}$ and some specific values of $k \geq 1$. This turns out to be the main difficulty in applying the above method.

In next subsection, we will use the last part of Theorem 6.3.11 to avoid some unnecessary computations. We will apply this theorem for $k=2$, F in C^4, and the remainder term $R_c(x_c)$ of order 4. This means that if we want to compute the Taylor's expansion of the reduced system to the order 3 (which is very common in such a context), we only need to compute G_2. So in application the last part of Theorem 6.3.11 will help to avoid a lot of computations.

(ii) $G \in V^m(X_c, D(A))$. Now we apply Proposition 6.3.9 recursively to (6.3.22). Set

$$u_1 := u.$$

For $m = 2, ..., k$, let $G_m \in V^m(X_c, D(A))$ be defined such that

$$[A, G_m](x_c) = \mathscr{P}_m \left[\frac{1}{m!} D^m F_{m-1}(0)(x_c, \ldots, x_c) \right] \text{ for each } x_c \in X_c.$$

We use the change of variables

$$u_{m-1} = u_m + G_m(\Pi_c u_m).$$

Then we consider F_m given by Proposition 6.3.9 and satisfying

$$F_m(u_m) = F_{m-1}(u_m) - [A, G_m](\Pi_c u_m) + O(\|u_m\|^{m+1}).$$

By applying Proposition 6.3.9, we have

$$\Pi_h D^j F_m(0) \mid_{X_c \times X_c \times \ldots \times X_c} = 0 \text{ for all } j = 1, ..., m,$$

and

$$\mathscr{G}\left(\Pi_c D^j F_m(0) \mid_{X_c \times X_c \times \ldots \times X_c}\right) \in \mathscr{C}_j^c \text{ for all } j = 1, ..., m.$$

Thus by using the change of variables locally around 0

$$u_k(t) = (I + G_k \Pi_c)^{-1} \ldots (I + G_3 \Pi_c)^{-1} (I + G_2 \Pi_c)^{-1} u(t),$$

we deduce that $u_k(t)$ is an integrated solution of system (6.3.24). Applying Theorem 6.1.21 and Lemma 6.1.20 to the above system, we obtain the following result which indicates that systems (6.3.22) and (6.3.24) are locally topologically equivalent around 0.

Theorem 6.3.13. *Let Assumptions 3.4.1, 3.5.2, 6.1.1, and 6.3.10 be satisfied. Then by using the change of variables locally around* 0

$$\begin{cases} u_k(t) = (I + G_k \Pi_c)^{-1} \ldots (I + G_3 \Pi_c)^{-1} (I + G_2 \Pi_c)^{-1} u(t) \\ \Leftrightarrow \\ u(t) = (I + G_2 \Pi_c)(I + G_3 \Pi_c) \ldots (I + G_k \Pi_c) u_k(t), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (6.3.22) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (6.3.24). Moreover, *the reduced system of Cauchy problem (6.3.24) is given by the ordinary differential equations on* X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k} \frac{1}{m!} \Pi_c D^m F_k(0)(x_c(t), ..., x_c(t)) + R_c(x_c(t)),$$

where

$$\mathscr{G}\left(\frac{1}{m!}\Pi_c D^m F_k(0) \mid_{X_c \times X_c \times ... \times X_c}\right) \in \mathscr{C}_m^c, \textit{ for all } m = 1, ..., k,$$

and the remainder term $R_c \in C^k(X_c, X_c)$ *satisfies*

$$D^j R_c(0) = 0 \textit{ for each } j = 1, ..., k,$$

or in other words $R_c(x_c(t))$ *is a remainder term of order* k.

If we assume in addition that $F \in C^{k+2}\left(\overline{D(A)}, X\right)$. *Then the reduced system of Cauchy problem (6.3.24) is given by the ordinary differential equations on* X_c :

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k+1} \frac{1}{m!} \Pi_c D^m F_k(0)(x_c(t), ..., x_c(t)) + R_c(x_c(t)),$$

the map $R_c \in C^{k+2}(X_c, X_c)$, *and* $R_c(x_c(t))$ *is a remainder term of order* $k+2$; *that is*

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c),$$

where $O(x_c)$ *is a function of* x_c *which remains bounded when* x_c *goes to* 0, *or equivalently,*

$$D^j R_c(0) = 0 \textit{ for each } j = 1, ..., k+1.$$

Proof. By Theorem 6.1.21 and Lemma 6.1.20, there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (6.3.24) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F_k[x_c(t) + \Psi_k(x_c(t))]$$

and

$$D^j \Psi_k(0) = 0 \text{ for } j = 1, ..., k.$$

By setting

$$R_c(x_c) = \Pi_c F_k[x_c + \Psi_k(x_c)] - \Pi_c F_k(x_c),$$

we obtain the first part of the Theorem. If we assume in addition that $F \in C^{k+2}\left(\overline{D(A)}, X\right)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus, $R_c \in C^{k+2}(X_c, X_c)$ and

$$\begin{aligned} R_c(x_c) &= \Pi_c \{F_k[x_c + \Psi_k(x_c)] - F_k(x_c)\} \\ &= \Pi_c \int_0^1 DF_k(x_c + s\Psi_k(x_c))(\Psi_k(x_c))\, ds. \end{aligned}$$

Set

$$h(x_c) := x_c + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$\begin{aligned} DF_k(h(x_c))(\Psi_k(x_c)) &= DF_k(0)(\Psi_k(x_c)) + \int_0^1 D^2F_k(lh(x_c))(h(x_c), \Psi_k(x_c))\,dl \\ &= \int_0^1 D^2F_k(lh(x_c))(h(x_c), \Psi_k(x_c))\,dl. \end{aligned}$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2F_k(l(x_c + s\Psi_k(x_c)))(x_c + s\Psi_k(x_c), \Psi_k(x_c))\,dlds$$

and $\Psi_k(x_c)$ is a term of order $k+1$, it follows that

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c).$$

The result follows. □

6.4 Remarks and Notes

(a) Center manifold theory. The classical center manifold theory was first established by Pliss [289] and Kelley [208] and was developed and completed in Carr [56], Sijbrand [319], Vanderbauwhede [343], etc. For the case of a single equilibrium, the center manifold theorem states that if a finite dimensional system has a nonhyperbolic equilibrium, then there exists a center manifold in a neighborhood of the nonhyperbolic equilibrium which is tangent to the generalized eigenspace associated to the corresponding eigenvalues with zero real parts, and the study of the general system near the nonhyperbolic equilibrium reduces to that of an ordinary differential equation restricted on the lower dimensional invariant center manifold. This usually means a considerable reduction of the dimension which leads to simple calculations and a better geometric insight. The center manifold theory has significant applications in studying other problems in dynamical systems, such as bifurcation, stability, perturbation, etc. It has also been used to study various applied problems in biology, engineering, physics, etc. We refer to, for example, Carr [56] and Hassard et al. [181].

There are two classical methods to prove the existence of center manifolds. The Hadamard (Hadamard [167]) method (the graph transformation method) is a geometric approach which bases on the construction of graphs over linearized spaces, see Hirsch et al. [188] and Chow et al. [65, 66]. The Liapunov-Perron (Liapunov [228], Perron [286]) method (the variation of constants method) is more analytic in nature, which obtains the manifold as a fixed point of a certain integral equation. The technique originated in Krylov and Bogoliubov [220] and was furthered developed by Hale [169, 171], see also Ball [36], Chow and Lu [67], Yi [378], etc. The smoothness of center manifolds can be proved by using the contraction mapping in a scale of Banach spaces (Vanderbauwhede and van Gils [344]), the Fiber contraction mapping technique (Hirsch et al. [188]), the Henry lemma (Henry [183], Chow and Lu [68]), among other methods (Chow et al. [64]). For further results and references on center manifolds, we refer to the monographs of Carr [56], Chow and Hale [62], Chow et al. [63], Sell and You [314], Wiggins [373], and the survey papers of Bates and Jones [39], Vanderbauwhede [343] and Vanderbauwhede and Iooss [345].

There have been several important extensions of the classical center manifold theory for invariant sets. For higher dimensional invariant sets, it is known that center manifolds exist for an invariant torus with special structure (Chow and Lu [69]), for an invariant set consisting of equilibria (Fenichel [140]), for some homoclinic orbits (Homburg [190], Lin [229] and Sandstede [306]), for skew-product flows (Chow and Yi [71]), for any piece of trajectory of maps (Hirsch et al. [188]), and for smooth invariant manifolds and compact invariant sets (Chow et al. [65, 66]).

Recently, great attention has been paid to the study of center manifolds in infinite dimensional systems and researchers have developed the center manifold theory for various infinite dimensional systems such as partial differential equations (Bates and Jones [39], Da Prato and Lunardi [84], Henry [183], Scheel [312]), semiflows in Banach spaces (Bates et al. [40], Chow and Lu [67], Gallay [148], Scarpellini [309], Vanderbauwhede [342], Vanderbauwhede and van Gils [344]), delay differential equations (Hale [172], Hale and Verduyn Lunel [175], Diekmann and van Gils [104, 105], Diekmann et al. [106], Hupkes and Verduyn Lunel [193]), infinite dimensional nonautonomous differential equations (Mielke [270, 271], Chicone and Latushkin [59]), and partial functional differential equations (Lin et al. [230], Faria et al. [135], Krisztin [219], Nguyen and Wu [277], Wu [374]). Infinite dimensional systems usually do not have some of the nice properties the finite dimensional systems have. For example, the initial value problem may not be well posed, the solutions may not be extended backward, the solutions may not be regular, the domain of operators may not be dense in the state space, etc. Therefore, the center manifold reduction of the infinite dimensional systems plays a very important role in the theory of infinite dimensional systems since it allows us to study ordinary differential equations reduced on the finite dimensional center manifolds. Vanderbauwhede and Iooss [345] described some minimal conditions which allow to generalize the approach of Vanderbauwhede [343] to infinite dimensional systems.

The goal of Section 6.1 was to combine the integrated semigroup theory with the techniques of Vanderbauwhede [342, 343], Vanderbauwhede and Van Gills [344] and Vanderbauwhede and Iooss [345] to develop a center manifold theory for abstract semilinear Cauchy problems with non-dense domain. The materials in Section 6.1 were taken from Magal and Ruan [248]. The existence of center-unstable manifold for abstract semilinear Cauchy problems with non-dense domain was established in Liu et al. [235].

(b) Hopf bifurcation theorem. The Hopf bifurcation theorem, proved by several researchers (see Andronov *et al.* [18], Hopf [191], Friedrichs [145], Hale [171]), gives a set of sufficient conditions to ensure that an autonomous ordinary differential equation with a parameter exhibits nontrivial periodic solutions for certain values of the parameter. The theorem has been used to study bifurcations in many applied subjects (see Marsden and McCraken [257] and Hassard et al. [181]).

In the 1970's, several Hopf bifurcation theorems were obtained for infinite dimensional systems in order to establish the bifurcation of periodic solutions of the Navier-Stokes equations. Such results are usually based on the so-called Liapunov-Schmidt or center manifold reduction approach. We refer to Iudovich [201], Sattinger [308], Iooss [200], Joseph and Sattinger [202], Marsden [256], Marsden and McCraken [257], Crandall and Rabinowitz [76], Henry [183], Da Prato and Lunardi [84], and Kielhőfer [213] for results on the subject. The Hopf bifurcation theorem has also been extended to functional differential equations (Hale and Verduyn Lunel[175], Diekmann et al. [106], Wu [374]), functional equations (Hale and De Oliveira [174]), and integral equations (Diekmann and van Gils [104], Diekmann et al. [106]). We also refer to Golubitsky and Rabinowitz [151] for a nice commentary on Hopf bifurcation theorem and more references.

In Section 6.2, which was adapted from Liu et al. [234], we applied the center manifold theorem developed in Section 6.1 to prove a Hopf bifurcation theorem for the abstract non-densely defined Cauchy problem. Since the problem is written as a Cauchy problem, the method may seem fairly classical, however the result is new and general, which can be applied to several types of equations. We will apply the main theorem to obtain a known Hopf bifurcation result for functional differential equations and a general Hopf bifurcation theorem for age structured models.

(c) Normal form theory. A normal form theorem was obtained first by Poincaré [291] and later by Siegel [317] for analytic differential equations. Simpler proofs of Poincaré's theorem and Siegel's theorem were given in Arnold [32], Meyer [267], Moser [272], and Zehnder [382]. For more results about normal form theory and its applications see, for example, the monographs by Arnold [32], Chow and Hale [62], Guckenheimer and Holmes [155], Meyer and Hall [**?**], Siegel and Moser [318], Chow et al. [63], Kuznetsov [223], and others.

Normal form theory has been extended to various classes of partial differential equations. In the context of autonomous partial differential equations we refer to Ashwin and Mei [33] (PDEs on the square), Eckmann et al. [125] (abstract parabolic equations), Faou et al. [130, 131] (Hamiltonian PDEs), Hassard, Kazarinoff and Wan [181] (Functional Differential Equations), Faria [133, 134] (PDEs with delay), Foias et al. [143] (Navier-Stokes equation), Kokubu [217] (reaction-diffusion equations), McKean and Shatah [262] (Schrödinger equation and heat equations), Nikolenko [278] (abstract semi-linear equations), Shatah [316] (Klein-Gordon equation), Zehnder [383] (abstract parabolic equations), etc. We also refer to Chow et al. [70] (and references therein) for a normal form theory in quasiperiodic partial differential equations.

In Section 6.3 we used the integrated semigroup theory, the semilinear Cauchy problem theory, and the center manifold theory to establish a normal form theory for the non-densely defined Cauchy problem. The goal was to provide a method for computing the required lower order terms of the Taylor expansion and the normal form of the reduced equations restricted on the center manifold. The main difficulty comes from the fact that the center manifold is defined by using implicit formulae in general. Here we showed that it is possible to find some appropriate changes of variables (in Banach spaces) to compute the Taylor expansion at any order and the normal form of the reduced system. The main results and computation procedures will be used to discuss Hopf bifurcation in age structured population models. The presentations in Section 6.3 were taken from Liu et al. [236].

Chapter 7
Functional Differential Equations

The goal of this chapter is to apply the theories developed in previous chapters to functional differential equations. In Section 7.1 retarded functional differential equations are re-written as abstract Cauchy problems and the integrated semigroup theory is used to study the existence of integrated solutions and to establish a general Hopf bifurcation theorem. Section 7.2 deals with neutral functional differential equations. In Section 7.3, firstly it is shown that a delayed transport equation for cell growth and division has asynchronous exponential growth; secondly it is demonstrated that partial functional differential equations can also be set up as an abstract Cauchy problem.

7.1 Retarded Functional Differential Equations

For $r \geq 0$, let $\mathscr{C} = C([-r,0];\mathbb{R}^n)$ be the Banach space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ endowed with the supremum norm

$$\|\varphi\| = \sup_{\theta \in [-r,0]} |\varphi(\theta)|_{\mathbb{R}^n}.$$

Consider the retarded functional differential equations (RFDE) of the form

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}(x_t) + f(t,x_t), \forall t \geq 0, \\ x_0 = \varphi \in C, \end{cases} \tag{7.1.1}$$

where $x_t \in \mathscr{C}$ satisfies $x_t(\theta) = x(t+\theta), B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, $\widehat{L} : \mathscr{C} \to \mathbb{R}^n$ is a bounded linear operator given by

$$\widehat{L}(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta),$$

here $\eta : [-r,0] \to M_n(\mathbb{R})$ is a map of bounded variation, and $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ is a continuous map.

In order to study the RFDE (7.1.1) by using the integrated semigroup theory, we need to rewrite (7.1.1) as an abstract non-densely defined Cauchy problem. Firstly, we regard RFDE (7.1.1) as a PDE. Define $u \in C([0,+\infty) \times [-r,0], \mathbb{R}^n)$ by

$$u(t,\theta) = x(t+\theta),\ \forall t \geq 0,\ \forall \theta \in [-r,0].$$

Note that if $x \in C^1([-r,+\infty), \mathbb{R}^n)$, then

$$\frac{\partial u(t,\theta)}{\partial t} = x'(t+\theta) = \frac{\partial u(t,\theta)}{\partial \theta}.$$

Hence, we must have

$$\frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \forall t \geq 0,\ \forall \theta \in [-r,0].$$

Moreover, for $\theta = 0$, we obtain

$$\begin{aligned}\frac{\partial u(t,0)}{\partial \theta} &= x'(t) = Bx(t) + \widehat{L}(x_t) + f(t,x_t) \\ &= Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)),\ \forall t \geq 0.\end{aligned}$$

Therefore, we deduce formally that u must satisfy a PDE

$$\begin{cases} \dfrac{\partial u(t,\theta)}{\partial t} - \dfrac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0, \\ u(0,.) = \varphi \in \mathscr{C}. \end{cases} \tag{7.1.2}$$

In order to rewrite the PDE (7.1.2) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary condition. This can be accomplished by adopting the following state space

$$X = \mathbb{R}^n \times \mathscr{C}$$

taken with the usual product norm

$$\left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| = |x|_{\mathbb{R}^n} + \|\varphi\|.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix},\ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \tag{7.1.3}$$

with

$$D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-r,0],\mathbb{R}^n).$$

Note that A is non-densely defined because

$$\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times \mathscr{C} \neq X.$$

We also define $L : \overline{D(A)} \to X$ by

$$L\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{L}(\varphi) \\ 0_{\mathscr{C}} \end{pmatrix}$$

and $F : \mathbb{R} \times \overline{D(A)} \to X$ by

$$F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f(t,\varphi) \\ 0_{\mathscr{C}} \end{pmatrix}.$$

Set

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}.$$

Now we can consider the PDE (7.1.2) as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t,v(t)),\ t \geq 0; \quad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}. \tag{7.1.4}$$

7.1.1 Integrated Solutions and Spectral Analysis

In this subsection we first study the integrated solutions of the Cauchy problem (7.1.4) in the special case

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} h(t) \\ 0 \end{pmatrix},\ t \geq 0,\ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}, \tag{7.1.5}$$

where $h \in L^1((0,\tau),\mathbb{R}^n)$. Recall that $v \in C([0,\tau],X)$ is an integrated solution of (7.1.5) if and only if

$$\int_0^t v(s)ds \in D(A),\ \forall t \in [0,\tau] \tag{7.1.6}$$

and

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A\int_0^t v(s)ds + \int_0^t \begin{pmatrix} h(s) \\ 0 \end{pmatrix} ds,\ \forall t \in [0,\tau]. \tag{7.1.7}$$

From (7.1.6) we note that if v is an integrated solution we must have

$$v(t) = \lim_{h \to 0^+} \frac{1}{h}\int_t^{t+h} v(s)ds \in \overline{D(A)}.$$

Hence $v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}$ with $u \in C([0,\tau], \mathscr{C})$. In order to obtain the uniqueness of integrated solutions of (7.1.5) we want to prove that A generates an integrated semigroup. So firstly we need to study the resolvent of A.

Theorem 7.1.1. *For the operator A defined in (7.1.3), the resolvent set of A satisfies*

$$\rho(A) = \rho(B),$$

where B is an $n \times n$ matrix defined in (7.1.1). Moreover, for each $\lambda \in \rho(A)$, we have the following explicit formula for the resolvent of A :

$$\begin{array}{l} (\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\ \Leftrightarrow \psi(\theta) = e^{\lambda\theta} (\lambda I - B)^{-1} [\varphi(0) + \alpha] + \int_\theta^0 e^{\lambda(\theta - s)} \varphi(s)\, ds. \end{array} \tag{7.1.8}$$

Proof. We first prove that $\rho(A) \subset \rho(B)$ for which we only need to show that $\sigma(B) \subset \sigma(A)$. Let $\lambda \in \sigma(B)$. Then, there exists $x \in \mathbb{C}^n \setminus \{0\}$ such that $Bx = \lambda x$. Consider

$$\varphi(\theta) = e^{\lambda\theta} x,$$

we have

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix} = \begin{pmatrix} -\lambda x + Bx \\ \lambda\varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \lambda\varphi \end{pmatrix}$$

Thus $\lambda \in \sigma(A)$. This implies that $\sigma(B) \subset \sigma(A)$. On the other hand, if $\lambda \in \rho(B)$ for $\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in X$, we must have $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \in D(A)$ such that

$$\begin{array}{l} (\lambda I - A) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \\ \quad \Leftrightarrow \begin{cases} \psi'(0) - B\psi(0) = \alpha \\ \quad \lambda\psi - \psi' = \varphi \end{cases} \\ \quad \Leftrightarrow \begin{cases} (\lambda I - B)\psi(0) = \alpha + \varphi(0) \\ \lambda\psi - \psi' = \varphi \end{cases} \\ \quad \Leftrightarrow \begin{cases} (\lambda I - B)\psi(0) = \alpha + \varphi(0) \\ \psi(\theta) = e^{\lambda(\theta - \widehat{\theta})} \psi\left(\widehat{\theta}\right) + \int_{\widehat{\theta}}^\theta e^{\lambda(\theta - l)} \varphi(l)\, dl, \forall \theta \geq \widehat{\theta} \end{cases} \\ \quad \Leftrightarrow \begin{cases} (\lambda I - B)\psi(0) = \alpha + \varphi(0) \\ \psi\left(\widehat{\theta}\right) = e^{\lambda\widehat{\theta}} \psi(0) - \int_0^{\widehat{\theta}} e^{\lambda(\widehat{\theta} - l)} \varphi(l)\, dl, \forall \widehat{\theta} \in [-r, 0], \end{cases} \\ \quad \Leftrightarrow \psi\left(\widehat{\theta}\right) = e^{\lambda\widehat{\theta}} (\lambda I - B)^{-1} [\alpha + \varphi(0)] - \int_0^{\widehat{\theta}} e^{\lambda(\widehat{\theta} - l)} \varphi(l)\, dl,\ \forall \widehat{\theta} \in [-r, 0]. \end{array}$$

Therefore, we obtain that $\lambda \in \rho(A)$ and the formula in (7.1.8) holds. $\square$

Since B is a matrix on $\mathbb{R}^n$, we have $\omega_0(B) := \sup_{\lambda \in \sigma(B)} \mathrm{Re}(\lambda)$ and the following lemma.

Lemma 7.1.2. *The linear operator $A : D(A) \subset X \to X$ is a Hille-Yosida operator. More precisely, for each $\omega_A > \omega_0(B)$ there exists $M_A \geq 1$ such that*

$$\left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}(X)} \leq \frac{M_A}{(\lambda - \omega_A)^n}, \ \forall n \geq 1, \forall \lambda > \omega_A. \tag{7.1.9}$$

Proof. Let $\omega_A > \omega_0(B)$ be given. We can define the equivalent norm on $\mathbb{R}^n$

$$|x| := \sup_{t \geq 0} e^{-\omega_A t} \left\|e^{Bt} x\right\|.$$

Then we have

$$\left|e^{Bt} x\right| \leq e^{\omega_A t} |x|, \forall t \geq 0$$

and

$$\|x\| \leq |x| \leq M_A \|x\|,$$

where

$$M_A := \sup_{t \geq 0} \left\|e^{(B - \omega_A I)t}\right\|_{M_n(\mathbb{R})}.$$

Moreover, for each $\lambda > \omega_A$, we have

$$\left|(\lambda I - B)^{-1} x\right| = \left|\int_0^{+\infty} e^{-\lambda s} e^{Bs} x ds\right| \leq \frac{|x|}{\lambda - \omega_A}.$$

We define the equivalent norm $|.|$ on X by

$$\left|\begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right| = |\alpha| + \|\varphi\|_{\omega_A},$$

where

$$\|\varphi\|_{\omega_A} := \sup_{\theta \in [-r,0]} \left|e^{-\omega_A \theta} \varphi(\theta)\right|.$$

Using (7.1.8) and the above results, we obtain

$$\begin{aligned}
&\left|(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}\right| \\
&\leq \sup_{\theta \in [-r,0]} \left[e^{-\omega_A \theta} e^{\lambda \theta} \left|(\lambda I - B)^{-1} [\varphi(0) + \alpha]\right| + e^{-\omega_A \theta} \int_\theta^0 e^{\lambda(\theta - s)} |\varphi(s)| ds\right] \\
&\leq \sup_{\theta \in [-r,0]} \left[e^{-\omega_A \theta} e^{\lambda \theta} \frac{1}{\lambda - \omega_A} [|\varphi(0)| + |\alpha|] + e^{-\omega_A \theta} e^{\lambda \theta} \int_\theta^0 e^{-(\lambda - \omega_A)s} ds \|\varphi\|_{\omega_A}\right] \\
&= \frac{1}{\lambda - \omega_A} |\alpha| + \sup_{\theta \in [-r,0]} [\frac{e^{-\omega_A \theta} e^{\lambda \theta}}{\lambda - \omega_A} |\varphi(0)| + \frac{e^{-\omega_A \theta} e^{\lambda \theta} \left[e^{-(\lambda - \omega_A)\theta} - 1\right]}{\lambda - \omega_A} \|\varphi\|_{\omega_A}] \\
&\leq \frac{1}{\lambda - \omega_A} \left[|\alpha| + \|\varphi\|_{\omega_A}\right]
\end{aligned}$$

$$= \frac{1}{\lambda - \omega_A} \left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\|.$$

Therefore, (7.1.9) holds and the proof is completed. $\square$

Since A is a Hille-Yosida operator, A generates a non-degenerated integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X. It follows from Corollary 3.6.3 that the abstract Cauchy problem (7.1.5) has at most one integrated solution.

Lemma 7.1.3. *Let $h \in L^1((0,\tau), \mathbb{R}^n)$ and $\varphi \in \mathscr{C}$ be given. Then there exists an unique integrated solution, $t \to v(t)$, of the Cauchy problem (7.1.5) which can be expressed explicitly by the following formula*

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}$$

with

$$u(t)(\theta) = x(t+\theta),\ \forall t \in [0,\tau],\ \forall \theta \in [-r,0], \tag{7.1.10}$$

where

$$x(t) = \begin{cases} \varphi(t), & t \in [-r,0], \\ e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}h(s)ds, & t \in [0,\tau]. \end{cases}$$

Proof. Since A is a Hille-Yosida operator, there is at most one integrated solution of the Cauchy problem (7.1.5). So it is sufficient to prove that u defined by (7.1.10) satisfies for each $t \in [0,\tau]$ the following

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_0^t u(l)dl \end{pmatrix} \in D(A) \tag{7.1.11}$$

and

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \int_0^t u(l)dl \end{pmatrix} + \begin{pmatrix} \int_0^t h(l)dl \\ 0 \end{pmatrix}. \tag{7.1.12}$$

Since

$$\int_0^t u(l)(\theta)\,dl = \int_0^t x(l+\theta)dl = \int_\theta^{t+\theta} x(s)ds$$

and $x \in C([-r,\tau], \mathbb{R}^n)$, $\int_0^t u(l)dl \in C^1([-r,0], \mathbb{R}^n)$. Therefore, (7.1.11) follows. Moreover,

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}$$

whenever $\varphi \in C^1([-r,0], \mathbb{R}^n)$. Hence

$$\begin{aligned} A \begin{pmatrix} 0 \\ \int_0^t u(l)dl \end{pmatrix} &= \begin{pmatrix} -[x(t) - x(0)] + B\int_0^t x(s)ds \\ x(t+.) - x(.) \end{pmatrix} \\ &= -\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \begin{pmatrix} -[x(t) - \varphi(0)] + B\int_0^t x(s)ds \\ x(t+.) \end{pmatrix}. \end{aligned}$$

Therefore, (7.1.12) is satisfied if and only if

$$x(t) = \varphi(0) + B\int_0^t x(s)ds + \int_0^t h(s)ds. \tag{7.1.13}$$

By using the usual variation of constants formula, we deduce that (7.1.13) is equivalent to

$$x(t) = e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}h(s)ds.$$

The proof is completed. □

Recall that $A_0 : D(A_0) \subset \overline{D(A)} \to \overline{D(A)}$, the part of A in $\overline{D(A)}$, is defined by

$$A_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A) : A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)} \right\}.$$

From the definition of A in (7.1.3) and the fact that $\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times \mathscr{C}$, we know that A_0 is the linear operator defined by

$$A_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A_0),$$

where

$$D(A_0) = \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times C^1\left([-r,0], \mathbb{R}^n\right) : -\varphi'(0) + B\varphi(0) = 0 \right\}.$$

Now by using the fact that A is a Hille-Yosida operator, we deduce that A_0 is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ and

$$v(t) = T_{A_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}$$

is the integrated solution of

$$\frac{dv(t)}{dt} = Av(t), \ t \geq 0; \ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}.$$

Using Lemma 7.1.3 with $h = 0$, we obtain the following result.

Lemma 7.1.4. *The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$ which is defined by*

$$T_{A_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{T}_{A_0}(t)(\varphi) \end{pmatrix}, \tag{7.1.14}$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases} e^{B(t+\theta)}\varphi(0), & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0. \end{cases}$$

Since A is a Hille-Yosida operator, we know that A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X, and $t \to S_A(t)\begin{pmatrix} x \\ \varphi \end{pmatrix}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} x \\ \varphi \end{pmatrix}, \ t \geq 0; \ v(0) = 0.$$

Since $S_A(t)$ is linear we have

$$S_A(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = S_A(t)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + S_A(t)\begin{pmatrix} x \\ 0 \end{pmatrix},$$

where

$$S_A(t)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \int_0^t T_{A_0}(l)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} dl$$

and $S_A(t)\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an integrated solution of

$$\frac{dv(t)}{dt} = Av(t) + \begin{pmatrix} x \\ 0 \end{pmatrix}, \ t \geq 0; \ v(0) = 0.$$

Therefore, by using Lemma 7.1.3 with $h(t) = x$ and the above results, we obtain the following result.

Lemma 7.1.5. *The linear operator A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X. Moreover,*

$$S_A(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{S}_A(t)(x,\varphi) \end{pmatrix}, \ \begin{pmatrix} x \\ \varphi \end{pmatrix} \in X,$$

where $\widehat{S}_A(t)$ is the linear operator defined by

$$\widehat{S}_A(t)(x,\varphi) = \widehat{S}_A(t)(0,\varphi) + \widehat{S}_A(t)(x,0)$$

with

$$\widehat{S}_A(t)(0,\varphi)(\theta) = \int_0^t \widehat{T}_{A_0}(s)(\varphi)(\theta)\,ds = \int_{-\theta}^t e^{B(s+\theta)}\varphi(0)ds + \int_0^{-\theta} \varphi(s+\theta)ds$$

and

$$\widehat{S}_A(t)(x,0)(\theta) = \begin{cases} \int_0^{t+\theta} e^{Bs}x ds, & t+\theta \geq 0, \\ 0, & t+\theta \leq 0. \end{cases}$$

Now we focus on the spectra of A and $A+L$. Since A is a Hille-Yosida operator, so is $A+L$. Moreover, $(A+L)_0 : D((A+L)_0) \subset \overline{D(A)} \to \overline{D(A)}$, the part of $A+L$ in

$\overline{D(A)}$, is a linear operator defined by

$$(A+L)_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D\left((A+L)_0\right),$$

where

$$D\left((A+L)_0\right) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times C^1\left([-r,0],\mathbb{R}^n\right) : \varphi'(0) = B\varphi(0) + \widehat{L}(\varphi) \right\}.$$

From Proposition 4.2.14 and Theorem 7.1.1, we know that

$$\sigma(B) = \sigma(A) = \sigma(A_0) \text{ and } \sigma(A+L) = \sigma\left((A+L)_0\right).$$

From (7.1.14), we have

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = e^{B(r+\theta)} e^{B(t-r)} \varphi(0), \ t \geq r, \theta \in [-r,0].$$

Therefore,

$$\widehat{T}_{A_0}(t) = L_2 L_1,$$

where $L_1 : \mathscr{C} \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \to \mathscr{C}$ are linear operators defined by

$$L_1 \varphi = e^{B(t-r)} \varphi(0), \ \varphi \in \mathscr{C}, t \geq r$$

and

$$L_2(x)(\theta) = e^{B(r+\theta)} x, \ x \in \mathbb{R}^n, \theta \in [-r,0],$$

respectively. Clearly L_1 is compact. Hence, we have

$$\omega_{0,\text{ess}}(A_0) = -\infty \text{ and } \sigma(B) = \sigma(A) = \sigma_P(A_0) = \sigma(A_0).$$

Therefore,

$$\omega_0(A_0) = \sup_{\lambda \in \sigma_P(A_0)} \operatorname{Re}(\lambda).$$

In the following lemma, we specify the point spectrum of $(A+L)_0$.

Lemma 7.1.6. *The point spectrum of* $(A+L)_0$ *is the set*

$$\sigma_P\left((A+L)_0\right) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

where

$$\Delta(\lambda) = \lambda I - B - \widehat{L}\left(e^{\lambda \cdot} I\right) = \lambda I - B - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta). \tag{7.1.15}$$

Proof. Let $\lambda \in \mathbb{C}$ be given. Then $\lambda \in \sigma_P\left((A+L)_0\right)$ if and only if there exists $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D\left((A+L)_0\right) \setminus \{0\}$ such that

$$(A+L)_0 \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \lambda \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}.$$

That is, $\lambda \in \sigma_P\left((A+L)_0\right)$ if and only if there exists $\varphi \in C^1\left([-r,0], \mathbb{C}^n\right) \setminus \{0\}$ such that

$$\varphi'(\theta) = \lambda \varphi(\theta), \ \forall \theta \in [-r,0] \tag{7.1.16}$$

and

$$\varphi'(0) = B\varphi(0) + \widehat{L}(\varphi). \tag{7.1.17}$$

Equation (7.1.16) is equivalent to

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \ \forall \theta \in [-r,0]. \tag{7.1.18}$$

Therefore,

$$\varphi \neq 0 \Leftrightarrow \varphi(0) \neq 0.$$

By combining (7.1.17) and (7.1.18), we obtain

$$\lambda \varphi(0) = B\varphi(0) + \widehat{L}\left(e^{\lambda \cdot}\varphi(0)\right).$$

The proof is completed. □

From the above discussion, we have the following proposition.

Proposition 7.1.7. *The linear operator $A+L : D(A) \to X$ is a Hille-Yosida operator and $(A+L)_0$ is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{(A+L)_0}(t)\right\}_{t\geq 0}$ of bounded linear operators on $\overline{D(A)}$. Moreover,*

$$T_{(A+L)_0}(t) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{T}_{(A+L)_0}(t)(\varphi) \end{pmatrix}$$

with

$$\widehat{T}_{(A+L)_0}(t)(\varphi)(\theta) = x(t+\theta), \ \forall t \geq 0, \forall \theta \in [-r,0],$$

where

$$x(t) = \begin{cases} \varphi(t), & \forall t \in [-r,0], \\ e^{Bt}\varphi(0) + \int_0^t e^{B(t-s)}\widehat{L}(x_s)\,ds, & \forall t \geq 0. \end{cases}$$

Furthermore,

$$\omega_{0,\mathrm{ess}}((A+L)_0) = -\infty, \quad \omega_0\left((A+L)_0\right) = \max_{\lambda \in \sigma_P((A+L)_0)} \mathrm{Re}(\lambda),$$

$$\sigma(A+L) = \sigma\left((A+L)_0\right) = \sigma_P\left((A+L)_0\right) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},$$

and each $\lambda_0 \in \sigma\left((A+L)_0\right) = \sigma(A+L)$ is a pole of $(\lambda I - (A+L))^{-1}$. For each $\gamma \in \mathbb{R}$, the subset $\{\lambda \in \sigma\left((A+L)_0\right) : \mathrm{Re}(\lambda) \geq \gamma\}$ is either empty or finite.

Proof. The first part of the result follows immediately from Lemma 7.1.3 applied with $h(t) = \widehat{L}(x_t)$. So it remains to prove that $\omega_{0,\mathrm{ess}}((A+L)_0) = -\infty$. But this property follows from the fact that $T_{(A+L)_0}(t)$ is compact for each t large enough. This is an immediate consequence of Theorem 4.7.3 (which applies because $LT_{A_0}(t)$ is compact for each $t>0$, and $T_{A_0}(t)$ is compact for $t \geq r$). $\square$

7.1.2 Projectors on the eigenspaces

Let $\lambda_0 \in \sigma(A+L)$ be given. From the above discussion we already knew that λ_0 is a pole of $(\lambda I-(A+L))^{-1}$of finite order $k_0 \geq 1$. This means that λ_0 is isolated in $\sigma(A+L)$ and the Laurent's expansion of the resolvent around λ_0 takes the following form

$$(\lambda I-(A+L))^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda-\lambda_0)^n B_n^{\lambda_0}. \tag{7.1.19}$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $A+L$ associated to λ_0. The goal of this subsection is to provide a method to compute $B_{-1}^{\lambda_0}$. Note that

$$(\lambda-\lambda_0)^{k_0} (\lambda I-(A+L))^{-1} = \sum_{m=0}^{+\infty} (\lambda-\lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0} = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left((\lambda-\lambda_0)^{k_0} (\lambda I-(A+L))^{-1} \right). \tag{7.1.20}$$

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results.

Lemma 7.1.8. *For each $\lambda \in \rho(A+L)$, we have the following explicit formula for the resolvent of $A+L$*

$$(\lambda I-(A+L))^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix}$$
$$\Leftrightarrow$$
$$\psi(\theta) = \int_\theta^0 e^{\lambda(\theta-s)} \varphi(s)\,ds + e^{\lambda\theta} \Delta(\lambda)^{-1} \left[\alpha + \varphi(0) + \widehat{L} \left(\int_.^0 e^{\lambda(.-s)} \varphi(s)\,ds \right) \right].$$

Proof. We consider the linear operator $A_\gamma : D(A) \subset X \to X$ defined by

$$A_\gamma \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + (B-\gamma I)\varphi(0) \\ \varphi' \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A),$$

and the bounded linear operator $L_\gamma \in \mathscr{L}(\overline{D(A)}, X)$ defined by

$$L_\gamma \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{L}(\varphi) + \gamma\varphi(0) \\ 0_{\mathscr{C}} \end{pmatrix}.$$

Then we have

$$A + L = A_\gamma + L_\gamma.$$

Moreover,

$$\omega_0(B - \gamma I) = \max_{\lambda\in\sigma(B-\gamma I)} \mathrm{Re}(\lambda) = \max_{\lambda\in\sigma(B)} \mathrm{Re}(\lambda) - \gamma = \omega_0(B) - \gamma.$$

Hence, by Theorem 7.1.1, for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B) - \gamma$ we have $\lambda \in \rho(A_\gamma)$ and

$$\begin{aligned}
&(\lambda I - A_\gamma)^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\
&\Leftrightarrow \\
&\psi(\theta) = e^{\lambda\theta} (\lambda I - (B - \gamma I))^{-1} [\varphi(0) + \alpha] + \int_\theta^0 e^{\lambda(\theta - s)} \varphi(s)\, ds. \quad (7.1.21)
\end{aligned}$$

Therefore, for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \omega_0(B) - \gamma$, we know that $\lambda I - (A_\gamma + L_\gamma)$ is invertible if and only if $I - L_\gamma(\lambda I - A_\gamma)^{-1}$ is invertible, and

$$(\lambda I - (A_\gamma + L_\gamma))^{-1} = (\lambda I - A_\gamma)^{-1} \left[I - L_\gamma (\lambda I - A_\gamma)^{-1} \right]^{-1}. \quad (7.1.22)$$

We also know that $\left[I - L_\gamma (\lambda I - A_\gamma)^{-1} \right] \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix}$ is equivalent to $\varphi = \widehat{\varphi}$ and

$$\begin{aligned}
\alpha - &\left[\widehat{L}\left(e^{\lambda\cdot} (\lambda I - (B - \gamma I))^{-1} \alpha \right) + \gamma (\lambda I - (B - \gamma I))^{-1} \alpha \right] \\
&= \widehat{\alpha} + \left[\begin{array}{c} \widehat{L}\left(e^{\lambda\cdot} (\lambda I - (B - \gamma I))^{-1} \widehat{\varphi}(0) + \int_\cdot^0 e^{\lambda(\cdot - s)} \widehat{\varphi}(s)\, ds \right) \\ + \gamma (\lambda I - (B - \gamma I))^{-1} \widehat{\varphi}(0) \end{array} \right].
\end{aligned}$$

Because

$$\begin{aligned}
\alpha - &\widehat{L}\left(e^{\lambda\cdot} (\lambda I - (B - \gamma I))^{-1} \alpha \right) - \gamma (\lambda I - (B - \gamma I))^{-1} \alpha \\
&= \left[\lambda I - (B - \gamma I) - \widehat{L}\left(e^{\lambda\cdot} I \right) - \gamma I \right] (\lambda I - (B - \gamma I))^{-1} \alpha \\
&= \left[\lambda I - B - \widehat{L}\left(e^{\lambda\cdot} I \right) \right] (\lambda I - (B - \gamma I))^{-1} \alpha \\
&= \Delta(\lambda) (\lambda I - (B - \gamma I))^{-1} \alpha,
\end{aligned}$$

we know that $I - L_\gamma (\lambda I - A_\gamma)^{-1}$ is invertible if and only if $\Delta(\lambda) = \lambda I - B - \widehat{L}(e^{\lambda\cdot} I)$ is invertible. Moreover,

$$\left[I - L_\gamma (\lambda I - A_\gamma)^{-1} \right]^{-1} \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}$$

is equivalent to $\varphi = \widehat{\varphi}$ and

$$\alpha = (\lambda I - (B-\gamma I))\,\Delta(\lambda)^{-1} \left[\begin{array}{r} \widehat{\alpha} + \widehat{L}\left(e^{\lambda \cdot} (\lambda I - (B-\gamma I))^{-1} \widehat{\varphi}(0) + \int_{\cdot}^{0} e^{\lambda(\cdot - s)} \widehat{\varphi}(s)\,ds \right) \\ + \gamma(\lambda I - (B-\gamma I))^{-1} \widehat{\varphi}(0) \end{array} \right]. \tag{7.1.23}$$

Recalling that $A+L = A_\gamma + L_\gamma$ and using (7.1.21), (7.1.22) and (7.1.23), we obtain for each $\gamma > 0$ large enough that

$$\begin{array}{l} (\lambda I - (A+L))^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} \\ \Leftrightarrow \\ \psi(\theta) = e^{\lambda\theta} (\lambda I - (B-\gamma I))^{-1} \varphi(0) + \int_{\theta}^{0} e^{\lambda(\theta - s)} \varphi(s)\,ds \\ \qquad + e^{\lambda\theta} \Delta(\lambda)^{-1} \left[\begin{array}{r} \alpha + \widehat{L}\left(e^{\lambda \cdot} (\lambda I - (B-\gamma I))^{-1} \varphi(0) + \int_{\cdot}^{0} e^{\lambda(\cdot - s)} \varphi(s)\,ds \right) \\ + \gamma(\lambda I - (B-\gamma I))^{-1} \varphi(0) \end{array} \right]. \end{array}$$

Now by taking the limit when $\gamma \to +\infty$, the result follows. □

The map $\lambda \to \Delta(\lambda)$ from $\mathbb{C}$ into $M_n(\mathbb{C})$ is differentiable and

$$\Delta^{(1)}(\lambda) := \frac{d\Delta(\lambda)}{d\lambda} = I - \int_{-r}^{0} d\eta(\theta)\,\theta e^{\lambda\theta}.$$

So the map $\lambda \to \Delta(\lambda)$ is analytic and

$$\Delta^{(n)}(\lambda) := \frac{d^n\Delta(\lambda)}{d\lambda^n} = -\int_{-r}^{0} d\eta(\theta)\,\theta^n e^{\lambda\theta},\ n \geq 2.$$

We know that the inverse function

$$\psi : L \to L^{-1}$$

of a linear operator $L \in \mathrm{Isom}(X)$ is differentiable, and

$$D\psi(L)\widehat{L} = -L^{-1} \circ \widehat{L} \circ L^{-1}.$$

Applying this result, we deduce that $\lambda \to \Delta(\lambda)^{-1}$ from $\rho(A+L)$ into $M_n(\mathbb{C})$ is differentiable, and $\frac{d}{d\lambda}\Delta(\lambda)^{-1} = -\Delta(\lambda)^{-1}\left(\frac{d}{d\lambda}\Delta(\lambda)\right)\Delta(\lambda)^{-1}$. Therefore, we obtain that the map $\lambda \to \Delta(\lambda)^{-1}$ is analytic and has a Laurent's expansion around λ_0 :

$$\Delta(\lambda)^{-1} = \sum_{n=-\widehat{k}_0}^{+\infty} (\lambda - \lambda_0)^n \Delta_n.$$

From the following lemma we know that $\widehat{k}_0 = k_0$.

Lemma 7.1.9. *Let $\lambda_0 \in \sigma(A+L)$ be given. Then the following statements are equivalent*

(a) λ_0 *is a pole of order* k_0 *of* $(\lambda I - (A+L))^{-1}$;
(b) λ_0 *is a pole of order* k_0 *of* $\Delta(\lambda)^{-1}$;
(c) $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1} \neq 0$ *and* $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0+1}\Delta(\lambda)^{-1} = 0.$

Proof. The proof follows from the explicit formula of the resolvent of $A+L$ obtained in Lemma 7.1.8. $\square$

Lemma 7.1.10. *The matrices* $\Delta_{-1},...,\Delta_{-k_0}$ *satisfy*

$$\Delta_{k_0}(\lambda_0)\begin{pmatrix}\Delta_{-1}\\ \Delta_{-2}\\ \vdots \\ \Delta_{-k_0+1}\\ \Delta_{-k_0}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

and

$$\begin{pmatrix}\Delta_{-k_0} & \Delta_{-k_0+1} & \cdots & \Delta_{-2} & \Delta_{-1}\end{pmatrix}\Delta_{k_0}(\lambda_0) = \begin{pmatrix}0 & \cdots & 0\end{pmatrix},$$

where

$$\Delta_{k_0}(\lambda_0) = \begin{pmatrix}\Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)!\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & 0 & \ddots & \ddots & \Delta^{(2)}(\lambda_0)/2!\\ \vdots & & \ddots & \ddots & \Delta^{(1)}(\lambda_0)\\ 0 & \cdots & \cdots & 0 & \Delta(\lambda_0)\end{pmatrix}.$$

Proof. We have

$$(\lambda-\lambda_0)^{k_0} I = \Delta(\lambda)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \Delta_{n-k_0}\right) = \left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \Delta_{n-k_0}\right)\Delta(\lambda).$$

Hence,

$$\begin{aligned}(\lambda-\lambda_0)^{k_0} I &= \left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \frac{\Delta^{(n)}(\lambda_0)}{n!}\right)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \Delta_{n-k_0}\right)\\ &= \sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}\end{aligned}$$

and

$$(\lambda-\lambda_0)^{k_0} I = \sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n \sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}.$$

By the uniqueness of the Taylor's expansion for analytic maps, we obtain for $n \in \{0,...,k_0-1\}$ that

$$0=\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}=\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}.$$

Therefore, the result follows. □

Now we look for an explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0. Set

$$\Psi_1(\lambda)(\varphi)(\theta):=\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds$$

and

$$\Psi_2(\lambda)\left(\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\right)(\theta):=e^{\lambda\theta}\left[\alpha+\varphi(0)+\widehat{L}\left(\int_.^0 e^{\lambda(.-s)}\varphi(s)\,ds\right)\right].$$

Then both maps are analytic and

$$(\lambda I-(A+L))^{-1}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}^n}\\ \Psi_1(\lambda)(\varphi)(\theta)+\Delta(\lambda)^{-1}\Psi_2(\lambda)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}(\theta)\end{pmatrix}.$$

We observe that the only singularity in the last expression is $\Delta(\lambda)^{-1}$. Since Ψ_1 and Ψ_2 are analytic, we have for $j=1,2$ that

$$\Psi_j(\lambda)=\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^j(\lambda_0),$$

where $|\lambda-\lambda_0|$ is small enough and $L_n^j(.):=\frac{d^n\Psi_j(.)}{d\lambda^n},\forall n\geq 0,\forall j=1,2$. Hence,

$$\begin{aligned}&\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left((\lambda-\lambda_0)^{k_0}\Psi_1(\lambda)\right)\\&=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\sum_{n=0}^{+\infty}\frac{(n+k_0)!}{(n+1)!}\frac{(\lambda-\lambda_0)^{n+1}}{n!}L_n^1(\lambda_0)\\&=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\Psi_2(\lambda)\right]\\&=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\left(\sum_{n=-k_0}^{+\infty}(\lambda-\lambda_0)^{n+k_0}\Delta_n\right)\left(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^2(\lambda_0)\right)\right]\\&=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)\left(\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^2(\lambda_0)\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} \sum_{j=0}^{n} (\lambda-\lambda_0)^{n-j} \Delta_{n-j-k_0} \frac{(\lambda-\lambda_0)^j}{j!} L_j^2(\lambda_0) \right] \\
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^2(\lambda_0) \right] \\
&= \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^2(\lambda_0).
\end{aligned}
$$

From the above results we obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0, which is given in the following proposition.

Proposition 7.1.11. *Each $\lambda_0 \in \sigma(A+L)$ is a pole of $(\lambda I-(A+L))^{-1}$ of order $k_0 \geq 1$. Moreover, k_0 is the only integer so that there exists $\Delta_{-k_0} \in M_n(\mathbb{R})$ with $\Delta_{-k_0} \neq 0$, such that*

$$\Delta_{-k_0} = \lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1}.$$

Furthermore, the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace of $A+L$ associated to λ_0 is defined by the following formula

$$B_{-1}^{\lambda_0} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{bmatrix} 0_{\mathbb{R}^n} \\ \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^2(\lambda_0) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \end{bmatrix}, \tag{7.1.24}$$

where

$$\Delta_{-j} = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-j)!} \frac{d^{k_0-j}}{d\lambda^{k_0-j}} \left((\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \right), j = 1, ..., k_0,$$

$$L_0^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}(\theta) = e^{\lambda\theta} \left[\alpha + \varphi(0) + \widehat{L}\left(\int_.^0 e^{\lambda(.-s)} \varphi(s)\, ds \right) \right],$$

and

$$
\begin{aligned}
L_j^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}(\theta) &= \frac{d^j}{d\lambda^j} \left[L_0^2(\lambda) \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}(\theta) \right] \\
&= \sum_{k=0}^{j} C_j^k \theta^k e^{\lambda\theta} \frac{d^{j-k}}{d\lambda^{j-k}} \left[\alpha + \varphi(0) + \widehat{L}\left(\int_.^0 e^{\lambda(.-s)} \varphi(s)\, ds \right) \right], \ j \geq 1,
\end{aligned}
$$

here

$$\frac{d^i}{d\lambda^i} \left[\alpha + \varphi(0) + \widehat{L}\left(\int_.^0 e^{\lambda(.-s)} \varphi(s)\, ds \right) \right] = \widehat{L}\left(\int_.^0 (.-s)^i e^{\lambda(.-s)} \varphi(s)\, ds \right), i \geq 1$$

In studying Hopf bifurcation it usually requires to consider the projector for a simple eigenvalue. Now we consider the case when λ_0 is a simple eigenvalue of

$A+L$. That is, λ_0 is pole of order 1 of the resolvent of $A+L$ and the dimension of the eigenspace of $A+L$ associated to the eigenvalue λ_0 is 1.

We know that λ_0 is a pole of order 1 of the resolvent of $A+L$ if and only if there exists $\Delta_{-1} \neq 0$, such that

$$\Delta_{-1} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) \Delta(\lambda)^{-1}.$$

From Lemma 7.1.10, we have $\Delta_{-1}\Delta(\lambda_0) = \Delta(\lambda_0)\Delta_{-1} = 0$. Hence

$$\Delta_{-1}\left[B + \widehat{L}\left(e^{\lambda_0 \cdot} I\right)\right] = \left[B + \widehat{L}\left(e^{\lambda_0 \cdot} I\right)\right]\Delta_{-1} = \lambda_0 \Delta_{-1}.$$

From the proof of Lemma 7.1.6, it can be checked that λ_0 is simple if and only if $\dim[\mathscr{N}(\Delta(\lambda_0))] = 1$. In that case, there exist $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$ such that

$$W_{\lambda_0}^T \Delta(\lambda_0) = 0 \text{ and } \Delta(\lambda_0) V_{\lambda_0} = 0. \tag{7.1.25}$$

Hence, by Lemma 7.1.10 (replacing $V_{\lambda_0} W_{\lambda_0}^T$ by $\delta V_{\lambda_0} W_{\lambda_0}^T$ for some $\delta \neq 0$ if necessary), we can always assume that

$$\Delta_{-1} = V_{\lambda_0} W_{\lambda_0}^T.$$

Then we can see that $B_{-1}^{\lambda_0} B_{-1}^{\lambda_0} = B_{-1}^{\lambda_0}$ if and only if

$$\Delta_{-1} = \Delta_{-1}\left[I + \widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0 \cdot} ds\right)\right]\Delta_{-1}.$$

Therefore, we obtain the following corollary.

Corollary 7.1.12. *$\lambda_0 \in \sigma(A+L)$ is a simple eigenvalue of $A+L$ if and only if*

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^2 \Delta(\lambda)^{-1} = 0$$

and

$$\dim[\mathscr{N}(\Delta(\lambda_0))] = 1.$$

Moreover, the projector on the eigenspace associated to λ_0 is

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{bmatrix} 0_{\mathbb{R}^n} \\ e^{\lambda_0 \theta}\Delta_{-1}\left[\alpha + \varphi(0) + \widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0(\cdot - s)}\varphi(s)\,ds\right)\right]\end{bmatrix},$$

where

$$\Delta_{-1} = V_{\lambda_0} W_{\lambda_0}^T$$

in which $V_{\lambda_0}, W_{\lambda_0} \in \mathbb{C}^n \setminus \{0\}$ are two vectors satisfying (7.1.25) and

$$\Delta_{-1} = \Delta_{-1}\left[I + \widehat{L}\left(\int_{\cdot}^{0} e^{\lambda_0 \cdot} ds\right)\right]\Delta_{-1}.$$

7.1.3 Hopf Bifurcation

Applying Theorem 6.1.21, a local center manifold theorem can be established for the RFDE (7.1.1). See Hale and Verduyn [175] and Guo and Wu [159]. Here we apply Theorem 6.2.7 to establish a Hopf bifurcation theorem for the RFDE.

Consider the following functional differential equation with a parameter

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + f(\mu, x_t),\ \forall t \geq 0, \\ x_0 = \varphi \in \mathscr{C}, \end{cases} \tag{7.1.26}$$

where $\mu \in \mathbb{R}, x_t \in \mathscr{C}$ satisfies $x_t(\theta) = x(t+\theta), B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, and $f : \mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ is a C^k-map with $k \geq 4$.

By setting $v(t) = \begin{pmatrix} 0 \\ x_t \end{pmatrix}$ we can rewrite equation (7.1.26) as the following abstract non-densely defined Cauchy problem on the Banach space $X = \mathbb{R}^n \times \mathscr{C}$:

$$\frac{dv(t)}{dt} = Av(t) + F(\mu, v(t)),\ t \geq 0,\ \ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)},$$

where $A : D(A) \subset X \to X$ is the linear operator defined by

$$A\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-r, 0], \mathbb{R}^n)$$

and $F : \mathbb{R} \times \overline{D(A)} \to X$ is defined by

$$F\left(\mu, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} f(\mu, \varphi) \\ 0_{\mathscr{C}} \end{pmatrix}.$$

We assume that $f(\mu, 0) = 0, \forall \mu \in \mathbb{R}$, and set

$$L\left(\mu, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix}\right) = \partial_x F(\mu, 0)\begin{pmatrix} 0_{\mathbb{R}^n} \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_\varphi f(\mu, 0)\psi \\ 0_{\mathscr{C}} \end{pmatrix} =: \begin{pmatrix} \widehat{L}(\mu, \psi) \\ 0_{\mathscr{C}} \end{pmatrix}.$$

By Proposition 7.1.7, we know that the linear operator $A + L(\mu, .) : D(A) \to X$ is a Hille-Yosida operator. Moreover, $\omega_{0,\mathrm{ess}}((A + L(\mu, .))_0) = -\infty$ and

$$\begin{aligned} \sigma(A + L(\mu, .)) &= \sigma((A + L(\mu, .))_0) \\ &= \sigma_P((A + L(\mu, .))_0) \end{aligned}$$

$$= \{\lambda \in \mathbb{C} : \det(\Delta(\mu,\lambda)) = 0\},$$

where

$$\Delta(\mu,\lambda) := \lambda I - B - \widehat{L}\left(\mu, e^{\lambda \cdot} I\right).$$

Hence, $A+L$ satisfies Assumptions 3.4.1, 3.5.2 and 6.2.1(c). In order to apply the Hopf Bifurcation Theorem 6.2.7 to system (7.1.26), we need to make the following assumption.

Assumption 7.1.13. Let $\varepsilon > 0$ and $f \in C^k((-\varepsilon,\varepsilon)\times\mathscr{C};\mathbb{R}^n)$ for some $k \geq 4$. Assume that there exists a continuously differentiable map $\lambda : (-\varepsilon,\varepsilon) \to \mathbb{C}$ such that for each $\mu \in (-\varepsilon,\varepsilon)$,

$$\det(\Delta(\mu,\lambda(\mu))) = 0$$

and $\lambda(\mu)$ is a simple eigenvalue of $(A+\partial_x F(\mu,0))_0$, which is equivalent to

$$\lim_{\lambda \to \lambda(\mu)} \frac{\det(\Delta(\mu,\lambda))}{(\lambda - \lambda(\mu))} \neq 0$$

and

$$\dim(\mathscr{N}(\Delta(\mu,\lambda(\mu)))) = 1.$$

Moreover, assume that

$$\operatorname{Im}(\lambda(0)) > 0,\ \operatorname{Re}(\lambda(0)) = 0,\ \frac{d\operatorname{Re}(\lambda(0))}{d\mu} \neq 0,$$

and

$$\{\lambda \in \Omega : \det(\Delta(\lambda,0)) = 0\} \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\}. \tag{7.1.27}$$

From Theorem 6.2.7 we can derive the following Hopf bifurcation theorem for functional differential equations.

Theorem 7.1.14. *Let Assumption 7.1.13 be satisfied. Then there exist a constant $\varepsilon^* > 0$ and three C^{k-1}-maps, $\varepsilon \to \mu(\varepsilon)$ from $(0,\varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to \varphi_\varepsilon$ from $(0,\varepsilon^*)$ into $\mathscr{C}$, and $\varepsilon \to T(\varepsilon)$ from $(0,\varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0,\varepsilon^*)$ there exists a $T(\varepsilon)$-periodic function $x_\varepsilon \in C^k(\mathbb{R},\mathbb{R}^n)$, which is a solution of (7.1.26) for the parameter value $\mu = \mu(\varepsilon)$ and the initial value $\varphi = \varphi_\varepsilon$. Moreover, we have the following properties*

(i) There exist a neighborhood N of 0 in $\mathbb{R}^n$ and an open interval I in $\mathbb{R}$ containing 0 such that for $\widehat{\mu} \in I$ and any periodic solution $\widehat{x}(t)$ in N with minimal period $\widehat{T}$ close to $\frac{2\pi}{\omega}$ of (7.1.26) for the parameter value $\widehat{\mu}$, there exists $\varepsilon \in (0,\varepsilon^*)$ such that $\widehat{x}(t) = x_\varepsilon(t+\theta)$ (for some $\theta \in [0,\gamma(\varepsilon))$), $\mu(\varepsilon) = \widehat{\mu}$, and $T(\varepsilon) = \widehat{T}$.

(ii) The map $\varepsilon \to \mu(\varepsilon)$ is a C^{k-1}-function and

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}), \forall \varepsilon \in (0,\varepsilon^*),$$

where $[\frac{k-2}{2}]$ *is the integer part of* $\frac{k-2}{2}$.

(iii) The period $T(\varepsilon)$ *of* $t \to u_\varepsilon(t)$ *is a* C^{k-1}*-function and*

$$T(\varepsilon) = \frac{2\pi}{\omega}[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \tau_{2n}\varepsilon^{2n}] + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where ω *is the imaginary part of* $\lambda(0)$ *defined in Assumption 7.1.13;*

(iv) The Floquet exponent $\beta(\varepsilon)$ *is a* C^{k-1} *function satisfying* $\beta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and having the Taylor expansion*

$$\beta(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \beta_{2n}\varepsilon^{2n} + O(\varepsilon^{k-1}),\ \forall \varepsilon \in (0, \varepsilon^*).$$

The periodic solution $x_\varepsilon(t)$ *is orbitally asymptotically stable with asymptotic phase if* $\beta(\varepsilon) < 0$ *and unstable if* $\beta(\varepsilon) > 0$.

Remark 7.1.15. In Assumption 7.1.13, if we only assume that $k \geq 2$ and replace condition (7.1.27) by

$$\{\lambda \in \mathbb{C} : \det(\Delta(0,\lambda)) = 0\} \cap i\omega\mathbb{Z} = \{i\omega, -i\omega\}$$

with $\omega = \mathrm{Im}(\lambda(0))$. Then by using Remark 6.2.8, we deduce that assertion (i) of Theorem 7.1.14 holds. So we derive a well known Hopf bifurcation theorem for delay differential equations (see Hale and Verduyn Lunel [175, Theorem 1.1, p. 332]).

By using the results in Section 6.3, we can also develop a normal form theory for the RFDEs. See Faria and Magalhães [136, 137] and Guo and Wu [159].

7.2 Neutral Functional Differential Equations

Consider the linear neutral functional differential equation (NFDE) in L^p spaces

$$\begin{cases} \frac{d}{dt}(x(t) - L_1(x_t)) = B(x(t) - L_1(x_t)) + L_2(x_t),\ t \geq 0, \\ x(0) = \widehat{x} \in \mathbb{R}^n,\ x_0 = \varphi \in L^p((-r,0),\mathbb{R}^n), \end{cases} \tag{7.2.1}$$

with $x_t \in L^p((-r,0),\mathbb{R}^n)$ satisfing $x_t(\theta) = x(t+\theta)$ for almost every $\theta \in (-r,0)$. Here $p \in [1,+\infty)$, $r \in [0,+\infty)$, $B \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, while L_j, $j = 1,2$, are bounded linear operators from $L^p((-r,0),\mathbb{R}^n)$ into $\mathbb{R}^n$ given by

$$L_j(\varphi) = \int_{-r}^{0} \eta_j(\theta)\varphi(\theta)\,d\theta,$$

here $\eta_j \in L^q((-r,0),M_n(\mathbb{R}))$ with $\frac{1}{p} + \frac{1}{q} = 1$, $j = 1,2$.

7.2.1 Spectral Theory

Set $u(t) = x_t$ for $t \geq 0$ and we get

$$\frac{d}{dt}[u(t,0) - L_1(u(t))] = B[u(t,0) - L_1(u(t))] + L_2(u(t)),\ t \geq 0.$$

Let $y(t) = u(t,0) - L_1(u(t))$. We obtain that

$$\frac{dy(t)}{dt} = By(t) + L_2(u(t)),\ t \geq 0$$

and

$$u(t,0) = L_1(u(t)) + y(t).$$

Therefore, u satisfies a PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial \theta} = 0,\ \forall \theta \in (-r,0), \\ u(t,0) = L_1(u(t)) + y(t), \\ \dfrac{dy(t)}{dt} = By(t) + L_2(u(t)), \\ y(0) = y_0 = \widehat{x} - L_1(\varphi) \in \mathbb{R}^n, \\ u(0,\cdot) = \varphi \in L^p((-r,0),\mathbb{R}^n). \end{cases} \tag{7.2.2}$$

Let $X = \mathbb{R}^n \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$ endowed with the product norm

$$\left\| \begin{pmatrix} z_1 \\ \varphi \\ z_2 \end{pmatrix} \right\| = |z_1|_{\mathbb{R}^n} + \|\varphi\|_{L^p((-r,0),\mathbb{R}^n)} + |z_2|_{\mathbb{R}^n}$$

and $X_0 = \{0_{\mathbb{R}^n}\} \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$. Set $v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \\ y(t) \end{pmatrix}$. We can consider (7.2.2) as an abstract non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + Lv(t) + \widehat{L}v(t),\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y_0 \end{pmatrix} \in \overline{D(A)}, \tag{7.2.3}$$

where $A : D(A) \subset X \to X$ is a linear operator defined by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ \varphi' \\ By \end{pmatrix}$$

with

$$D(A) := \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n,$$

$L, \widehat{L} : X_0 \to X$ are defined by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} \text{ and } \widehat{L} \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} L_1(\varphi) \\ 0 \\ L_2(\varphi) \end{pmatrix},$$

respectively. Note that $\overline{D(A)} = X_0$.

Lemma 7.2.1. *The resolvent sets of A and $A+L$ satisfy*

$$\rho(A) = \rho(A+L) = \rho(B).$$

We have the following explicit formulas for the resolvents of A and $A+L$:

$$\begin{aligned} &(\lambda I - A)^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda \theta} \alpha + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl, \ \forall \theta \in (-r, 0) \\ \widehat{y} = (\lambda I - B)^{-1} y \end{cases} \end{aligned} \tag{7.2.4}$$

and

$$\begin{aligned} &(\lambda I - (A+L))^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \\ &\Leftrightarrow \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda \theta} [(\lambda I - B)^{-1} y + \alpha] + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl, \ \forall \theta \in (-r, 0) \\ \widehat{y} = (\lambda I - B)^{-1} y. \end{cases} \end{aligned} \tag{7.2.5}$$

Proof. We only prove the result for $A+L$, the proof for A is similar. To prove $\rho(A+L) \subset \rho(B)$, we only need to show that $\sigma(B) \subset \sigma(A+L)$. Let $\lambda \in \sigma(B)$. Then, there exists $\widehat{y} \in \mathbb{C}^n \setminus \{0\}$ such that $B\widehat{y} = \lambda \widehat{y}$. Consider $\widehat{\varphi}(\theta) = e^{\lambda\theta}\widehat{y}$, we have

$$(A+L) \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} -\widehat{\varphi}(0) + \widehat{y} \\ \widehat{\varphi}' \\ B\widehat{y} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \lambda \widehat{\varphi} \\ \lambda \widehat{y} \end{pmatrix}.$$

Thus $\lambda \in \sigma(A+L)$. This implies that $\sigma(B) \subset \sigma(A+L)$. On the other hand, if $\lambda \in \rho(B)$, for $\begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \in X$ we must have $\begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \in D(A)$ such that

$$
\begin{aligned}
&(\lambda I-(A+L))\begin{pmatrix}0_{\mathbb{R}^n}\\ \widehat{\varphi}\\ \widehat{y}\end{pmatrix}=\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\\
\Leftrightarrow &\begin{cases}\widehat{\varphi}(0)-\widehat{y}=\alpha\\ \lambda\widehat{\varphi}-\widehat{\varphi}'=\varphi\\ \lambda\widehat{y}-B\widehat{y}=y\end{cases}\\
\Leftrightarrow &\begin{cases}\widehat{\varphi}\left(\widehat{\theta}\right)=e^{\lambda\widehat{\theta}}[(\lambda I-B)^{-1}y+\alpha]+\int_{\widehat{\theta}}^{0}e^{\lambda(\widehat{\theta}-l)}\varphi(l)\,dl,\ \forall\widehat{\theta}\in(-r,0)\\ \widehat{y}=(\lambda I-B)^{-1}y.\end{cases}
\end{aligned}
$$

Therefore, we obtain that $\lambda\in\rho(A+L)$ and the formula in (7.2.5) holds. $\square$

Since B is a matrix on $\mathbb{R}^n$, we have $\omega_0(B):=\max_{\lambda\in\sigma(B)}\operatorname{Re}(\lambda)$ and the following lemma.

Lemma 7.2.2. *For each $\omega_A>\omega_0(B)$, one has $(\omega_A,+\infty)\subset\rho(A)$ and there exists $M_A\geq 1$ such that*

$$
\left\|(\lambda I-A)^{-n}\right\|_{\mathscr{L}(X_0)}\leq\frac{M_A}{(\lambda-\omega_A)^n},\ \forall n\geq 1,\ \forall\lambda>\omega_A. \tag{7.2.6}
$$

Moreover,

$$
\lim_{\lambda\to+\infty}(\lambda I-A)^{-1}x=0,\forall x\in X.
$$

Proof. Let $\omega_A>\omega_0(B)$. From Lemma 7.2.1 we obtain that $(\omega_A,+\infty)\subset\rho(B)=\rho(A)$. We can define the equivalent norm on $\mathbb{R}^n$ as follows

$$
|y|:=\sup_{t\geq 0}e^{-\omega_A t}\left\|e^{Bt}y\right\|,\ y\in\mathbb{R}^n.
$$

Then we have

$$
\left|e^{Bt}y\right|\leq e^{\omega_A t}|y|,\forall t\geq 0
$$

and

$$
\|y\|\leq|y|\leq M_A\|y\|,
$$

where

$$
M_A:=\sup_{t\geq 0}\left\|e^{(B-\omega_A I)t}\right\|_{M_n(\mathbb{R})}.
$$

Moreover, for each $\lambda>\omega_A$, we have

$$
\left|(\lambda I-B)^{-1}y\right|=\left|\int_0^{+\infty}e^{-\lambda s}e^{Bs}y\,ds\right|\leq\frac{|y|}{\lambda-\omega_A}.
$$

We define an equivalent norm $|.|$ on X by

$$
\left|\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right|=|\alpha|+\|\varphi\|_{\omega_A}+|y|,
$$

where

$$\|\varphi\|_{\omega_A} := \left|e^{-\omega_A \cdot}\varphi(\cdot)\right|_{L^p} = \left(\int_{-r}^{0}\left|e^{-\omega_A\theta}\varphi(\theta)\right|^p d\theta\right)^{1/p}.$$

Using (7.2.4) and the above results, we obtain for $\begin{pmatrix}0\\ \varphi\\ y\end{pmatrix} \in X_0$ that

$$\begin{aligned}\left|(\lambda I - A)^{-1}\begin{pmatrix}0\\ \varphi\\ y\end{pmatrix}\right| &\leq \left|e^{-\omega_A \cdot}\int_{\cdot}^{0} e^{\lambda(\cdot - l)}\varphi(l)\,dl\right|_{L^p} + \left|(\lambda I - B)^{-1}y\right| \\ &\leq \left|e^{(\lambda-\omega_A)\cdot}\right|_{L^1}\left|e^{-\omega_A \cdot}\varphi(\cdot)\right|_{L^p} + \frac{1}{\lambda-\omega_A}|y| \\ &\leq \frac{1}{\lambda-\omega_A}\left[\|\varphi\|_{\omega_A} + |y|\right].\end{aligned}$$

Therefore, (7.2.6) holds. The last part of the proof is trivial. □

As an immediate consequence of the above lemma and by applying Proposition 3.4.3, we obtain the following lemma.

Lemma 7.2.3. *A_0, the part of A in X_0, is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ of bounded linear operators on X_0, which is defined by*

$$T_{A_0}(t)\begin{pmatrix}0\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix}0\\ \widehat{T}_{A_0}(t)\varphi\\ e^{Bt}y\end{pmatrix}, \tag{7.2.7}$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(\theta) = \begin{cases}\varphi(t+\theta), & \text{if } t+\theta \leq 0,\\ 0, & \text{if } t+\theta > 0.\end{cases}$$

Moreover, A generates an integrated semigroup $\{S_A(t)\}_{t\geq 0}$ on X, which is defined by

$$S_A(t)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix}0\\ \alpha 1_{[-t,0]}(.) + \int_0^t \widehat{T}_{A_0}(l)\varphi dl\\ \int_0^t e^{Bl}y dl\end{pmatrix}.$$

Set

$$X_1 = \mathbb{R}^n \times \{0_{L^p}\} \times \{0_{\mathbb{R}^n}\}.$$

Then we have

$$X = X_1 \oplus X_0.$$

By using the same argument as in the proof of Theorem 3.8.6 we obtain the following result.

Lemma 7.2.4. *For each $\tau > 0$, each $h_1 \in L^p((0,\tau), X_1)$ and each $h_2 \in L^1((0,\tau), X_0)$, there exists a unique integrated solution of the Cauchy problem*

$$\frac{dv(t)}{dt} = Av(t) + h(t),\ t \in [0,\tau],\ h = h_1 + h_2 \text{ and } v(0) = v_0 := \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y_0 \end{pmatrix},$$

which is given by

$$v(t) = T_{A_0}(t)v_0 + \frac{d}{dt}(S_A * h)(t), \forall t \in [0,\tau].$$

Moreover, we have the following estimate

$$\|v(t)\| \le M_A e^{\omega_A t}\|v_0\| + \left(\int_0^t \|h_1(s)\|^p ds\right)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)}\|h_2(s)\| ds, \forall t \in [0,\tau].$$

Furthermore, $v(t)$ can be expressed as

$$v(t) = \begin{pmatrix} 0 \\ u(t) \\ y(t) \end{pmatrix}$$

with

$$\begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \widehat{T}_{A_0}(t)\varphi \\ e^{Bt}y_0 \end{pmatrix} + \begin{pmatrix} h_1(t+.)1_{[-t,0]}(.) + \int_0^t \widehat{T}_{A_0}(t-s)h_{21}(s)ds \\ \int_0^t e^{B(t-s)}h_{22}(s)ds \end{pmatrix},$$

here $h_2(t) = (0, h_{21}(t), h_{22}(t))$.

By using the same argument as in the proof of Theorem 3.8.6 and noting that if $h \in C^1([0,\tau], X)$, we have

$$\begin{aligned} \left\| \frac{d}{dt}(S_A * h)(t) \right\| &= \left\| \frac{d}{dt}(S_A * Ph)(t) + \frac{d}{dt}(S_A * (I-P)h)(t) \right\| \\ &\le \left(\int_0^t \|Ph(s)\|^p ds\right)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)}\|(I-P)h(s)\| ds,\ \forall t \in [0,\tau], \end{aligned}$$

where $P : X \to X$ is defined by

$$Px = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix},\ \forall x = \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} \in X.$$

Let

$$\Gamma(t,h) = \left(\int_0^t \|Ph(s)\|^p ds\right)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)}\|(I-P)h(s)\| ds,\ \forall t \in [0,\tau].$$

We obtain that

$$
\begin{aligned}
\Gamma(t,h) &= \left(\int_0^t \|Ph(s)\|^p\, ds\right)^{1/p} + M_A \int_0^t e^{\omega_A(t-s)} \|(I-P)h(s)\|\, ds \\
&\leq t^{1/p} \sup_{s\in[0,t]} \|Ph(s)\| + M_A \left(\int_0^t e^{q\omega_A(t-s)} ds\right)^{1/q} \left(\int_0^t \|(I-P)h(s)\|^p\, ds\right)^{1/p} \\
&\leq t^{1/p} \|P\| \sup_{s\in[0,t]} \|h(s)\| + M_A \left(\int_0^t e^{q\omega_A(t-s)} ds\right)^{1/q} t^{1/p} \|I-P\| \sup_{s\in[0,t]} \|h(s)\| \\
&\leq \delta(t) \sup_{s\in[0,t]} \|h(s)\|,\ \forall t \in [0,\tau],
\end{aligned}
$$

where $1/p + 1/q = 1$ and $\delta(t) = t^{1/p}\|P\| + t^{1/p} M_A \left(\int_0^t e^{q\omega_A(t-s)} ds\right)^{1/q} \|I-P\|$, which satisfies $\lim_{t\to 0^+} \delta(t) = 0$. Hence, we get $\|L+\widehat{L}\|_{\mathscr{L}(X_0,X)} \delta(t) < 1$ for t small enough. Therefore, by using the perturbation result Theorem 3.5.1 we know that $A+L+\widehat{L}$ statisfies the same properties as A. In particular, $(A+L+\widehat{L})_0$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L+\widehat{L})_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0.

From the definition of $A+L+\widehat{L}$ in (7.2.3) and the fact that $D(A) := \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$ and $\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times L^p((-r,0),\mathbb{R}^n) \times \mathbb{R}^n$, we know that

$$
\begin{aligned}
& D\left((A+L+\widehat{L})_0\right) \\
&= \left\{ \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in \{0_{\mathbb{R}^n}\} \times W^{1,p}((-r,0),\mathbb{R}^n) \times \mathbb{R}^n \,\middle|\, -\varphi(0) + y + L_1(\varphi) = 0 \right\}.
\end{aligned}
$$

Lemma 7.2.5. *The point spectrum of* $(A+L+\widehat{L})_0$ *is the set*

$$
\sigma_P\left(\left(A+L+\widehat{L}\right)_0\right) = \{\lambda \in \mathbb{C} : \det(\Delta(\lambda)) = 0\},
$$

where

$$
\begin{aligned}
\Delta(\lambda) &= (\lambda I - B)\left[I - L_1\left(e^{\lambda \cdot} I\right)\right] - L_2\left(e^{\lambda \cdot} I\right) \\
&= (\lambda I - B)\left[I - \int_{-r}^0 e^{\lambda\theta} \eta_1(\theta)\, d\theta\right] - \int_{-r}^0 e^{\lambda\theta} \eta_2(\theta)\, d\theta.
\end{aligned}
\tag{7.2.8}
$$

Proof. Let $\lambda \in \mathbb{C}$ be given. Then $\lambda \in \sigma_P((A+L+\widehat{L})_0)$ if and only if there exist $\varphi \in W^{1,p}((-r,0),\mathbb{C}^n) \setminus \{0\}$ and $y \in \mathbb{C}^n$ such that

$$
\varphi'(\theta) = \lambda\varphi(\theta), \forall \theta \in (-r,0),
$$

$$
By + L_2(\varphi) = \lambda y \text{ and } \varphi(0) = y + L_1(\varphi).
$$

Hence, we obtain that

$$
\varphi(\theta) = e^{\lambda\theta}\varphi(0),\ \lambda y - By - L_2\left(e^{\lambda \cdot}\varphi(0)\right) = 0
$$

and

$$y = \varphi(0) - L_1(e^{\lambda\cdot}\varphi(0)).$$

Therefore,

$$\varphi \neq 0 \Leftrightarrow \varphi(0) \neq 0 \text{ and } (\lambda I - B)\left[\varphi(0) - L_1(e^{\lambda\cdot}\varphi(0))\right] - L_2\left(e^{\lambda\cdot}\varphi(0)\right) = 0.$$

The proof is complete. □

From the above discussion and results, we obtain the following proposition.

Proposition 7.2.6. *$(A+L+\widehat{L})_0$ is the infinitesimal generator of a strongly continuous semigroup $\{T_{(A+L+\widehat{L})_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0. Moreover,*

$$\begin{aligned}
&\omega_{0,\mathrm{ess}}\left((A+L+\widehat{L})_0\right) = \omega_{0,\mathrm{ess}}(A_0) = -\infty,\\
&\omega_0\left(\left(A+L+\widehat{L}\right)_0\right) = \max_{\lambda\in\sigma_P\left(\left(A+L+\widehat{L}\right)_0\right)} \mathrm{Re}(\lambda),\\
&\sigma\left(A+L+\widehat{L}\right) = \sigma\left(\left(A+L+\widehat{L}\right)_0\right) = \sigma_P\left(\left(A+L+\widehat{L}\right)_0\right)\\
&\qquad = \{\lambda\in\mathbb{C} : \det(\Delta(\lambda)) = 0\}
\end{aligned}$$

and each $\lambda_0 \in \sigma(A+L+\widehat{L})$ is a pole of $(\lambda I - (A+L+\widehat{L}))^{-1}$. For each $\gamma\in\mathbb{R}$, the subset $\{\lambda\in\sigma((A+L+\widehat{L})_0) : \mathrm{Re}(\lambda)\geq\gamma\}$ is either empty or finite.

Proof. We only need to prove that $\omega_{0,\mathrm{ess}}((A+L+\widehat{L})_0) = \omega_{0,\mathrm{ess}}(A_0) = -\infty$. From (7.2.7) it is easy to know that for $t > r$, $T_{A_0}(t)$ is compact. Hence $\omega_{0,\mathrm{ess}}(A_0) = -\infty$. Since for each $t > 0$, $(L+\widehat{L})T_{A_0}(t)$ is compact, the result follows by applying Theorem 4.7.3. □

7.2.2 Projectors on the eigenspaces

Let $\lambda_0 \in \sigma(A+L+\widehat{L})$ be given. From Proposition 7.2.6 we already know that λ_0 is a pole of $(\lambda I - (A+L+\widehat{L}))^{-1}$of finite order $k_0 \geq 1$. This means that λ_0 is isolated in $\sigma(A+L+\widehat{L})$ and the Laurent's expansion of the resolvent around λ_0 takes the following form

$$\left(\lambda I - \left(A+L+\widehat{L}\right)\right)^{-1} = \sum_{n=-k_0}^{+\infty} (\lambda-\lambda_0)^n B_n^{\lambda_0}. \tag{7.2.9}$$

The bounded linear operator $B_{-1}^{\lambda_0}$ is the projector on the generalized eigenspace of $A+L+\widehat{L}$ associated to λ_0. Now we give a method to compute $B_{-1}^{\lambda_0}$. Note that

$$(\lambda-\lambda_0)^{k_0}\left(\lambda I - \left(A+L+\widehat{L}\right)\right)^{-1} = \sum_{m=0}^{+\infty} (\lambda-\lambda_0)^m B_{m-k_0}^{\lambda_0}.$$

So we have the following approximation formula

$$B_{-1}^{\lambda_0} = \lim_{\lambda \to \lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left((\lambda - \lambda_0)^{k_0} \left(\lambda I - \left(A + L + \widehat{L} \right) \right)^{-1} \right).$$

In order to give an explicit formula for $B_{-1}^{\lambda_0}$, we need the following results.

Lemma 7.2.7. *For each $\lambda \in \rho(A+L+\widehat{L})$, we have the following explicit formula for the resolvent of $A+L+\widehat{L}$,*

$$\begin{array}{l} \left(\lambda I - \left(A + L + \widehat{L} \right) \right)^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \\ \Leftrightarrow \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda \theta} \Phi_\lambda + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl \\ \widehat{y} = \Phi_\lambda - L_1 \left(e^{\lambda \cdot} \Phi_\lambda \right) - L_1 \left(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l)\, dl \right) - \alpha, \end{cases} \end{array} \tag{7.2.10}$$

where Φ_λ is defined by

$$\Phi_\lambda = \Delta(\lambda)^{-1} \begin{bmatrix} (\lambda I - B) \left(L_1 \left(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l)\, dl \right) + \alpha \right) \\ + L_2 \left(\int_\cdot^0 e^{\lambda(\cdot - l)} \varphi(l)\, dl \right) + y \end{bmatrix} \tag{7.2.11}$$

with $\Delta(\lambda)$ defined in (7.2.8).

Proof. If $\lambda \in \rho(A+L+\widehat{L})$ and $\gamma > 0$ is large enough such that $\operatorname{Re}(\lambda) > \omega_0(B) - \gamma$, then we obtain that $\lambda \in \rho(B - \gamma I)$. Consider the linear operators $A_\gamma : D(A) \subset X \to X$ and $L_\gamma : X_0 \to X$ defined respectively by

$$A_\gamma \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} -\varphi(0) + y \\ \varphi' \\ (B - \gamma I) y \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in D(A)$$

and

$$L_\gamma \left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \right) = \begin{pmatrix} L_1(\varphi) \\ 0 \\ L_2(\varphi) + \gamma y \end{pmatrix}, \ \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \\ y \end{pmatrix} \in X_0.$$

From Lemma 7.2.1, we know that $\lambda \in \rho\left(A_\gamma\right)$ and

$$\begin{array}{l} \left(\lambda I - A_\gamma \right)^{-1} \begin{pmatrix} \alpha \\ \varphi \\ y \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \widehat{\varphi} \\ \widehat{y} \end{pmatrix} \\ \Leftrightarrow \begin{cases} \widehat{\varphi}(\theta) = e^{\lambda \theta} [((\lambda + \gamma) I - B)^{-1} y + \alpha] + \int_\theta^0 e^{\lambda(\theta - l)} \varphi(l)\, dl \\ \widehat{y} = [(\lambda + \gamma) I - B]^{-1} y. \end{cases} \end{array} \tag{7.2.12}$$

Hence, we know that $\lambda I-\left(A_\gamma+L_\gamma\right)$ is invertible if and only if $I-L_\gamma\left(\lambda I-A_\gamma\right)^{-1}$ is invertible, and

$$\left(\lambda I-\left(A_\gamma+L_\gamma\right)\right)^{-1}=\left(\lambda I-A_\gamma\right)^{-1}\left[I-L_\gamma\left(\lambda I-A_\gamma\right)^{-1}\right]^{-1}. \tag{7.2.13}$$

We also know that $\left[I-L_\gamma\left(\lambda I-A_\gamma\right)^{-1}\right]\begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\varphi}\\ \widetilde{y}\end{pmatrix}=\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}$ is equivalent to

$$\widetilde{\varphi}=\varphi,$$

$$\widetilde{\alpha}-L_1(e^{\lambda\cdot}\widetilde{\alpha})-L_1\left(e^{\lambda\cdot}y_\lambda\right)=L_1\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha \tag{7.2.14}$$

and (note that $\widetilde{y}-\gamma[(\lambda+\gamma)I-B]^{-1}\widetilde{y}=(\lambda I-B)\,y_\lambda$)

$$-L_2(e^{\lambda\cdot}\widetilde{\alpha})-L_2(e^{\lambda\cdot}y_\lambda)+(\lambda I-B)\,y_\lambda=L_2\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+y, \tag{7.2.15}$$

where $y_\lambda=((\lambda+\gamma)I-B)^{-1}\widetilde{y}$. By computing $(\lambda I-B)\times$ (7.2.14) + (7.2.15), we get

$$\begin{aligned}(\lambda I-B)&\left[\widetilde{\alpha}-L_1\left(e^{\lambda\cdot}\widetilde{\alpha}\right)\right]-L_2\left(e^{\lambda\cdot}\widetilde{\alpha}\right)+(\lambda I-B)\left[y_\lambda-L_1\left(e^{\lambda\cdot}y_\lambda\right)\right]-L_2\left(e^{\lambda\cdot}y_\lambda\right)\\ &=(\lambda I-B)\left(L_1\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right)+L_2\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+y,\end{aligned}$$

i.e.

$$\begin{aligned}&\Delta(\lambda)\left(\widetilde{\alpha}+y_\lambda\right)\\ &\quad=(\lambda I-B)\left(L_1\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right)+L_2\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+y.\end{aligned}$$

We know from (7.2.14) that

$$\widetilde{\alpha}-L_1\left(e^{\lambda\cdot}\left(\widetilde{\alpha}+y_\lambda\right)\right)=L_1\left(\int_\cdot^0 e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha.$$

Therefore, $I-L_\gamma\left(\lambda I-A_\gamma\right)^{-1}$ is invertible if and only if $\Delta(\lambda)$ is invertible. Moreover,

$$\left[I-L_\gamma\left(\lambda I-A_\gamma\right)^{-1}\right]^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}=\begin{pmatrix}\widetilde{\alpha}\\ \widetilde{\varphi}\\ \widetilde{y}\end{pmatrix}$$

is equivalent to

$$\widetilde{\varphi}=\varphi, \tag{7.2.16}$$

$$\widetilde{\alpha} = L_1\left(e^{\lambda\cdot}\Phi_\lambda\right) + L_1\left(\int_{\cdot}^{0} e^{\lambda(\cdot-l)}\varphi(l)\,dl\right) + \alpha \tag{7.2.17}$$

and

$$\widetilde{y} = ((\lambda+\gamma)I - B)\left[\Phi_\lambda - \widetilde{\alpha}\right], \tag{7.2.18}$$

where Φ_λ is defined by (7.2.11). Note that $A + L + \widehat{L} = A_\gamma + L_\gamma$. By using (7.2.12), (7.2.13) and (7.2.16)-(7.2.18), we obtain that

$$\left(\lambda I - \left(A + L + \widehat{L}\right)\right)^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \left(\lambda I - \left(A_\gamma + L_\gamma\right)\right)^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix} = \begin{pmatrix}0_{\mathbb{R}^n}\\ \widehat{\varphi}\\ \widehat{y}\end{pmatrix}$$
$$\Leftrightarrow \begin{cases}\widehat{\varphi}(\theta) = e^{\lambda\theta}\Phi_\lambda + \int_\theta^0 e^{\lambda(\theta-l)}\varphi(l)\,dl\\ \widehat{y} = \Phi_\lambda - L_1\left(e^{\lambda\cdot}\Phi_\lambda\right) - L_1\left(\int_{\cdot}^{0} e^{\lambda(\cdot-l)}\varphi(l)\,dl\right) - \alpha.\end{cases}$$

This completes the proof. $\square$

The map $\lambda \to \Delta(\lambda)$ from $\mathbb{C}$ into $M_n(\mathbb{C})$ is differentiable and

$$\Delta^{(1)}(\lambda) := \frac{d\Delta(\lambda)}{d\lambda} = I - \int_{-r}^{0}\left(e^{\lambda\theta} + \lambda\theta e^{\lambda\theta}\right)\eta_1(\theta)\,d\theta$$
$$+B\int_{-r}^{0}\theta e^{\lambda\theta}\eta_1(\theta)\,d\theta - \int_{-r}^{0}\theta e^{\lambda\theta}\eta_2(\theta)\,d\theta.$$

So the map $\lambda \to \Delta(\lambda)$ is analytic and

$$\Delta^{(n)}(\lambda) := \frac{d^n\Delta(\lambda)}{d\lambda^n} = -\int_{-r}^{0}\left(n\theta^{n-1}e^{\lambda\theta} + \lambda\theta^n e^{\lambda\theta}\right)\eta_1(\theta)\,d\theta$$
$$+B\int_{-r}^{0}\theta^n e^{\lambda\theta}\eta_1(\theta)\,d\theta - \int_{-r}^{0}\theta^n e^{\lambda\theta}\eta_2(\theta)\,d\theta,\ n \geq 2.$$

We know that the inverse function

$$\psi : L \to L^{-1}$$

of a linear operator $L \in \mathrm{Isom}(X)$ is differentiable, and

$$D\psi(L)\widehat{L} = -L^{-1} \circ \widehat{L} \circ L^{-1}.$$

Applying this result, we deduce that $\lambda \to \Delta(\lambda)^{-1}$ from $\rho(A + L + \widehat{L})$ into $M_n(\mathbb{C})$ is differentiable, and $\frac{d}{d\lambda}\Delta(\lambda)^{-1} = -\Delta(\lambda)^{-1}\left(\frac{d}{d\lambda}\Delta(\lambda)\right)\Delta(\lambda)^{-1}$. Therefore, we obtain that the map $\lambda \to \Delta(\lambda)^{-1}$ is analytic and has a Laurent's expansion around λ_0 :

$$\Delta(\lambda)^{-1} = \sum_{n=-\widehat{k}_0}^{+\infty}(\lambda - \lambda_0)^n\Delta_n. \tag{7.2.19}$$

Lemma 7.2.8. *Let $\lambda_0 \in \sigma(A+L+\widehat{L})$. Then the following statements are equivalent*

(a) λ_0 *is a pole of order* k_0 *of* $(\lambda I-(A+L+\widehat{L}))^{-1}$;
(b) λ_0 *is a pole of order* k_0 *of* $\Delta(\lambda)^{-1}$;
(c) $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\neq 0$ *and* $\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k_0+1}\Delta(\lambda)^{-1}=0.$

Proof. The proof follows from the explicit formula of the resolvent of $A+L+\widehat{L}$ obtained in Lemma 7.2.7. □

Lemma 7.2.9. *The matrices $\Delta_{-1},\dots,\Delta_{-k_0}$ in (7.2.19) satisfy*

$$\Delta_{k_0}(\lambda_0)\begin{pmatrix}\Delta_{-1}\\ \Delta_{-2}\\ \vdots\\ \Delta_{-k_0+1}\\ \Delta_{-k_0}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

and

$$\left(\Delta_{-k_0}\ \Delta_{-k_0+1}\ \cdots\ \Delta_{-2}\ \Delta_{-1}\right)\Delta_{k_0}(\lambda_0)=\left(0\ \cdots\ 0\right),$$

where

$$\Delta_{k_0}(\lambda_0)=\begin{pmatrix}\Delta(\lambda_0) & \Delta^{(1)}(\lambda_0) & \Delta^{(2)}(\lambda_0)/2! & \cdots & \Delta^{(k_0-1)}(\lambda_0)/(k_0-1)!\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & 0 & \ddots & \ddots & \Delta^{(2)}(\lambda_0)/2!\\ \vdots & & \ddots & \ddots & \Delta^{(1)}(\lambda_0)\\ 0 & \cdots & \cdots & 0 & \Delta(\lambda_0)\end{pmatrix}.$$

Proof. We have

$$(\lambda-\lambda_0)^{k_0}I=\Delta(\lambda)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)=\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)\Delta(\lambda).$$

Hence,

$$\begin{aligned}(\lambda-\lambda_0)^{k_0}I&=\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\frac{\Delta^{(n)}(\lambda_0)}{n!}\right)\left(\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\Delta_{n-k_0}\right)\\&=\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}\end{aligned}$$

and

$$(\lambda-\lambda_0)^{k_0}I=\sum_{n=0}^{+\infty}(\lambda-\lambda_0)^n\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}.$$

By the uniqueness of the Taylor's expansion for analytic maps, we obtain for $n \in \{0,...,k_0-1\}$ that

$$0=\sum_{k=0}^{n}\Delta_{k-k_0}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}=\sum_{k=0}^{n}\frac{\Delta^{(n-k)}(\lambda_0)}{(n-k)!}\Delta_{k-k_0}.$$

Therefore, the result follows. $\square$

Set

$$\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}:=(\lambda I-B)\left(L_1\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right)+L_2\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+y,$$

$$\Psi_2(\lambda)(\varphi)(\theta):=\int_{\theta}^{0}e^{\lambda(\theta-l)}\varphi(l)\,dl$$

and

$$\Psi_3(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}:=L_1\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha.$$

Then all maps are analytic and

$$\begin{aligned}&\left(\lambda I-\left(A+L+\widehat{L}\right)\right)^{-1}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\\ &=\begin{pmatrix}0_{\mathbb{R}^n}\\ e^{\lambda\cdot}\Delta(\lambda)^{-1}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}+\Psi_2(\lambda)(\varphi)(\cdot)\\ \Delta(\lambda)^{-1}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}-L_1\left(e^{\lambda\cdot}\Delta(\lambda)^{-1}\Psi_1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right)-\Psi_3(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\end{pmatrix}.\end{aligned} \tag{7.2.20}$$

We observe that the only singularity in (7.2.20) is $\Delta(\lambda)^{-1}$. Since Ψ_1 ,Ψ_2 and Ψ_3 are analytic, we have for $j=1,2,3$ that

$$\Psi_j(\lambda)=\sum_{n=0}^{+\infty}\frac{(\lambda-\lambda_0)^n}{n!}L_n^j(\lambda_0), \tag{7.2.21}$$

where $|\lambda-\lambda_0|$ is small enough and $L_n^j(.):=\frac{d^n\Psi_j(.)}{d\lambda^n},\forall n\geq 0,\forall j=1,2,3$. Hence, we have

$$\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-1)!}\frac{d^{k_0-1}}{d\lambda^{k_0-1}}\left[(\lambda-\lambda_0)^{k_0}\Psi_i(\lambda)\right]$$

$$
\begin{aligned}
&= \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \sum_{n=0}^{+\infty} \frac{(n+k_0)!}{(n+1)!} \frac{(\lambda-\lambda_0)^{n+1}}{n!} L_n^i(\lambda_0) \\
&= 0,\ i = 2,3.
\end{aligned}
$$

From (7.2.19) and (7.2.21), we obtain that

$$
\begin{aligned}
&\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[(\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \Psi_1(\lambda) \right] \\
&\quad = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\left(\sum_{n=-k_0}^{+\infty} (\lambda-\lambda_0)^{n+k_0} \Delta_n \right) \left(\sum_{n=0}^{+\infty} \frac{(\lambda-\lambda_0)^n}{n!} L_n^1(\lambda_0) \right) \right] \\
&\quad = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\left(\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \Delta_{n-k_0} \right) \left(\sum_{n=0}^{+\infty} \frac{(\lambda-\lambda_0)^n}{n!} L_n^1(\lambda_0) \right) \right] \\
&\quad = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} \sum_{j=0}^{n} (\lambda-\lambda_0)^{n-j} \Delta_{n-j-k_0} \frac{(\lambda-\lambda_0)^j}{j!} L_j^1(\lambda_0) \right] \\
&\quad = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^1(\lambda_0) \right] \\
&\quad = \sum_{j=0}^{k_0-1} \frac{1}{j!} \Delta_{-1-j} L_j^1(\lambda_0)
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[e^{\lambda\theta} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1} \Psi_1(\lambda) \right] \\
&\quad = \lim_{\lambda\to\lambda_0} \frac{1}{(k_0-1)!} \frac{d^{k_0-1}}{d\lambda^{k_0-1}} \left[e^{\lambda\theta} \left(\sum_{n=0}^{+\infty} (\lambda-\lambda_0)^{n} \sum_{j=0}^{n} \Delta_{n-j-k_0} \frac{1}{j!} L_j^1(\lambda_0) \right) \right] \\
&\quad = \sum_{i=0}^{k_0-1} \frac{1}{i!} \theta^i e^{\lambda_0\theta} \sum_{j=0}^{k_0-1-i} \frac{1}{j!} \Delta_{-1-j-i} L_j^1(\lambda_0).
\end{aligned}
$$

From the above results we obtain the explicit formula for the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace associated to λ_0.

Proposition 7.2.10. *Each $\lambda_0 \in \sigma(A+L+\widehat{L})$ is a pole of $(\lambda I - (A+L+\widehat{L}))^{-1}$ of order $k_0 \geq 1$. Moreover, k_0 is the only integer such that there exists $\Delta_{-k_0} \in M_n(\mathbb{R})$ with $\Delta_{-k_0} \neq 0$, such that*

$$
\Delta_{-k_0} = \lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^{k_0} \Delta(\lambda)^{-1}.
$$

Furthermore, the projector $B_{-1}^{\lambda_0}$ on the generalized eigenspace of $(A+L+\widehat{L})$ associated to λ_0 is defined by the following formula

$$B_{-1}^{\lambda_0}\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}=\begin{bmatrix}0_{\mathbb{R}^n}\\ \widehat{\varphi}\\ \sum_{j=0}^{k_0-1}\frac{1}{j!}\Delta_{-1-j}L_j^1(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}-L_1(\widehat{\varphi})\end{bmatrix},$$

where

$$\widehat{\varphi}(\theta)=\sum_{i=0}^{k_0-1}\theta^i e^{\lambda_0\theta}\frac{1}{i!}\sum_{j=0}^{k_0-1-i}\frac{1}{j!}\Delta_{-1-j-i}L_j^1(\lambda_0)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix},$$

$$\Delta_{-j}=\lim_{\lambda\to\lambda_0}\frac{1}{(k_0-j)!}\frac{d^{k_0-j}}{d\lambda^{k_0-j}}\left((\lambda-\lambda_0)^{k_0}\Delta(\lambda)^{-1}\right),\ \ j=1,...,k_0,$$

$$L_0^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}=(\lambda I-B)\left(L_1\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right)+L_2\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+y$$

and

$$\begin{aligned}L_j^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}&=\frac{d^j}{d\lambda^j}\left[L_0^1(\lambda)\begin{pmatrix}\alpha\\ \varphi\\ y\end{pmatrix}\right]\\ &=(\lambda I-B)L_1\left(\int_{\cdot}^{0}(.-l)^j e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)\\ &\quad+j\frac{d^{j-1}}{d\lambda^{j-1}}\left[L_1\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right]\\ &\quad+L_2\left(\int_{\cdot}^{0}(.-l)^j e^{\lambda(.-l)}\varphi(l)\,dl\right),\quad j\geq 1,\end{aligned}$$

here

$$\frac{d^i}{d\lambda^i}\left[L_1\left(\int_{\cdot}^{0}e^{\lambda(\cdot-l)}\varphi(l)\,dl\right)+\alpha\right]=L_1\left(\int_{\cdot}^{0}(\cdot-l)^i e^{\lambda(\cdot-l)}\varphi(l)\,dl\right),\ \ i\geq 1.$$

Applying the results in Chapter 6, one can establish the center manifold theorem, Hopf bifurcation theorem, and normal form theory for neutral functional differential equations.

7.3 Partial Functional Differential Equations

In this section we first show that a delayed transport equation for cell growth and division has asynchronous exponential growth using Theorem 4.6.2 and Corollary 4.6.6. Then we demonstrate that partial functional differential equations can also be set up as an abstract Cauchy problem.

7.3.1 A Delayed Transport Model of Cell Growth and Division

Assume that each cell grows linearly so that its mass $m(t)$ at time t after birth satisfies $m(t+\tau) = m(\tau) + t$ for $t, \tau > 0$. Each cell divides into exactly two cells of equal mass after a random length of time $1+T$ comprised of a constant deterministic phase and an exponentially distributed phase with probability

$$\Pr\{T > t\} = e^{-pt},\ t \geq 0,$$

where p is a constant. Let $u(x,t)$ be defined so that

$$\int_{x_1}^{x_2} u(x,t)dx$$

is the rate per unit time at which cells divide with mass at division between x_1 and x_2 with $1 \leq x_1 \leq x_2$. The function $u(x,t)$ satisfies the delayed transport equation (Hannsgen et al. [177, 178])

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + \dfrac{\partial u(x,t)}{\partial x} = -pu(x,t) + 4pu(2(x-1), t-1),\ t \geq 0,\ x \geq 1, \\ u(x,t) = \phi(x,t),\ -1 \leq t \leq 0,\ x \geq 0, \\ u(x,t) = 0 \text{ for } t \geq 0, 0 \leq x \leq 1. \end{cases} \tag{7.3.1}$$

Regarding the basic properties of the solutions of model (7.3.1), Hannsgen et al. [177, 178] proved the following results.

Proposition 7.3.1. *Let $\phi \in C^1([0,\infty)\times[-1,0])$ with $\phi(0,t) = 0,\ -1 \leq t \leq 0, \phi(x,0) = 0, 0 \leq x \leq 1$. If ϕ is nonnegative, then model (7.3.1) has a unique solution which is also nonnegative. Let $\lambda = \lambda_0$ be the unique real solution of the equation $\lambda + p = 2pe^{-\lambda}$ and let*

$$h(x) = \begin{cases} N \sum\limits_{n=0}^{\infty} (-2)^n c_n e^{-2^n(\lambda_0+p)(x-2)} & x \geq 2, \\ 0 & 0 \leq x \leq 2, \end{cases} \tag{7.3.2}$$

where

$$c_0 = 1,\ c_n = \frac{1}{(2-1)(4-1)\cdots(2^n-1)},\ n = 1, 2, \ldots,$$

and N is a normalizing constant. Then

$$h(x) > 0 \ \text{for}\ \ x > 2,\ \int_2^{\infty} h(x)dx = 1,\ \text{ and } h(x) \sim Ne^{-(\lambda_0+p)(x-2)} \text{ as } x \to \infty.$$

If ϕ satisfies $0 \leq \phi(x,t) \leq m_0(x)$ with $m_0 \in C([0,\infty)) \cap L^1(0,\infty)$ and m_0 nonincreasing, then $e^{-\lambda_0 t}u(x,t)$ converges weakly to $c(\phi)h(.)$ in $L^1(1,\infty)$, where

$$c(\phi) = \frac{\int_1^{\infty} \phi(x,0)dx + (\lambda_0+p)\int_{-1}^{0}\int_0^{\infty} e^{-\lambda_0\theta}\phi(x,\theta)dxd\theta}{1+\lambda_0+p}. \tag{7.3.3}$$

Let $0 < \tau < \frac{p}{2}$, let

$$Y = \{f \in C([0,\infty)) : f(0) = 0 \text{ and } \lim_{x\to\infty} e^{\tau x}|f(x)| = 0\}$$

with norm $\|f\|_Y = \sup_{x\geq 0} e^{\tau x}|f(x)|$, and let

$$X = \{\phi \in C)[-1,0];Y) : \phi(0)(x) = 0 \text{ for } x \in [0,1]\}$$

with norm $\|\phi\|_X = \sup_{-1\leq\theta\leq 0} \|\phi(\theta)\|_Y$. Define a linear operator $B : Y \to Y$ by

$$Bf = -f' - pf, \ D(B) = \{f \in Y : f' \in Y\}.$$

Define a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators in Y with infinitesimal generator B by

$$(S(t)f)(x) = \begin{cases} e^{-pt}f(x-t) & x \geq t, \\ 0 & 0 \leq x \leq t. \end{cases} \tag{7.3.4}$$

Define a map $F : X \to Y$ by

$$(F\phi)(x) = \begin{cases} 4p\phi(-1)(2(x-1)) & x \geq 1, \\ 0 & 0 \leq x \leq 1. \end{cases} \tag{7.3.5}$$

Let $\phi \in X$. For $u : [-1,\infty) \to Y$, define $u_t \in X$ for $t \geq 0$ by $u_t(\theta) = u(t+\theta),\ -1 \leq \theta \leq 0$. Consider

$$u(t) = S(t)\phi(0) + \int_0^t S(t-s)Fu_s ds, \ t \geq 0; \ u_0 = \phi. \tag{7.3.6}$$

By the results in Travis and Webb [340], we know that the problem (7.3.5) has a unique solution. Moreover, define the linear operator $A : X \to X$ by

$$A\phi = \phi', \ D(A) = \{\phi \in X : \phi' \in X, \phi(0) \in D(B), \text{ and } \phi'(0) = B\phi(0) + F\phi\}$$

and a family of operators $\{T(t)\}_{t\geq 0}$ in X by

$$T(t)\phi = u_t, \ t \geq 0.$$

Then $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators in X with infinitesimal generator A. Furthermore,

$$(T(t)\phi)(0)(x) = u(t)(0)(x) \equiv u(x,t)$$

is the unique solution of model (7.3.1) for $\phi \in D(A)$.

Now we apply Theorem 4.6.2 and Corollary 4.6.6 to show that the solutions of model (7.3.1) have asynchronous exponential growth.

Theorem 7.3.2. *Let λ_0 be the unique real solution of $\lambda + p = 2pe^{-\lambda}$. Then $\{T(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant λ_0. Moreover,*

$$\lim_{t\to\infty} e^{-\lambda_0 t}T(t)\phi = c(\phi)\phi_0$$

for all $\phi \in X$, *where* $\phi_0(\theta) = e^{\lambda_0\theta}h,\ -1 \le \theta \le 0,\ c(\phi)$ *is defined as in (7.3.3), and* h *is defined as in (7.3.2).*

Proof. Since $\lambda_0 > 0$, we prove the theorem in three steps: (I) $\omega_{0,\mathrm{ess}}(A) < 0$; (II) $\lambda_0 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ and λ_0 is a simple pole of $(\lambda I - A)^{-1}$; (III) P_0 defined in Definition 4.6.1 is given by $P_0\phi = c(\phi)\phi_0$.

(I) We use Proposition 4.6.3 to show that $\omega_{0,\mathrm{ess}}(A) \le 2\tau - p$. More specifically, we show that $T(t) = U(t) + V(t)$ for $t \ge 2$, where $\|U(t)\| \le Ce^{(2\tau-p)}$ and $V(t)$ is compact. From problem (7.3.6) we have for $t \ge 0,\ x \ge 0$ that

$$\begin{aligned}
u(t)(x) &= e^{-pt}\phi(0)(x-t) + \int_0^t e^{-p(t-s)}Fu_s(x-t+s)ds \\
&= e^{-pt}\phi(0)(x-t) + 4p\int_0^t e^{-p(t-s)}u(s-1)(2(x-t+s-1))ds \\
&= e^{-pt}\phi(0)(x-t) + 4p\int_{-1}^{t-1} e^{-p(t-l-1)}u(l)(2(x-t+l))dl. \qquad (7.3.7)
\end{aligned}$$

It follows from (7.3.7) that for $t \ge 1, x \ge 0$

$$\begin{aligned}
u(t)(x) &= e^{-pt}\phi(0)(x-t) + 4p\int_{-1}^{0} e^{-p(t-l-1)}\phi(l)(2(x-t+l))dl \\
&\quad +4p\int_0^{t-1} e^{-p(t-l-1)}\Big\{e^{-pl}\phi(0)(2(x-t+l)-l) \\
&\qquad +4p\int_{-1}^{l-1} e^{-p(t-w-1)}u(w)[2(2(x-t+l)-l+w)]dw\Big\}dl. \qquad (7.3.8)
\end{aligned}$$

Now (7.3.8) implies that for $\phi \in X,\ t \ge 2,\ -1 \le \theta \le 0,\ x \ge 0$

$$\begin{aligned}
(T(t)\phi)(\theta)(x) &= u(t+\theta)(x) \\
&= e^{-p(t+\theta)}\phi(0)(x-t-\theta) \\
&\quad +4p\int_{-1}^{0} e^{-p(t+\theta-l-1)}\phi(l)(2(x-t-\theta+l))dl \\
&\quad +4p\int_{2(x-t-\theta)}^{2x-t-\theta-1} e^{-p(t+\theta-1)}\phi(0)(v)dv \\
&\quad +16p^2\int_{2(x-t-\theta)}^{2x-t-\theta-1}\int_{-1}^{2(t+\theta-x)+v-1} e^{-p(t+\theta-w-2)}u(w)[2(v+w)]dwdv \\
&\equiv (U_1(t)\phi)(\theta)(x) + (U_2(t)\phi)(\theta)(x) + (U_2(t)\phi)(\theta)(x) + (V(t)\phi)(\theta)(x).
\end{aligned}$$

Notice that $U_1(t)(X) \subset X$ and

$$\|U_1(t)\phi\|_X = \sup_{-1\le\theta\le 0}\sup_{x\ge 0} e^{\tau x}e^{-p(t+\theta)}|\phi(0)(x-t-\theta)|$$

$$\leq \sup_{-1\leq\theta\leq 0} e^{(\tau-p)(t+\theta)}\|\phi(0)\|_Y$$
$$\leq e^{(\tau-p)(t-1)}\|\phi\|_X.$$

Since for $t\geq 0,\ 0\leq x\leq 1,\ -1\leq l\leq 0$, we have $2(x-t+l)\leq 0$ and $\phi(l)[2(x-t+l)]=0$, so $U_2(t)(X)\subset X$. Moreover,

$$\begin{aligned}\|U_2(t)\phi\|_X &= \sup_{-1\leq\theta\leq 0}\sup_{x\geq 0} e^{\tau x}4p\int_{-1}^{0} e^{-p(t+\theta-l-1)}e^{-2\tau(x-t-\theta+l)}\|\phi(l)\|_Y dl\\ &\leq \frac{4pe^p e^{(2\tau-p)(t-1)}}{p-2\tau}\|\phi\|_X.\end{aligned}$$

Next, since for $t\geq 0,\ 0\leq x\leq 1,\ 2(x-1)\leq v\leq 2x-t-1$, we have $v\leq 0$ and $\phi(0)(v)=0$, so $U_3(t)(X)\subset X$. Furthermore,

$$\begin{aligned}\|U_3(t)\phi\|_X &= \sup_{-1\leq\theta\leq 0}\sup_{x\geq 0} e^{\tau x}4p\int_{2(x-t-\theta)}^{2x-t-\theta-1} e^{-p(t+\theta-l-1)}e^{-\tau v}\|\phi(0)\|_Y dl\\ &\leq \frac{4pe^p e^{(2\tau-p)(t-1)}}{\tau}\|\phi\|_X.\end{aligned}$$

Let $U=U_1+U_2+U_3$ and we have shown that $|U(t)|\leq Ce^{(2\tau-p)t},\ t\geq 2$, where C is independent of t.

Since $V(t)=T(t)-U(t)$, we have $V(t)(X)\subset X$. We now show that $V(t)$ is compact for each $t\geq 2$. Fix $t\geq 2$ and let Q be a bounded subset of X. It suffices to show that (i) $V(t)\phi$ is uniformly bounded for $\phi\in Q$; (ii) $(V(t)\phi)(\theta)$ is equicontinuous in θ for $\phi\in Q$; (iii) $(V(t)\phi)(\theta)$ is equicontinuous in x in bounded intervals for $\phi\in Q,\ -1\leq\theta\leq 0$; and (iv) for $\varepsilon>0$ there exists $x_\varepsilon>0$ such that $e^{\tau x}|(V(t)\phi)(\theta)(x)|<\varepsilon$ for $x>x_\varepsilon$. Note that for $\phi\in Q,\ -1\leq\theta\leq 0,\ x\geq 0,\ 2(x-t-\theta)\leq v\leq 2x-t-\theta-1$, and $-1\leq w\leq 2(t+\theta-x)+v-1$, we have $-1\leq w\leq t-2$, so that

$$\|u(w)\|_Y=\|(T(w)\phi)(0)\|_Y\leq \sup_{0\leq s\leq t-2}|T(s)|\|\phi\|_X. \tag{7.3.9}$$

Consequently, for $\phi\in Q,\ -1\leq\theta\leq 0,\ x\geq 0$

$$\begin{aligned}&e^{\tau x}\|(V(t)\phi)(\theta)(x)\|_X\\ &\leq 16p^2e^{\tau x}e^{-p(t+\theta-2)}\int_{2(x-t-\theta)}^{2x-t-\theta-1}\int_{-1}^{2(t+\theta-x)+v-1} e^{pw}e^{-2\tau(v+w)}\|u(w)\|_Y dwdv\\ &\leq Ce^{-3\tau x},\end{aligned} \tag{7.3.10}$$

where C is independent of ϕ,θ, and x. Properties (i)-(iii) follow immediately from (7.3.8), (7.3.9), and the formula for $(V(t)\phi)(\theta)(x)$.

(II) Let $\hat{\lambda}_0=\sup\{\mathrm{Re}\lambda:\lambda\in\sigma(A)\}$. We use Proposition 4.6.5 (i) and (iv) to show that $\hat{\lambda}_0=\lambda_0$. Notice that X is a Banach lattice, where $\phi_1\geq\phi_2$ means that $\phi_1(\theta)(x)\geq$

$\phi_2(\theta)(x),\ -1 \le \theta \le 0,\ x \ge 0$. Moreover, from (7.3.6) it follows that $T(t)\phi \ge 0$ for all $t \ge 0$ and $\phi \ge 0$. Suppose that $\phi \in X, \phi \ne 0$, and $A\phi = \lambda\phi$. Then, $\phi(\theta) = e^{\lambda\theta}\phi(0),\ -1 \le \theta \le 0$, where $\phi(0) \in D(B)$ and $\phi'(0) = B\phi(0) + F\phi$. Consequently,

$$\lambda\phi(0)(x) = -\phi'(0)(x) - p\phi(0)(x) + 4pe^{-\lambda}\phi(0)(2(x-1)),\ \ x \ge 1,$$

which means that

$$\phi(0)(x) = 4pe^{-\lambda}\int_1^x e^{-(\lambda+p)(x-y)}\phi(0)(2(y-1))dy,\ \ x \ge 1. \tag{7.3.11}$$

It was shown in Hannsgen and Tyson [177] that a solution of (7.3.11) is $\phi(0)(x) = h(x)$ in (7.3.2) and $\lambda = \lambda_0$. By Proposition 7.3.1, we have $h \in Y$. Thus, $\lambda_0 \in \sigma_p(A)$ and $A\phi_0 = \lambda_0\phi_0$, where $\phi_0(\theta) = e^{\lambda_0\theta}h$. Since $\omega_{0,\mathrm{ess}}(A) < 0$ and $\lambda_0 > 0$, we have $\omega_{0,\mathrm{ess}}(A) < \hat{\lambda}_0$. By Proposition 4.6.5 (i) we know that $\hat{\lambda}_0 > \mathrm{Re}\lambda$ for all $\lambda \in \sigma(A)$.

Now consider the scalar delay differential equation

$$\frac{dN}{dt} = -pN(t) + 2pN(t-1),\ t \ge 0;\ \ N_0 = \bar{\phi} \in \bar{X} \equiv C([-1,0]). \tag{7.3.12}$$

We know that (see Hale [170]) there is a strongly continuous semigroup $\{\bar{T}(t)\}_{t\ge 0}$ of bounded linear operators in $\bar{X}$ defined by $\bar{T}(t)\bar{\phi} = N_t,\ \bar{\phi} \in \bar{X},\ t \ge 0$. For $\phi \in X$, define

$$\bar{\phi}(\theta) = \int_0^\infty \phi(\theta)(x)dx,\ -1 \le \theta \le 0.$$

From equation (7.3.1) we have

$$(\bar{T}(t)\bar{\phi})(\theta) = \int_0^\infty (\bar{T}(t)\phi)(\theta)(x)dx.$$

Since $D(A)$ is dense in X, the last formula holds for all $\phi \in X,\ t \ge 0$. The infinitesimal generator of $\{\bar{T}(t)\}_{t\ge 0}$ is

$$\bar{A}\bar{\phi} = \bar{\phi}',\ D(\bar{A}) = \{\bar{\phi} \in \bar{X} : \bar{\phi}' \in \bar{X}\ \text{ and }\ \bar{\phi}'(0) = -p\bar{\phi}(0) + 2p\bar{\phi}(-1)\}.$$

Note that $\bar{T}(t)$ is compact for $t \ge 1$, we have $\omega_{0,\mathrm{ess}}(A) = -\infty$. Also, $\sigma_p(\bar{A}) = \{\lambda : \lambda + p = 2pe^{-\lambda}\}$. Thus, for $\lambda \notin \sigma_p(\bar{A}),\ \bar{\phi} \in \bar{X}$

$$(\lambda I - \bar{A})^{-1}\bar{\phi}(\theta) = \frac{e^{\lambda\theta}(\bar{\phi}(0) + 2p\int_{-1}^0 e^{-\lambda(1+\sigma)}\bar{\phi}(\sigma)d\sigma)}{\lambda + p - 2pe^{-\lambda}} + \int_\theta^0 e^{\lambda(\theta-\sigma)}\bar{\phi}(\sigma)d\sigma.$$

By Theorem 4.6.2 and the Residue Theorem, we have for all $\bar{\phi} \in \bar{X}$ that

$$\lim_{t\to\infty} \|e^{-\lambda_0 t}\bar{T}(t)\bar{\phi} - \bar{P}_0\bar{\phi}\|_{\bar{X}} = 0, \tag{7.3.13}$$

$$\bar{P}_0\bar{\phi}(\theta) = \frac{e^{\lambda\theta}(\bar{\phi}(0) + 2p\int_{-1}^0 e^{-\lambda(1+\sigma)}\bar{\phi}(\sigma)d\sigma)}{1 + \lambda_0 + p}. \tag{7.3.14}$$

Define $f \in X^*$ by

$$\langle f, \phi \rangle \equiv \int_{-1}^{0} \int_{0}^{\infty} \phi(\theta)(x) dx d\theta, \ \phi in X,$$

and observe that f is strictly positive. It follows from (7.3.13) and (7.3.14) that

$$\lim_{t\to\infty} \langle f, e^{-\lambda_0 t} T(t)\phi \rangle = \int_{-1}^{0} (\bar{P}_0 \bar{\phi})(\theta) d\theta, \tag{7.3.15}$$

which is positive if $\phi \geq 0, \ \phi \neq 0$. By Proposition 4.6.5 (iv) we know that $\lambda_0 = \hat{\lambda}_0$ and by Proposition 4.6.5 (v) it follows that λ_0 is a simple pole of $(\lambda - A)^{-1}$.

(III) Suppose that $A\phi = \lambda_0 \phi$. From (7.3.11) we have that

$$\begin{aligned}
\int_1^\infty |\phi(0)(x)| dx &= 4pe^{-\lambda_0} \int_1^\infty \left| \int_1^x e^{-(\lambda_0+p)(x-y)} \phi(0)(2(y-1)) dy \right| dx \\
&= 4pe^{-\lambda_0} \int_1^\infty \int_1^x e^{-(\lambda_0+p)(x-y)} |\phi(0)(2(y-1))| dy dx \\
&= 4pe^{-\lambda_0} \int_1^\infty \int_y^\infty e^{-(\lambda_0+p)(x-y)} |\phi(0)(2(y-1))| dy dx \\
&= \frac{4pe^{-\lambda_0}}{\lambda_0 + p} \int_1^\infty |\phi(0)(2(y-1))| dy \\
&= \int_1^\infty |\phi(0)(x)| dx.
\end{aligned}$$

The above inequality must be an equality, so that for $x \geq 1$

$$\left| \int_1^x e^{-(\lambda_0+p)(x-y)} \phi(0)(2(y-1)) dy \right| = \int_1^x e^{-(\lambda_0+p)(x-y)} |\phi(0)(2(y-1))| dy.$$

Thus, $\phi(0) = \text{const}|\phi(0)(x)|$, so is $h - \phi(0)$. If $\int_1^\infty \phi(0)(x)dx = 1$, then $\int_1^\infty (h(x) - \phi(0)(x))dx = 0$, which means that $h = \phi(0)$ and $\phi_0 = \phi$. By Theorem 4.6.2, we know that

$$\lim_{t\to\infty} e^{-\lambda_0 t} T(t)\phi = P_0 \phi, \ \forall \phi \in X.$$

Let $\phi \in X$. Since $\mathcal{N}_{\lambda_0}(A) = \mathcal{N}(\lambda_0 I - A)$, it follows that P_0 has rank one and there exists a constant c such that $P_0 \phi = c\phi_0$. From (7.3.3) and (7.3.15), we have

$$c\langle f, \phi_0 \rangle = \int_{-1}^{0} (\bar{P}_0 \bar{\phi})(\theta) d\theta = c(\phi) \langle f, \phi_0 \rangle.$$

Since $\phi_0 \geq 0$ and f is strictly positive, we must have $c = c(\phi)$.

Now apply Theorem 4.6.2 and Corollary 4.6.6, we know that the solutions of model (7.3.1) have asynchronous exponential growth and complete the proof. □

7.3.2 Partial Functional Differential Equations

Let $B : D(B) \subset Y \to Y$ be a linear operator on a Banach space $(Y, \|\ \|_Y)$. Assume that it is a Hille-Yosida operator; that is, there exist two constants, $\omega_B \in \mathbb{R}$ and $M_B > 0$, such that $(\omega_B, +\infty) \subset \rho(B)$ and

$$\left\|(\lambda I - B)^{-n}\right\| \leq \frac{M_B}{(\lambda - \omega_B)^n},\ \forall \lambda > \omega_B,\ \forall n \geq 1.$$

Set

$$Y_0 := \overline{D(B)}.$$

Consider B_0, the part of B in Y_0, which is defined by

$$B_0 y = By \text{ for each } y \in D(B_0)$$

with

$$D(B_0) := \{y \in D(B) : By \in Y_0\}.$$

For $r \geq 0$, set

$$C_B := C([-r, 0]; Y)$$

which is endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|_Y.$$

Consider the partial functional differential equations (PFDE):

$$\begin{cases} \dfrac{dy(t)}{dt} = By(t) + \widehat{L}(y_t) + f(t, y_t),\ \forall t \geq 0, \\ y_0 = \varphi \in C_B, \end{cases} \tag{7.3.16}$$

where $y_t \in C_B$ satisfies $y_t(\theta) = y(t+\theta), \theta \in [-r, 0]$, $\widehat{L} : C_B \to Y$ is a bounded linear operator, and $f : \mathbb{R} \times C_B \to Y$ is a continuous map. Since B is a Hille-Yosida operator, it is well known that B_0, the part of B in Y_0, generates a C_0-semigroup of bounded linear operators $\left\{T_{B_0}(t)\right\}_{t\geq 0}$ on Y_0, and B generates an integrated semigroup $\{S_B(t)\}_{t\geq 0}$ on Y. The solution of the Cauchy problem (7.3.16) must be understood as a fixed point of

$$y(t) = T_{B_0}(t)\varphi(0) + \frac{d}{dt}\int_0^t S_B(t-s)\left[\widehat{L}(y_s) + f(s, y_s)\right] ds.$$

Since $\left\{T_{B_0}(t)\right\}_{t\geq 0}$ acts on Y_0, we observe that it is necessary to assume that

$$\varphi(0) \in Y_0 \Rightarrow \varphi \in C_B.$$

Define $u \in C([0, +\infty) \times [-r, 0], Y)$ by

$$u(t,\theta) = y(t+\theta),\ \forall t \geq 0,\ \forall \theta \in [-r,0].$$

Note that if $y \in C^1([-r,+\infty),Y)$, then

$$\frac{\partial u(t,\theta)}{\partial t} = y'(t+\theta) = \frac{\partial u(t,\theta)}{\partial \theta}.$$

Hence, we must have

$$\frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0,\ \forall t \geq 0,\ \forall \theta \in [-r,0].$$

Moreover, for $\theta = 0$, we obtain

$$\frac{\partial u(t,0)}{\partial \theta} = y'(t) = By(t) + \widehat{L}(y_t) + f(t,y_t) = Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0.$$

Therefore, we deduce formally that u must satisfy a PDE

$$\begin{cases} \dfrac{\partial u(t,\theta)}{\partial t} - \dfrac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \dfrac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \widehat{L}(u(t,.)) + f(t,u(t,.)), \forall t \geq 0, \\ u(0,.) = \varphi \in C_B. \end{cases} \tag{7.3.17}$$

Consider the state space

$$X = Y \times C$$

taken with the usual product norm

$$\left\| \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\| = \|y\|_Y + \|\varphi\|_\infty.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix},\ \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A) \tag{7.3.18}$$

with

$$D(A) = \{0_Y\} \times \{\varphi \in C^1([-r,0],Y),\ \ \varphi(0) \in D(B)\}.$$

Note that A is non-densely defined because

$$X_0 := \overline{D(A)} = \{0_Y\} \times C_B \neq X.$$

We also define $L : X_0 \to X$ by

$$L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} := \begin{pmatrix} \widehat{L}(\varphi) \\ 0_C \end{pmatrix}$$

and $F : \mathbb{R} \times X_0 \to X$ by

$$F\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) := \begin{pmatrix} f(t, \varphi) \\ 0_C \end{pmatrix}.$$

Set

$$v(t) := \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}.$$

Now we can consider the PDE (7.3.17) as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t, v(t)),\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0. \tag{7.3.19}$$

In order to study the semilinear PFDE

$$\begin{cases} \dfrac{dy(t)}{dt} = By(t) + \widehat{L}(y_t) + f(y_t),\ \forall t \geq 0, \\ y_0^{\varphi} = \varphi \in C_B = \{\varphi \in C([-r, 0]; Y) :\ \ \varphi(0) \in \overline{D(B)}\}, \end{cases} \tag{7.3.20}$$

we considered the associated abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(v(t)),\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}, \tag{7.3.21}$$

where

$$F\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} f(\varphi) \\ 0 \end{pmatrix}.$$

We can check that the integrated solutions of (7.3.21) are the usual solutions of the PFDE (7.3.20).

Now we investigate the properties of the semiflows generated by the PFDE by using the known results on non-densely defined semi-linear Cauchy problems. In particular when f is Lipschitz continuous, from the results of Thieme [328], for each $\varphi \in C_B$ we obtain a unique solution $t \to y^{\varphi}(t)$ on $[-r, +\infty)$ of (7.3.20), and we can define a nonlinear C^0-semigroup $\{U(t)\}_{t \geq 0}$ on C_B by

$$U(t)\varphi = y_t^{\varphi}.$$

From the results in Magal [242], one may also consider the case where f is Lipschitz on bounded sets of C_B.

In order to describe the local asymptotic behavior around some equilibrium, we assume that $\overline{y} \in D(B)$ is an equilibrium of the PFDE (7.3.20); that is,

$$0 = B\overline{y} + L\left(\overline{y}1_{[-r,0]}\right) + f\left(\overline{y}1_{[-r,0]}\right).$$

Then by the stability result of Theorem 5.7.1, we obtain the following stability results for PFDE.

Theorem 7.3.3 (Exponential Stability). *Assume that $f : C_B \to \mathbb{R}^n$ is continuously differentiable in some neighborhood of $\bar{y}1_{[-r,0]}$ and $Df\left(\bar{y}1_{[-r,0]}\right) = 0$. Assume in addition that*

$$\omega_{0,\text{ess}}((A+L)_0) < 0$$

and each $\lambda \in \mathbb{C}$ such that

$$N(\Delta(\lambda)) \neq 0$$

has strictly negative real part. Then there exist $\eta, M, \gamma \in [0,+\infty)$ such that for each $\varphi \in C$ with $\left\|\varphi - \bar{y}1_{[-r,0]}\right\|_\infty \leq \eta$, the PFDE (7.3.20) has a unique solution $t \to y^\varphi(t)$ on $[-r,+\infty)$ satisfying

$$\left\|y_t^\varphi - \bar{y}1_{[-r,0]}\right\|_\infty \leq Me^{-\gamma t}\left\|\varphi - \bar{y}1_{[-r,0]}\right\|_\infty, \ \forall t \geq 0.$$

The above theorem is well known in the context of RFDE and PFDE (see, for example, Hale and Verduyn Lunel [175, Corollary 6.1, p. 215] and Wu [374, Corollary 1.11, p. 71]).

If we denote $\Pi_c : X \to X$ the bounded linear operator of projection

$$\Pi_c = B_{-1}^{\lambda_1} + \dots + B_{-1}^{\lambda_m}$$

where $\{\lambda_1, \lambda_2, \dots, \lambda_m\} = \sigma_C(A+L) := \{\lambda \in \sigma(A+L) : \text{Re}(\lambda) = 0\}$. Then

$$X_c = \Pi_c(X)$$

is the direct sum of the generalized eigenspaces associated to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$. Moreover,

$$\Pi_c(X) \subset X_0$$

and Π_c commutes with the resolvent of $(A+L)$. Set

$$X_h = R(I - \Pi_c) \ (\nsubseteq X_0).$$

Then we have the following state space decomposition

$$X = X_c \oplus X_h \text{ and } X_0 = X_{0c} \oplus X_{0h},$$

where

$$X_{0c} = X_c \cap X_0 = X_c \text{ and } X_{0h} = X_h \cap X_0 \neq X_h.$$

Then we can split the original abstract Cauchy problem (7.3.21) into the following system

$$\begin{cases} \dfrac{du_c(t)}{dt} = (A+L)_c u_c(t) + \Pi_c F(u_c(t) + u_h(t)), \\ \dfrac{du_h(t)}{dt} = (A+L)_h u_h(t) + \Pi_h F(u_c(t) + u_h(t)), \end{cases} \tag{7.3.22}$$

where $(A+L)_c$, the part of $A+L$ in X_c, is a bounded linear operator (since $\dim(X_c) < +\infty$), and $(A+L)_h$, the part of $A+L$ in X_h, is a non-densely defined

Hille-Yosida operator. So the first equation of (7.3.22) is an ordinary differential equation and the second equation of (7.3.22) is a new non-densely defined Cauchy problem with

$$\sigma\left((A+L)_h\right) = \sigma\left((A+L)\right) \setminus \sigma_C\left(A+L\right).$$

Assume for simplicity that f is C^k in some neighbordhood of the equilibrium 0_{C_B} and that

$$f(0) = 0 \text{ and } Df(0) = 0.$$

Then we can find (see Theorem 6.1.10) a manifold

$$M = \left\{x_c + \psi\left(x_c\right) : x_c \in X_c\right\},$$

where the map $\psi : X_c \to X_h \cap \overline{D(A)}$ is C^k with

$$\psi(0) = 0,\ D\psi(0) = 0,$$

and M is locally invariant by the semiflow generated by (7.3.21).

More precisely, we can find a neighborhood Ω of 0 in C_B such that if $I \subset \mathbb{R}$ is an interval and $u_c : I \to X_c$ is a solution of the ordinary differential equation

$$\frac{du_c(t)}{dt} = (A+L)_c\, u_c(t) + \Pi_c F\left(u_c(t) + \psi\left(u_c(t)\right)\right) \tag{7.3.23}$$

satisfying

$$u(t) := u_c(t) + \psi\left(u_c(t)\right) \in \Omega,\ \forall t \in I,$$

then $u(t)$ is an integrated solution of (7.3.21), that is,

$$u(t) = u(s) + A\int_s^t u(l)dl + \int_s^t F(u(l))dl,\ \forall t, s \in I \text{ with } t \geq s.$$

Conversely, if $u : \mathbb{R} \to X_0$ is an integrated solution of (7.3.21) satisfying

$$u(t) \in \Omega,\ \forall t \in \mathbb{R},$$

then $u_c(t) = \Pi_c u(t)$ is a solution of the ordinary differential equation (7.3.23). This result leads to the Hopf bifurcation results for PFDE (see Wu [374]). A normal form theory can be established similarly for the PFDE.

7.4 Remarks and Notes

Since the state space for functional differential equations (FDE) is infinitely dimensional, techniques and methods from functional analysis and operator theory have been further developed and extensively used to study such equations (Hale and Verduyn Lunel [175], Diekmann et al. [106], Engel and Nagel [126]). In particular, the semigroup theory of operators on a Banach space has been successfully used to study the dynamical behavior of FDE (Adimy and Arino [5], Diekmann et al. [98], Frasson and Verduyn Lunel [144], Thieme [328], Verduyn Lunel [347], Webb [357, 362, 363]). In studying bifurcation problems, such as Hopf bifurcation, for FDE, we need to compute explicitly the flow on the center manifold. To do that, we need to know detailed information about the underlying center manifold of the linearized equation.

To obtain explicit formulas for the projectors on the generalized eigenspaces associated to some eigenvalues for linear functional differential equations (FDE)

$$\begin{cases} \dfrac{dx(t)}{dt} = Bx(t) + \widehat{L}(x_t), \forall t \geq 0 \\ x_0 = \varphi \in C([-r,0], \mathbb{R}^n), \end{cases}$$

the usual method is based on the formal adjoint approach, see Hale and Verduyn Lunel [175]. The method was recently further studied in the monograph of Diekmann et al. [98] using the so called sun start adjoint spaces, see also Kaashoek and Verduyn Lunel [203], Frasson and Verduyn Lunel [144], Diekmann et al. [98] and the references cited therein. The key idea is to formulate the problem as an abstract Cauchy problem and many approaches have been used. In the 1970s, Webb [357], Travis and Webb [340, 341] viewed the problem as a nonlinear Cauchy problem and focused on many aspects of the problem by using this approach. Another approach is a direct method, that is to use the variation of constants formula and work directly with the system (see Arino and Sanchez [30] and Kappel [204]).

We used an integrated semigroup formulation for the problem. Adimy [3, 4], Adimy and Arino [5], and Thieme [328] were the first to apply integrated semigroup theory to study FDEs. This approach has been extensively developed by Arino's team in the 1990s (see Ezzinbi and Adimy [129] for a survey on this topic). Here we used a formulation of the FDE that is an intermediate between the formulations of Adimy [3, 4] and Thieme [328]. In fact, compared with Adimy's approach we did not use any Radon measure to give a sense of the value of $x_t(\theta)$ at $\theta = 0$, while compared to Thieme's approach we kept only one equation. Our approach is more closely related to the one by Travis and Webb [340, 341]. With such a setting, we can apply the results in Chapter 6 to establish center manifold theorem, Hopf bifurcation theorem, and normal form theory for FEDs. In Section 7.1 we only presented results on Hopf bifurcation for retarded functional differential equations. The presentations in Section 7.1 were taken from Liu et al. [233, 234]. We refer to the books of Hale and Verduyn Lunel [175], Hassard et al. [181], Diekmann et al. [106], and Guo and Wu [159] for more results on center manifold theory, Hopf bifurcation and normal form theory in the context of functional differential equations. See also Faria and Magalhães [136, 137] for normal forms for FDEs with applications to Hopf bifurcation.

The materials in Section 7.2 were taken from Ducrot et al. [111]. Hopf bifurcation in NFDEs has been studied by many authors, see for example, Krawcewicz et al. [218], Wei and Ruan [369], Weedermann [368], Guo and Lamb [158], and Wang and Wei [353]. However, there are very few results on the center manifold and normal form theories for NFDEs (Ait Babram et al. [9], Weedermann [367]). We can use the settings in Section 7.2 and apply the results in Chapter 6 to establish center manifold and normal form theories for NFDEs.

The presentation of subsection 7.3.1 was taken from Webb [363] who gave necessary and sufficient conditions for a semigroup to have asynchronous exponential growth (Theorem 4.6.2) and applied it to a delayed transport equation for cell growth and division. Piazzera and Tonetto [288] showed that the solutions of a population equation with age structure and delayed birth process have asynchronous exponential growth. Yan et al. [377] studied a size-structured cannibalism model with environment feedback and delayed birth process and proved that the model has asynchronous exponential growth, among other properties. Subsection 7.3.2 demonstrates that partial functional differential equations can also be set up as an abstract Cauchy problem. Thus, we can study center manifolds, Hopf bifurcation and normal forms in partial functional differential equations, see Wu [374]. We refer to Adimy et al. [6], Ezzinbi and Adimy [129], Faria [134], Faria [135], Fitzgibbon [141], Lin et al. [230], Martin and Smith [259, 260], Memory [264, 264], Nguyen and Wu [277], Ruan et al. [301], Ruan and Zhang [302], Ruess [304, 305], Travis and Webb [340, 341], Yoshida [379], and the monograph of Wu [374] on studies related to partial functional differential equations.

Chapter 8
Age-structured Models

In this chapter we apply the results obtained in the previous chapters to age-structured models. In Section 8.1, a Hopf bifurcation theorem is established for the general age-structured systems. Section 8.2 deals with a susceptible-infectious epidemic model with age of infection, uniform persistence of the model is established, local and global stability of the disease-free equilibrium is studied by spectral analysis, and global stability of the unique endemic equilibrium is discussed by constructing a Liapunov functional. Section 8.3 focuses on a scalar age-structured model, detailed results on the existence of integrated solutions, local stability of equilibria, Hopf bifurcation, and normal forms are presented.

8.1 General Age-structured Models

Consider a general class of age-structured models (Webb [362], Iannelli [195])

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial a}=-D(a)u(t,a)+M(\mu,u(t,.))(a),\ a\geq 0,t\geq 0, \\ u(t,0)=B\left(\mu,u(t,.)\right) \\ u(0,.)=u_0\in L^1\left((0,+\infty),\mathbb{R}^n\right), \end{cases} \tag{8.1.1}$$

where $\mu\in\mathbb{R}$ is a parameter, $D(.)=\operatorname{diag}(d_1(.),\ldots,d_n(.))\in L^\infty((0,+\infty),M_n(\mathbb{R}^+))$, $M:\mathbb{R}\times L^1((0,+\infty),\mathbb{R}^n)\to L^1((0,+\infty),\mathbb{R}^n)$ is the mortality function, and $B:\mathbb{R}\times L^1((0,+\infty),\mathbb{R}^n)\to\mathbb{R}^n$ is the birth function. We make the following assumptions.

Assumption 8.1.1. Assume that there exists $\gamma\in L^\infty\left((0,+\infty),M_{p\times n}(\mathbb{R})\right)$ for integer $p\geq 1$ such that:

(a) (Birth function) The map $B:\mathbb{R}\times L^1\left((0,+\infty),\mathbb{R}^n\right)\to\mathbb{R}^n$ is 4-time continuously differentiable and has the following form

$$B(\mu,\varphi)=\int_0^{+\infty}\beta\left(\mu,\int_0^{+\infty}\gamma(s)\varphi(s)ds\right)(a)\varphi(a)da+\Theta\left(\mu,\int_0^{+\infty}\gamma(s)\varphi(s)ds\right),$$

where $\Theta : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^n$ and $\beta : \mathbb{R} \times \mathbb{R}^p \to L^\infty\left((0,+\infty), M_n(\mathbb{R})\right)$ are C^4 maps.

(b) (Mortality function) The map $M : \mathbb{R} \times L^1\left((0,+\infty),\mathbb{R}^n\right) \to L^1\left((0,+\infty),\mathbb{R}^n\right)$ is 4-time continuously differentiable and has the following form

$$M(\varphi)(a) = \widehat{M}\left(\mu, \int_0^{+\infty} \gamma(s)\varphi(s)ds\right)(a)\varphi(a),$$

in which $\widehat{M} : \mathbb{R} \times \mathbb{R}^p \to L^\infty\left((0,+\infty), M_n(\mathbb{R})\right)$.

We assume that there exists a smooth branch of equilibria from which bifurcation will occur.

Assumption 8.1.2. Assume that there exists a parameterized curve $\mu \to \overline{u}(\mu)(.)$ from $(-\varepsilon,\varepsilon)$ into $L^1\left((0,+\infty),\mathbb{R}^n\right)$ such that

$$\overline{u}(\mu) \in W^{1,1}\left((0,+\infty),\mathbb{R}^n\right),$$

$$\frac{\partial \overline{u}(\mu)(a)}{\partial a} = -D(a)\overline{u}(\mu)(a) + M(\mu,\overline{u}(\mu))(a) \text{ for almost every } a \geq 0,$$

and

$$\overline{u}(\mu)(0) = B(\mu,\overline{u}(\mu)).$$

Consider the Banach space

$$X = \mathbb{R}^n \times L^1\left((0,+\infty),\mathbb{R}^n\right),$$

the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - D\varphi \end{pmatrix}$$

with

$$D(A) = \{0\} \times W^{1,1}\left((0,+\infty),\mathbb{R}^n\right),$$

and the map $F : \mathbb{R} \times \overline{D(A)} \to X$ defined by

$$F\left(\mu, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} B(\mu,\varphi) \\ M(\mu,\varphi) \end{pmatrix}.$$

Observe that A is non-densely defined since

$$\overline{D(A)} = \{0\} \times L^1\left((0,+\infty),\mathbb{R}^n\right) \neq X.$$

Setting $v(t) = \begin{pmatrix} 0 \\ u(t,.) \end{pmatrix}$, we can rewrite system (8.1.1) as the following non-densely defined abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + F(\mu, v(t)),\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in \overline{D(A)}. \tag{8.1.2}$$

Therefore, the theories developed in the previous chapters can be applied to study the existence of integrated solutions, stability of equilibria, center manifolds, Hopf bifurcation, and normal forms of system (8.1.2) (thus system (8.1.1)). Here we prove a general Hopf bifurcation theorem for the general age-structured system (8.1.1).

Notice that for $\lambda > \max_{i=1,\ldots,n} \sup_{a\geq 0} \text{ess}\,(-d_i(a))$, we have $\lambda \in \rho(A)$ and (since the matrices $D(a)$ and $D(l)$ commute)

$$(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow \varphi(a) = e^{-\int_0^a \lambda + D(l)dl}\alpha + \int_0^a e^{-\int_s^a \lambda+D(l)dl}\psi(s)ds.$$

The linear operator A is a Hille-Yosida operator and A_0, the part of A in $\overline{D(A)}$, is the infinitesimal generator of the strongly continuous semigroup $\{T_{A_0}(t)\}$ of bounded linear operators on $\overline{D(A)}$,

$$T_{A_0}(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ \widehat{T}_{A_0}(t)\varphi\end{pmatrix},$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(a) = \begin{cases}\exp\left(-\int_{a-t}^a D(l)dl\right)\varphi(a-t) & \text{if } a \geq t,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that

$$A\begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix} + F\left(\mu, \begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix}\right) = 0.$$

So we make the following change of variables

$$w(t) := v(t) - \begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix}$$

and obtain that

$$\frac{dw(t)}{dt} = Aw(t) + G(\mu, w(t)),\ t \geq 0,\ \ w(0) = x \in \overline{D(A)}, \tag{8.1.3}$$

where

$$G(\mu, x) = F\left(\mu, x + \begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix}\right) - F\left(\mu, \begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix}\right).$$

Note that

$$G(\mu, 0) = 0, \forall \mu \in (-\varepsilon, \varepsilon).$$

The linearized equation of (8.1.2) around the equilibrium $\begin{pmatrix}0\\ \overline{u}(\mu)\end{pmatrix}$ (or (8.1.3) around the equilibrium 0) is given by

$$\frac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + \partial_x F\left(\mu, \begin{pmatrix} 0 \\ \overline{u}(\mu) \end{pmatrix}\right)\widehat{v}(t),\ t \geq 0,\ \ \widehat{v}(0) = x \in \overline{D(A)}.$$

To simplify the notation, set

$$\overline{x}_\mu = \begin{pmatrix} 0 \\ \overline{u}(\mu) \end{pmatrix}.$$

We first estimate the essential growth rate of the strongly continuous semigroup generated by $\left(A + \partial_x F\left(\mu, \overline{x}_\mu\right)\right)_0$, the part of $A + \partial_x F\left(\mu, \overline{x}_\mu\right) : D(A) \subset X \to X$ in $\overline{D(A)}$. We observe that

$$\partial_x F\left(\mu, \overline{x}_\mu\right)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_x B\left(\mu, \overline{u}(\mu)\right)(\varphi) \\ L_1(a)\varphi(a) + L_2\left(\int_0^{+\infty}\gamma(s)\varphi(s)ds\right)(a) \end{pmatrix},$$

where $L_1 \in L^\infty\left((0,+\infty), M_n(\mathbb{R})\right)$ is defined by

$$L_1(a) := \widehat{M}\left(\mu, \int_0^{+\infty}\gamma(s)\overline{u}(\mu)(s)ds\right)(a)$$

and

$$L_2(\widehat{\gamma})(a) = \partial_{\widehat{\gamma}} H\left(\mu, \int_0^{+\infty}\gamma(s)\overline{u}(\mu)(s)ds\right)(a)(\widehat{\gamma}),$$

here $H : \mathbb{R} \times \mathbb{R}^p \to L^\infty\left((0,+\infty), \mathbb{R}^n\right)$ is a map defined by

$$H(\mu, \widehat{\gamma})(a) = \widehat{M}(\mu, \widehat{\gamma})(a)\overline{u}(\mu)(a).$$

We split $\partial_x F\left(\mu, \overline{x}_\mu\right)$ into the sum of two operators

$$\partial_x F\left(\mu, \overline{x}_\mu\right)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = B\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + C\begin{pmatrix} 0 \\ \varphi \end{pmatrix},$$

where $B : \overline{D(A)} \subset X \to X$ is the bounded linear operator

$$B\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ L_1(.)\varphi(.) \end{pmatrix}$$

and $C : \overline{D(A)} \subset X \to X$ is the compact bounded linear operator defined by

$$C\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \partial_x B\left(\mu, \overline{u}(\mu)\right)(\varphi) \\ L_2\left(\int_0^{+\infty}\gamma(s)\varphi(s)ds\right)(a) \end{pmatrix}.$$

We consider the linear non-autonomous semiflow $\{U(a,s)\}_{a \geq s \geq 0} \subset M_n(\mathbb{R})$ on $\mathbb{R}^n$ generated by

$$\begin{aligned} \frac{dU(a,s)x}{da} &= (-D(a) + L_1(a))\,U(a,s)x \text{ for almost every } a \geq s \geq 0, \\ U(s,s)x &= x \in \mathbb{R}^n. \end{aligned}$$

This is, $a \to U(a,s)x$ is the unique solution from $(s,+\infty)$ into $\mathbb{R}^n$ of the integral equation

$$U(a,s)x = e^{-\int_s^a D(r)dr}x + \int_s^a e^{-\int_l^a D(r)dr} L_1(l)U(l,s)xdl.$$

We make an assumption on the estimation of $U(a,s)$.

Assumption 8.1.3. Assume that there exist two constants, $\nu > 0$ and $M \geq 1$, such that

$$\|U(a,s)\| \leq Me^{-\nu(a-s)}, \forall a \geq s \geq 0.$$

Set

$$\Omega := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\nu\}.$$

Here we use the same approach as in Thieme [330], Magal [242], or Magal and Ruan [245]. More precisely, for each $\lambda \in \Omega$ set

$$\begin{gathered} R_\lambda \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ \Leftrightarrow \varphi(a) = e^{-\lambda a}U(a,0)\alpha + \int_0^a e^{-\lambda(a-s)}U(a,s)\psi(s)ds. \end{gathered} \tag{8.1.4}$$

Then R_λ is a pseudo resolvent and

$$R_\lambda = (\lambda I - (A+B))^{-1}.$$

Moreover, for each $\omega_{A+B} \in (0,\nu)$, we can find $M_{A+B} \geq 1$ such that

$$\left\|(\lambda I - (A+B))^{-1}\right\| \leq \frac{M_{A+B}}{(\lambda + \omega_{A+B})^n}, \ \forall \lambda > -\omega_{A+B}, \forall n \geq 1. \tag{8.1.5}$$

Then $(A+B)_0$, the part of $A+B$ in $\overline{D(A)}$, generates a strongly continuous semigroup $\{T_{(A+B)_0}(t)\}_{t\geq 0}$ on $\overline{D(A)}$, which is defined by

$$T_{(A+B)_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{(A+B)_0}(t)\varphi \end{pmatrix},$$

where

$$\widehat{T}_{(A+B)_0}(t)(\varphi)(a) = \begin{cases} U(a,a-t)\varphi(a-t) & \text{if } a \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

From (8.1.5) we obtain

$$\left\|T_{(A+B)_0}(t)\right\| \leq M_{A+B}e^{-\omega_{A+B}t}, \ \forall t \geq 0.$$

Thus, for each $\omega_{A+B} \in (0,\nu)$,

$$\omega_{0,\mathrm{ess}}((A+B)_0) \leq \omega_0((A+B)_0) \leq -\omega_{A+B}.$$

Now since C is a compact bounded linear operator, we can apply the perturbation results in Theorem 4.7.3 to obtain that

$$\omega_{0,\text{ess}}\left((A+B+C)_0\right) \leq -\omega_{A+B},\ \forall \omega_{A+B} \in (0,\nu).$$

We obtain the following proposition.

Proposition 8.1.4. *Let Assumptions 8.1.1-8.1.3 be satisfied. Then*

$$\omega_{0,\text{ess}}\left((A+\partial_x F\left(0,\bar{x}_0\right))_0\right) < 0;$$

that is, the essential growth rate of the C_0−semigroup $\{T_{(A+\partial_x F(0,\bar{x}_0))_0}(t)\}_{t\geq 0}$ is strictly negative.

In order to apply Theorem 6.2.7, it remains to precise the spectral properties of $\left(A+\partial_x F\left(\mu,\bar{x}_\mu\right)\right)_0$. Let $\lambda \in \Omega$ be given. Since $(\lambda I-(A+B))$ is invertible, it follows that $\left(\lambda I-\left(A+\partial_x F\left(\mu,\bar{x}_\mu\right)\right)\right) = (\lambda I-(A+B+C))$ is invertible if and only if $I-C(\lambda I-(A+B))^{-1}$ is invertible. Moreover, when $I-C(\lambda I-(A+B))^{-1}$ is invertible we have

$$(\lambda I-(A+B+C))^{-1} = (\lambda I-(A+B))^{-1}\left[I-C(\lambda I-(A+B))^{-1}\right]^{-1}. \quad (8.1.6)$$

Here in order to compute the resolvent and to derive a characteristic equation, we need more details. We have

$$\begin{aligned}\partial_\varphi B(\mu,\bar{u}(\mu))(\varphi) = &\int_0^{+\infty}\beta_{\mu,\bar{u}(\mu)}(a)\varphi(a)da \\ &+\int_0^{+\infty}L_3\left(\mu,\int_0^{+\infty}\gamma(s)\varphi(s)ds\right)(a)da \\ &+D\Theta\left(\mu,\int_0^{+\infty}\gamma(s)\bar{u}(\mu)(s)ds\right)\left(\int_0^{+\infty}\gamma(s)\varphi(s)ds\right),\end{aligned}$$

where

$$\beta_{\mu,\bar{u}(\mu)}(a) = \beta\left(\mu,\int_0^{+\infty}\gamma(s)\bar{u}(\mu)(s)ds\right)(a)$$

and $L_3 : \mathbb{R}\times\mathbb{R}^p \to L^\infty\left((0,+\infty),M_n(\mathbb{R})\right)$ is given by

$$L_3(\mu,\widehat{\gamma})(a) = \partial_{\widehat{\gamma}}\widehat{H}(\mu,\int_0^{+\infty}\gamma(s)\bar{u}(\mu)(s)ds)(a)(\widehat{\gamma}),$$

in which $\widehat{H} : \mathbb{R}\times\mathbb{R}^p \to L^\infty\left((0,+\infty),\mathbb{R}^n\right)$ is a map defined by

$$\widehat{H}(\mu,\widehat{\gamma})(a) = \beta(\mu,\widehat{\gamma})(a)\bar{u}(\mu)(a).$$

Consider

$$\left(I-C(\lambda I-(A+B))^{-1}\right)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix} = \begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}$$

or

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} - C \begin{pmatrix} 0 \\ e^{-\lambda \cdot} U(.,0)\alpha + \int_0^{\cdot} e^{-\lambda(.-s)} U(.,s)\varphi(s)ds \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\varphi} \end{pmatrix}.$$

We obtain the system

$$\begin{cases} \alpha - \partial_\varphi B(\mu, \overline{u}(\mu)) \left(e^{-\lambda \cdot} U(.,0)\alpha + \int_0^{\cdot} e^{-\lambda(.-s)} U(.,s)\varphi(s)ds \right) = \widehat{\alpha} \\ \varphi(a) - L_2 \left(\int_0^{+\infty} \gamma(s) \left[e^{-\lambda s} U(s,0)\alpha + \int_0^s e^{-\lambda(s-l)} U(s,l)\varphi(l)dl \right] ds \right)(a) = \widehat{\varphi}(a). \end{cases} \tag{8.1.7}$$

Set

$$\Gamma_1(\varphi) = \int_0^{+\infty} \beta_{\mu,\overline{u}(\mu)}(a)\varphi(a)da$$

and

$$\Gamma_2(\varphi) = \int_0^{+\infty} \gamma(a)\varphi(a)da.$$

We obtain

$$\left[I - \partial_\varphi B(\mu, \overline{u}(\mu)) e^{-\lambda \cdot} U(.,0).) \right] \alpha - \partial_\varphi B(\mu, \overline{u}(\mu)) \left(\int_0^{\cdot} e^{-\lambda(.-s)} U(.,s)\varphi(s)ds \right) = \widehat{\alpha}.$$

By applying

$$\varphi \to \Gamma_1 \left(\int_0^s e^{-\lambda(s-l)} U(s,l)\varphi(l)dl \right) =: x_1$$

and

$$\varphi \to \Gamma_2 \left(\int_0^s e^{-\lambda(s-l)} U(s,l)\varphi(l)dl \right) =: x_2$$

to both sides of the second equation of system (8.1.7), we obtain

$$\begin{aligned} &\left[I - \partial_\varphi B(\mu, \overline{u}(\mu)) \left(e^{-\lambda \cdot} U(.,0).) \right] \alpha \\ &\quad - \left[x_1 + \int_0^{+\infty} L_3(\mu, x_2)(a)da + D\Theta\left(\mu, \int_0^{+\infty} \gamma(s)\overline{u}(\mu)(s)ds\right)(x_2) \right] = \widehat{\alpha}, \\ &x_1 - \Gamma_1 \left(\int_0^{\cdot} e^{-\lambda(.-l)} U(.,l) L_2(x_2)(l)dl \right) \\ &\quad - \Gamma_1 \left(\int_0^{\cdot} e^{-\lambda(.-l)} U(.,l) L_2(\Gamma_2\left(e^{-\lambda \cdot} U(.,0)\alpha\right))(l)dl \right) = \widehat{x}_1, \\ &x_2 - \Gamma_2 \left(\int_0^{\cdot} e^{-\lambda(.-l)} U(.,l) L_2(x_2)(l)dl \right) \\ &\quad - \Gamma_2 \left(\int_0^{\cdot} e^{-\lambda(.-l)} U(.,l) L_2(\Gamma_2\left(e^{-\lambda \cdot} U(.,0)\alpha\right))(l)dl \right) = \widehat{x}_2, \end{aligned} \tag{8.1.8}$$

where

$$\widehat{x}_1 := \Gamma_1 \left(\int_0^s e^{-\lambda(s-l)} U(s,l)\widehat{\varphi}(l)dl \right)$$

and

$$\widehat{x}_2 := \Gamma_2 \left(\int_0^s e^{-\lambda(s-l)} U(s,l)\widehat{\varphi}(l)dl \right).$$

The above system can be rewritten as a finite dimensional system of linear equations

$$\Delta(\mu,\lambda)\begin{pmatrix}\alpha\\ x_1\\ x_2\end{pmatrix}=\begin{pmatrix}\widehat{\alpha}\\ \widehat{x}_1\\ \widehat{x}_2\end{pmatrix}$$

and we obtain the characteristic equation

$$\det(\Delta(\mu,\lambda)):=0.$$

When $\Delta(\mu,\lambda)$ is invertible, we have

$$\Delta(\mu,\lambda)^{-1}=\frac{1}{\det(\Delta(\mu,\lambda))}\left[\mathrm{cof}(\Delta(\mu,\lambda))\right]^T,$$

where $\mathrm{cof}(\Delta(\mu,\lambda))$ is the matrix of cofactors of $\Delta(\mu,\lambda)$. So we obtain

$$\begin{pmatrix}\alpha\\ x_1\\ x_2\end{pmatrix}=\frac{1}{\det(\Delta(\mu,\lambda))}\left[\mathrm{cof}(\Delta(\mu,\lambda))\right]^T\begin{pmatrix}\widehat{\alpha}\\ \widehat{x}_1\\ \widehat{x}_2\end{pmatrix}. \tag{8.1.9}$$

Denote

$$\left[\mathrm{cof}(\Delta(\mu,\lambda))\right]^T=\begin{pmatrix}N_{11} & N_{12} & N_{13}\\ N_{21} & N_{22} & N_{23}\\ N_{31} & N_{32} & N_{33}\end{pmatrix},$$

where the blocks N_{ij} are such that the system (8.1.9) can be rewritten as

$$\begin{pmatrix}\alpha\\ x_1\\ x_2\end{pmatrix}=\det(\Delta(\mu,\lambda))^{-1}\begin{pmatrix}N_{11}\widehat{\alpha}+N_{12}\widehat{x}_1+N_{13}\widehat{x}_2\\ N_{21}\widehat{\alpha}+N_{22}\widehat{x}_1+N_{23}\widehat{x}_2\\ N_{31}\widehat{\alpha}+N_{32}\widehat{x}_1+N_{33}\widehat{x}_2\end{pmatrix}. \tag{8.1.10}$$

Finally, by using the second equation of (8.1.7), we have

$$\varphi(a)=\widehat{\varphi}(a)+L_2(x_2)(a)+L_2\left(\int_0^{+\infty}\gamma(s)e^{-\lambda s}U(s,0)(\alpha)\,ds\right)(a). \tag{8.1.11}$$

Therefore, we derive that if $\det(\Delta(\mu,\lambda))\neq 0$, then $I-C(\lambda I-(A+B))^{-1}$ is invertible, and by using equations (8.1.7), (8.1.10), and (8.1.11), we obtain the following explicit formula

$$\begin{array}{l}\left(I-C(\lambda I-(A+B))^{-1}\right)\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}=\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}\\ \Leftrightarrow\begin{cases}\alpha=\det(\Delta(\mu,\lambda))^{-1}\left(N_{11}\widehat{\alpha}+N_{12}\widehat{x}_1+N_{13}\widehat{x}_2\right),\\ \varphi(a)=\widehat{\varphi}(a)+\det(\Delta(\mu,\lambda))^{-1}L_2(N_{31}\widehat{\alpha}+N_{32}\widehat{x}_1+N_{33}\widehat{x}_2)(a)\\ \quad+\det(\Delta(\mu,\lambda))^{-1}L_2\left(\int_0^{+\infty}\gamma(s)e^{-\lambda s}U(s,0)\left(N_{11}\widehat{\alpha}+N_{12}\widehat{x}_1+N_{13}\widehat{x}_2\right)ds\right)(a).\end{cases}\end{array} \tag{8.1.12}$$

We observe that the only singularity in the above expression comes from $\det(\Delta(\mu,\lambda))^{-1}$ when λ approaches an eigenvalue. By the above discussion, we obtain the following result.

Lemma 8.1.5. *Let Assumptions 8.1.1-8.1.3 be satisfied. Then we have the following:*

(i) $\sigma(A+B+C)\cap\Omega=\sigma_p(A+B+C)\cap\Omega=\{\lambda\in\Omega:\det(\Delta(\mu,\lambda))=0\}$;

(ii) *If* $\lambda\in\rho(A+B+C)$, *we have the following formula for the resolvent*

$$\begin{aligned}
&(\lambda I-(A+B+C))^{-1}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}=\begin{pmatrix}0\\ \psi\end{pmatrix}\\
&\Leftrightarrow \psi(a)=e^{-\lambda a}U(a,0)\det(\Delta(\mu,\lambda))^{-1}(N_{11}\widehat{\alpha}+N_{12}\widehat{x}_1+N_{13}\widehat{x}_2)\\
&\quad+\int_0^a e^{-\lambda(a-s)}U(a,s)\Bigg[\widehat{\varphi}(a)+\det(\Delta(\mu,\lambda))^{-1}L_2(N_{31}\widehat{\alpha}+N_{32}\widehat{x}_1+N_{33}\widehat{x}_2)(a)\\
&\quad+\det(\Delta(\mu,\lambda))^{-1}L_2\left(\int_0^{+\infty}\gamma(s)e^{-\lambda s}U(s,0)\begin{pmatrix}N_{11}\widehat{\alpha}+\\ N_{12}\widehat{x}_1+N_{13}\widehat{x}_2\end{pmatrix}ds\right)(a)\Bigg]ds,
\end{aligned} \tag{8.1.13}$$

where

$$\widehat{x}_1=\int_0^{+\infty}\beta_{\mu,\overline{u}(\mu)}(a)\left(\int_0^a e^{-\lambda(a-l)}U(a,l)\widehat{\varphi}(l)dl\right)da,$$
$$\widehat{x}_2=\int_0^{+\infty}\gamma(a)\left(\int_0^a e^{-\lambda(a-l)}U(a,l)\widehat{\varphi}(l)dl\right)da$$

and N_{ij} is defined in (8.1.9);

(iii) *If* $\lambda_0\in\{\lambda\in\Omega:\det\Delta(\mu,\lambda)=0\}=\Omega\cap\sigma(A+B+C)$, *then λ_0 is isolated. Moreover, λ_0 is a simple pole of the resolvent of $A+B+C$ if*

$$\lim_{\lambda\to\lambda_0}\frac{\det(\Delta(\mu,\lambda))}{\lambda-\lambda_0}\neq 0.$$

Furthermore, λ_0 is a simple eigenvalue if in addition to condition (8.1.9) the dimension of the eigenspace of $A+B+C$ associated to λ_0 is 1, *which is equivalent to*

$$\dim(\mathcal{N}(\Delta(\lambda_0,\mu)))=1.$$

Proof. Assume that $\lambda\in\Omega$ and $\det(\Delta(\lambda,\mu))\neq 0$. From (8.1.3), (8.1.5), and (8.1.11) we obtain (8.1.12). Therefore, we obtain that $\{\lambda\in\Omega:\det(\Delta(\lambda,\mu))\neq 0\}\subset\rho(A+B+C)$ and

$$\sigma(A+B+C)\cap\Omega\subset\{\lambda\in\Omega:\det(\Delta(\lambda,\mu))=0\}.$$

Conversely, assume that $\lambda\in\Omega$ and $\det(\Delta(\lambda,\mu))=0$. We claim that we can find $\begin{pmatrix}0\\ \psi\end{pmatrix}\in D(A)\setminus\{0\}$ such that

$$(A+B+C)\begin{pmatrix}0\\ \psi\end{pmatrix}=\lambda\begin{pmatrix}0\\ \psi\end{pmatrix}. \tag{8.1.14}$$

Indeed, set

$$\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} := (\lambda I - (A+B)) \begin{pmatrix} 0 \\ \psi \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} 0 \\ \psi \end{pmatrix} = (\lambda I - (A+B))^{-1} \begin{pmatrix} \alpha \\ \varphi \end{pmatrix}.$$

So we can find a solution of (8.1.14) if and only if we can find $\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in X \setminus \{0\}$ satisfying

$$\left[I - C (\lambda I - (A+B))^{-1} \right] \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = 0.$$

Now from the above discussion this is equivalent to find $(\alpha, x_1, x_2)^T \neq 0$ satisfying

$$\Delta(\lambda, \mu)(\alpha, x_1, x_2)^T = 0,$$

where

$$x_1 = \int_0^{+\infty} \beta_{\mu, \overline{u}(\mu)}(a) \left(\int_0^a e^{-\lambda(a-l)} U(a,l) \varphi(l) dl \right) da,$$
$$x_2 = \int_0^{+\infty} \gamma(a) \left(\int_0^a e^{-\lambda(a-l)} U(a,l) \varphi(l) dl \right) da.$$

But by the assumption $\det(\Delta(\lambda, \mu)) = 0$, we can find $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(A) \setminus \{0\}$ satisfying (8.1.13), which yields $\lambda \in \sigma_p(A+B+C)$. Hence, $\{\lambda \in \Omega : \det(\Delta(\lambda, \mu)) = 0\} \subset \sigma_p(A+B+C)$ and (i) follows. Assertion (iii) follows from (8.1.12) and the same argument as in the above proof. The proof is complete. $\square$

From the above discussion we know that Assumptions 3.4.1, 3.5.2 and 6.2.1(c) hold. In order to apply the Hopf Bifurcation Theorem 6.2.7 to system (8.1.2), we only need to make the following assumption.

Assumption 8.1.6. There exists a continuously differentiable map $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{C}$ such that for each $\mu \in (-\varepsilon, \varepsilon)$,

$$\det(\Delta(\mu, \lambda(\mu))) = 0,$$

and $\lambda(\mu)$ is a simple eigenvalue of $\left(A + \partial_x F(\mu, \overline{x}_\mu)\right)_0$ which is equivalent to verify that

$$\lim_{\lambda \to \lambda(\mu)} \frac{\det(\Delta(\mu, \lambda))}{(\lambda - \lambda(\mu))} \neq 0$$

and

$$\dim(\mathscr{N}(\Delta(\mu, \lambda(\mu)))) = 1.$$

Moreover, assume that

$$\operatorname{Im}(\lambda(0)) > 0, \ \operatorname{Re}(\lambda(0)) = 0, \ \frac{d\operatorname{Re}(\lambda(0))}{d\mu} \neq 0,$$

and

$$\{\lambda \in \Omega : \det(\Delta(\lambda, 0)) = 0\} \cap i\mathbb{R} = \left\{\lambda(0), \overline{\lambda(0)}\right\}. \tag{8.1.15}$$

If $\lambda(\mu)$ is a solution of the characteristic equation, so is $\overline{\lambda(\mu)}$. So from the above assumption we obtain a pair of conjugated simple eigenvalues. Now by using Theorem 6.2.7, we derive the following Hopf bifurcation theorem for the age-structured system (8.1.1).

Theorem 8.1.7. *Let the Assumptions 8.1.1-8.1.3 and 8.1.6 be satisfied. Then there exist a constant $\varepsilon^* > 0$ and three C^{k-1} maps, $\varepsilon \to \mu(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, $\varepsilon \to u_{0,\varepsilon}$ from $(0, \varepsilon^*)$ into $L^1((0, +\infty), \mathbb{R}^n)$, and $\varepsilon \to T(\varepsilon)$ from $(0, \varepsilon^*)$ into $\mathbb{R}$, such that for each $\varepsilon \in (0, \varepsilon^*)$ there exists a $T(\varepsilon)$-periodic function $u_\varepsilon \in C^k\left(\mathbb{R}, L^1((0, +\infty), \mathbb{R}^n)\right)$, which is a solution of (8.1.1) for the parameter value $\mu = \mu(\varepsilon)$ and the initial value $u_0 = u_{0,\varepsilon}$. Moreover, we have the following properties;*

(i) There exist a neighborhood N of 0 in $L^1((0, +\infty), \mathbb{R}^n)$ and an open interval I in $\mathbb{R}$ containing 0 such that for $\widehat{\mu} \in I$ and any periodic solution $\widehat{u}(t)$ in N, with minimal period $\widehat{T}$ close to $\frac{2\pi}{\mathrm{Im}(\lambda(0))}$, of (8.1.1) for the parameter value $\widehat{\mu}$, there exists $\varepsilon \in (0, \varepsilon^*)$ such that $\widehat{u}(t) = u_\varepsilon(t + \theta)$ (for some $\theta \in [0, p(\varepsilon))$), $\mu(\varepsilon) = \widehat{\mu}$, and $T(\varepsilon) = \widehat{T}$;

(ii) The map $\varepsilon \to \mu(\varepsilon)$ is a C^{k-1} function and

$$\mu(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \mu_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where $[\frac{k-2}{2}]$ is the integer part of $\frac{k-2}{2}$;

(iii) The period $p(\varepsilon)$ of $t \to u_\varepsilon(t)$ is a C^{k-1} function and

$$T(\varepsilon) = \frac{2\pi}{\mathrm{Im}(\lambda(0))}[1 + \sum_{n=1}^{[\frac{k-2}{2}]} \tau_{2n} \varepsilon^{2n}] + O(\varepsilon^{k-1}), \forall \varepsilon \in (0, \varepsilon^*),$$

where $\mathrm{Im}(\lambda(0))$ is defined in Assumption 8.1.6;

(iv) The Floquet exponent $\beta(\varepsilon)$ is a C^{k-1} function satisfying $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and having the Taylor expansion

$$\beta(\varepsilon) = \sum_{n=1}^{[\frac{k-2}{2}]} \beta_{2n} \varepsilon^{2n} + O(\varepsilon^{k-1}), \ \forall \varepsilon \in (0, \varepsilon^*).$$

The periodic solution $x_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\varepsilon) < 0$ and unstable if $\beta(\varepsilon) > 0$.

Remark 8.1.8. If we only assume that $k \geq 2$, and the condition (8.1.15) is replaced by

$$\{\lambda \in \mathbb{C} : \det(\Delta(0, \lambda)) = 0\} \cap i\omega\mathbb{Z} = \{i\omega, -i\omega\}$$

with $\omega = \mathrm{Im}(\lambda(0))$, then by using Remark 6.2.8, we deduce that assertion (i) of Theorem 8.1.7 holds.

8.2 A Susceptible-Infectious Model with Age of Infection

Consider a population divided into two subgroups: susceptible individuals $S(t)$ at time t and infected individuals $I(t,a)$ at time t with the age of infection $a \geq 0$; that is, the time since the infection began. For two given age values a_1 and a_2 with $0 \leq a_1 < a_2 \leq +\infty$, the number of infected individuals with age of infection a between a_1 and a_2 is

$$\int_{a_1}^{a_2} i(t,a)\,da.$$

The infection-age allows different interpretations for values of a. For example, an individual may be exposed (infected but not yet infectious to susceptibles) from age $a = 0$ to $a = a_1$ and infectious (infected and infectious to susceptibles) from age a_1 to age a_2. Consider an infection-age model with a mass action law incidence function

$$\begin{cases} \dfrac{dS(t)}{dt} = \gamma - \nu_S S(t) - \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) da, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I(a)\, i(t,a), \\ i(t,0) = \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) da, \\ S(0) = S_0 \geq 0,\ i(0,.) = i_0 \in L^1_+(0,+\infty). \end{cases} \tag{8.2.1}$$

The parameter $\gamma > 0$ is the influx rate of the susceptible class and $\nu_S > 0$ is the exit (or/and mortality) rate of susceptible individuals. The function $\beta(a)$ can be interpreted as the probability to be infectious (capable of transmitting the disease) with age of infection $a \geq 0$. The quantity

$$\int_0^{+\infty} \beta(a) i(t,a) da$$

is the number of infectious individuals within the subpopulation (I). The function $\beta(a)$ allows variable probability of infectiousness as the disease progresses within an infected individual. $\eta > 0$ is the rate at which an infectious individual infects the susceptible individuals. Finally, $\nu_I(a)$ is the exit (or/and mortality, or/and recovery) rate of infected individuals with an age of infection $a \geq 0$. As a consequence the quantity

$$l_{\nu_I}(a) := \exp\left(-\int_0^a \nu_I(l)\,dl\right)$$

is the probability for an individual to stay in the class (I) after a period of time $a \geq 0$. We make the following assumption.

Assumption 8.2.1. The function $a \to \beta(a)$ is bounded and uniformly continuous from $[0,+\infty)$ to $[0,+\infty)$, the function $a \to \nu_I(a)$ belongs to $L^\infty_+(0,+\infty)$ and satisfies $\nu_I(a) \geq \nu_S$ for almost every $a \geq 0$.

Consider the two cases of $\beta(a)$ described in Figure 8.1, in which an incubation period of 10 time units (hours or days) was used depending on the time scale. In case (A), after the incubation period the infectiousness function $\beta(a)$ increases with the age of infection. This situation corresponds to a disease which becomes more and more transmissible with the age of infection. In case (B), after the incubation period the infectiousness of infected individuals increases, passes through a maximum at $a = 20$, and then decreases and is eventually equal to 0 for large values of $a \geq 0$.

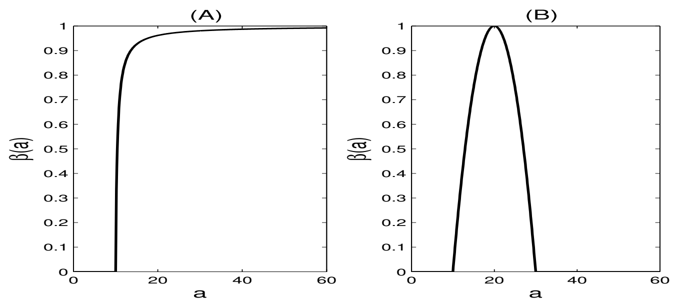

Fig. 8.1: Two cases of $\beta(a)$, the probability to be infectious as a function of infection-age a.

The basic reproduction number $\mathscr{R}_0$, defined as the number of secondary infections produced by a single infected individual, is given by

$$\mathscr{R}_0 = \eta \frac{\gamma}{\nu_S} \int_0^{+\infty} \beta(a) l_{\nu_I}(a) da.$$

System (8.2.1) has at most two equilibria. The disease-free equilibrium

$$\left(\bar{S}_F,\, 0\right) = \left(\frac{\gamma}{\nu_S},\, 0\right)$$

always exists. Moreover, when $\mathscr{R}_0 > 1$, there exists a unique endemic equilibrium

$$(\bar{S}_E, \bar{\imath}_E)$$

(i.e. with $\bar{\imath}_E \in L^1_+(0,+\infty) \setminus \{0\}$) defined by

$$\bar{S}_E = 1/\left(\eta \int_0^{+\infty} \beta(a) l_{\nu_I}(a) da\right) = \frac{\bar{S}_F}{\mathscr{R}_0},\ \ \bar{\imath}_E(a) = l_{\nu_I}(a)\bar{\imath}_E(0)$$

with $\bar{\imath}_E(0) := \gamma - \nu_S \bar{S}_E$.

8.2.1 Integrated Solutions and Attractors

Without loss of generality, we can add the equation of the recovered class

$$\begin{cases} \dfrac{dR(t)}{dt} = \int_0^{+\infty} (\nu_I(a) - \nu_S)\, i(t,a)da - \nu_S R(t), \forall t \geq 0, \\ R(0) \geq 0 \end{cases}$$

to system (8.2.1). Since by assumption $\nu_I(a) \geq \nu_S$ for almost every $a \geq 0$, we deduce that $R(0) \geq 0 \Rightarrow R(t) \geq 0, \forall t \geq 0$. Now, by setting

$$N(t) = S(t) + \int_0^{+\infty} i(t,a)da + R(t),$$

we can see that $N(t)$ satisfies the following ordinary differential equation

$$\frac{dN(t)}{dt} = \gamma - \nu_S N(t). \tag{8.2.2}$$

So $N(t)$ converges to $\dfrac{\gamma}{\nu_S}$. Moreover, since $R(t) \geq 0, \forall t \geq 0$, we obtain the following estimate

$$S(t) + \int_0^{+\infty} i(t,a)da \leq N(t), \ \ \forall t \geq 0. \tag{8.2.3}$$

Note that in the Volterra integral formulation, system (8.2.1) can be formulated as follows

$$\frac{dS(t)}{dt} = \gamma - \nu_S S(t) - \eta S(t) \int_0^{+\infty} \beta(a) i(t,a)da,$$
$$i(t,a) = \begin{cases} \exp\left(\int_{a-t}^{a} \nu_I(l)\,dl\right) i_0(a-t) & \text{if } a-t \geq 0, \\ \exp\left(\int_0^a \nu_I(l)\,dl\right) b(t-a) & \text{if } a-t \leq 0, \end{cases}$$

where $t \to b(t)$ is the unique continuous function satisfying

$$b(t) = \eta S(t) \begin{bmatrix} \int_0^t \beta(a) \exp\left(\int_0^a \nu_I(l)\,dl\right) b(t-a)da \\ + \int_t^{+\infty} \beta(a) \exp\left(\int_{a-t}^{a} \nu_I(l)\,dl\right) i_0(a-t)da \end{bmatrix}. \tag{8.2.4}$$

In order to take into account the boundary condition, we extend the state space and consider

$$\widehat{X} = \mathbb{R} \times L^1(0,+\infty)$$

and the linear operator $\widehat{A} : D\left(\widehat{A}\right) \subset \widehat{X} \to \widehat{X}$ on $\widehat{X}$ defined by

$$\widehat{A}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \nu_I \varphi \end{pmatrix} \quad \text{with} \quad D\left(\widehat{A}\right) = \{0\} \times W^{1,1}(0,+\infty).$$

If $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$, then $\lambda \in \rho\left(\widehat{A}\right)$ and we have following explicit formula for the resolvent of $\widehat{A}$:

$$\left(\lambda I - \widehat{A}\right)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow \varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda dl}\alpha + \int_0^a e^{-\int_s^a \nu_I(l)+\lambda dl}\psi(s)ds.$$

Note that system (8.2.1) can be written as

$$\begin{cases} \dfrac{dS(t)}{dt} = \gamma - \nu_S S(t) - \eta S(t)\int_0^{+\infty}\beta(a)i(t,a)da,\\ \dfrac{d}{dt}\begin{pmatrix}0\\ i(t,.)\end{pmatrix} = \widehat{A}\begin{pmatrix}0\\ i(t,.)\end{pmatrix} + \begin{pmatrix}\eta S(t)\int_0^{+\infty}\beta(a)i(t,a)da\\ 0\end{pmatrix}\\ S(0) = S_0 \geq 0,\\ i(0,.) = i_0 \in L^1_+(0,+\infty). \end{cases} \tag{8.2.5}$$

Set

$$X = \mathbb{R}\times\mathbb{R}\times L^1(0,+\infty),\ \ X_+ = \mathbb{R}_+\times\mathbb{R}_+\times L^1(0,+\infty),$$

and let $A : D(A) \subset X \to X$ be the linear operator defined by

$$A\begin{pmatrix}S\\ \begin{pmatrix}0\\ i\end{pmatrix}\end{pmatrix} = \begin{pmatrix}-\nu_S S\\ \widehat{A}\begin{pmatrix}0\\ i\end{pmatrix}\end{pmatrix} = \begin{bmatrix}-\nu_S & 0\\ 0 & \widehat{A}\end{bmatrix}\begin{pmatrix}S\\ \begin{pmatrix}0\\ i\end{pmatrix}\end{pmatrix}$$

with

$$D(A) = \mathbb{R}\times D(\widehat{A}).$$

Then $\overline{D(A)} = \mathbb{R}\times\{0\}\times L^1(0,+\infty)$ is not dense in X. Consider the nonlinear map $F : \overline{D(A)} \to X$ defined by

$$F\begin{pmatrix}S\\ \begin{pmatrix}0\\ i\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\gamma - \eta S\int_0^{+\infty}\beta(a)i(a)da\\ \begin{pmatrix}\eta S\int_0^{+\infty}\beta(a)i(a)da\\ 0\end{pmatrix}\end{pmatrix}.$$

Define

$$X_0 := \overline{D(A)} = \mathbb{R}\times\{0\}\times L^1(0,+\infty)$$

and

$$X_{0+} := \overline{D(A)}\cap X_+ = \mathbb{R}_+\times\{0\}\times L^1_+(0,+\infty).$$

We can rewrite system (8.2.5) as the following abstract Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F(u(t)),\ \ t \geq 0;\ \ u(0) = x \in \overline{D(A)}. \tag{8.2.6}$$

By using the fact that the nonlinearities are Lipschitz continuous on bounded sets, (8.2.3), and Theorem 5.2.7, we obtain the following proposition.

Proposition 8.2.2. *There exists a uniquely determined semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that for each $x = \begin{pmatrix} S_0 \\ \begin{pmatrix} 0 \\ i_0 \end{pmatrix} \end{pmatrix} \in X_{0+}$, there exists a unique continuous map $U \in C([0,+\infty), X_{0+})$ which is an integrated solution of the Cauchy problem (8.2.6); that is,*

$$\int_0^t U(s)xds \in D(A),\ \forall t \geq 0,$$

and

$$U(t)x = x + A\int_0^t U(s)xds + \int_0^t F(U(s)x)\,ds,\ \forall t \geq 0.$$

Moreover,

$$\limsup_{t\to+\infty} S(t) \leq \frac{\gamma}{\nu_S}.$$

By using the results in Magal and Thieme [251] (see also Thieme and Vrabie [339]) and by using the fact that $a \to \beta(a)$ is uniformly continuous, we have the following result.

Proposition 8.2.3. *The semiflow $\{U(t)\}_{t\geq 0}$ is asymptotically smooth; that is, for any nonempty closed bounded set $\mathscr{B} \subset X_{0+}$ for which $U(t)\mathscr{B} \subset \mathscr{B}$, there is a compact set $\mathscr{J} \subset \mathscr{B}$ such that $\mathscr{J}$ attracts $\mathscr{B}$.*

Moreover, by using (8.2.3), we have the following proposition.

Proposition 8.2.4. *The semiflow $\{U(t)\}_{t\geq 0}$ is bounded dissipative; that is, there is a bounded set $\mathscr{B}_0 \subset X_{0+}$ such that $\mathscr{B}_0$ attracts each bounded set in X_{0+}.*

By Propositions 8.2.3 and 8.2.4 and the results of Hale [173], we obtain the following result.

Proposition 8.2.5. *There exists a compact set $\mathscr{A} \subset X_{0+}$ such that*

(i) *$\mathscr{A}$ is invariant under the semiflow $U(t)$; that is,*

$$U(t)\mathscr{A} = \mathscr{A}, \forall t \geq 0;$$

(ii) *$\mathscr{A}$ attracts the bounded sets of X_{0+} under U; that is, for each bounded set $\mathscr{B} \subset X_{0+}$,*

$$\lim_{t\to+\infty} \delta(U(t)\mathscr{B}, \mathscr{A}) = 0,$$

where the semi-distance $\delta(.,.)$ is defined as

$$\delta(\mathscr{B}, \mathscr{A}) = \sup_{x\in\mathscr{B}} \inf_{y\in\mathscr{A}} \|x - y\|;$$

(iii) *$\mathscr{A}$ is locally asymptotically stable.*

8.2.2 Local and Global Stability of the Disease-free Equilibrium

The linearized system at the disease-free equilibrium $\left(\bar{S}_F, 0\right)$ is

$$\begin{cases} \dfrac{dS(t)}{dt} = -\nu_S S(t) - \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) da, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I(a)\, i(t,a), \\ i(t,0) = \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) da, \\ S(0) = S_0 \geq 0, \\ i(0,.) = i_0 \in L^1_+(0,+\infty). \end{cases}$$

Notice that the dynamics of i do not depend on S and so, in order to study the uniform persistence of the disease, we only need to focus on the linear subsystem

$$\begin{cases} \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I(a)\, i(t,a), \\ i(t,0) = \eta \bar{S}_F \int_0^{+\infty} \beta(a) i(t,a) da, \\ i(0,.) = i_0 \in L^1_+(0,+\infty), \end{cases}$$

where $\bar{S}_F = \dfrac{\gamma}{\nu_S}$. Define

$$\hat{B}_\kappa \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \kappa \int_0^{+\infty} \beta(a)\phi(a) da \\ 0 \end{pmatrix}$$

with $\kappa = \eta \frac{\gamma}{\nu_S}$. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$, define the characteristic function $\Delta(\lambda)$ as

$$\Delta(\lambda) = 1 - \kappa \int_0^{+\infty} \beta(a) e^{-\int_0^a \nu_I(l) + \lambda dl} da.$$

Moreover, since $\lambda I - \widehat{A}$ is invertible, we deduce that $\lambda I - (\widehat{A} + \hat{B}_\kappa)$ is invertible if and only if $I - \hat{B}_\kappa (\lambda I - \widehat{A})^{-1}$ is invertible or

$$\lambda \in \rho\left(\widehat{A} + \hat{B}_\kappa\right) \Leftrightarrow 1 \in \rho\left(\hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1}\right),$$

and we have

$$\left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1} = \left(\lambda I - \widehat{A}\right)^{-1} \left[I - \hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1}\right]^{-1}.$$

But we also have

$$\begin{pmatrix} \alpha \\ \psi \end{pmatrix} - \hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$
$$\Leftrightarrow \begin{cases} \alpha - \left[\kappa \int_0^{+\infty} \beta(a) \left[e^{-\int_0^a \nu_I(l)+\lambda dl}\alpha + \int_0^a e^{-\int_s^a \nu_I(l)+\lambda dl}\psi(s)ds\right] da\right] = \gamma \\ \psi = \varphi \end{cases}.$$

We can isolate α only if $\Delta(\lambda) \neq 0$. So for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$, the linear operator $I - \hat{B}_\kappa(\lambda I - \widehat{A})^{-1}$ is invertible if and only if $\Delta(\lambda) \neq 0$, and we have

$$\left[I - \hat{B}_\kappa \left(\lambda I - \widehat{A}\right)^{-1}\right]^{-1} \begin{pmatrix} \gamma \\ \varphi \end{pmatrix}$$
$$= \begin{pmatrix} \Delta(\lambda)^{-1} \left[\kappa \int_0^{+\infty} \beta(a) \int_0^a e^{-\int_s^a \nu_I(l)+\lambda dl}\varphi(s)dsda + \gamma\right] \\ \varphi \end{pmatrix}.$$

It follows that for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\nu_S$ and $\Delta(\lambda) \neq 0$, we have

$$\left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$
$$\Leftrightarrow$$
$$\varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda dl} \left\{\Delta(\lambda)^{-1} \left[\kappa \int_0^{+\infty} \beta(a) \int_0^a e^{-\int_s^a \nu_I(l)+\lambda dl}\psi(s)dsda + \alpha\right]\right\}$$
$$+ \int_0^a e^{-\int_s^a \nu_I(l)+\lambda dl}\psi(s)ds.$$

Assume that $\mathscr{R}_0 = \kappa \int_0^{+\infty} \beta(a)e^{-\int_0^a \nu_I(l)dl}da > 1$. Then we can find $\lambda_0 \in \mathbb{R}$ such that

$$\kappa \int_0^{+\infty} \beta(a)e^{-\int_0^a (\nu_I(l)+\lambda_0)dl}da = 1,$$

and $\lambda_0 > 0$ is a dominant eigenvalue of $\widehat{A} + \hat{B}_\kappa$ (see Webb [363]). Moreover, we have

$$\frac{d\Delta(\lambda_0)}{d\lambda} = \kappa \int_0^{+\infty} a\beta(a)e^{-\int_0^a \nu_I(l)+\lambda_0 dl}da > 0,$$

and the expression

$$\widehat{\Pi} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0) \left(\lambda I - \left(\widehat{A} + \hat{B}_\kappa\right)\right)^{-1} \begin{pmatrix} \alpha \\ \psi \end{pmatrix}$$

exists and satisfies

$$\widehat{\Pi} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$
$$\Leftrightarrow \varphi(a) = e^{-\int_0^a \nu_I(l)+\lambda_0 dl} \left\{\left(\frac{d\Delta(\lambda_0)}{d\lambda}\right)^{-1} \left[\kappa \int_0^{+\infty} \beta(a) \int_0^a e^{-\int_s^a \nu_I(l)+\lambda_0 dl}\psi(s)dsda + \alpha\right]\right\}.$$

The linear operator $\widehat{\Pi} : \widehat{X} \to \widehat{X}$ is the projector onto the generalized eigenspace of $\widehat{A} + \hat{B}_\kappa$ associated with the eigenvalue λ_0. Define $\Pi : X \to X$ by

$$\Pi \begin{pmatrix} S \\ \alpha \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\Pi} \begin{pmatrix} \alpha \\ \psi \end{pmatrix} \end{pmatrix}.$$

Define

$$\widehat{M}_0 = \left\{ i \in L^1_+ (0, +\infty) : \int_0^{\bar{a}} i(a)da > 0 \right\}$$

and

$$M_0 = [0, +\infty) \times \widehat{M}_0 = \{x \in X_{0+} : \Pi x \neq 0\}, \quad \partial M_0 = [0, +\infty) \times L^1_+ (0, +\infty) \setminus M_0.$$

Then $X_{0+} = M_0 \cup \partial M_0$.

Lemma 8.2.6. *The subsets M_0 and ∂M_0 are both positively invariant under the semiflow $\{U(t)\}_{t \geq 0}$; that is,*

$$U(t)M_0 \subset M_0 \text{ and } U(t)\partial M_0 \subset \partial M_0.$$

Moreover, for each $x \in \partial M_0$,

$$U(t)x \to \bar{x}_F \text{ as } t \to +\infty,$$

where $\bar{x}_F = \begin{pmatrix} \bar{S}_F \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}$ is the disease-free equilibrium of $\{U(t)\}_{t \geq 0}$.

When $\mathscr{R}_0 \leq 1$, we have the following result extending the results proved in Thieme and Castillo-Chavez [337, Theorem 2] and D'Agata et al. [80, Proposition 3.10].

Theorem 8.2.7. *Assume that $\mathscr{R}_0 \leq 1$. Then the disease-free equilibrium $\left(\bar{S}_F, 0\right)$ is global asymptotically stable for the semiflow generated by system (8.2.1).*

Proof. Since $\mathscr{R}_0 \leq 1$, we first observe that

$$\mathscr{R}_0 = \eta \frac{\gamma}{\nu_S} \int_0^{+\infty} \beta(a) l_{\nu_I}(a) da \leq 1 \Leftrightarrow \frac{\gamma}{\nu_S} \leq \bar{S}_E. \tag{8.2.7}$$

Set

$$\Gamma_I(a) = \eta \bar{S}_E \int_a^{+\infty} e^{-\int_a^s \nu_I(l)dl} \beta(s) ds, \; \forall a \geq 0.$$

Since

$$\eta \bar{S}_E = \left(\int_0^{+\infty} e^{-\int_0^s \nu_I(l)dl} \beta(s) ds \right)^{-1},$$

we have

$$\begin{cases} \Gamma_I'(a) = \nu_I(a)\Gamma_I(a) - \eta \bar{S}_E \beta(a) \text{ for almost every } a \geq 0, \\ \Gamma_I(0) = 1. \end{cases}$$

Define

$$D((A+F)_0) = \left\{ x \in D(A) : Ax + F(x) \in \overline{D(A)} \right\}.$$

Let $x \in D((A+F)_0) \cap X_{0+}$. Then we know (see Thieme [328] or Magal [242]) that $i(t,.) \in W^{1,1}(0,+\infty), \forall t \geq 0$, and for each $\forall t \geq 0$,

$$i(t,0) = S(t)\eta \int_0^{+\infty} \beta(a) i(t,a) da, \forall t \geq 0,$$

the map $t \to i(t,.)$ belongs to $C^1\left([0,+\infty), L^1(0,+\infty)\right)$, and $\forall t \geq 0$,

$$\frac{di(t,.)}{dt} = -\frac{\partial i(t,.)}{\partial a} - \nu_I(a) i(t,a) \text{ for almost every } a \in (0,+\infty).$$

So $\forall t \geq 0$,

$$\frac{d \int_0^{+\infty} \Gamma_I(a) i(t,a) da}{dt} = -\int_0^{+\infty} \Gamma_I(a) \frac{\partial i_R(t,a)}{\partial a} da - \int_0^{+\infty} \Gamma_I(a) \nu_I(a) i_I(t,a) da.$$

By using the fact that $i(t,.) \in W^{1,1}(0,+\infty)$, we deduce that $i(t,a) \to 0$ as $a \to +\infty$. By integrating by part we obtain that

$$\begin{aligned} \frac{d \int_0^{+\infty} \Gamma_I(a) i(t,a) da}{dt} &= -\left[\Gamma_I(a) i(t,a)\right]_0^{+\infty} + \int_0^{+\infty} \Gamma_I'(a) i(t,a) da \\ &\quad - \int_0^{+\infty} \Gamma_I(a) \nu_I(a) i_R(t,a) da \\ &= i(t,0) - \eta \bar{S}_E \int_0^{+\infty} \beta(a) i(t,a) da \\ &= \eta \left(S(t) - \bar{S}_E\right) \int_0^{+\infty} \beta(a) i(t,a) da. \end{aligned} \tag{8.2.8}$$

The density of $D((A+F)_0) \cap X_{0+}$ in X_{0+} implies that the above equality holds for any initial value $x \in X_{0+}$.

Let $x \in \mathscr{A}$, the attractor for system (8.2.5), be fixed. Since there exists a complete orbit $\left\{ u(t) = \begin{pmatrix} S(t) \\ \begin{pmatrix} 0 \\ i(t,.) \end{pmatrix} \end{pmatrix} \right\}_{t \in \mathbb{R}} \subset \mathscr{A}$, it follows from the $S(t)$ equation in system (8.2.5) that for each $t < 0$,

$$\begin{aligned} S(0) &= e^{-\int_t^0 \nu_S + \int_0^{+\infty} \beta(a) i(l,a) da dl} S(t) + \int_t^0 e^{-\int_t^s \nu_S + \int_0^{+\infty} \beta(a) i(l,a) da dl} \gamma ds \\ &\leq e^{-\int_t^0 \nu_S + \int_0^{+\infty} \beta(a) i(l,a) da dl} S(t) + \int_t^0 e^{-\int_t^s \nu_S dl} \gamma ds. \end{aligned}$$

Taking the limit when $t \to -\infty$, we have $S(0) \leq \frac{\gamma}{\nu_S}$. Since the above argument holds for any $x \in \mathscr{A}$, we have that

$$S(t) \leq \frac{\gamma}{\nu_S}, \ \forall t \in \mathbb{R}. \tag{8.2.9}$$

Combining (8.2.7), (8.2.8) and (8.2.9), we know that $t \to \int_0^{+\infty} \Gamma_I(a) i(t,a) da$ is non-increasing along the complete orbit.

Now assume that $\mathscr{A} \nsubseteq M_0$. Let $x \in \mathscr{A} \setminus M_0$. By using the definition of Γ_I and the definition of M_0, it follows that

$$\int_0^{+\infty} \Gamma_I(a) i(0,a) da > 0.$$

Since $t \to \int_0^{+\infty} \Gamma_I(a) i(t,a) da$ is non-increasing, it follows that

$$\int_0^{+\infty} \Gamma_I(a) i(t,a) da \geq \int_0^{+\infty} \Gamma_I(a) i(0,a) da > 0, \ \forall t \leq 0.$$

Thus, the α-limit set of the complete orbit passing through x satisfies

$$\alpha(x) := \cap_{t \leq 0} \overline{\cup_{s \leq t} \{u(t)\}} \subset \mathscr{A} \cap M_0.$$

Moreover, there exists a constant $C > 0$ such that for each $\hat{x} = \begin{pmatrix} \widehat{S} \\ \begin{pmatrix} 0 \\ \hat{\imath} \end{pmatrix} \end{pmatrix} \in \alpha(x)$, we have

$$\int_0^{+\infty} \Gamma_I(a) \hat{\imath}(a) da = C > 0 \tag{8.2.10}$$

and

$$\widehat{S} \leq \frac{\gamma}{\nu_S}. \tag{8.2.11}$$

Let $\left\{ \widehat{u}(t) = \begin{pmatrix} \hat{S}(t) \\ \begin{pmatrix} 0 \\ \hat{\imath}(t,.) \end{pmatrix} \end{pmatrix} \right\}_{t \geq 0}$ be the solution of the Cauchy problem (8.2.6) with initial value $\hat{x} \in \alpha(x)$. Then (8.2.10) implies that $\hat{x} \in M_0$, and by using (8.2.4) we deduce that there exists $t_1 > 0$ such that

$$\int_0^{+\infty} \beta(a) \hat{\imath}(t,a) da > 0, \ \forall t \geq t_1.$$

Now by using the invariance of the α-limit set $\alpha(x)$ by the semiflow generated by (8.2.6), the $S(t)$ equation, and (8.2.11), we have for each $t_2 > t_1$ that

$$\hat{S}(t) < \frac{\gamma}{\nu_S}, \ \forall t \geq t_2.$$

Finally, since by (8.2.7) we have $\frac{\gamma}{\nu_S} \leq \bar{S}_E$, and by using (8.2.8) we obtain

$$\frac{d\int_0^{+\infty}\Gamma_I(a)\hat{\imath}(t,a)da}{dt} < 0, \, \forall t \geq t_2,$$

so the map $t \to \int_0^{+\infty}\Gamma_I(a)\hat{\imath}(t,a)da$ is not constant. This contradiction assures that $\mathscr{A} \subset \partial M_0$. It follows that

$$\mathscr{A} = \{\bar{x}_F\},$$

the result follows. □

8.2.3 Uniform Persistence

By applying the results in Magal and Zhao [252] (or Magal [243]) and Proposition 8.2.5, we have the following proposition.

Proposition 8.2.8. *Assume that* $\mathscr{R}_0 > 1$. *The semiflow* $\{U(t)\}_{t\geq 0}$ *is uniformly persistent with respect to the pair* $(\partial M_0, M_0)$; *that is, there exists* $\varepsilon > 0$ *such that*

$$\liminf_{t\to+\infty} \|\Pi U(t)x\| \geq \varepsilon, \;\; \forall x \in M_0.$$

Moreover, there exists a compact subset $\mathscr{A}_0$ *of* M_0 *which is a global attractor for* $\{U(t)\}_{t\geq 0}$ *in* M_0; *that is,*

(i) $\mathscr{A}_0$ *is invariant under* U; *i.e.,*

$$U(t)\mathscr{A}_0 = \mathscr{A}_0, \forall t \geq 0;$$

(ii) For each compact subset $\mathscr{C} \subset M_0$,

$$\lim_{t\to+\infty} \delta(U(t)\mathscr{C}, \mathscr{A}_0) = 0.$$

Furthermore, the subset $\mathscr{A}_0$ *is locally asymptotically stable.*

Proof. Since the disease-free equilibrium $\bar{x}_F = \begin{pmatrix} \bar{S}_F \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}$ is globally asymptotically stable in ∂M_0, to apply Theorem 4.1 in Hale and Waltman [176], we only need to study the behavior of the solutions starting in M_0 in some neighborhood of $\bar{x}_F$. It is sufficient to prove that there exists $\varepsilon > 0$, such that for each $x = \begin{pmatrix} S_0 \\ \begin{pmatrix} 0 \\ i_0 \end{pmatrix} \end{pmatrix} \in \{y \in M_0 : \|\bar{x}_F - y\| \leq \varepsilon\}$, there exists $t_0 \geq 0$ such that

$$\|\bar{x}_F - U(t_0)x\| > \varepsilon.$$

This will show that $\{y \in X_{0+} : \|\bar{x}_F - y\| \leq \varepsilon\}$ is an isolating neigborhood of $\{\bar{x}_F\}$ (i.e. there exists a neigborhood of $\{\bar{x}_F\}$ in which $\{\bar{x}_F\}$ is the largest invariant set for

U), and

$$W^s(\{\overline{x}_F\}) \cap M_0 = \varnothing,$$

where

$$W^s(\{\overline{x}_F\}) = \left\{ x \in X_{0+} : \lim_{t\to+\infty} U(t)x = \overline{x}_F \right\}.$$

Assume by contradiction that for each $n \geq 0$, we can find $x_n = \begin{pmatrix} S_0^n \\ \begin{pmatrix} 0 \\ i_0^n \end{pmatrix} \end{pmatrix} \in \left\{ y \in M_0 : \|\overline{x}_F - y\| \leq \dfrac{1}{n+1} \right\}$ such that

$$\|\overline{x}_F - U(t)x_n\| \leq \frac{1}{n+1}, \ \forall t \geq 0. \tag{8.2.12}$$

Set

$$\begin{pmatrix} S^n(t) \\ \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} \end{pmatrix} := U(t)x_n.$$

We have

$$\left| S^n(t) - \bar{S}_F \right| \leq \frac{1}{n+1}, \forall t \geq 0.$$

Moreover, the map $t \to \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix}$ is an integrated solution of the Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} = \widehat{A}\begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix} + F_2(S^n(t), \begin{pmatrix} 0 \\ i^n(t,.) \end{pmatrix}), \ t \geq 0; \ \begin{pmatrix} 0 \\ i^n(0,.) \end{pmatrix} = \begin{pmatrix} 0 \\ i_0^n \end{pmatrix}.$$

Since $\widehat{A}$ is resolvent positive and F_2 monotone non-increasing, we deduce that

$$i^n(t,.) \geq \tilde{\imath}^n(t,.), \tag{8.2.13}$$

where $t \to \tilde{\imath}^n(t,.)$ is a solution of the linear Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix} = \widehat{A}\begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix} + F_2(\bar{S}_F + \tfrac{1}{n+1}, \begin{pmatrix} 0 \\ \tilde{\imath}^n(t,.) \end{pmatrix}), \ t \geq 0; \ \begin{pmatrix} 0 \\ \tilde{\imath}^n(0,.) \end{pmatrix} = \begin{pmatrix} 0 \\ i_0^n \end{pmatrix},$$

or $\tilde{\imath}^n(t,a)$ is a solution of the PDE

$$\begin{cases} \dfrac{\partial \tilde{\imath}^n(t,a)}{\partial t} + \dfrac{\partial \tilde{\imath}^n(t,a)}{\partial a} = -\nu_I(a)\,\tilde{\imath}^n(t,a), \\ \tilde{\imath}^n(t,0) = \eta\left(\bar{S}_F - \dfrac{1}{n+1}\right)\int_0^{+\infty}\beta(a)\tilde{\imath}^n(t,a)da, \\ \tilde{\imath}^n(0,.) = i_0^n \in L_+^1(0,+\infty). \end{cases}$$

We observe that

$$F_2(\bar{S}_F - \frac{1}{n+1}, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}) = \hat{B}_{\kappa_n} \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

with

$$\kappa_n = \eta \left(\bar{S}_F - \frac{1}{n+1} \right).$$

Now since $\mathscr{R}_0 > 1$, we deduce that for all $n \geq 0$ large enough, the dominated eigenvalue of the linear operator $\widehat{A} + \hat{B}_{\kappa_n} : D(A) \subset X \to X$ satisfies the characteristic equation

$$\eta \left(\bar{S}_F - \frac{1}{n+1} \right) \int_0^{+\infty} \beta(a) e^{-\int_0^a \nu_I(l) + \lambda_{0n} dl} da = 1.$$

It follows that $\lambda_{0n} > 0$ for all $n \geq 0$ large enough. Now $x_n \in M_0$, we have

$$\widehat{\Pi}_n \begin{pmatrix} 0 \\ i_0^n \end{pmatrix} \neq 0,$$

where $\widehat{\Pi}_n$ is the projector on the eigenspace associated to the dominante eigenvalue λ_{0n}. It follows that

$$\lim_{t \to +\infty} \|\tilde{\imath}^n(t,.)\| = +\infty,$$

and by using (8.2.13) we obtain

$$\lim_{t \to +\infty} \|i^n(t,.)\| = +\infty.$$

We obtain a contradiction with (8.2.12), and the result follows. □

8.2.4 Local and Global Stabilities of the Endemic Equilibrium

We first have the following result on the local stability of the endemic equilibrium.

Proposition 8.2.9. *Assume that $\mathscr{R}_0 > 1$. Then the endemic equilibrium $\bar{x}_E = \begin{pmatrix} \bar{S}_E \\ \begin{pmatrix} 0 \\ \bar{\imath}_E \end{pmatrix} \end{pmatrix}$ is locally asymptotically stable for $\{U(t)\}_{t \geq 0}$.*

Proof. The linearized equation of (8.2.6) around the endemic equilibrium $\bar{x}_E$ is

$$\frac{dv(t)}{dt} = Av(t) + DF(\bar{x}_E)(v(t)), \; t \geq 0, \; v(0) = x \in \overline{D(A)},$$

which corresponds to the following PDE

$$\begin{cases} \dfrac{dx(t)}{dt} = -\nu_S x(t) - \eta \bar{S}_E \int_0^{+\infty} \beta(a) y(t,a) da - x(t) \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da, \\ \dfrac{\partial y(t,a)}{\partial t} + \dfrac{\partial y(t,a)}{\partial a} = -\nu_I(a) y(t,a), \\ y(t,0) = \eta \bar{S}_E \int_0^{+\infty} \beta(a) y(t,a) da + x(t) \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da, \\ x(0) = x_0 \in \mathbb{R}, \\ y(0,.) = y_0 \in L^1(0,+\infty). \end{cases}$$

Since the semigroup $\left\{T_{A_0}(t)\right\}_{t \geq 0}$ generated by A_0, the part of A in $\overline{D(A)}$, satisfies

$$\left\| T_{A_0}(t) \right\| \leq \hat{M} e^{-\nu_S t}, \ \forall t \geq 0,$$

for some constant $\hat{M} > 0$, it follows that the essential growth rate $\omega_{\text{ess}}(A_0)$ of $\left\{T_{A_0}(t)\right\}_{t \geq 0}$ is no more than $-\nu_S$. Let $\{T_{(A+DF(\bar{x}_E))_0}(t)\}_{t \geq 0}$ be the linear C_0-semigroup generated by $(A + DF(\bar{x}_E))_0$, the part of $A + DF(\bar{x}_E) : D(A) \subset X \to X$ in $\overline{D(A)}$. Since $DF(\bar{x}_E)$ is a compact bounded linear operator, it follows from Theorem 4.7.3 that

$$\omega_{\text{ess}}((A + DF(\bar{x}_E))_0) \leq -\nu_S.$$

It remains to study the point spectrum of $(A + DF(\bar{x}_E))_0$. Consider the exponential solutions (i. e. solutions of the form $u(t) = e^{\lambda t} x$ with $x \neq 0$) to derive the characteristic equation and obtain the following system

$$\begin{cases} \lambda x = -\nu_S x - \eta \bar{S}_E \int_0^{+\infty} \beta(a) y(a) da - x \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da \\ \lambda y(a) + \dfrac{dy(a)}{da} = -\nu_I(a) y(a), \\ y(0) = \eta \bar{S}_E \int_0^{+\infty} \beta(a) y(a) da + x \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da, \end{cases}$$

where $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > -\nu_S$ and $(x,y) \in \mathbb{R} \times W^{1,1}(0,+\infty) \setminus \{0\}$. By integrating $y(a)$ we obtain the system of two equations for $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > -\nu_S$,

$$\left(\lambda + \nu_S + \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da \right) x = -\eta \bar{S}_E y(0) \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} da$$

and

$$\left(1 - \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} da \right) y(0) = +x\eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da,$$

where

$$\bar{S}_E := \left(\eta \int_0^{+\infty} \beta(a) l_{\nu_I}(a) da \right)^{-1}, \ \eta \int_0^{+\infty} \beta(a) \bar{\iota}_E(a) da = \gamma \bar{S}_E^{-1} - \nu_S,$$

and $(x, y(0)) \in \mathbb{R}^2 \setminus \{0\}$. We have

$$1 = \eta \bar{S}_E \int_0^{+\infty} \beta(a) l_{\nu_I}(a) e^{-a\lambda} da$$

$$-\frac{\left(\gamma\bar{S}_E^{-1}-\nu_S\right)}{\left(\lambda+\nu_S+\gamma\bar{S}_E^{-1}-\nu_S\right)}\eta\bar{S}_E\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\lambda}da$$
$$=\eta\bar{S}_E\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\lambda}da\left[1-\frac{\left(\gamma\bar{S}_E^{-1}-\nu_S\right)}{\left(\lambda+\gamma\bar{S}_E^{-1}\right)}\right].$$

Thus, it remains to study the characteristic equation for $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)>-\nu_S$,

$$1=\eta\bar{S}_E\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\lambda}da\left[\frac{(\lambda+\nu_S)}{\left(\lambda+\gamma\bar{S}_E^{-1}\right)}\right]. \tag{8.2.14}$$

By considering the real and the imaginary parts of λ, we obtain

$$\frac{\left[\left(\mathrm{Re}(\lambda)+\gamma\bar{S}_E^{-1}\right)+i\mathrm{Im}(\lambda)\right]\left[\left(\mathrm{Re}(\lambda)+\nu_S\right)-i\mathrm{Im}(\lambda)\right]}{\left[\left(\mathrm{Re}(\lambda)+\nu_S\right)^2+\mathrm{Im}(\lambda)^2\right]}$$
$$=\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\left[\cos(a\mathrm{Im}(\lambda))+i\sin(a\mathrm{Im}(\lambda))\right]da\right].$$

By identifying the real and imaginary parts, we obtain for the real part that

$$\left(\mathrm{Re}(\lambda)+\gamma\bar{S}_E^{-1}\right)\left(\mathrm{Re}(\lambda)+\nu_S\right)+\mathrm{Im}(\lambda)^2$$
$$=\left[\left(\mathrm{Re}(\lambda)+\nu_S\right)^2+\mathrm{Im}(\lambda)^2\right]\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\cos(a\mathrm{Im}(\lambda))da\right].$$

Thus,

$$\left(\gamma\bar{S}_E^{-1}-\nu_S\right)\left(\mathrm{Re}(\lambda)+\nu_S\right)$$
$$=\left[\left(\mathrm{Re}(\lambda)+\nu_S\right)^2+\mathrm{Im}(\lambda)^2\right]\left[\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\cos(a\mathrm{Im}(\lambda))da\right]-1\right].$$

Assume that there exists $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$ satisfying (8.2.14). Since $\bar{S}_E=\left(\eta\int_0^{+\infty}\beta(a)l_{\nu_I}(a)da\right)^{-1}$, we deduce that

$$\eta\bar{S}_E\left[\int_0^{+\infty}\beta(a)l_{\nu_I}(a)e^{-a\mathrm{Re}(\lambda)}\cos(a\mathrm{Im}(\lambda))da\right]\leq 1,$$

and since $\mathscr{R}_0=\eta\frac{\gamma}{\nu_S}\int_0^{+\infty}\beta(a)l_{\nu_I}(a)da=\frac{\gamma}{\nu_S}\frac{1}{\bar{S}_E}>1$, we obtain $\gamma\bar{S}_E^{-1}-\nu_S>0$. Thus,

$$\left(\gamma\bar{S}_E^{-1}-\nu_S\right)\left(\mathrm{Re}(\lambda)+\nu_S\right)>0.$$

It follows that the characteristic equation (8.2.14) has no root with non-negative real part. The proof is complete. □

Assume that $\mathscr{R}_0>1$. By using Proposition 8.2.8 (since $\mathscr{A}_0$ is invariant under U), we can find a complete orbit $\{u(t)\}_{t\in\mathbb{R}}\subset\mathscr{A}_0$ of $\{U(t)\}_{t\geq 0}$; that is,

$$u(t)=U(t-s)u(s),\forall t,s\in\mathbb{R}\text{ with }t\geq s.$$

So we have

$$u(t) = \begin{pmatrix} S(t) \\ \begin{pmatrix} 0 \\ i(t,.) \end{pmatrix} \end{pmatrix} \in \mathscr{A}_0, \ \forall t \in \mathbb{R},$$

and $\{(S(t), i(t,.))\}_{t\in\mathbb{R}}$ is a complete orbit of system (8.2.1). Moreover, by using the same arguments as in Lemma 3.6 and Proposition 4.3 in D'Agata et al. [80], we have the following lemma.

Lemma 8.2.10. *There exist constants $M > \varepsilon > 0$ such that for each complete orbit $\left\{ \begin{pmatrix} S(t) \\ \begin{pmatrix} 0 \\ i(t,.) \end{pmatrix} \end{pmatrix} \right\}_{t\in\mathbb{R}}$ of U in $\mathscr{A}_0$, we have*

$$\varepsilon \leq S(t) \leq M, \ \forall t \in \mathbb{R},$$

and

$$\varepsilon \leq \int_0^{+\infty} \beta(a) i(t,a) da \leq M, \ \forall t \in \mathbb{R}.$$

Moreover, $O = \overline{\cup_{t\in\mathbb{R}} \{(S(t), i(t,.))\}}$ is compact in $\mathbb{R} \times L^1(0,+\infty)$.

To make a change of variable, using Volterra integral formulation (8.2.3) of the solution we have

$$i(t,a) = \exp\left(\int_0^a -\nu_I(r) dr\right) b(t-a),$$

where

$$b(t) = \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) da.$$

Set

$$u(t,a) := \exp\left(\int_0^a (\nu_I(r) - \nu_S) dr\right) i(t,a) = e^{-\nu_S a} b(t-a),$$

$$\widehat{l}(a) := \exp\left(-\int_0^a (\nu_I(r) - \nu_S) dr\right),$$

and

$$\widehat{\beta}(a) := \beta(a) \widehat{l}(a).$$

Then we have

$$i(t,a) = \widehat{l}(a) u(t,a)$$

and $(S(t), u(t,a))_{t\in\mathbb{R}}$ is a complete orbit of the following system

$$
\begin{cases}
\dfrac{dS(t)}{dt} = \gamma - \nu_S S(t) - \eta S(t) \int_0^{+\infty} \widehat{\beta}(a) u(t,a) da, \\
\dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\nu_I u(t,a) \\
u(t,0) = \eta S(t) \int_0^{+\infty} \widehat{\beta}(a) u(t,a) da, \\
S(0) = S_0 \geq 0, \\
u(0,.) = u_0 \in L_+^1 (0,+\infty).
\end{cases}
\tag{8.2.15}
$$

Moreover, by using (8.2.15) we deduce that

$$
\frac{d\left[S(t) + \int_0^{+\infty} u(t,a) da\right]}{dt} = \gamma - \nu_S \left[S(t) + \int_0^{+\infty} u(t,a) da\right], \tag{8.2.16}
$$

and since $t \to \left[S(t) + \int_0^{+\infty} u(t,a) da\right]$ is a bounded complete orbit of the above ordinary differential equation, we have

$$
\gamma = \nu_S \left[S(t) + \int_0^{+\infty} u(t,a) da\right], \; \forall t \in \mathbb{R}.
$$

For the sake of simplicity, we make the following assumption.

Assumption 8.2.11. Assume that

$$
\nu_I(a) = \nu_S, \forall a \geq 0, \text{ and } \gamma = \nu_S.
$$

We now prove the global stability of the endemic equilibrium.

Theorem 8.2.12. *Assume that $\mathscr{R}_0 > 1$ and Assumption 8.2.11 is satisfied. Then the equilibrium $(\bar{S}_E, \bar{\imath}_E)$ is globally asymptotically stable.*

Proof. Under the Assumption 8.2.11 system (8.2.1) becomes

$$
\begin{cases}
\dfrac{dS(t)}{dt} = \nu_S - \nu_S S(t) - \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) da, \\
\dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\nu_I i(t,a), \\
i(t,0) = \eta S(t) \int_0^{+\infty} \beta(a) i(t,a) da, \\
S(0) = S_0 \geq 0, \; i(0,.) = i_0 \in L_+^1 (0,+\infty).
\end{cases}
\tag{8.2.17}
$$

In this special case, the endemic equilibrium satisfies the following system of equations

$$
\begin{aligned}
0 &= \nu_S - \nu_S \bar{S}_E - \eta \bar{S}_E \int_0^{+\infty} \beta(a) \bar{\imath}_E(a) da \\
\bar{\imath}_E(a) &= e^{-\nu_S a} \bar{\imath}_E(0)
\end{aligned}
\tag{8.2.18}
$$

with

$$
1 = \eta \bar{S}_E \int_0^{+\infty} \beta(a) e^{-\nu_S a} da.
$$

Moreover, by Lemma 8.2.10 we can consider a complete orbit $\{(S(t), i(t,.))\}_{t\in\mathbb{R}}$ of system (8.2.17) satisfying

$$\varepsilon \leq S(t) \leq M, \ \ \varepsilon \leq \int_0^{+\infty} \beta(a) i(t,a) da \leq M, \ \ \forall t \in \mathbb{R}.$$

Moreover, $O = \overline{\cup_{t\in\mathbb{R}} \{(S(t), i(t,.))\}}$ is compact in $\mathbb{R} \times L^1(0,+\infty)$. Furthermore, we have

$$\frac{i(t,a)}{\bar{\imath}_E(a)} = \frac{b(t-a)}{\bar{\imath}_E(0)} = \frac{\eta S(t-a) \int_0^{+\infty} \beta(l) i(t-a,l) dl}{\bar{\imath}_E(0)}.$$

Thus,

$$\frac{\eta}{\bar{\imath}_E(0)} \varepsilon^2 \leq \frac{i(t,a)}{\bar{\imath}_E(a)} \leq \frac{\eta}{\bar{\imath}_E(0)} M^2.$$

To construct a Liapunov functional, let

$$g(x) = x - 1 - \ln x.$$

Note that $g'(x) = 1 - \dfrac{1}{x}$. Thus, g is decreasing on $(0,1]$ and increasing on $[1,\infty)$. The function g has only one extremum which is a global minimum at 1, satisfying $g(1) = 0$. We first define expressions $V_S(t)$ and $V_i(t)$, and calculate their derivatives. Then, we will analyze the Liapunov functional $V = V_S + V_i$. Let

$$V_S(t) = g\left(\frac{S(t)}{\bar{S}_E}\right).$$

Then

$$\begin{aligned}
\frac{dV_S}{dt} &= g'\left(\frac{S(t)}{\bar{S}_E}\right) \frac{1}{\bar{S}_E} \frac{dS}{dt} \\
&= \left(1 - \frac{\bar{S}_E}{S(t)}\right) \frac{1}{\bar{S}_E} \left[\nu_S - \nu_S S(t) - \textstyle\int_0^\infty \eta \beta(l) i(t,l) S(t) dl\right] \\
&= \left(1 - \frac{\bar{S}_E}{S(t)}\right) \frac{1}{\bar{S}_E} \left[\nu_S \left(\bar{S}_E - S(t)\right) + \textstyle\int_0^\infty \eta \beta(l) \left(\bar{\imath}_E(l) \bar{S}_E - i(t,l) S(t)\right) dl\right] \\
&= -\nu_S \frac{\left(S(t) - \bar{S}_E\right)^2}{S(t) \bar{S}_E} + \textstyle\int_0^\infty \eta \beta(l) \bar{\imath}_E(l) \left(1 - \dfrac{i(t,l)}{\bar{\imath}_E(l)} \dfrac{S(t)}{\bar{S}_E} - \dfrac{\bar{S}_E}{S(t)} + \dfrac{i(t,l)}{\bar{\imath}_E(l)}\right) dl.
\end{aligned} \tag{8.2.19}$$

Let

$$V_i(t) = \int_0^\infty \alpha(a) g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right) da,$$

where

$$\alpha(a) := \int_a^\infty \eta \beta(l) \bar{\imath}_E(l) dl. \tag{8.2.20}$$

Then

$$
\begin{aligned}
\frac{dV_i}{dt} &= \frac{d}{dt}\int_0^\infty \alpha(a) g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right) da\\
&= \frac{d}{dt}\int_0^\infty \alpha(a) g\left(\frac{b(t-a)}{\bar{\imath}_E(0)}\right) da\\
&= \frac{d}{dt}\int_{-\infty}^t \alpha(t-s) g\left(\frac{b(s)}{\bar{\imath}_E(0)}\right) ds\\
&= \alpha(0) g\left(\frac{b(t)}{\bar{\imath}_E(0)}\right) + \int_{-\infty}^t \alpha'(t-s) g\left(\frac{b(s)}{\bar{\imath}_E(0)}\right) da,
\end{aligned}
$$

and thus

$$
\frac{dV_i}{dt} = \alpha(0) g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) + \int_0^\infty \alpha'(a) g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right) da. \tag{8.2.21}
$$

Moreover, by the definition of α we have

$$
\alpha(0) g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) = \int_0^\infty \eta\beta(l)\bar{\imath}_E(l) g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) dl. \tag{8.2.22}
$$

Noting additionally that $\alpha'(a) = -\eta\beta(a)\bar{\imath}_E(a)$, we may combine equations (8.2.21) and (8.2.22) to get

$$
\frac{dV_i}{dt} = \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[g\left(\frac{i(t,0)}{\bar{\imath}_E(0)}\right) - g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\right] da.
$$

Filling in for the function g, we obtain

$$
\frac{dV_i}{dt} = \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{i(t,a)}{\bar{\imath}_E(a)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\right] da. \tag{8.2.23}
$$

Let

$$
V(t) = V_S(t) + V_i(t).
$$

Then by combining (8.2.19) and (8.2.23), we have

$$
\begin{aligned}
\frac{dV}{dt} = &-\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E}\\
&+\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\begin{bmatrix} 1 - \dfrac{i(t,a)}{\bar{\imath}_E(a)}\dfrac{S(t)}{\bar{S}_E} - \dfrac{\bar{S}_E}{S(t)} + \dfrac{i(t,0)}{\bar{\imath}_E(0)}\\ -\ln\dfrac{i(t,0)}{\bar{\imath}_E(0)} + \ln\dfrac{i(t,a)}{\bar{\imath}_E(a)} \end{bmatrix} da.
\end{aligned} \tag{8.2.24}
$$

Now we show that $\dfrac{dV}{dt}$ is non-positive. To do this, we demonstrate that two of the terms in (8.2.24) cancel out:

$$\begin{aligned}
&\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\right]da \\
&\quad = \frac{1}{\bar{S}_E}\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\bar{S}_E\,da\frac{i(t,0)}{\bar{\imath}_E(0)} - \frac{1}{\bar{S}_E}\int_0^\infty \eta\beta(a)i(t,a)S(t)\,da \\
&\quad = \frac{1}{\bar{S}_E}\bar{\imath}_E(0)\frac{i(t,0)}{\bar{\imath}_E(0)} \;-\; \frac{1}{\bar{S}_E}i(t,0) \\
&\quad = 0.
\end{aligned} \tag{8.2.25}$$

Using this to simplify equation (8.2.24) gives

$$\begin{aligned}
\frac{dV}{dt} &= -\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E} \\
&\qquad + \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[1 - \frac{\bar{S}_E}{S(t)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\right]da.
\end{aligned} \tag{8.2.26}$$

Noting that $\bar{\imath}_E(0)/i(t,0)$ is independent of a, we may multiply both sides of (8.2.25) by this quantity to obtain

$$\int_0^\infty \eta\beta(a)\bar{\imath}_E(a)\left[1 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right]da = 0. \tag{8.2.27}$$

We now add (8.2.27) to (8.2.26), and also add and subtract $\ln(S(t)/\bar{S}_E)$ to get

$$\frac{dV}{dt} = -\nu_S\frac{\left(S(t)-\bar{S}_E\right)^2}{S(t)\bar{S}_E} + \int_0^\infty \eta\beta(a)\bar{\imath}_E(a)C(a)\,da,$$

where

$$\begin{aligned}
C(a) &= 2 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)} - \frac{\bar{S}_E}{S(t)} - \ln\frac{i(t,0)}{\bar{\imath}_E(0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)} + \ln\frac{S(t)}{\bar{S}_E} - \ln\frac{S(t)}{\bar{S}_E} \\
&= \left(1 - \frac{\bar{S}_E}{S(t)} + \ln\frac{\bar{S}_E}{S(t)}\right) + \left(1 - \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)} + \ln\frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right) \\
&= -\left[g\left(\frac{\bar{S}_E}{S(t)}\right) + g\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\frac{S(t)}{\bar{S}_E}\frac{\bar{\imath}_E(0)}{i(t,0)}\right)\right] \\
&\le 0.
\end{aligned}$$

Thus, $\dfrac{dV}{dt} \le 0$ with equality if and only if

$$\frac{\bar{S}_E}{S(t)} = 1 \qquad \text{and} \qquad \frac{i(t,a)}{\bar{\imath}_E(a)}\frac{\bar{\imath}_E(0)}{i(t,0)} = 1. \tag{8.2.28}$$

Using (8.2.18), the second condition in (8.2.28) is equivalent to

$$i(t,a) = i(t,0)e^{-\nu_S a},\ \ \forall a \ge 0. \tag{8.2.29}$$

Next we look for the largest invariant set Q for which (8.2.28) holds. In Q, we must have $S(t) = \bar{S}_E$ for all t and so $\frac{dS}{dt} = 0$. Combining this with (8.2.29), we obtain

$$\begin{aligned}
0 &= \nu_S - \nu_S \bar{S}_E - \textstyle\int_0^\infty \eta\beta(a) i(t,a)\, da\, \bar{S}_E \\
&= \nu_S - \nu_S \bar{S}_E - \textstyle\int_0^\infty \eta\beta(a) i(t,0) e^{-\nu_S a}\, da\, \bar{S}_E \\
&= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \textstyle\int_0^\infty \eta\beta(a) \bar{\imath}_E(0) e^{-\nu_S a}\, da\, \bar{S}_E \\
&= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \textstyle\int_0^\infty \eta\beta(a) \bar{\imath}_E(a)\, da\, \bar{S}_E \\
&= \nu_S - \nu_S \bar{S}_E - \frac{i(t,0)}{\bar{\imath}_E(0)} \left(\nu_S - \nu_S \bar{S}_E\right) \\
&= \left(1 - \frac{i(t,0)}{\bar{\imath}_E(0)}\right) \left(\nu_S - \nu_S \bar{S}_E\right).
\end{aligned}$$

Since $\bar{S}_E$ is not equal to 1, we must have $i(t,0) = \bar{\imath}_E(0)$ for all t. Thus, the set Q consists of only the endemic equilibrium.

Assume that $\mathscr{A}_0$ is larger than $\{\bar{x}_E\}$. Then there exists $x \in \mathscr{A}_0 \setminus \{\bar{x}_E\}$, and we can find a complete orbit $\{u(t)\}_{t\in\mathbb{R}} \subset \mathscr{A}_0$ of U, passing through x at $t = 0$, with alpha limit set $\alpha(x)$. Since

$$u(0) = x \neq \bar{x}_E, \tag{8.2.30}$$

we deduce that $t \to V(u(t))$ is a non-increasing map. Thus, V is a constant functional on $\alpha(x)$. Since $\alpha(x)$ is invariant under U, it follows that

$$\alpha(x) = \{\bar{x}_E\}. \tag{8.2.31}$$

Recalling from Proposition 8.2.9 that the endemic equilibrium is locally asymptotically stable, equation (8.2.31) implies $x = \bar{x}_E$ which contradicts (8.2.30) and completes the proof. □

8.2.5 Numerical Examples

We present three examples to illustrate the infection-age model (8.2.1). In these examples the infection-age is used to track the period of incubation, the period of infectiousness, the appearance of symptoms, and the quarantine of infectives.

Example 8.2.13. In the first example we interpret infection-age as an exposed period (infected but not yet infectious) from $a = 0$ to $a = a_1$ and an infectious period from $a = a_1$ to $a = a_2$. The total numbers of exposed individuals $E(t)$ and infectious individuals $I(t)$ at time t are

$$E(t) = \int_0^{a_1} \imath(t,a)da, \qquad I(t) = \int_{a_1}^{a_2} \imath(t,a)da.$$

This interpretation of the model is typical of a disease such as influenza, in which there is an initial non-infectious period followed by a period of increasing then decreasing infectiousness. We investigate the role of quarantine in controlling an epidemic using infection-age to track quarantined individuals. Consider a population of initially γ/ν_S susceptible individuals with an on-going influx at a rate γ and efflux at a rate ν_S. These rates influence the extinction or endemicity of the epidemic; specifically, the continuing arrival of new susceptibles enables the disease to persist, which might otherwise extinguish.

Set $\gamma = 365$, $\nu_S = 1/365$ (time units are days). For a population of approximately 100,000 people these rates may be interpreted in terms of daily immigration and emigration of the population. Set $a_1 = 5$ and $a_2 = 21$. We use the form of the transmission function $\beta(a)$ in Fig. 8.2(A) (see Fig. 8.2):

$$\beta(a) = \begin{cases} 0.0 & \text{if } 0.0 \le a \le 5.0, \\ 0.66667(a-5.0)^2 e^{-.6(a-5.0)} & \text{if } a > 5.0. \end{cases}$$

Set the transmission rate $\eta = 1.5 \times 10^{-5}$ and $\nu_I(a) = \nu_Q(a) + \nu_H(a) + \nu_S$, where

$$\nu_Q(a) = \begin{cases} -\log(.95) & \text{if } 0.0 \le a \le 5.0, \\ 0.0 & \text{if } a > 5.0, \end{cases} \quad \nu_H(a) = \begin{cases} 0.0 & \text{if } 0.0 \le a \le 5.0, \\ -\log(.5) & \text{if } a > 5.0. \end{cases}$$

The function $\nu_Q(a)$ represents quarantine of exposed individuals at a rate of 5% per day and the function $\nu_H(a)$ represents hospitalized (or removed) infectious individuals at a rate of 50% per day. It is assumed that exposed individuals are asymptomatic (only asymptomatic individuals are quarantined) and only infectious individuals are symptomatic (symptomatic individuals are hospitalized or otherwise isolated from susceptibles). These assumptions were valid for the SARS epidemic in 2003, but may not hold for other influenza epidemics. In fact, in the 1918 influenza pandemic the infectious period preceded the symptomatic period by several days, resulting in much higher transmission. We assume that the initial susceptible population is $S(0) = \gamma/\nu_S = 133,225$ and initial age distribution of infectives (see Fig. 8.2) is $\iota(0,a) = 50.0(a+2.0)e^{-0.4(a+2.0)}, a \ge 0.0$. For these parameters $\mathscr{R}_0 < 1.0$ and the epidemic is extinguished in approximately one year (Fig. 8.3).

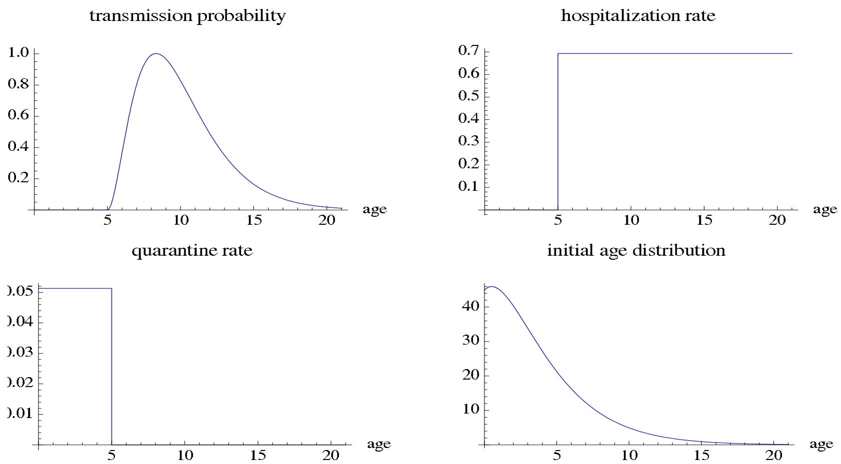

Fig. 8.2: The period of infectiousness begins at day 5 and lasts 16 days. The transmission probability peaks at 8.33333 days. Symptoms appear at day 5, which coincides with the beginning of the infectious period. Infected individuals are hospitalized (or otherwise removed) at a rate of 50% per day after day 5. Pre-symptomatic individuals are quarantined at a rate of 5% per day during the pre-symptomatic period. From the initial infection-age distribution we obtain $E(0) = 179.9$ and $I(0) = 72.5$.

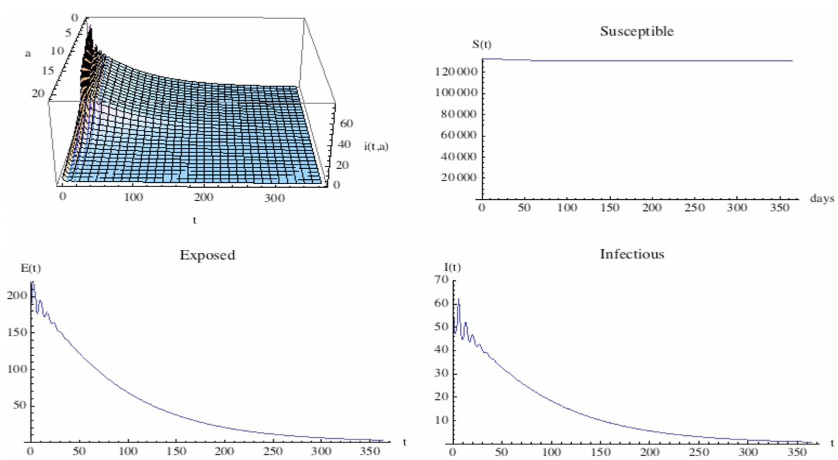

Fig. 8.3: With quarantine of asymptomatic individuals at a rate of 5% per day, the disease is extinguished and the susceptible population converges to the disease-free steady state $\bar{S}_F = \gamma/\nu_S = 133,225$, $\bar{I}_F = 0.0$. $\mathscr{R}_0 = 0.939$.

Example 8.2.14. In the second example we simulate Example 8.2.13 without quarantine measures implemented (that is, all parameters and initial conditions are as in Example 8.2.13 except that $\nu_Q(a) \equiv 0.0$). Without quarantine control the disease becomes endemic (Fig. 8.4). In this case $\mathscr{R}_0 > 1$ and the solutions converge to the endemic equilibrium. From Fig. 8.4 it is seen that the solutions oscillate as they converge to the disease equilibrium over a period of years. The on-going source γ of susceptibles allows the disease to persist albeit at a relatively low level. At equilibrium the population of susceptibles is significantly lower than the disease-free susceptible population.

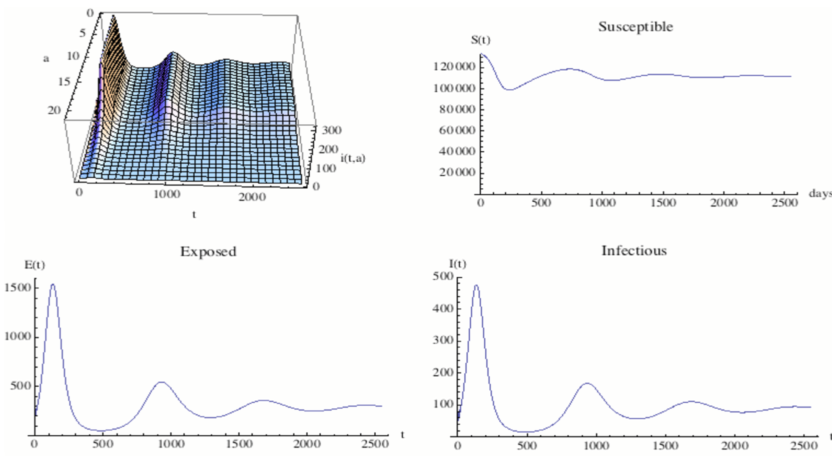

Fig. 8.4: Without quarantine implemented, the disease becomes endemic and the populations converge with damped oscillations to the disease steady state $\bar{S}_E = \frac{\gamma}{\mathscr{R}_0 \nu_S} = 102{,}480$, $\bar{E} = 369$, $\bar{I} = 120$, and $\bar{\iota}_E(a) = l_{\nu_I}(a)\bar{\iota}_E(0)$, $l_{\nu_I}(a) = \exp(-\int_0^a \nu_I(l)\,dl)$, $\bar{\iota}_E(0) = \gamma - \nu_S \bar{S}_E = 85.8$. $\mathscr{R}_0 = 1.26$.

Example 8.2.15. In the third example we assume that the infectious period and the symptomatic period are not coincident, as in the above two examples. In this case the severity of the epidemic may be much greater, since the transmission potential of some infectious individuals will not be known during some part of their period of infectiousness. We illustrate this case in an example in which the infectious period overlaps the asymptomatic period by one day. All parameters and initial conditions are as in Example 8.2.13, except that symptoms first appear on day 6, which means that hospitalization (or removal) of infectious individuals does not begin until 1 day after the period of infectiousness begins. It is also assumed that quarantine of infected individuals does not end until day 6 (Fig. 8.5). In this scenario the epidemic, even with quarantine measures implemented as in Example 8.2.13, becomes endemic. The epidemic populations exhibit extreme oscillations with the infected populations attaining very low values and the population converging to the steady state (Fig. 8.6).

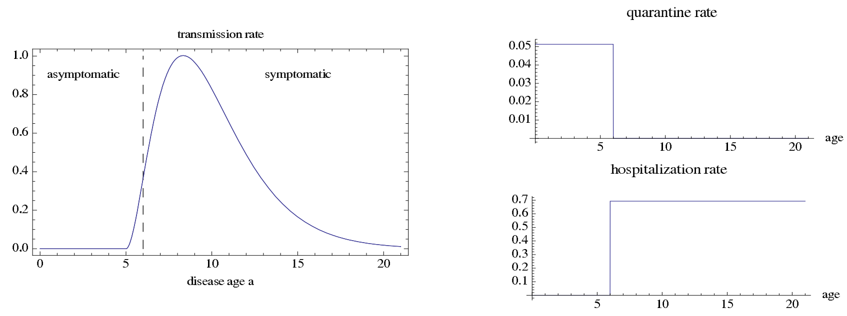

Fig. 8.5: The period of infectiousness overlaps by 1 day with the asymptomatic period. Quarantine of asymptomatic individuals ends on day 6. Hospitalization of symptomatic individuls begins on day 6.

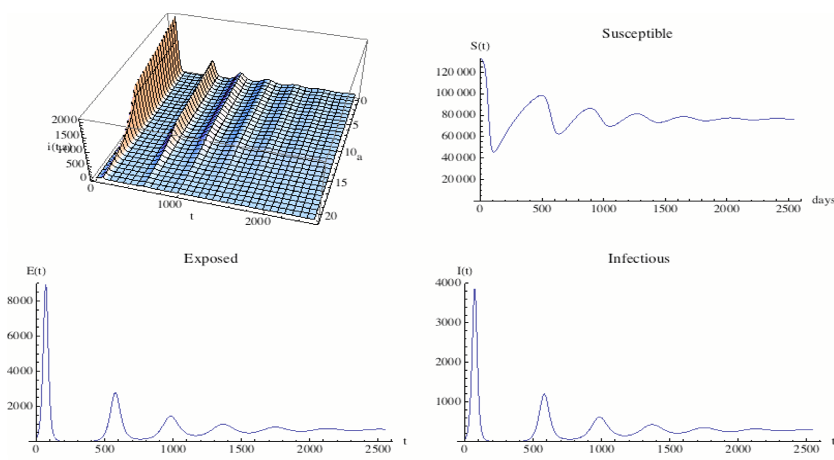

Fig. 8.6: When the infectious period overlaps the asymptomatic period, even with quarantine implemented, the disease becomes endemic and the populations converge with damped oscillations to the disease steady state $\bar{S}_E = \frac{\gamma}{\mathscr{R}_0 \nu_S} = 74,834$, $\bar{E} = 690$, $\bar{I} = 303$, and $\bar{\imath}_E(a) = l_{\nu_I}(a)\bar{\imath}_E(0)$, $l_{\nu_I}(a) = \exp(-\int_0^a \nu_I(l)\,dl)$, $\bar{\imath}_E(0) = \gamma - \nu_S \bar{S}_E = 160.0$. $\mathscr{R}_0 = 1.78$.

8.3 A Scalar Age-structured Model

In this section we focus on a scalar age-structured model. Let $u(t,a)$ denote the density of a population at time t with age a. Consider the following age structured model

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu u(t,a),\ a \in (0,+\infty), \\ u(t,0) = \alpha h \left(\int_0^{+\infty} \gamma(a) u(t,a) da \right), \\ u(0,.) = \varphi \in L^1_+ \left((0,+\infty); \mathbb{R} \right), \end{cases} \tag{8.3.1}$$

where $\mu > 0$ is the mortality rate of the population, the function $h(\cdot)$ describes the fertility of the population, $\alpha \geq 0$ is considered as a bifurcation parameter.

We first make an assumption on the functions $h(\cdot)$ and $\gamma(\cdot)$.

Assumption 8.3.1. Assume that $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = xe^{-\beta x},\ \beta > 0,\ \ \forall x \in \mathbb{R},$$

and $\gamma \in L^\infty_+ \left((0,+\infty), \mathbb{R} \right)$ with

$$\int_0^{+\infty} \gamma(a) e^{-\mu a} da = 1.$$

8.3.1 Existence of Integrated Solutions

Set

$$Y = \mathbb{R} \times L^1 \left((0,+\infty); \mathbb{R} \right), \quad Y_0 = \{0\} \times L^1 \left((0,+\infty); \mathbb{R} \right),$$
$$Y_+ = \mathbb{R}_+ \times L^1 \left((0,+\infty); \mathbb{R}_+ \right),\ Y_{0+} = Y_0 \cap Y_+.$$

Assume that Y is endowed with the product norm

$$\|x\| = |\alpha| + \|\varphi\|_{L^1((0,+\infty);\mathbb{R})},\ \ \forall x = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in Y.$$

Denote by

$$Y^{\mathbb{C}} = Y + iY \text{ and } Y_0^{\mathbb{C}} = Y_0 + iY_0$$

the complexified Banach space of Y and Y_0, respectively. We can identify $Y^{\mathbb{C}}$ to

$$Y = \mathbb{C} \times L^1 \left((0,+\infty); \mathbb{C} \right)$$

endowed with the product norm

$$\|x\| = |\alpha| + \|\varphi\|_{L^1((0,+\infty);\mathbb{C})},\ \ \forall x = \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \in Y^{\mathbb{C}}.$$

From now on, for each $x \in Y$, we denote by

$$\bar{x} = \begin{pmatrix} \overline{\alpha} \\ \overline{\varphi} \end{pmatrix},\ \ \mathrm{Re}(x) = \frac{x + \bar{x}}{2}, \text{ and } \ \mathrm{Im}(x) = \frac{x - \bar{x}}{2}.$$

Consider the linear operator $A : D(A) \subset Y \to Y$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) = \{0\} \times W^{1,1}((0,+\infty);\mathbb{R}).$$

Moreover, for each $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\mu$, we have $\lambda \in \rho(A)$ and

$$\begin{aligned} &(\lambda I - A)^{-1}\begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(a) = e^{-(\lambda+\mu)a}\alpha + \int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds. \end{aligned}$$

Note that

$$\lambda \in \rho(A) \Leftrightarrow \overline{\lambda} \in \rho(A)$$

and

$$(\lambda I - A)^{-1} x = \overline{\left(\overline{\lambda} I - A\right)^{-1} \overline{x}}, \ \forall x \in Y, \ \forall \lambda \in \rho(A).$$

It is well known that A is a Hille-Yosida operator. Moreover, A_0 is the part of A in Y_0 generates a C_0-semigroup of bounded linear operators $\left\{T_{A_0}(t)\right\}_{t\geq 0}$, which is defined by

$$T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{A_0}(t)\varphi \end{pmatrix},$$

where

$$\widehat{T}_{A_0}(t)(\varphi)(a) = \begin{cases} e^{-\mu t}\varphi(a-t) & \text{if } a \geq t, \\ 0 & \text{if } a \leq t. \end{cases}$$

$\{S_A(t)\}_{t\geq 0}$ is the integrated semigroup generated by A and is defined by

$$S_A(t)\begin{pmatrix} \alpha \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ L(t)\alpha + \int_0^t \widehat{T}_{A_0}(s)\varphi ds \end{pmatrix},$$

where

$$L(t)(\alpha)(a) = \begin{cases} 0 & \text{if } a \geq t, \\ e^{-\mu a}\alpha & \text{if } a \leq t. \end{cases}$$

Define $H : Y_0 \to Y$ and $H_1 : Y_0 \to \mathbb{R}$ by

$$H\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} H_1\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ 0 \end{pmatrix}, \ H_1\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = h\left(\int_0^{+\infty} \gamma(a)\varphi(a)da\right).$$

Then, by identifying $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, the problem (8.3.1) can be considered as the following Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + \alpha H(v(t)) \text{ for } t \geq 0, \;\; v(t) = y \in Y_{0+}. \tag{8.3.2}$$

Since h is Lipschitz continuous on $[0,+\infty)$, the following lemma is a consequence of the results in Proposition 5.4.1.

Lemma 8.3.2. *Let Assumption 8.3.1 be satisfied. Then for each $\alpha \geq 0$, there exists a family of continuous maps $\{U_\alpha(t)\}_{t\geq 0}$ on Y_{0+} such that for each $y \in Y_{0+}$, the map $t \to U_\alpha(t)y$ is the unique integrated solution of (8.3.2), that is,*

$$U_\alpha(t)y = y + A\int_0^t U_\alpha(s)y ds + \int_0^t \alpha H(U_\alpha(l)y)dl, \; \forall t \geq 0,$$

or equivalently

$$U_\alpha(t)y = T_{A_0}(t)y + \frac{d}{dt}\left(S_A * \alpha H(U_\alpha(.)y)\right)(t), \; \forall t \geq 0.$$

Moreover, $\{U_\alpha(t)\}_{t\geq 0}$ is a continuous semiflow, that is, $U(0) = Id$,

$$U_\alpha(t)U_\alpha(s) = U_\alpha(t+s), \forall t,s \geq 0,$$

and the map $(t,x) \to U_\alpha(t)x$ is continuous from $[0,+\infty)\times Y_{0+}$ into Y_{0+}.

8.3.2 Spectral Analysis

We recall that $\overline{y} \in Y_{0+}$ is an equilibrium of $\{U_\alpha(t)\}_{t\geq 0}$ if and only if

$$\overline{y} \in D(A) \text{ and } A\overline{y} + \alpha H(\overline{y}) = 0.$$

Here if $\alpha > 1$, equation (8.3.1) has two non-negative equilibria given by

$$\overline{v} = \begin{pmatrix} 0 \\ \overline{u} \end{pmatrix} \text{ with } \overline{u}(a) = Ce^{-\mu a},$$

where C is a solution of

$$C = \alpha h\left(C\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right) \text{ with } C \geq 0.$$

But by Assumption 8.3.1 we have $\int_0^{+\infty}\gamma(a)e^{-\mu a}da = 1$, so

$$C = 0 \text{ or } C = \overline{C}(\alpha) := \beta^{-1}\ln(\alpha).$$

From now on we set

$$\overline{v}_\alpha = \begin{pmatrix} 0 \\ \overline{u}_\alpha \end{pmatrix} \text{ with } \;\overline{u}_\alpha(a) = \overline{C}(\alpha)\,e^{-\mu a}, \;\forall \alpha > 1. \tag{8.3.3}$$

We have

$$\alpha H(\overline{v}_\alpha) = \begin{pmatrix} \overline{C}(\alpha) \\ 0 \end{pmatrix},$$

$$\alpha DH(\psi)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \alpha h'\left(\int_0^{+\infty}\gamma(a)\psi(a)da\right)\int_0^{+\infty}\gamma(a)\varphi(a)da \\ 0 \end{pmatrix},$$

so

$$\alpha DH(\overline{v}_\alpha)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \eta(\alpha)\int_0^{+\infty}\gamma(a)\varphi(a)da \\ 0 \end{pmatrix},$$

where

$$\begin{aligned}\eta(\alpha) &= \alpha h'\left(\int_0^{+\infty}\gamma(a)e^{-\mu a}da\overline{C}(\alpha)\right) \\ &= \alpha\left(1-\beta\overline{C}(\alpha)\right)\exp\left(-\beta\overline{C}(\alpha)\right) \\ &= 1-\ln(\alpha).\end{aligned}$$

We also have for $k \geq 1$ that

$$\alpha D^k H(\psi)\left(\begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix},\ldots,\begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}\right) = \begin{pmatrix} \alpha h^{(k)}\left(\int_0^{+\infty}\gamma(a)\psi(a)da\right)\prod_{i=1}^{k}\int_0^{+\infty}\gamma(a)\varphi_i(a)da \\ 0 \end{pmatrix}.$$

The characteristic equation of the problem is

$$1 = \eta(\alpha)\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}da \text{ with } \lambda\in\mathbb{C} \text{ and } \operatorname{Re}(\lambda) > -\mu. \tag{8.3.4}$$

Set

$$\Omega = \{\lambda\in\mathbb{C} : \operatorname{Re}(\lambda) > -\mu\}$$

and consider the map $\Delta : \mathbb{R}\times\Omega\to\mathbb{C}$ defined by

$$\Delta(\alpha,\lambda) = 1-\eta(\alpha)\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}da. \tag{8.3.5}$$

One can prove that Δ is holomorphic. Moreover, for each $k \geq 1$ and each $\lambda\in\Omega$, we have

$$\frac{d^k\Delta(\alpha,\lambda)}{d\lambda^k} = (-1)^{k+1}\eta(\alpha)\int_0^{+\infty}a^k\gamma(a)e^{-(\lambda+\mu)a}da.$$

To simplify the notation, we set

$$B_\alpha x = Ax+\alpha DH(\overline{v}_\alpha)x \text{ with } D(B_\alpha) = D(A)$$

and identify B_α to

$$B_\alpha^{\mathbb{C}}(x+iy) = B_\alpha^{\mathbb{C}}x+iB_\alpha^{\mathbb{C}}y,\ \forall (x+iy)\in D\left(B_\alpha^{\mathbb{C}}\right) := D(A)+iD(A).$$

Note that the part of B_α in $D(B_\alpha)$ is the generator of the linearized equation at $\overline{v}_\alpha$.

Lemma 8.3.3. *Let Assumption 8.3.1 be satisfied. Then the linear operator $B_\alpha : D(A) \subset Y \to Y$ is a Hille-Yosida operator and*

$$\omega_{\text{ess}}\left((B_\alpha)_0\right) \leq -\mu.$$

Proof. Since $\alpha DH(\overline{v}_\alpha)$ is a bounded linear operator, it follows that $B_\alpha^{\mathbb{C}}$ is a Hille-Yosida operator. Moreover, by applying Theorem 3 in Thieme [331] (or Theorem 1.2 in Ducrot et al. [110] or Theorem 4.7.3) to $B_\alpha + \varepsilon I$ for each $\varepsilon \in (0,\mu)$, we deduce that $\omega_{\text{ess}}\left((B_\alpha)_0\right) \leq -\mu$. □

Lemma 8.3.4. *Let Assumption 8.3.1 be satisfied. Then the linear operator $B_\alpha : D(A) \subset Y \to Y$ is a Hille-Yosida operator and we have the following:*

(i) $\sigma\left(B_\alpha^{\mathbb{C}}\right) \cap \Omega = \{\lambda \in \Omega : \Delta(\alpha,\lambda) = 0\}$;

(ii) *If $\lambda \in \Omega \cap \rho\left(B_\alpha^{\mathbb{C}}\right)$, we have the following explicit formula for the resolvent*

$$\begin{aligned}
&\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} \begin{pmatrix} \delta \\ \psi \end{pmatrix} \\
\Leftrightarrow\ &\varphi(a) = \int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds \\
&\qquad + \Delta(\alpha,\lambda)^{-1}\left[\delta + \eta(\alpha)\int_0^{+\infty}\chi_\lambda(s)\psi(s)ds\right]e^{-(\lambda+\mu)a},
\end{aligned} \tag{8.3.6}$$

where

$$\chi_\lambda(s) = \int_s^{+\infty}\gamma(l)e^{-(\lambda+\mu)(l-s)}dl, \ \forall s \geq 0.$$

Proof. Assume that $\lambda \in \Omega$ and $\Delta(\alpha,\lambda) \neq 0$. Then we have

$$\begin{aligned}
&\left(\lambda I - B_\alpha^{\mathbb{C}}\right)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \delta \\ \psi \end{pmatrix} \\
&\Leftrightarrow (\lambda I - A)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \delta \\ \psi \end{pmatrix} + \alpha DH(\overline{v}_\alpha)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\
&\Leftrightarrow \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = (\lambda I - A)^{-1}\begin{pmatrix} \delta \\ \psi \end{pmatrix} + (\lambda I - A)^{-1}\alpha DH(\overline{v}_\alpha)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\
&\Leftrightarrow \varphi(a) = e^{-(\lambda+\mu)a}\delta + \int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds \\
&\qquad + e^{-(\lambda+\mu)a}\eta(\alpha)\int_0^{+\infty}\gamma(a)\varphi(a)da.
\end{aligned}$$

Thus

$$\Delta(\alpha,\lambda)\int_0^{+\infty}\gamma(a)\varphi(a)da = \int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}\delta + \int_0^{+\infty}\gamma(a)\int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)dsda,$$

so

$$\varphi(a) = e^{-(\lambda+\mu)a}\left[1+\eta(\alpha)\Delta(\alpha,\lambda)^{-1}\int_0^{+\infty}\gamma(l)e^{-(\lambda+\mu)l}dl\right]\delta$$
$$+\int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds$$
$$+\eta(\alpha)e^{-(\lambda+\mu)a}\Delta(\alpha,\lambda)^{-1}\int_0^{+\infty}\gamma(l)\int_0^l e^{-(\lambda+\mu)(l-s)}\psi(s)dsdl.$$

But we have

$$1+\eta(\alpha)\Delta(\alpha,\lambda)^{-1}\int_0^{+\infty}\gamma(a)e^{-(\lambda+\mu)a}=\Delta(\alpha,\lambda)^{-1}$$

and

$$\int_0^{+\infty}\gamma(l)\int_0^l e^{-(\lambda+\mu)(l-s)}\psi(s)dsdl=\int_0^{+\infty}\int_s^{+\infty}\gamma(l)e^{-(\lambda+\mu)(l-s)}dl\psi(s)ds.$$

Hence (ii) follows. We conclude that

$$\{\lambda\in\Omega:\Delta(\alpha,\lambda)\neq 0\}\subset\rho\left(\lambda I-B_\alpha^{\mathbb{C}}\right)\cap\Omega,$$

which implies that

$$\sigma\left(\lambda I-B_\alpha^{\mathbb{C}}\right)\cap\Omega\subset\{\lambda\in\Omega:\Delta(\alpha,\lambda)=0\}.$$

Assume that $\lambda\in\Omega$ is given such that $\Delta(\alpha,\lambda)=0$. Then for $\varphi(.)=e^{-(\lambda+\mu).}$ we have

$$B_\alpha^{\mathbb{C}}\begin{pmatrix}0\\ \varphi\end{pmatrix}=\lambda\begin{pmatrix}0\\ \varphi\end{pmatrix},$$

so $\lambda I-B_\alpha^{\mathbb{C}}$ is not invertible. We deduce that

$$\{\lambda\in\Omega:\Delta(\alpha,\lambda)=0\}\subset\sigma\left(\lambda I-B_\alpha^{\mathbb{C}}\right)\cap\Omega,$$

and (i) follows. $\square$

The following lemma is well known (see, for example, Dolbeault [108, Theorem 2.1.2, p. 43].

Lemma 8.3.5. *Let f be an holomorphic map from an open connected subset $\Omega\subset\mathbb{C}$ and let $z_0\in\mathbb{C}$. Then the following assertions are equivalent:*

(i) $f=0$ *on* Ω;
(ii) f *is null in a neighborhood of* z_0;
(iii) *For each* $k\in\mathbb{N}, f^{(k)}(z_0)=0$.

Lemma 8.3.6. *Let Assumption 8.3.1 be satisfied. Then we have the following:*

(i) *If* $\lambda_0 \in \sigma\left(B_\alpha^{\mathbb{C}}\right) \cap \Omega$, *then* λ_0 *is isolated in* $\sigma\left(B_\alpha^{\mathbb{C}}\right)$;

(ii) *If* $\lambda_0 \in \sigma\left(B_\alpha^{\mathbb{C}}\right) \cap \Omega$ *and if* $k \geq 1$ *is the smallest integer such that* $\dfrac{d^k\Delta(\alpha,\lambda_0)}{d\lambda^k} \neq 0$, *then* λ_0 *a pole of order* k *of* $\left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1}$. *Moreover, if* $k=1$, *then* λ_0 *is a simple isolated eigenvalue of* $B_\alpha^{\mathbb{C}}$ *and the projector on the eigenspace associated to* λ_0 *is defined by*

$$\widehat{\Pi}_{\lambda_0}\begin{pmatrix}\delta\\ \psi\end{pmatrix} = \begin{pmatrix} 0 \\ \frac{d\Delta(\alpha,\lambda_0)}{d\lambda}^{-1}\left[\delta + \eta(\alpha)\int_0^{+\infty}\chi_{\lambda_0}(s)\psi(s)ds\right]e^{-(\lambda_0+\mu).}\end{pmatrix};$$

(iii) *For* $\forall x \in Y^{\mathbb{C}}$,

$$\overline{\widehat{\Pi}_{\lambda_0}x} = \widehat{\Pi}_{\overline{\lambda_0}}\overline{x}.$$

Proof. Since Ω is open and connected, we can apply Lemma 8.3.5 to Δ, and since for each $\lambda > 0$ large enough $\Delta(\lambda) > 0$, we have that for each $\lambda \in \Omega$, there exists $m \geq 0$ such that $\frac{d^m\Delta(\alpha,\lambda)}{d\lambda^m} \neq 0$. Moreover, for each $\lambda_0 \in \Omega$,

$$\Delta(\alpha,\lambda) = \sum_{k\geq 0}\frac{(\lambda-\lambda_0)^k}{k!}\frac{d^k\Delta(\alpha,\lambda_0)}{d\lambda^k}$$

whenever $|\lambda - \lambda_0|$ is small enough. It follows that each root of Δ is isolated. Moreover, assume that there exists $\lambda_0 \in \Omega$ such that $\Delta(\alpha,\lambda_0) = 0$. Let $m_0 \geq 1$ be the smallest integer such that $\frac{d^{m_0}\Delta(\alpha,\lambda_0)}{d\lambda^{m_0}} \neq 0$. Then we have

$$\Delta(\alpha,\lambda) = (\lambda-\lambda_0)^{m_0}g(\lambda)$$

with

$$g(\lambda) = \sum_{k=m_0}^{\infty}\frac{(\lambda-\lambda_0)^{k-m_0}}{k!}\frac{d^k\Delta(\alpha,\lambda_0)}{d\lambda^k}$$

whenever $|\lambda-\lambda_0|$ is small enough. So the multiplicity of λ_0 is k. Now by using Lemma 8.3.4 we deduce that if $\lambda_0 \in \sigma\left(B_\alpha^{\mathbb{C}}\right) \cap \Omega$, then λ_0 is isolated in $\sigma\left(B_\alpha^{\mathbb{C}}\right)$. Moreover, by using (8.3.6) we have for $k \geq 1$ that

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^k\left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\\
&= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^k\Delta(\alpha,\lambda)^{-1}\left[\delta+\eta(\alpha)\int_0^{+\infty}\chi_\lambda(s)\psi(s)ds\right]\begin{pmatrix}0\\ e^{-(\lambda+\mu).}\end{pmatrix}\\
&= \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^{k-m_0}\frac{1}{g(\lambda)}\left[\delta+\eta(\alpha)\int_0^{+\infty}\chi_\lambda(s)\psi(s)ds\right]\begin{pmatrix}0\\ e^{-(\lambda+\mu).}\end{pmatrix},
\end{aligned}$$

so

$$\lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^k\left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1}\begin{pmatrix}0\\ \psi\end{pmatrix} = 0 \text{ if } k > m_0. \tag{8.3.7}$$

But λ_0 is isolated, we have

$$\left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} = \sum_{k=-\infty}^{\infty} (\lambda - \lambda_0)^k D_k,$$

where

$$D_k = \frac{1}{2\pi i} \int_{S_{\mathbb{C}}(\lambda_0,\varepsilon)^+} (\lambda - \lambda_0)^{-k-1} \left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} d\lambda \tag{8.3.8}$$

for $\varepsilon > 0$ small enough and each $k \in \mathbb{Z}$. By combining (8.3.7) and (8.3.8), we obtain when $\varepsilon \to 0$ that

$$D_{-k} = 0 \text{ for each } k \geq m_0 + 2.$$

It follows that λ_0 is a pole of the resolvent and

$$\left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} = \sum_{k=-m_0-1}^{\infty} (\lambda - \lambda_0)^k D_k.$$

Noticing that

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{m_0+1} \left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} = D_{-m_0-1}$$

and using (8.3.7) once more, we deduce that $D_{-m_0-1} = 0$. Finally, we have

$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{m_0} \left(\lambda I - B_\alpha^{\mathbb{C}}\right)^{-1} = D_{-m_0}$$

and

$$D_{-m_0} \begin{pmatrix} \delta \\ \psi \end{pmatrix} = \frac{1}{g(\lambda_0)} \left[\delta + \eta(\alpha) \int_0^{+\infty} \chi_{\lambda_0}(s) \psi(s) ds\right] \begin{pmatrix} 0 \\ e^{-(\lambda_0+\mu)\cdot} \end{pmatrix}.$$

Therefore, λ_0 is a pole of order $m_0 \geq 1$. □

8.3.3 Hopf Bifurcation

To study Hopf bifurcation in equation (8.3.1), we make the following assumption.

Assumption 8.3.7. Assume that $\alpha^* > 1$ and $\theta^* > 0$ such that $i\theta^*$ *and* $-i\theta^*$ are simple eigenvalues of B_{α^*} and

$$\sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(B_{\alpha^*}) \setminus \{i\theta^*, -i\theta^*\}\} < 0.$$

Under Assumption 8.3.7 we have

$$\overline{\frac{d\Delta(\alpha^*, -i\theta^*)}{d\lambda}} = \frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda} \neq 0.$$

Moreover, by using assertion (iii) in Lemma 8.3.6, we can define $\widehat{\Pi}_c : Y \to Y$ as

$$\widehat{\Pi}_c \begin{pmatrix} \delta \\ \varphi \end{pmatrix} = \widehat{\Pi}_{i\theta^*} \begin{pmatrix} \delta \\ \varphi \end{pmatrix} + \widehat{\Pi}_{-i\theta^*} \begin{pmatrix} \delta \\ \varphi \end{pmatrix}, \ \forall \begin{pmatrix} \delta \\ \varphi \end{pmatrix} \in Y.$$

By using Theorem 4.5.8 and Lemma 4.5.2, we deduce the following result.

Lemma 8.3.8. *Let Assumptions 8.3.1 and 8.3.7 be satisfied. Then*

$$\sigma\left(B_{\alpha^*}\mid_{\widehat{\Pi}_c(Y)}\right) = \{i\theta^*, -i\theta^*\}, \ \sigma\left(B_{\alpha^*}\mid_{\left(I-\widehat{\Pi}_c\right)(Y)}\right) = \sigma(B_{\alpha^*}) \setminus \{i\theta^*, -i\theta^*\},$$

and

$$\omega_0\left(B_{\alpha^*}\mid_{\left(I-\widehat{\Pi}_c\right)(Y)}\right) < 0.$$

We have

$$\begin{aligned} \widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{bmatrix} 0 \\ \frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda}^{-1} e^{-(i\theta^*+\mu).} + \frac{d\Delta(\alpha^*, -i\theta^*)}{d\lambda}^{-1} e^{-(-i\theta^*+\mu).} \end{bmatrix} \\ &= \left|\frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda}\right|^{-2} \begin{bmatrix} 0 \\ \operatorname{Re}(\Delta(\alpha^*, i\theta^*))\widehat{e}_1 + \operatorname{Im}(\Delta(\alpha^*, i\theta^*))\widehat{e}_2 \end{bmatrix} \end{aligned}$$

with

$$\widehat{e}_1 = \left[e^{-(i\theta^*+\mu).} + e^{-(-i\theta^*+\mu).}\right], \ \widehat{e}_2 = \frac{\left(e^{-(i\theta^*+\mu).} - e^{-(-i\theta^*+\mu).}\right)}{i}.$$

Set

$$\widehat{\Pi}_s := \left(I - \widehat{\Pi}_c\right).$$

Then we have

$$\begin{aligned} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \left(I - \widehat{\Pi}_c\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -\frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda}^{-1} e^{-(i\theta^*+\mu).} - \frac{d\Delta(\alpha^*, -i\theta^*)}{d\lambda}^{-1} e^{-(-i\theta^*+\mu).} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -\left|\frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda}\right|^{-2} [\operatorname{Re}(\Delta(\alpha^*, i\theta^*))\widehat{e}_1 + \operatorname{Im}(\Delta(\alpha^*, i\theta^*))\widehat{e}_2] \end{pmatrix}. \end{aligned}$$

In order to compute the second derivative of the center manifold at 0, we need the following lemma.

Lemma 8.3.9. *Let Assumptions 8.3.1 and 8.3.7 be satisfied. Then for each* $\lambda \in i\mathbb{R} \setminus \{-i\theta^*, i\theta^*\}$,

$$\left(\lambda I - B_{\alpha^*}^{\mathbb{C}}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1} \frac{e^{-(i\theta^*+\mu).}}{\lambda - i\theta^*} - \frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1} \frac{e^{-(-i\theta^*+\mu).}}{\lambda + i\theta^*} + \Delta(\alpha^*,\lambda)^{-1} e^{-(\lambda+\mu).} \end{pmatrix}$$

Moreover, if $\lambda = i\theta^*$, we have

$$\left(i\theta^* I - B_{\alpha^*}^{\mathbb{C}} \mid_{\widehat{\Pi}_s(Y)}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1} \frac{e^{-(-i\theta^*+\mu).}}{2i\theta^*} + \frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-2} \left[-\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda} \cdot - \frac{1}{2} \frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2} \right] e^{-(i\theta^*+\mu).} \end{pmatrix}$$

and if $\lambda = -i\theta^*$, we have

$$\left(-i\theta^* I - B_{\alpha^*}^{\mathbb{C}} \mid_{\widehat{\Pi}_s(Y)}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1} \frac{e^{-(i\theta^*+\mu).}}{-2i\theta^*} + \frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-2} \left[-\frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda} \cdot - \frac{1}{2} \frac{d^2\Delta(\alpha^*,-i\theta^*)}{d\lambda^2} \right] e^{-(-i\theta^*+\mu).} \end{pmatrix}.$$

Proof. For each $\lambda \in \rho\left(B_{\alpha^*}^{\mathbb{C}}\right)$, we have

$$\left(\lambda I - B_{\alpha^*}^{\mathbb{C}}\right)^{-1} \begin{pmatrix} 0 \\ e^{-(\pm i\theta^*+\mu).} \end{pmatrix} = (\lambda \pm i\theta^*)^{-1} \begin{pmatrix} 0 \\ e^{-(\pm i\theta^*+\mu).} \end{pmatrix}.$$

Hence,

$$\left(\lambda I - B_{\alpha^*}^{\mathbb{C}} \mid_{\widehat{\Pi}_s(Y)}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\lambda I - B_{\alpha^*}^{\mathbb{C}}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1} \frac{e^{-(i\theta^*+\mu).}}{\lambda - i\theta^*} - \frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1} \frac{e^{-(-i\theta^*+\mu).}}{\lambda + i\theta^*} + \Delta(\alpha^*,\lambda)^{-1} e^{-(\lambda+\mu).} \end{pmatrix}.$$

Thus,

$$\left(0I - B_{\alpha^*}^{\mathbb{C}} \mid_{\widehat{\Pi}_s(Y)}\right)^{-1} \widehat{\Pi}_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1} \frac{e^{-(i\theta^*+\mu).}}{-i\theta^*} - \frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1} \frac{e^{-(-i\theta^*+\mu).}}{i\theta^*} + \Delta(\alpha^*,0)^{-1} e^{-\mu.} \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \left|\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda} i\theta^*\right|^2 \left[\operatorname{Re}\left(\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda} i\theta^*\right) e_1 + \operatorname{Im}\left(\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda} i\theta^*\right) e_2 \right] + \Delta(\alpha^*,0)^{-1} e^{-\mu.} \end{pmatrix}.$$

Moreover, we have

$$\left(i\theta^* I - B^{\mathbb{C}}_{\alpha^*}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1}\widehat{\Pi}_s\begin{pmatrix}1\\0\end{pmatrix} = \lim_{\substack{\lambda\to i\theta^*\\ \text{with } \lambda\in\rho\left(B^{\mathbb{C}}_{\alpha}\right)}}\left(\lambda I - B^{\mathbb{C}}_{\alpha^*}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1}\widehat{\Pi}_s\begin{pmatrix}1\\0\end{pmatrix},$$

so

$$\begin{aligned}&\left(i\theta^* I - B^{\mathbb{C}}_{\alpha^*}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1}\widehat{\Pi}_s\begin{pmatrix}1\\0\end{pmatrix}\\ &= \lim_{\substack{\lambda\to i\theta^*\\ \text{with } \lambda\in\rho\left(B^{\mathbb{C}}_{\alpha}\right)}}\begin{pmatrix}0\\ -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1}\frac{e^{-(i\theta^*+\mu)\cdot}}{\lambda-i\theta^*} - \frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1}\frac{e^{-(-i\theta^*+\mu)\cdot}}{\lambda+i\theta^*} + \Delta(\alpha^*,\lambda)^{-1}e^{-(\lambda+\mu)\cdot}\end{pmatrix}.\end{aligned}$$

Notice that

$$\begin{aligned}&-\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-1}\frac{e^{-(i\theta^*+\mu)\cdot}}{\lambda-i\theta^*} + \Delta(\alpha^*,\lambda)^{-1}e^{-(\lambda+\mu)\cdot}\\ &= \frac{(\lambda-i\theta^*)^2}{\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}(\lambda-i\theta^*)\Delta(\alpha^*,\lambda)}\frac{\left[-\Delta(\alpha^*,\lambda)e^{-(i\theta^*+\mu)\cdot} + (\lambda-i\theta^*)\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}e^{-(\lambda+\mu)\cdot}\right]}{(\lambda-i\theta^*)^2}\end{aligned}$$

and

$$\frac{(\lambda-i\theta^*)^2}{\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}(\lambda-i\theta^*)\Delta(\alpha^*,\lambda)} = \frac{1}{\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}\frac{\Delta(\alpha^*,\lambda)}{\lambda-i\theta^*}} \to \frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-2} \text{ as } \lambda\to i\theta^*.$$

We have

$$\Delta(\alpha,\lambda) = (\lambda-i\theta^*)\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda} + \frac{(\lambda-i\theta^*)^2}{2}\frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2} + (\lambda-i\theta^*)^3 g(\lambda-i\theta^*)$$

with $g(0) = \frac{1}{3!}\frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2}$. Therefore,

$$\begin{aligned}&\frac{\left[-\Delta(\alpha^*,\lambda)e^{-(i\theta^*+\mu)\cdot} + (\lambda-i\theta^*)\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}e^{-(\lambda+\mu)\cdot}\right]}{(\lambda-i\theta^*)^2}\\ &= \frac{-(\lambda-i\theta^*)\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}\left[e^{-(i\theta^*+\mu)\cdot} - e^{-(\lambda+\mu)\cdot}\right]}{(\lambda-i\theta^*)^2}\\ &\quad + \frac{-\left[\frac{(\lambda-i\theta^*)^2}{2}\frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2} + (\lambda-i\theta^*)^3 g(\lambda-i\theta^*)\right]e^{-(i\theta^*+\mu)\cdot}}{(\lambda-i\theta^*)^2}\\ &\to -\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}e^{-(i\theta^*+\mu)\cdot} - \frac{1}{2}\frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2}e^{-(i\theta^*+\mu)\cdot} \text{ as } \lambda\to i\theta^*.\end{aligned}$$

Finally, it implies that

$$
\begin{aligned}
&\left(i\theta^* I - B^{\mathbb{C}}_{\alpha^*}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1}\widehat{\Pi}_s\begin{pmatrix}1\\0\end{pmatrix}\\
&=\begin{pmatrix} 0 \\ -\frac{d\Delta(\alpha^*,-i\theta^*)}{d\lambda}^{-1}\dfrac{e^{-(-i\theta^*+\mu)\cdot}}{2i\theta^*}+\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}^{-2}\left[-\frac{d\Delta(\alpha^*,i\theta^*)}{d\lambda}\cdot-\frac{1}{2}\frac{d^2\Delta(\alpha^*,i\theta^*)}{d\lambda^2}\right]e^{-(i\theta^*+\mu)\cdot}\end{pmatrix}.
\end{aligned}
$$

The case $\lambda = -i\theta^*$ can be proved similarly. This completes the proof. $\square$

In order to apply the Center Manifold Theorem 6.1.21 to the above system, we include the parameter α into the state variable. So we consider the system

$$
\begin{cases}
\dfrac{dv(t)}{dt} = Av(t) + \alpha(t)H(v(t)),\\
\dfrac{d\alpha(t)}{dt} = 0,\\
v(0) = v_0 \in Y_0,\ \alpha(0) = \alpha_0 \in \mathbb{R}.
\end{cases}
\tag{8.3.9}
$$

Making a change of variables

$$
\alpha = \widehat{\alpha} + \alpha^* \quad \text{and } v = \widehat{v} + \overline{v}_{\alpha^*},
$$

we obtain the system

$$
\begin{aligned}
&\frac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + (\widehat{\alpha}(t) + \alpha^*)\left[H(\widehat{v}(t) + \overline{v}_{(\widehat{\alpha}(t)+\alpha^*)}) - H(\overline{v}_{(\widehat{\alpha}(t)+\alpha^*)})\right],\\
&\frac{d\widehat{\alpha}(t)}{dt} = 0.
\end{aligned}
\tag{8.3.10}
$$

Set

$$
\mathscr{X} = Y \times \mathbb{R},\ \mathscr{X}_0 = \overline{D(A)} \times \mathbb{R}
$$

and

$$
\widehat{H}(\widehat{\alpha},\widehat{v}) = (\widehat{\alpha} + \alpha^*)\left[H(\widehat{v} + \overline{v}_{(\widehat{\alpha}+\alpha^*)}) - H(\overline{v}_{(\widehat{\alpha}+\alpha^*)})\right].
$$

We have

$$
\partial_v\widehat{H}(\widehat{\alpha},\widehat{v})(w) = (\widehat{\alpha} + \alpha^*)DH(\widehat{v} + \overline{v}_{(\widehat{\alpha}+\alpha^*)})(w)
$$

and

$$
\begin{aligned}
\partial_{\widehat{\alpha}}\widehat{H}(\widehat{\alpha},\widehat{v})(\widetilde{\alpha}) = \widetilde{\alpha}\Big\{ & H(\widehat{v} + \overline{v}_{(\widehat{\alpha}+\alpha^*)}) - H(\overline{v}_{(\widehat{\alpha}+\alpha^*)})\\
&+ (\widehat{\alpha} + \alpha^*)\left[DH(\widehat{v} + \overline{v}_{(\widehat{\alpha}+\alpha^*)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha^*)}}{d\widehat{\alpha}}\right)\right.\\
&\left.- DH(\overline{v}_{(\widehat{\alpha}+\alpha^*)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha^*)}}{d\widehat{\alpha}}\right)\right]\Big\}.
\end{aligned}
$$

So $\partial_v\widehat{H}(0,0) = \alpha^* DH(\overline{v}_{\alpha^*})$ and $\partial_{\widehat{\alpha}}\widehat{H}(0,0) = 0$.

Consider the linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{X} \to \mathscr{X}$ defined by

$$\mathscr{A}\begin{pmatrix}\widehat{v}\\ \widehat{\alpha}\end{pmatrix} = \begin{pmatrix}(A+\alpha^* DH(\overline{v}_{\alpha^*}))\widehat{v}\\ 0\end{pmatrix} \tag{8.3.11}$$

with $D(\mathscr{A}) = D(A)\times\mathbb{R}$ and the map $F : \overline{D(\mathscr{A})} \to \mathscr{X}$ defined by

$$F\begin{pmatrix}v\\ \widehat{\alpha}\end{pmatrix} = \begin{pmatrix}F_1\begin{pmatrix}\widehat{v}\\ \widehat{\alpha}\end{pmatrix}\\ 0\end{pmatrix},$$

where $F_1 : \mathscr{X} \to \mathbb{R}$ is defined by

$$F_1\begin{pmatrix}\widehat{v}\\ \widehat{\alpha}\end{pmatrix} = (\widehat{\alpha}+\alpha^*)\left[H(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha^*)}) - H(\overline{v}_{(\widehat{\alpha}+\alpha^*)})\right] - \alpha^* DH(\overline{v}_{\alpha^*})(\widehat{v}).$$

Then we have

$$F\begin{pmatrix}0\\ \widehat{\alpha}\end{pmatrix} = 0,\ \forall\widehat{\alpha} > 1-\alpha^*,\ \text{ and } DF(0) = 0.$$

Now we can apply Theorem 6.1.21 to the system

$$\frac{dw(t)}{dt} = \mathscr{A}w(t) + F(w(t)),\quad w(0) = w_0 \in D(\mathscr{A}). \tag{8.3.12}$$

We have for $\lambda \in \rho(\mathscr{A})\cap\Omega = \Omega\setminus(\sigma(B_{\alpha^*})\cup\{0\})$ that

$$(\lambda I - \mathscr{A})^{-1}\begin{pmatrix}\delta\\ \psi\\ r\end{pmatrix} = \begin{pmatrix}(\lambda I - B_{\alpha^*})^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\\ \dfrac{r}{\lambda}\end{pmatrix}.$$

By using a similar argument as in the proof of Lemma 8.3.6 and employing Lemma 8.3.5, we obtain the following lemma.

Lemma 8.3.10. *Let Assumptions 8.3.1 and 8.3.7 be satisfied. Then*

$$\sigma(\mathscr{A}) = \sigma(B_\alpha)\cup\{0\}.$$

Moreover, the eigenvalues 0 *and* $\pm i\theta^*$ *of* $\mathscr{A}$ *are simple. The corresponding projectors* $\Pi_0, \Pi_{\pm i\theta^*} : \mathscr{X}+i\mathscr{X} \to \mathscr{X}+i\mathscr{X}$ *are defined respectively by*

$$\Pi_0\begin{pmatrix}v\\ r\end{pmatrix} = \begin{pmatrix}0\\ r\end{pmatrix},$$
$$\Pi_{\pm i\theta^*}\begin{pmatrix}v\\ r\end{pmatrix} = \begin{pmatrix}\widehat{\Pi}_{\pm i\theta^*}v\\ 0\end{pmatrix}$$

In this context, the projector $\Pi_c : \mathscr{X} \to \mathscr{X}$ is defined by

$$\Pi_c(x) = (\Pi_0 + \Pi_{i\theta^*} + \Pi_{-i\theta^*})(x), \ \forall x \in \mathscr{X}.$$

Note that we have

$$\overline{\Pi_{i\theta^*}(x)} = \Pi_{-i\theta^*}(\bar{x}), \ \forall x \in \mathscr{X} + i\mathscr{X},$$

so the above projector Π_c maps $\mathscr{X}$ into $\mathscr{X}$. Set

$$\mathscr{X}_c = \mathscr{R}(\Pi_c(\mathscr{X}))$$

and define the basis of $\mathscr{X}_c$ by

$$e_1 = \begin{pmatrix} 0_{\mathbb{R}} \\ e^{-(\mu+i\theta^*)\cdot} + e^{-(\mu-i\theta^*)\cdot} \\ 0_{\mathbb{R}} \end{pmatrix}, \ e_2 = \begin{pmatrix} 0_{\mathbb{R}} \\ \frac{e^{-(\mu+i\theta^*)\cdot} - e^{-(\mu-i\theta^*)\cdot}}{i} \\ 0_{\mathbb{R}} \end{pmatrix}, \ e_3 = \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \\ 1 \end{pmatrix}$$

and

$$\mathscr{A} e_1 = -\theta^* e_2, \ \mathscr{A} e_2 = \theta^* e_1, \ \mathscr{A} e_3 = 0.$$

Then the matrix of $\mathscr{A}_c$ in the basis $\{e_1, e_2, e_3\}$ of $\mathscr{X}_c$ is given by

$$M = \begin{bmatrix} 0 & -\theta^* & 0 \\ \theta^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.3.13}$$

Moreover, we have

$$\begin{aligned} \Pi_c \begin{pmatrix} 1 \\ 0_{L^1} \\ 0_{\mathbb{R}} \end{pmatrix} &= \begin{pmatrix} \widehat{\Pi}_{+i\theta^*} \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} + \widehat{\Pi}_{-i\theta^*} \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} \\ 0_{\mathbb{R}} \end{pmatrix} \\ &= \begin{pmatrix} 0_{\mathbb{R}} \\ \frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda}^{-1} e^{-(i\theta^*+\mu)\cdot} + \frac{d\Delta(\alpha^*, -i\theta^*)}{d\lambda}^{-1} e^{-(-i\theta^*+\mu)\cdot} \\ 0_{\mathbb{R}} \end{pmatrix}. \end{aligned}$$

Thus,

$$\Pi_c \begin{pmatrix} \delta \\ 0_{L^1} \\ r \end{pmatrix} = \delta \left| \frac{d\Delta(\alpha^*, i\theta^*)}{d\lambda} \right|^{-2} (\mathrm{Re}(\Delta(\alpha^*, i\theta^*)) e_1 + \mathrm{Im}(\Delta(\alpha^*, i\theta^*)) e_2) + re_3.$$

Therefore, we can apply Theorem 6.1.21. Let $\Gamma : \mathscr{X}_{0c} \to \mathscr{X}_{0s}$ be the map defined in Theorem 6.1.21. Since $\mathscr{X}_s \subset Y \times \{0_{\mathbb{R}}\}$ and $\{e_1, e_2, e_3\}$ is a basis of $\mathscr{X}_c$, it follows that

$$\Psi(x_1 e_1 + x_2 e_2 + x_3 e_3) = \begin{pmatrix} \Psi_1(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ 0_{\mathbb{R}} \end{pmatrix}.$$

Since $F \in C^\infty(\mathscr{X}_0, \mathscr{X})$, we can assume that $\Psi \in C_b^3(\mathscr{X}_{0c}, \mathscr{X}_{0s})$, and the reduced system is given by

$$
\begin{aligned}
\frac{dx_c(t)}{dt} &= \mathscr{A}_0 \mid_{\mathscr{X}_c} x_c(t) + \Pi_c F\left(x_c(t) + \Psi\left(x_c(t)\right)\right) \\
&= \mathscr{A}_0 \mid_{\mathscr{X}_c} x_c(t) + F_1\left(x_c(t) + \Psi\left(x_c(t)\right)\right) \Pi_c \begin{pmatrix} 1 \\ 0_{L^1} \\ 0_{\mathbb{R}} \end{pmatrix}, \\
D\Gamma(0) &= 0, \\
\Gamma \begin{pmatrix} 0_Y \\ \widehat{\alpha} \end{pmatrix} &= 0 \text{ for all } \widehat{\alpha} \in \mathbb{R} \text{ with } |\widehat{\alpha}| \text{ small enough.}
\end{aligned}
$$

The system expressed in the basis $\{e_1, e_2, e_3\}$ of $\mathscr{X}_c$ is given by

$$
\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = M \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + G\left(x_1(t), x_2(t), x_3(t)\right) V, \tag{8.3.14}
$$

where M is given by (8.3.13),

$$
V = \left| \frac{d\Delta\left(\alpha^*, i\theta^*\right)}{d\lambda} \right|^{-2} \begin{pmatrix} \operatorname{Re}\left(\Delta\left(\alpha^*, i\theta^*\right)\right) \\ \operatorname{Im}\left(\Delta\left(\alpha^*, i\theta^*\right)\right) \\ 0 \end{pmatrix}
$$

and

$$
G\left(x_1, x_2, x_3\right) = F_1 \circ (I + \Psi)\left(x_1 e_1 + x_2 e_2 + x_3 e_3\right).
$$

Here x_3 corresponds to the parameter of the system. Note that we can compute explicitly the third order Taylor expansion of the reduced system around 0 and have

$$
\begin{aligned}
& DG(x_c) = DF_1\left(x_c + \Psi(x_c)\right)\left(I + D\Psi(x_c)\right), \\
& D^2 G(x_c)\left(x_c^1, x_c^2\right) \\
& \quad = D^2 F_1\left(x_c + \Psi(x_c)\right)\left(\left(I + D\Psi(x_c)\right)\left(x_c^1\right), \left(I + D\Psi(x_c)\right)\left(x_c^2\right)\right) \\
& \qquad + DF_1\left(x_c + \Psi(x_c)\right) D^2\Psi(x_c)\left(x_c^1, x_c^2\right), \\
& D^3 G(x_c)\left(x_c^1, x_c^2, x_c^3\right) \\
& \quad = D^3 F_1\left(x_c + \Psi(x_c)\right)\left(\left(I + D\Psi(x_c)\right)\left(x_c^1\right), \left(I + D\Psi(x_c)\right)\left(x_c^2\right), \left(I + D\Psi(x_c)\right)\left(x_c^3\right)\right) \\
& \qquad + D^2 F_1\left(x_c + \Psi(x_c)\right)\left(\left(D^2\Psi(x_c)\right)\left(x_c^1, x_c^3\right), \left(I + D\Psi(x_c)\right)\left(x_c^2\right)\right) \\
& \qquad + D^2 F_1\left(x_c + \Psi(x_c)\right)\left(\left(I + D\Psi(x_c)\right)\left(x_c^1\right), D^2\Psi(x_c)\left(x_c^2, x_c^3\right)\right) \\
& \qquad + D^2 F_1\left(x_c + \Psi(x_c)\right)\left(D^2\Psi(x_c)\left(x_c^1, x_c^2\right), \left(I + D\Psi(x_c)\right)\left(x_c^3\right)\right) \\
& \qquad + DF_1\left(x_c + \Psi(x_c)\right) D^3\Psi(x_c)\left(x_c^1, x_c^2, x_c^3\right).
\end{aligned}
$$

Since $DF_1(0) = 0$, $\Psi(0) = 0$, and $D\Psi(0) = 0$, we obtain that

$$
DG(0) = 0,\ D^2 G(0)\left(x_c^1, x_c^2\right) = D^2 F_1(0)\left(x_c^1, x_c^2\right)
$$

and

$$\begin{aligned} D^2G(x_c)\left(x_c^1,x_c^2,x_c^3\right) = {} & D^3F_1(0)\left(x_c^1,x_c^2,x_c^3\right) \\ & +D^2F_1(0)\left(D^2\Psi(0)\left(x_c^1,x_c^3\right),x_c^2\right) \\ & +D^2F_1(0)\left(x_c^1,D^2\Psi(0)\left(x_c^2,x_c^3\right)\right) \\ & +D^2F_1(0)\left(D^2\Psi(0)\left(x_c^1,x_c^2\right),x_c^3\right). \end{aligned}$$

Moreover, by computing the Taylor expansion to the order 3 of the problem, we have

$$\begin{aligned} G(h) = {} & \frac{1}{2!}D^2G(0)(h,h)+\frac{1}{3!}D^3G(0)(h,h,h) \\ & +\frac{1}{4!}\int_0^1(1-t)^4D^4F_1(th)(h,h,h,h)\,dt. \end{aligned}$$

Notice that we can compute explicitly that

$$\frac{1}{2!}D^2G(0)(h,h)+\frac{1}{3!}D^3G(0)(h,h,h).$$

Because F_1 is explicit, we only need to compute $D^2\Psi(0)$. For each $x,y\in\mathscr{X}_c$,

$$D^2\Psi(0)(x,y)=\lim_{\lambda\to+\infty}\int_0^{+\infty}T_{A_0}(l)\Pi_{0s}\lambda(\lambda-A)^{-1}D^{(2)}F(0)\left(e^{-A_{0c}l}x,e^{-A_{0c}l}y\right)dl.$$

Using the fact that

$$\begin{aligned} e^{A_ct}e_1 &= \cos(\theta^*t)e_1-\sin(\theta^*t)e_2, \\ e^{A_ct}e_2 &= \sin(\theta^*t)e_1+\cos(\theta^*t)e_2, \\ e^{A_ct}e_3 &= e_3 \end{aligned}$$

and

$$\cos(\theta^*t)=\frac{\left(e^{i\theta^*t}+e^{-i\theta^*t}\right)}{2},\ \sin(\theta^*t)=\frac{\left(e^{i\theta^*t}-e^{-i\theta^*t}\right)}{2i},$$

and following Lemma 8.3.9 and the same method at the end of Chapter 6 (i.e. the same method as in the proof of (iii) in Theorem 6.1.21), we can obtain an explicit formula for $D^2\Psi(0)(e_i,e_j)$: For $i,j=1,2$,

$$D^2\Psi(0)(e_i,e_j)=\sum_{\substack{\lambda\in\Lambda_{i,j},\\ k,l=1,2}}\begin{pmatrix} c_{ij}(\lambda)\left(\lambda I-B_\alpha^{\mathbb{C}}\mid_{\widehat{\Pi}_s(Y)}\right)^{-1}\widehat{\Pi}_s\begin{pmatrix}1\\0_{L_1}\end{pmatrix}D^2F_1(e_k,e_l) \\ 0 \end{pmatrix},$$

where $\Lambda_{i,j}$ is a finite subset included in $i\mathbb{R}$. So we can compute $D^2\Psi(0)$ and thus have proven that the system (8.3.14) on the center manifold is C^3 in its variables.

Next, we need to study the eigenvalues of the characteristic equation (8.3.4). Assume that the parameter $\alpha>e$ and consider

$$\Delta(\alpha,\lambda) = 1 - \eta(\alpha)\int_0^{+\infty} \gamma(a) e^{-(\lambda+\mu)a} da$$

with

$$\eta(\alpha) = 1 - \ln(\alpha).$$

We have

$$\frac{\partial \Delta(\alpha,\lambda)}{\partial \alpha} = -\frac{1}{\alpha}\left[\int_0^{+\infty} \gamma(a) e^{-(\lambda+\mu)a} da\right].$$

If $\Delta(\alpha,\lambda) = 0$ and $\alpha > e$, then

$$\frac{\partial \Delta(\alpha,\lambda)}{\partial \alpha} = \frac{1}{\alpha\eta(\alpha)} < 0.$$

In addition to Assumption 8.3.7, we also make the following assumptions.

Assumption 8.3.11. Assume that there is a number $\alpha^* > e$ such that

a) If $\lambda \in \Omega$ and $\Delta(\alpha,\lambda) = 0$, then $\mathrm{Re}\left(\frac{\partial \Delta(\alpha,\lambda)}{\partial \lambda}\right) > 0$;
b) There exists a constant $C > 0$ such that for each $\alpha \in [e,\alpha^*]$,

$$\mathrm{Re}(\lambda) \geq -\mu \text{ and } \Delta(\alpha,\lambda) = 0 \Rightarrow |\lambda| \leq C;$$

c) There exists $\theta^* > 0$ such that $\Delta(\alpha^*, i\theta^*) = 0$ and $\Delta(\alpha^*, i\theta) \neq 0, \forall\theta \in [0,+\infty)\setminus\{\theta^*\}$;
d) For each $\alpha \in [e,\alpha^*)$, $\Delta(\alpha, i\theta) \neq 0, \forall\theta \in [0,+\infty)$.

Note that if $\alpha = e$, we have $\Delta(\alpha,\lambda) = 1$, so there is no eigenvalue. By the continuity of $\Delta(\alpha,\lambda)$ and using Assumption 8.3.11 b), we deduce that there exists $\alpha_1 \in [e,\alpha^*]$ such that

$$\Delta(\alpha,\lambda) \neq 0, \forall\lambda \in \Omega,\ \ \forall\alpha \in [e,\alpha_1).$$

Because of Assumption 8.3.11 a), we can apply locally the implicit function theorem and deduce that if $\widehat{\alpha} > e$, $\widehat{\lambda} \in \Omega$, and $\Delta\left(\widehat{\alpha},\widehat{\lambda}\right) = 0$, then there exist two constants $\varepsilon > 0$, $r > 0$, and a continuously differentiable map $\widehat{\lambda} : (\widehat{\alpha}-\varepsilon, \widehat{\alpha}+\varepsilon) \to \mathbb{C}$, such that

$$\Delta(\alpha,\lambda) = 0 \ \text{ and } (\alpha,\lambda) \in (\widehat{\alpha}-\varepsilon, \widehat{\alpha}+\varepsilon) \times B_{\mathbb{C}}(0,r) \Leftrightarrow \lambda = \widehat{\lambda}(\alpha).$$

Moreover, we have

$$\Delta\left(\widehat{\alpha}, \widehat{\lambda}(\alpha)\right) = 0$$

and

$$\frac{\partial \Delta\left(\widehat{\alpha},\widehat{\lambda}(\alpha)\right)}{\partial \alpha} + \frac{\partial \Delta\left(\widehat{\alpha},\widehat{\lambda}(\alpha)\right)}{\partial \lambda}\frac{d\widehat{\lambda}(\alpha)}{d\alpha} = 0.$$

Thus,

$$\frac{d\widehat{\lambda}(\alpha)}{d\alpha} = \frac{1}{\frac{\partial \Delta\left(\widehat{\alpha},\widehat{\lambda}(\alpha)\right)}{\partial \lambda}} \frac{-1}{\alpha\eta(\alpha)}.$$

However,

$$\operatorname{Re}\left(\frac{\partial \Delta\left(\widehat{\alpha},\widehat{\lambda}(\alpha)\right)}{\partial \lambda}\right) > 0 \Leftrightarrow \operatorname{Re}\left(\frac{1}{\frac{\partial \Delta\left(\widehat{\alpha},\widehat{\lambda}(\alpha)\right)}{\partial \lambda}}\right) > 0,$$

so

$$\frac{d\operatorname{Re}\left(\widehat{\lambda}(\alpha)\right)}{d\alpha} > 0.$$

Summarizing the above analysis, we have the following Lemma.

Lemma 8.3.12. *Let Assumptions 8.3.1, 8.3.7 and 8.3.11 be satisfied. Then we have the following:*

(a) For each $\alpha \in [e, \alpha^*)$, the characteristic equation $\Delta(\alpha,\lambda) = 0$ has no roots with positive real part;

(b) There exist constants $\varepsilon > 0$, $\eta > 0$, and a continuously differentiable map $\widehat{\lambda}$: $(\alpha^* - \varepsilon, \alpha^* + \varepsilon) \to \mathbb{C}$, such that

$$\Delta\left(\alpha,\widehat{\lambda}(\alpha)\right) = 0, \ \forall \alpha \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$$

with

$$\widehat{\lambda}(\alpha^*) = i\theta^* \text{ and } \frac{d}{d\alpha}\operatorname{Re}\left(\widehat{\lambda}(\alpha^*)\right) > 0,$$

and for each $\alpha \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$, if

$$\Delta(\alpha,\lambda) = 0, \ \lambda \neq \widehat{\lambda}(\alpha), \text{ and } \lambda \neq \overline{\widehat{\lambda}(\alpha)},$$

then

$$\operatorname{Re}(\lambda) < -\eta.$$

In order to find the critical values of the parameter α and verify the transversality condition, we need to be more specific about the function $\gamma(a)$. We make the following assumption.

Assumption 8.3.13. Assume that

$$\gamma(a) = \begin{cases} \delta(a-\tau)^n e^{-\zeta(a-\tau)} & \text{if } a \geq \tau \\ 0 & \text{if } a \in [0,\tau) \end{cases} \tag{8.3.15}$$

for some integer $n \geq 1$, $\tau \geq 0$, $\zeta > 0$, and

$$\delta = \left(\int_{\tau}^{+\infty} (a-\tau)^n e^{-\zeta(a-\tau)} da\right)^{-1} > 0.$$

Note that if $n \geq 1$, then γ satisfies the conditions in Assumption 8.3.1. We have for $\lambda \in \Omega$ that

$$\begin{aligned}\int_0^{+\infty} \gamma(a) e^{-(\mu+\lambda)a} da &= \int_\tau^{+\infty} \gamma(a) e^{-(\mu+\lambda)a} da \\ &= \delta e^{-(\mu+\lambda)\tau} \int_\tau^{+\infty} (a-\tau)^n e^{-(\mu+\zeta+\lambda)(a-\tau)} da \\ &= \delta e^{-(\mu+\lambda)\tau} \int_0^{+\infty} l^n e^{-(\mu+\zeta+\lambda)l} dl.\end{aligned}$$

Set

$$I_n(\lambda) = \int_0^{+\infty} l^n e^{-(\mu+\zeta+\lambda)l} dl \quad \text{for each } n \geq 0 \quad \text{and each } \lambda \in \Omega.$$

Then we have

$$\begin{aligned}\Delta(\alpha, \lambda) &= 1 - \eta(\alpha) \int_0^{+\infty} \gamma(a) e^{-(\lambda+\mu)a} da \\ &= 1 - \eta(\alpha) \delta e^{-(\mu+\lambda)\tau} I_n(\lambda).\end{aligned}$$

Then by integrating by part we have for $n \geq 1$ that

$$\begin{aligned}I_n(\lambda) &= \int_0^{+\infty} l^n e^{-(\mu+\zeta+\lambda)l} dl \\ &= \left[\frac{l^n e^{-(\mu+\zeta+\lambda)l}}{-(\mu+\zeta+\lambda)} \right]_0^{+\infty} - \int_0^{+\infty} \frac{n l^{n-1} e^{-(\mu+\zeta+\lambda)l}}{(\mu+\zeta+\lambda)} dl \\ &= \frac{n}{(\mu+\zeta+\lambda)} I_{n-1}(\lambda)\end{aligned}$$

and

$$I_0(\lambda) = \int_0^{+\infty} e^{-(\mu+\zeta+\lambda)l} dl = \frac{1}{(\mu+\zeta+\lambda)}.$$

Therefore,

$$I_n(\lambda) = \frac{n!}{(\mu+\zeta+\lambda)^{n+1}}, \ \forall n \geq 0$$

with $0! = 1$.

The characteristic equation (8.3.4) becomes

$$1 = \eta(\alpha) \delta n! \frac{e^{-\tau(\mu+\lambda)}}{(\mu+\zeta+\lambda)^{n+1}}, \quad \mathrm{Re}(\lambda) > -\mu. \tag{8.3.16}$$

When $n = 0$, the above characteristic equation (8.3.16) is well known in the context of delay differential equations (see Hale and Verduyn Lunel [175], p.341). Note also that when $\tau = 0$, (8.3.16) becomes trivial. Indeed, assume that $\tau = 0$ and $\eta < 0$, then we have

$$(\mu+\zeta+\lambda)^{n+1} = -|\eta|\,\delta n! = |\eta|\,\delta n! e^{i(2k+1)\pi} \text{ for } k=0,1,2,...$$

so

$$\lambda = -(\mu+\zeta) + \sqrt[n+1]{|\eta|\,\delta n!} e^{i\frac{(2k+1)}{n+1}\pi} \text{ for } k=0,1,2,...$$

We have

$$\begin{aligned}
\frac{d\Delta(\alpha,\lambda)}{d\lambda} &= \eta \int_0^{+\infty} a\gamma(a) e^{-(\lambda+\mu)a} da \\
&= \eta\delta e^{-(\lambda+\mu)\tau} \int_\tau^{+\infty} a(a-\tau)^n e^{-(\mu+\zeta+\lambda)(a-\tau)} da \\
&= \eta\delta e^{-(\lambda+\mu)\tau} \left[\int_\tau^{+\infty} (a-\tau)^{n+1} e^{-(\mu+\zeta+\lambda)(a-\tau)} da \right. \\
&\qquad \left. +\tau \int_\tau^{+\infty} (a-\tau)^n e^{-(\mu+\zeta+\lambda)(a-\tau)} da \right] \\
&= \eta\delta e^{-(\lambda+\mu)\tau} [I_{n+1} + \tau I_n] \\
&= \eta\delta e^{-(\lambda+\mu)\tau} \left[\frac{n+1}{(\mu+\zeta+\lambda)} + \tau \right] I_n \\
&= \left[\frac{n+1}{(\mu+\zeta+\lambda)} + \tau \right] [1-\Delta(\alpha,\lambda)].
\end{aligned}$$

If $\Delta(\alpha,\lambda)=0$, it follows that

$$\frac{d\Delta(\alpha,\lambda)}{d\lambda} = \left[\frac{n+1}{(\mu+\zeta+\lambda)} + \tau \right] \neq 0 \text{ and } \mathrm{Re}\left(\frac{d\Delta(\alpha,\lambda)}{d\lambda} \right) > 0.$$

Hence, all eigenvalues are simple and we can apply the implicit function theorem around each solution of the characteristic equation.

Notice that

$$|\mu+\zeta+\lambda|^2 = |\eta(\alpha)\,\delta n!|^{\frac{2}{n+1}} e^{-\frac{2\tau}{n+1}(\mu+\zeta+Re(\lambda))}.$$

So

$$\mathrm{Im}(\lambda)^2 = |\eta(\alpha)\,\delta n!|^{\frac{2}{n+1}} e^{-\frac{2\tau}{n+1}(\mu+\zeta+\mathrm{Re}(\lambda))} - (\mu+\zeta+\mathrm{Re}(\lambda))^2. \tag{8.3.17}$$

Thus, there exists $\delta_1 > 0$ such that $-\mu < \mathrm{Re}(\lambda) \le \delta_1$. This implies that the characteristic equation (8.3.16) satisfies Assumption 8.3.11 b). Using (8.3.17) we also know that for each real number δ, there is at most one pair of complex conjugate eigenvalues such that $\mathrm{Re}(\lambda) = \delta$.

Lemma 8.3.14. *Let Assumption 8.3.13 be satisfied. Then Assumptions 8.3.1, 8.3.7 and 8.3.11 are satisfied.*

Proof. In order to prove the above lemma it is sufficient to prove that for $\alpha > e$ large enough there exists $\lambda \in \mathbb{C}$ such that

$$\Delta(\alpha,\lambda)=0 \text{ and } \operatorname{Re}(\lambda)>0.$$

The characteristic equation can be rewritten as follows

$$(\xi+\lambda)^{n+1}=-\chi(\alpha)e^{-\tau(\xi+\lambda)},\ \operatorname{Re}(\lambda)\geq 0,$$

where

$$\chi(\alpha)=(\ln(\alpha)-1)\delta n!e^{\tau\xi}=\ln\left(\frac{\alpha}{e}\right)\delta n!e^{\tau\xi}>0 \text{ and } \xi=\mu+\zeta>0.$$

Replacing λ by $\widehat{\lambda}=\tau(\xi+\lambda)$ and $\chi(\alpha)$ by $\widehat{\chi}(\alpha)=\tau^{n+1}\chi(\alpha)$, we obtain

$$\widehat{\lambda}^{n+1}=-\widehat{\chi}(\alpha)e^{-\widehat{\lambda}} \text{ and } \operatorname{Re}\left(\widehat{\lambda}\right)\geq\tau\xi.$$
$$\Leftrightarrow \widehat{\lambda}^{n+1}=\widehat{\chi}(\alpha)e^{-\widehat{\lambda}+(2k+1)\pi i} \text{ and } \operatorname{Re}\left(\widehat{\lambda}\right)\geq\tau\xi, k\in\mathbb{Z}.$$

So we must find $\widehat{\lambda}=a+ib$ with $a>\tau\xi$ such that

$$\begin{cases} a=\widehat{\chi}(\alpha)^{\frac{1}{n+1}}e^{-a}\cos\left(\frac{b+(2k+1)\pi}{n+1}\right), \\ b=\widehat{\chi}(\alpha)^{\frac{1}{n+1}}e^{-a}\sin\left(-\frac{b+(2k+1)\pi}{n+1}\right) \end{cases} \tag{8.3.18}$$

for some $k\in\mathbb{Z}$.

From the first equation of system (8.3.18) we must have

$$\frac{a}{\widehat{\chi}(\alpha)^{\frac{1}{n+1}}e^{-a}}\in[0,1) \text{ and } \cos\left(\frac{b+(2k+1)\pi}{n+1}\right)>0.$$

Moreover, system (8.3.18) can also be written as

$$\tan\left(\frac{b+(2k+1)\pi}{n+1}\right)=-\frac{b}{a},$$

and

$$ae^{a}=\widehat{\chi}(\alpha)^{\frac{1}{n+1}}\cos\left(\frac{b+(2k+1)\pi}{n+1}\right).$$

Set

$$\widehat{b}=\frac{b+(2k+1)\pi}{n+1}.$$

Then

$$b=(n+1)\widehat{b}-(2k+1)\pi.$$

The problem becomes to find $\widehat{\theta}\in\mathbb{R}\backslash\left\{\frac{\pi}{2}+m\pi : m\in\mathbb{Z}\right\}$ such that

$$\cos(\widehat{\theta})>0,\ \tan\left(\widehat{\theta}\right)=-\frac{(n+1)\widehat{\theta}-(2k+1)\pi}{a},\ k\in\mathbb{Z}, \tag{8.3.19}$$

and

$$ae^a = \widehat{\chi}(\alpha)^{\frac{1}{n+1}} \cos\left(\widehat{\theta}\right). \tag{8.3.20}$$

Fix $a > \tau\xi = \tau(\mu+\xi)$, then it is clear that we can find $\widehat{\theta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that (8.3.19) is satisfied. Moreover, $\widehat{\chi}(e) = 0$ and $\widehat{\chi}(\alpha) \to +\infty$ as $\alpha \to +\infty$. Thus, we can find $\widehat{\alpha} > e$, in turn we can find $\alpha > e$, such that (8.3.20) is satisfied. The result follows. □

Therefore, by the Hopf Bifurcation Theorem 6.2.7 and Proposition 6.1.22 we have the following result.

Theorem 8.3.15. *Let Assumptions 8.3.1 and 8.3.13 be satisfied and assume that* $\tau > 0$. *Then the characteristic equation (8.3.16) with* $\alpha = \alpha_k$, $k \in \mathbb{N}\setminus\{0\}$, *has a unique pair of purely imaginary roots* $\pm i\omega_k$, *where*

$$1 = n!\eta(\alpha_k)\frac{e^{-\mu\tau}}{\left(\sqrt{(\mu+\varsigma)^2+\omega_k^2}\right)^{n+1}}$$

and $\omega_k > 0$ *is the unique solution of*

$$-\left(\omega\tau + (n+1)\arctan\frac{\omega}{\varsigma+\mu}\right) = \pi - 2k\pi,$$

such that the age-structured model (8.3.1) undergoes a Hopf bifurcation at the equilibrium $u = \overline{u}_{\alpha_k}$ *given by (8.3.3). In particular, a non-trivial periodic solution bifurcates from the equilibrium* $u = \overline{u}_{\alpha_k}$ *when* $\alpha = \alpha_k$.

To carry out some numerical simulations, we consider the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = -\mu u(t,a),\ t \geq 0,\ a \geq 0 \\ u(t,0) = h\left(\int_0^{+\infty} b(a)u(t,a)da\right) \\ u(0,a) = u_0(a) \end{cases}$$

with the initial value function

$$u_0(a) = a\exp(-0.08a),$$

the fertility rate function

$$h(x) = \alpha x \exp(-\beta x)$$

and the birth rate function (see Fig. 8.7)

$$b(a) = \begin{cases} \delta\exp(-\gamma(a-\tau))(a-\tau), & \text{if } a \geq \tau, \\ 0, & \text{if } a \in [0,\tau]. \end{cases}$$

where

$$\mu = 0.1,\ \beta = 1,\ \delta = 1,\ \gamma = 1,\ \tau = 5.$$

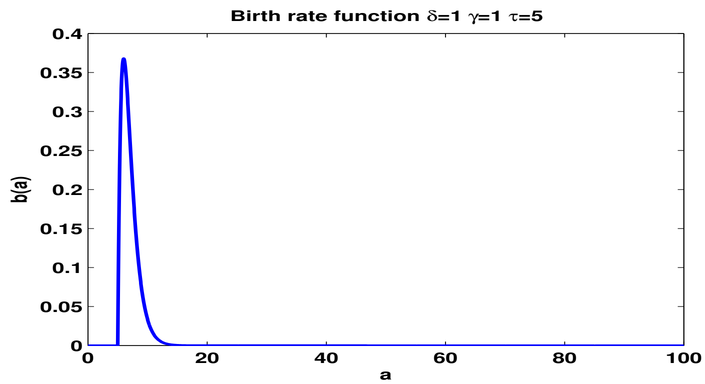

Fig. 8.7: The birth rate function $b(a)$ with $\delta = 1, \gamma = 1$, and $\tau = 5$.

The equilibrium is given by

$$\overline{u}(a) = Ce^{-\mu a}, \ \ a \geq 0, \ \ C = h\left(\int_0^{+\infty} b(a)e^{-\mu a}Cda\right).$$

We choose $\alpha \geq 0$ as the bifurcation parameter. When $\alpha = 10$, the solution converges to the equilibrium (see Fig. 8.8 upper figure). When $\alpha = 20$, the equilibrium loses its stability, a Hopf bifurcation occurs and there is a time periodic solution (see Fig. 8.8 lower figure).

8.3.4 Direction and Stability of Hopf bifurcation

In this section, we apply the normal form theory of Chapter 6 to the Cauchy problem (8.3.2) and to compute up to the third terms of the Taylor's expansion for the reduced system on the center manifold, from which explicit formulae are given to determine the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions.

Consider systems (8.3.9) and (8.3.12). We apply the method described in Theorem 6.3.11 with $k = 2$. The main point is to compute $L_2 \in \mathscr{L}_s\left(\mathscr{X}_c^2, \mathscr{X}_h \cap D(\mathscr{A})\right)$ by solving the following equation for each $(w_1, w_2) \in \mathscr{X}_c^2$:

$$\frac{d}{dt}\left[L_2(e^{\mathscr{A}_c t}w_1, e^{\mathscr{A}_c t}w_2)\right](0) = \mathscr{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 F\left(0\right)(w_1, w_2). \quad (8.3.21)$$

Note that

$$\frac{d}{dt}\left[L_2(e^{\mathscr{A}_c t}w_1, e^{\mathscr{A}_c t}w_2)\right](0) = L_2\left(\mathscr{A}_c w_1, w_2\right) + L_2\left(w_1, \mathscr{A}_c w_2\right).$$

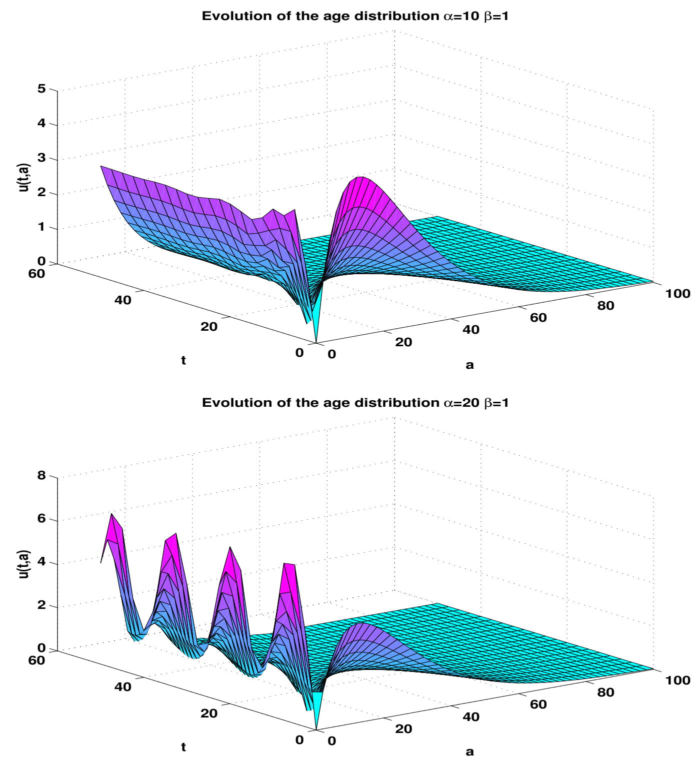

Fig. 8.8: The age distribution of $u(t,a)$, which converges to the equilibrium when $\alpha = 10$ (upper) and is time periodic when $\alpha = 20$ (lower).

So system (8.3.21) can be rewritten as

$$L_2(\mathscr{A}_c w_1, w_2) + L_2(w_1, \mathscr{A}_c w_2) = \mathscr{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 F(0)(w_1, w_2). \quad (8.3.22)$$

We first observe that

$$D^2 F(0)(w_1, w_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^2 W(0)(w_1, w_2) \end{pmatrix}$$

and

$$D^3 F(0)(w_1, w_2, w_3) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^3 W(0)(w_1, w_2, w_3) \end{pmatrix}$$

for each

$$w_1 := \begin{pmatrix} \widehat{\alpha}_1 \\ v_1 \end{pmatrix},\ w_2 := \begin{pmatrix} \widehat{\alpha}_2 \\ v_2 \end{pmatrix},\ w_3 := \begin{pmatrix} \widehat{\alpha}_3 \\ v_3 \end{pmatrix} \in \overline{D(\mathscr{A})},$$

with $v_i = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_i \end{pmatrix}$, $i = 1,2,3$, where

$$
\begin{aligned}
&D^2W(0)(w_1,w_2) \\
&\quad = D^2W(0)\left(\begin{pmatrix}\widehat{\alpha}_1 \\ v_1\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}_2 \\ v_2\end{pmatrix}\right) \\
&\quad = \alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_2) + \widehat{\alpha}_2 DH\left(\overline{v}_{\alpha_k}\right)(v_1) + \widehat{\alpha}_1 DH\left(\overline{v}_{\alpha_k}\right)(v_2) \\
&\qquad + \widehat{\alpha}_2 \alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_1 \alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right),
\end{aligned}
\tag{8.3.23}
$$

and

$$
\begin{aligned}
&D^3W(0)(w_1,w_2,w_3) \\
&\quad = D^3W(0)\left(\begin{pmatrix}\widehat{\alpha}_1 \\ v_1\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}_2 \\ v_2\end{pmatrix}, \begin{pmatrix}\widehat{\alpha}_3 \\ v_3\end{pmatrix}\right) \\
&\quad = \widehat{\alpha}_1 D^2H\left(\overline{v}_{\alpha_k}\right)(v_2,v_3) + \widehat{\alpha}_2 D^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_3) + \widehat{\alpha}_3 D^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_2) \\
&\qquad + 2\widehat{\alpha}_2\widehat{\alpha}_3 D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) + 2\widehat{\alpha}_1\widehat{\alpha}_3 D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + 2\widehat{\alpha}_1\widehat{\alpha}_2 D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_3, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) + \widehat{\alpha}_2\widehat{\alpha}_3\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1, \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_1\widehat{\alpha}_3\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2, \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right) + \widehat{\alpha}_1\widehat{\alpha}_2\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_3, \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)(v_1,v_2,v_3) + \widehat{\alpha}_3\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_2\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,v_3, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) + \widehat{\alpha}_1\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_2,v_3, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_2\widehat{\alpha}_3\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_1\widehat{\alpha}_3\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\
&\qquad + \widehat{\alpha}_1\widehat{\alpha}_2\alpha_k D^3H\left(\overline{v}_{\alpha_k}\right)\left(v_3, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right),
\end{aligned}
\tag{8.3.24}
$$

in which $(k = 1,2,3)$

$$D^k H\left(\overline{v}_{\alpha_k}\right)\left(\binom{0}{\varphi_1},\ldots,\binom{0}{\varphi_k}\right)=\begin{pmatrix} h^{(k)}\left(\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\prod_{i=1}^{k}\int_0^{+\infty}\gamma(a)\,\varphi_i(a)\,da \\ 0\end{pmatrix},$$

$$h^{(1)}(x)=(1-\beta x)\exp(-\beta x),$$

$$h^{(2)}(x)=\left(\beta^2 x-2\beta\right)\exp(-\beta x),$$

$$h^{(3)}(x)=\left(-\beta^3 x+3\beta^2\right)\exp(-\beta x),$$

$$\overline{v}_{\alpha_k}=\binom{0}{\overline{u}_{\alpha_k}}=\begin{pmatrix} 0 \\ \frac{\ln\left(\alpha_k\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da}\exp(-\mu\cdot)\end{pmatrix},$$

$$\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da=\frac{\ln\left(\alpha_k\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right)}{\beta},\quad \frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}=\frac{1}{\widehat{\alpha}+\alpha_k}\times\frac{\exp(-\mu\cdot)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da},$$

$$\int_0^{+\infty}\gamma(a)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}(a)da=\frac{1}{\beta\alpha_k},\quad \frac{d^2\overline{u}_{\widehat{\alpha}+\alpha_k}}{d(\widehat{\alpha})^2}=-\frac{1}{(\widehat{\alpha}+\alpha_k)^2}\times\frac{\exp(-\mu\cdot)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da},$$

$$\int_0^{+\infty}\gamma(a)\left.\frac{d^2\overline{u}_{\widehat{\alpha}+\alpha_k}}{d(\widehat{\alpha})^2}\right|_{\widehat{\alpha}=0}(a)da=-\frac{1}{\beta(\alpha_k)^2}.$$

To simplify the computation, we use the eigenfunctions of $\mathscr{A}$ in $\mathscr{X}_c$ and consider

$$\widehat{e}_1:=\begin{pmatrix}1\\ \begin{pmatrix}0_{\mathbb{R}}\\ 0_C\end{pmatrix}\end{pmatrix},\ \widehat{e}_2:=\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu+i\omega_k).}\end{pmatrix}\end{pmatrix},\ \widehat{e}_3=:\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu-i\omega_k).}\end{pmatrix}\end{pmatrix}.$$

We have

$$\mathscr{A}\widehat{e}_1=0,\ \mathscr{A}\widehat{e}_2=i\omega_k\widehat{e}_2,\ \text{and}\ \mathscr{A}\widehat{e}_3=-i\omega_k\widehat{e}_3.$$

In order to simplify the notation, from now on we set

$$\chi:=\int_0^{+\infty}\gamma(a)\,e^{-\mu a}da=\frac{n!\exp(-\mu\tau)}{(\mu+\varsigma)^{n+1}}.\tag{8.3.25}$$

(i) Computation of $L_2(\widehat{e}_1,\widehat{e}_1)$: We have

$$\Pi_h D^2F(0)(\widehat{e}_1,\widehat{e}_1)=0$$

and

$$\mathscr{A}_c\widehat{e}_1=0.$$

By (8.3.22) we have

$$L_2(\mathscr{A}_c\widehat{e}_1,\widehat{e}_1)+L_2(\widehat{e}_1,\mathscr{A}_c\widehat{e}_1)=\mathscr{A}_h L_2(\widehat{e}_1,\widehat{e}_1)+\frac{1}{2!}\Pi_h D^2F(0)(\widehat{e}_1,\widehat{e}_1).$$

So

$$0=\mathscr{A}_h L_2(\widehat{e}_1,\widehat{e}_1).$$

Since 0 belongs to the resolvent set of $\mathscr{A}_h$, we obtain

$$L_2(\widehat{e}_1,\widehat{e}_1)=0.\tag{8.3.26}$$

(ii) Computation of $L_2(\widehat{e}_1,\widehat{e}_2)$: Since $\mathscr{A}_c\widehat{e}_1 = 0$ and $\mathscr{A}_c\widehat{e}_2 = i\omega_k\widehat{e}_2$, the equation

$$L_2\left(\mathscr{A}_c\widehat{e}_1,\widehat{e}_2\right)+L_2\left(\widehat{e}_1,\mathscr{A}_c\widehat{e}_2\right)=\mathscr{A}_hL_2(\widehat{e}_1,\widehat{e}_2)+\frac{1}{2!}\Pi_hD^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_2\right)$$

is equivalent to

$$\left(i\omega_k-\mathscr{A}_h\right)L_2\left(\widehat{e}_1,\widehat{e}_2\right)=\frac{1}{2!}\Pi_hD^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_2\right),$$

where

$$\begin{aligned}
&D^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_2\right)\\
&=\begin{pmatrix} 0_{\mathbb{R}} \\ D^2W\left(0\right)\left(\begin{pmatrix}1\\0_X\end{pmatrix},\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu+i\omega_k)\cdot}\end{pmatrix}\end{pmatrix}\right)\end{pmatrix}\\
&=\begin{pmatrix} 0_{\mathbb{R}} \\ DH\left(\overline{v}_{\alpha_k}\right)\begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu+i\omega_k)\cdot}\end{pmatrix}\end{pmatrix}\\
&\quad+\begin{pmatrix} 0_{\mathbb{R}} \\ \alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu+i\omega_k)\cdot}\end{pmatrix},\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\end{pmatrix}\\
&=\begin{pmatrix} 0_{\mathbb{R}} \\ h^{(1)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\int_0^{+\infty}\gamma(a)\,e^{-(\mu+i\omega_k)a}da \\ +\alpha_kh^{(2)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\int_0^{+\infty}\gamma(a)\,e^{-(\mu+i\omega_k)a}da\int_0^{+\infty}\gamma(a)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}(a)da \\ 0_{L^1}\end{pmatrix}.
\end{aligned}$$

Thus, we have

$$D^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_2\right)=c_{12}\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}1\\0_{L^1}\end{pmatrix}\end{pmatrix}$$

with

$$\begin{aligned}
c_{12}&=h^{(1)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\int_0^{+\infty}\gamma(a)\,e^{-(\mu+i\omega_k)a}da\\
&\quad+\alpha_kh^{(2)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\int_0^{+\infty}\gamma(a)\,e^{-(\mu+i\omega_k)a}da\int_0^{+\infty}\gamma(a)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}(a)da\\
&=\left(1-\beta\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\exp\left(-\beta\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\frac{\chi}{1-\ln\left(\alpha\chi\right)}\\
&\quad+\alpha_k\left(\beta^2\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da-2\beta\right)\exp\left(-\beta\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\chi}{1-\ln(\alpha\chi)} \times \frac{1}{\alpha_k}\frac{1}{\beta} \\
&= \frac{\chi}{1-\ln(\alpha\chi)} \left[\left(1-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right) \exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right) \right. \\
&\quad \left. + \left(\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da - 2\right) \exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right) \right] \\
&= -\frac{\chi}{1-\ln(\alpha\chi)} \exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right) \\
&= -\frac{\chi}{1-\ln(\alpha\chi)} \left(\alpha_k\int_0^{+\infty}\gamma(a)\,e^{-\mu a}da\right)^{-1} \\
&= -\frac{1}{\alpha_k(1-\ln(\alpha\chi))}.
\end{aligned}
$$

So

$$
L_2(\widehat{e}_1,\widehat{e}_2) = -\frac{1}{2\alpha_k(1-\ln(\alpha\chi))}\left(i\omega_k - \mathscr{A}_h\right)^{-1} \Pi_h \begin{pmatrix} 0_{\mathbb{R}} \\ 1 \\ 0_{L^1} \end{pmatrix}.
$$

By using a similar method together with Lemmas 8.3.9 and 8.3.10, we obtain the following results:

$$
L_2(\widehat{e}_1,\widehat{e}_2) = L_2(\widehat{e}_2,\widehat{e}_1) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{1,2} \end{pmatrix} \end{pmatrix}, \tag{8.3.27}
$$

$$
L_2(\widehat{e}_1,\widehat{e}_3) = L_2(\widehat{e}_3,\widehat{e}_1) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{1,3} \end{pmatrix} \end{pmatrix}, \tag{8.3.28}
$$

$$
L_2(\widehat{e}_2,\widehat{e}_3) = L_2(\widehat{e}_3,\widehat{e}_2) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{2,3} \end{pmatrix} \end{pmatrix}, \tag{8.3.29}
$$

$$
L_2(\widehat{e}_2,\widehat{e}_2) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{2,2} \end{pmatrix} \end{pmatrix}, \tag{8.3.30}
$$

$$
L_2(\widehat{e}_3,\widehat{e}_3) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{3,3} \end{pmatrix} \end{pmatrix}, \tag{8.3.31}
$$

where

$$
\psi_{1,2}(a) = \overline{\psi_{1,3}(a)} := -\frac{1}{2\alpha_k(1-\ln(\alpha_k\chi))}
$$

$$\times \begin{pmatrix} -\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(-i\omega_k+\mu)a}}{2i\omega_k} + \\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-2} \left[-\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda} a - \frac{1}{2} \frac{d^2\Delta(\alpha_k,i\omega_k)}{d\lambda^2} \right] e^{-(i\omega_k+\mu)a} \end{pmatrix},$$

$$\begin{aligned} \psi_{2,2}(a) = \overline{\psi_{3,3}(a)} &:= \frac{\beta\chi(\ln(\alpha_k\chi)-2)}{2(1-\ln(\alpha_k\chi))^2} \\ &\times \begin{pmatrix} -\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1} \frac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \\ \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1} \frac{e^{-(-i\omega_k+\mu)a}}{3i\omega_k} + \Delta(\alpha_k,2i\omega_k)^{-1} e^{-(2i\omega_k+\mu)a} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \psi_{2,3}(a) = &\frac{\beta\chi(\ln(\alpha_k\chi)-2)}{2(1-\ln(\alpha_k\chi))^2} \\ &\times \begin{pmatrix} \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1} \frac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \\ \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1} \frac{e^{-(-i\omega_k+\mu)a}}{i\omega_k} + \Delta(\alpha_k,0)^{-1} e^{-\mu a} \end{pmatrix}. \end{aligned}$$

By using (8.3.26)-(8.3.31), and the fact that

$$e_1 = \widehat{e}_1,\ e_2 = \widehat{e}_2 + \widehat{e}_3,\ \text{and } e_3 = \frac{\widehat{e}_2 - \widehat{e}_3}{i},$$

we obtain the following lemma.

Lemma 8.3.16. *The symmetric and bilinear map $L_2 : \mathscr{X}_c^2 \to \mathscr{X}_h \cap D(\mathscr{A})$ is defined by*

(a) $L_2(e_1,e_1) = 0$;

(b) $L_2(e_1,e_2)$ *and* $L_2(e_2,e_1)$ *are defined by*

$$L_2(e_1,e_2) = L_2(e_2,e_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Re}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(c) $L_2(e_1,e_3)$ *and* $L_2(e_3,e_1)$ *are defined by*

$$L_2(e_1,e_3) = L_2(e_3,e_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Im}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(d) $L_2(e_2,e_2)$ *is defined by*

$$L_2(e_2,e_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{pmatrix};$$

(e) $L_2(e_2, e_3)$ *and* $L_2(e_3, e_2)$ *are defined by*

$$L_2(e_2,e_3) = L_2(e_3,e_2) = \left(\begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ 2\mathrm{Im}\psi_{2,2} \end{pmatrix} \right);$$

(f) $L_2(e_3,e_3) = \left(\begin{matrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ -2\mathrm{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{matrix} \right).$

Define $G_2 : \mathscr{X} \to \mathscr{X}_h \cap D(\mathscr{A})$ by

$$G_2(\Pi_c w) := L_2(\Pi_c w, \Pi_c w), \ \forall w \in \mathscr{X},$$

the changes of variables $\xi_2 : \mathscr{X} \to \mathscr{X}$ and $\xi_2^{-1} : \mathscr{X} \to \mathscr{X}$ by

$$\xi_2(w) := w - G_2(\Pi_c w) \text{ and } \xi_2^{-1}(w) := w + G_2(\Pi_c w), \ \forall w \in \mathscr{X},$$

and $F_2 : \overline{D(\mathscr{A})} \to \mathscr{X}$ by

$$\begin{aligned} F_2(w) := {} & F\left(\xi_2^{-1}(w)\right) + \mathscr{A} G_2(\Pi_c w) - DG_2(\Pi_c w)\mathscr{A}_c \Pi_c w \\ & - DG_2(\Pi_c w)\Pi_c F\left(\xi_2^{-1}(w)\right). \end{aligned}$$

By applying Theorem 6.3.11 to (8.3.12) for $k = 2$, we obtain the following theorem.

Theorem 8.3.17. *By using the change of variables*

$$w_2(t) = w(t) - G_2(\Pi_c w(t)) \Leftrightarrow w(t) = w_2(t) + G_2(\Pi_c w_2(t)),$$

the map $t \to w(t)$ *is an integrated solution of the Cauchy problem (8.3.12) if and only if* $t \to w_2(t)$ *is an integrated solution of the Cauchy problem*

$$\begin{cases} \dfrac{dw_2(t)}{dt} = \mathscr{A} w_2(t) + F_2(w_2(t)), \ t \geq 0, \\ w_2(0) = w_2 \in \overline{D(\mathscr{A})}. \end{cases} \tag{8.3.32}$$

Moreover, the reduced equation of the Cauchy problem (8.3.32) is given by the ordinary differential equations on $\mathbb{R} \times \mathscr{Y}_c$ *(where* $\mathscr{Y}_c := \widehat{\Pi}_c(\mathscr{X})$*):*

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{dy_c(t)}{dt} = B_{\alpha_k}|_{\widehat{\Pi}_c(Y)} y_c(t) + \widehat{\Pi}_c W(I + G_2) \begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix} + \widehat{R}_c \begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix}, \end{cases} \tag{8.3.33}$$

where $\widehat{R}_c \in C^4(\mathbb{R} \times \mathscr{Y}_c, \mathscr{Y}_c)$*, and* $\widehat{R}_c \begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix}$ *is a remainder term of order* 4; *that is,*

$$\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix} = \|(\widehat{\alpha}, y_c)\|^4 O(\widehat{\alpha}, y_c),$$

where $O(\widehat{\alpha}, y_c)$ *is a function of* $(\widehat{\alpha}, y_c)$ *which remains bounded when* $(\widehat{\alpha}, y_c)$ *goes to* 0, *or equivalently,*

$$D^j\widehat{R}_c(0) = 0 \text{ for each } j = 1,2,3.$$

Furthermore,

$$\frac{\partial^j\widehat{R}_c(0)}{\partial^j\widehat{\alpha}} = 0, \ \forall j = 1,2,3,4,$$

which implies that

$$\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix} = O\left(\widehat{\alpha}^3\|y_c\| + \widehat{\alpha}^2\|y_c\|^2 + \widehat{\alpha}\|y_c\|^3 + \|y_c\|^4\right).$$

In the following theorem we compute the Taylor's expansion of the reduced system (8.3.33) by using the formula obtained for L_2 in Lemma 8.3.16.

Theorem 8.3.18. *The reduced system (8.3.33) expressed in terms of the basis* $\{e_1, e_2, e_3\}$ *has the following form*

$$\begin{cases}\dfrac{d\widehat{\alpha}(t)}{dt} = 0,\\ \dfrac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = M_c\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} + \left(\widetilde{H}_2 + \widetilde{H}_3 + \widehat{R}_c\right)\begin{pmatrix}\widehat{\alpha}(t)\\ \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\end{pmatrix},\end{cases} \tag{8.3.34}$$

where

$$M_c = \begin{bmatrix}0 & \omega_k\\ -\omega_k & 0\end{bmatrix};$$

the map $\widetilde{H}_2 : \mathbb{R}^3 \to \mathbb{R}^2$ *is defined by*

$$\widetilde{H}_2\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix} = \chi_2(\widehat{\alpha}, x, y)\left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)\end{bmatrix},$$

in which

$$\chi_2(\widehat{\alpha}, x, y) = -\frac{2}{\alpha_k[1-\ln(\alpha_k\chi)]}\widehat{\alpha}x + \frac{2\chi\beta(\ln(\alpha_k\chi)-2)}{[1-\ln(\alpha_k\chi)]^2}x^2;$$

the map $\widetilde{H}_3 : \mathbb{R}^3 \to \mathbb{R}^2$ *is defined by*

$$\widetilde{H}_3\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix} = \chi_3(\widehat{\alpha}, x, y)\left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)\end{bmatrix},$$

in which

$$\begin{aligned}
&\chi_3\left(\widehat{\alpha},x,y\right)\\
&=\left(-\frac{2\widehat{\alpha}}{\alpha_k\chi}+\frac{4\beta\left(\ln\left(\alpha_k\chi\right)-2\right)x}{1-\ln\left(\alpha_k\chi\right)}\right)\times\left[\left(x^2-y^2\right)\int_0^{+\infty}\gamma(a)\operatorname{Re}\psi_{2,2}\left(a\right)da\right.\\
&\quad+\left(x^2+y^2\right)\int_0^{+\infty}\gamma(a)\,\psi_{2,3}\left(a\right)da+2xy\int_0^{+\infty}\gamma(a)\operatorname{Im}\psi_{2,2}\left(a\right)da\\
&\quad\left.+2\widehat{\alpha}x\int_0^{+\infty}\gamma(a)\operatorname{Re}\psi_{1,2}\left(a\right)da+2\widehat{\alpha}y\int_0^{+\infty}\gamma(a)\operatorname{Im}\psi_{1,2}\left(a\right)da\right]\\
&\quad+\frac{1}{\left(\alpha_k\right)^2\left(1-\ln\left(\alpha_k\chi\right)\right)}\widehat{\alpha}^2x+\frac{2\beta\chi}{\alpha_k\left(1-\ln\left(\alpha_k\chi\right)\right)^2}\widehat{\alpha}x^2\\
&\quad+\frac{4\beta^2\left(-\ln\left(\alpha_k\chi\right)+3\right)\chi^2}{3\left(1-\ln\left(\alpha_k\chi\right)\right)^3}x^3;
\end{aligned}$$

and the remainder term $\widehat{R}_c\in C^4\left(\mathbb{R}^3,\mathbb{R}^2\right)$ *satisfies*

$$\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\x\\y\end{pmatrix}=O\left(\widehat{\alpha}^3\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|+\widehat{\alpha}^2\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2+\widehat{\alpha}\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^3+\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^4\right).\quad(8.3.35)$$

Proof. We firstly prove that the reduced system (8.3.33) expressed in terms of the basis $\{e_1,e_2,e_3\}$ has the following form

$$\begin{cases}\dfrac{d\widehat{\alpha}(t)}{dt}=0,\\ \dfrac{d}{dt}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=M_c\begin{pmatrix}x(t)\\y(t)\end{pmatrix}+\left(\widehat{H}_2+\widehat{H}_3+\widehat{R}_c\right)\begin{pmatrix}\widehat{\alpha}(t)\\\begin{pmatrix}x(t)\\y(t)\end{pmatrix}\end{pmatrix},\end{cases}\quad(8.3.36)$$

where the map $\widehat{H}_2:\mathbb{R}^3\to\mathbb{R}^2$ is defined by

$$\widehat{H}_2\begin{pmatrix}\widehat{\alpha}\\x\\y\end{pmatrix}=\widetilde{\psi}\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\operatorname{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\\operatorname{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}\quad(8.3.37)$$

with

$$\widetilde{\psi}=-\frac{\widehat{\alpha}}{\alpha_k\chi}\int_0^{+\infty}\gamma(a)\,\psi\left(a\right)da+\frac{\beta\left(\ln\left(\alpha_k\chi\right)-2\right)}{2\chi}\left(\int_0^{+\infty}\gamma(a)\,\psi\left(a\right)da\right)^2$$

and

$$\begin{aligned}
&\int_0^{+\infty}\gamma(a)\,\psi\left(a\right)da\\
&\quad=x\frac{2\chi}{1-\ln\left(\alpha_k\chi\right)}+2\left(x^2-y^2\right)\int_0^{+\infty}\gamma(a)\operatorname{Re}\psi_{2,2}\left(a\right)da
\end{aligned}$$

$$+2\left(x^2+y^2\right)\int_0^{+\infty}\gamma(a)\,\psi_{2,3}(a)\,da+4xy\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{2,2}(a)\,da$$
$$+4\widehat{\alpha}x\int_0^{+\infty}\gamma(a)\,\mathrm{Re}\psi_{1,2}(a)\,da+4\widehat{\alpha}y\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{1,2}(a)\,da.$$

The map $\widehat{H}_3:\mathbb{R}^3\to\mathbb{R}^2$ is defined by

$$\widehat{H}_3\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix}=\widehat{\psi}\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}, \tag{8.3.38}$$

where

$$\widehat{\psi}=\frac{1}{(\alpha_k)^2\left(1-\ln(\alpha_k\chi)\right)}\widehat{\alpha}^2x+\frac{2\beta\chi}{\alpha_k\left(1-\ln(\alpha_k\chi)\right)^2}\widehat{\alpha}x^2+\frac{4\beta^2\left(-\ln(\alpha_k\chi)+3\right)\chi^2}{3\left(1-\ln(\alpha_k\chi)\right)^3}x^3.$$

By using the Taylor's expansion of W around 0, the reduced system (8.3.33) can be rewritten as follows:

$$\begin{aligned}\frac{d\widehat{\alpha}(t)}{dt}&=0,\\ \frac{dy_c(t)}{dt}&=B_{\alpha_k}|_{\widehat{\Pi}_c(X)}y_c(t)+\frac{1}{2!}\widehat{\Pi}_cD^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}\right)^2\\ &\quad+\frac{1}{3!}\widehat{\Pi}_cD^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}\right)^3+\widetilde{R}_c\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}.\end{aligned}$$

Set

$$y_c=\begin{pmatrix}0\\ xb_1+yb_2\end{pmatrix}=\begin{pmatrix}0\\ x\left(e^{-(\mu+i\omega_k)\cdot}+e^{-(\mu-i\omega_k)\cdot}\right)+y\left(\frac{e^{-(\mu+i\omega_k)\cdot}-e^{-(\mu-i\omega_k)\cdot}}{i}\right)\end{pmatrix}.$$

Since we consider $\{e_1,e_2,e_3\}$ as the basis for $\mathscr{X}_c=\mathscr{R}(\Pi_c)$, i.e., $\left\{\binom{0}{b_1},\binom{0}{b_2}\right\}$ is a basis of $\mathscr{Y}_c:=\widehat{\Pi}_c(\mathscr{X})$, we obtain that

$$M_c=\begin{bmatrix}0&\omega_k\\ -\omega_k&0\end{bmatrix}.$$

Now we compute $\widehat{H}_2(\mathscr{X})$**.** We have

$$\begin{aligned}(I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}&=\widehat{\alpha}e_1+xe_2+ye_3+L_2\left(\widehat{\alpha}e_1+xe_2+ye_3,\widehat{\alpha}e_1+xe_2+ye_3\right)\\ &=\widehat{\alpha}e_1+xe_2+ye_3+\widehat{\alpha}^2L_2(e_1,e_1)+x^2L_2(e_2,e_2)+y^2L_2(e_3,e_3)\\ &\quad+2\widehat{\alpha}xL_2(e_1,e_2)+2\widehat{\alpha}yL_2(e_1,e_3)+2xyL_2(e_2,e_3).\end{aligned}$$

By Lemma 8.3.16, it follows that

$$
\begin{aligned}
&(I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}=\begin{pmatrix}\widehat{\alpha}\\ \begin{pmatrix}0\\ \psi\end{pmatrix}\end{pmatrix}\\
&\Leftrightarrow \psi(a)=x\left(e^{-(\mu+i\omega_k)a}+e^{-(\mu-i\omega_k)a}\right)+y\left(\frac{e^{-(\mu+i\omega_k)a}-e^{-(\mu-i\omega_k)a}}{i}\right)\\
&\quad +2\left(x^2-y^2\right)\mathrm{Re}\left(\psi_{2,2}(a)\right)+2\left(x^2+y^2\right)\psi_{2,3}(a)\\
&\quad +4\widehat{\alpha}x\mathrm{Re}\left(\psi_{1,2}(a)\right)+4\widehat{\alpha}y\mathrm{Im}\left(\psi_{1,2}(a)\right)+4xy\mathrm{Im}\left(\psi_{2,2}(a)\right).
\end{aligned}
\tag{8.3.39}
$$

By (8.3.23) we deduce that

$$
\begin{aligned}
&\frac{1}{2!}D^2W(0)\begin{pmatrix}\widehat{\alpha}\\ \begin{pmatrix}0\\ \psi\end{pmatrix}\end{pmatrix}^2\\
&=\widehat{\alpha}DH\left(\overline{v}_{\alpha_k}\right)\begin{pmatrix}0\\ \psi\end{pmatrix}+\frac{1}{2}\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0\\ \psi\end{pmatrix},\begin{pmatrix}0\\ \psi\end{pmatrix}\right)\\
&\quad+\widehat{\alpha}\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0\\ \psi\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{\alpha_k}\times\frac{\exp(-\mu\cdot)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da}\end{pmatrix}\right)=\begin{pmatrix}\widetilde{\psi}\\ 0\end{pmatrix},
\end{aligned}
$$

where

$$
\widetilde{\psi}=-\frac{\widehat{\alpha}}{\alpha_k\chi}\int_0^{+\infty}\gamma(a)\,\psi(a)\,da+\frac{\beta\left(\ln\left(\alpha_k\chi\right)-2\right)}{2\chi}\left(\int_0^{+\infty}\gamma(a)\,\psi(a)\,da\right)^2
$$

with

$$
\begin{aligned}
&\int_0^{+\infty}\gamma(a)\,\psi(a)\,da\\
&=x\frac{2\chi}{1-\ln\left(\alpha_k\chi\right)}+2\left(x^2-y^2\right)\int_0^{+\infty}\gamma(a)\,\mathrm{Re}\psi_{2,2}(a)\,da\\
&\quad+2\left(x^2+y^2\right)\int_0^{+\infty}\gamma(a)\,\psi_{2,3}(a)\,da+4xy\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{2,2}(a)\,da\\
&\quad+4\widehat{\alpha}x\int_0^{+\infty}\gamma(a)\,\mathrm{Re}\psi_{1,2}(a)\,da+4\widehat{\alpha}y\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{1,2}(a)\,da.
\end{aligned}
$$

By projecting on $\mathscr{X}_c$ and using Lemma 8.3.8 and the same identification as above, we obtain

$$
\begin{aligned}
&\frac{1}{2!}\widehat{\Pi}_c D^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}\right)^2=\widetilde{\psi}\widehat{\Pi}_c\begin{pmatrix}1\\ 0\end{pmatrix}\\
&=\widetilde{\psi}\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}0\\ \mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_1+\mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_2\end{bmatrix},
\end{aligned}
$$

and (8.3.37) follows. Set

$$\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}=\widetilde{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}$$
$$+\frac{1}{3!}\widehat{\Pi}_c\left\{D^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}\right)^3-D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3\right\}.$$

Then by (8.3.39) and (8.3.24), we deduce that the remainder term satisfies the order condition (8.3.35). Thus, it only remains to compute $\frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3$. In order to compute $\widehat{H}_3(\mathscr{X})$, we consider

$$\begin{aligned}
&D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3\\
&=3\widehat{\alpha}D^2H\left(\overline{v}_{\alpha_k}\right)(y_c,y_c)+6\left(\widehat{\alpha}\right)^2D^2H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\quad+3\left(\widehat{\alpha}\right)^2\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right)+\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)(y_c,y_c,y_c)\\
&\quad+3\widehat{\alpha}\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(y_c,y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\quad+3\left(\widehat{\alpha}\right)^2\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0},\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right).
\end{aligned}$$

Using the same notation as above for y_c and after some computation, we deduce that

$$\frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3=\begin{pmatrix}\widehat{\psi}\\ 0\end{pmatrix}$$

with

$$\begin{aligned}
\widehat{\psi}&=\frac{1}{6}h^{(2)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\left(3A\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^2+\frac{3\left(\widehat{\alpha}\right)^2}{\alpha_k\beta}\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)\\
&\quad+\frac{1}{6}h^{(3)}\left(\int_0^{+\infty}\gamma(a)\overline{u}_{\alpha_k}(a)\,da\right)\times\left[\alpha_k\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^3\right.\\
&\quad\left.+\frac{3\widehat{\alpha}}{\beta}\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^2+\frac{3\left(\widehat{\alpha}\right)^2}{\alpha_k\beta^2}\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right]\\
&=\frac{1}{(\alpha_k)^2(1-\ln(\alpha_k\chi))}\widehat{\alpha}^2x+\frac{2\beta\chi}{\alpha_k(1-\ln(\alpha_k\chi))^2}\widehat{\alpha}x^2\\
&\quad+\frac{4\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{3(1-\ln(\alpha_k\chi))^3}x^3.
\end{aligned}$$

By Lemma 8.3.8, we obtain

$$\begin{aligned}\frac{1}{3!}\widehat{\Pi}_c D^3 W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3 &= \widehat{\psi}\widehat{\Pi}_c\begin{pmatrix}1\\0\end{pmatrix}\\ &= \widehat{\psi}\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}0\\ Re\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_1+\mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_2\end{bmatrix}\end{aligned}$$

and (8.3.38) follows. Moreover, (8.3.36) can be rewritten as (8.3.34). □

From Theorem 8.3.18, dropping the auxiliary equation for the parameter, we obtain the following equations

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = M_c\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\\ &+\chi_2\left(\widehat{\alpha},x,y\right)\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}\\ &+\chi_3\left(\widehat{\alpha},x,y\right)\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}\\ &+\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix},\end{aligned} \tag{8.3.40}$$

where M_c, $\chi_2\left(\widehat{\alpha},x,y\right)$, $\chi_3\left(\widehat{\alpha},x,y\right)$ and $\widehat{R}_c$ are defined in Theorem 8.3.18 and $\widehat{\alpha}$ is the parameter here.

We now study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions following the Hopf bifurcation theorem presented in Hassard et al. [181, page 16]. We first make some preliminary remarks. Rewrite system (8.3.40) as follows

$$\frac{dX}{dt} = F(X,\widehat{\alpha}), \tag{8.3.41}$$

where the equilibrium point is $X = 0 \in \mathbb{R}^2$ and the critical value of the bifurcation parameter $\widehat{\alpha}$ is 0. Since the equilibrium solutions belong to the center mainfold, we have for each $|\widehat{\alpha}|$ small enough that

$$F(0,\widehat{\alpha}) = 0.$$

Notice that $\partial_x F\left(0,\widehat{\alpha}\right)$ is unknown whenever $\widehat{\alpha} \neq 0$. The system (8.3.40) only provides an approximation of order 2 for $\partial_x F\left(0,\widehat{\alpha}\right)$ with respect to $\widehat{\alpha}$. Nevertheless by using Proposition 6.1.22, we know that the eigenvalues of $\partial_x F\left(0,\widehat{\alpha}\right)$ $\lambda(\widehat{\alpha})$ are the roots of the original characteristic equation

$$1 = \eta(\alpha)\int_0^{+\infty}\gamma(a)e^{-(\mu+\lambda)a}da \Leftrightarrow 1 = \eta\left(\widehat{\alpha}+\alpha_k\right)\int_0^{+\infty}\gamma(a)e^{-(\mu+\lambda)a}da \tag{8.3.42}$$

with

$$\eta(\alpha)=\frac{1-\ln\left(\alpha\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right)}{\int_0^{+\infty}\gamma(a)e^{-\mu a}da}=\frac{1-\ln(\alpha\chi)}{\chi},$$

and

$$\lambda(0)=\pm\omega_k i.$$

The implicit function theorem implies that the characteristic equation has a unique pair of complex conjugate roots $\lambda(\widehat{\alpha}),\overline{\lambda(\widehat{\alpha})}$ close to $i\omega_k,-i\omega_k$ for $\widehat{\beta}$ in a neighborhood of 0. Here $\lambda(\widehat{\alpha})=a(\widehat{\alpha})+ib(\widehat{\alpha})$, $a(0)=0$ and $ib(0)=i\omega_k$ (where $\omega_k>0$ are provided by Theorem 8.3.15 for $k\in\mathbb{N}$). From (8.3.42), we have

$$\eta'(\widehat{\alpha}+\alpha_k)\int_0^{+\infty}\gamma(a)e^{-(\mu+\lambda(\widehat{\alpha}))a}da-\eta(\widehat{\alpha}+\alpha_k)\int_0^{+\infty}a\gamma(a)e^{-(\mu+\lambda)a}da\frac{d\lambda(\widehat{\alpha})}{d\widehat{\alpha}}=0$$

and

$$\begin{aligned}\int_0^{+\infty}a\gamma(a)e^{-(\mu+\lambda)a}da&=\int_\tau^{+\infty}a(a-\tau)^n e^{-(\mu+\lambda)a}da\\&=\int_\tau^{+\infty}(a-\tau)^{n+1}e^{-(\mu+\lambda)a}da+\tau\int_\tau^{+\infty}(a-\tau)^n e^{-(\mu+\lambda)a}da\\&=(n+1)!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+2}}+\tau n!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+1}}\\&=\left[\frac{(n+1)}{(\varsigma+\lambda+\mu)}+\tau\right]n!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+1}}\\&=\left[\frac{(n+1)}{(\varsigma+\lambda+\mu)}+\tau\right]\int_0^{+\infty}\gamma(a)e^{-(\mu+\lambda)a}da.\end{aligned}$$

Thus

$$\eta'(\widehat{\alpha}+\alpha_k)-\eta(\widehat{\alpha}+\alpha_k)\left[\frac{(n+1)}{(\varsigma+\lambda+\mu)}+\tau\right]\frac{d\lambda(\widehat{\alpha})}{d\widehat{\alpha}}=0$$

and

$$\begin{aligned}\frac{d\lambda(0)}{d\widehat{\alpha}}&=\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\left[\frac{(n+1)}{(\varsigma+i\omega_k+\mu)}+\tau\right]^{-1}=\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\frac{(\varsigma+i\omega_k+\mu)}{(n+1)+\tau(\varsigma+i\omega_k+\mu)}\\&=\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\frac{(\varsigma+i\omega_k+\mu)\left[\left[(n+1)+\tau(\varsigma+\mu)\right]-i\tau\omega_k\right]}{\left[(n+1)+\tau(\varsigma+\mu)\right]^2+\tau^2\omega_k^2},\end{aligned}$$

$$\begin{aligned}a'(0)&=\mathrm{Re}\left[\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\left[\frac{(n+1)}{(\varsigma+i\omega_k+\mu)}+\tau\right]^{-1}\right]\\&=\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\mathrm{Re}\left[\frac{(\varsigma+i\omega_k+\mu)}{(n+1)+\tau(\varsigma+i\omega_k+\mu)}\right].\end{aligned}$$

It follows that

$$a'(0) = \frac{\alpha_k \chi}{[\ln(\alpha_k \chi) - 1]} \frac{(\varsigma+\mu)[(n+1)+\tau(\varsigma+\mu)]+\tau\omega_k^2}{[(n+1)+\tau(\varsigma+\mu)]^2+\tau^2\omega_k^2} > 0. \tag{8.3.43}$$

Finally, the spectrum of $\partial_x F(0,\widehat{\alpha})$ is

$$\sigma(\partial_x F(0,\widehat{\alpha})) = \left\{\lambda(\widehat{\alpha}), \overline{\lambda(\widehat{\alpha})}\right\}.$$

Using a procedure as in the proof of Lemma 3.3 on p.92 in Kuznetsov [223] and introducing a complex variable z, we rewrite system (8.3.41) for sufficiently small $|\widehat{\alpha}|$ as a single equation:

$$\dot{z} = \lambda(\widehat{\alpha})z + g(z,\bar{z};\widehat{\alpha}), \tag{8.3.44}$$

where

$$\lambda(\widehat{\alpha}) = a(\widehat{\alpha}) + ib(\widehat{\alpha}), \;\; g(z,\bar{z},\widehat{\alpha}) = \sum_{i+j=2}^{3} \frac{1}{i!j!} g_{ij}(\widehat{\alpha}) z^i \bar{z}^j + O(|z|^3).$$

One can verify that system (8.3.41) satisfies

(1) $F(0,\widehat{\alpha}) = 0$ for $\widehat{\alpha}$ in an open interval containing 0, and $0 \in \mathbb{R}^2$ is an isolated stationary point of F;
(2) $F(X,\widehat{\alpha})$ is jointly $C^{L+2}(L \geq 2)$ in X and $\widehat{\alpha}$ in a neighborhood of $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$;
(3) $A(\widehat{\alpha}) = D_X F(0,\widehat{\alpha})$ has a pair of complex conjugate eigenvalues λ and $\bar{\lambda}$ such that $\lambda(\widehat{\alpha}) = a(\widehat{\alpha}) + ib(\widehat{\alpha})$, where $b(0) = \omega_0 > 0$, $a(0) = 0$, $a'(0) \neq 0$,

then by Hassard et al. [181, Theorem II, p. 16], there exist an $\varepsilon_p > 0$ and a $C^{L+1}-$function

$$\widehat{\alpha}(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \widehat{\alpha}_{2i} \varepsilon^{2i} + O(\varepsilon^{L+1}), \; 0 < \varepsilon < \varepsilon_p, \tag{8.3.45}$$

such that for each $\varepsilon \in (0,\varepsilon_p)$ system (8.3.41) has a family of periodic solutions $P_\varepsilon(t)$ with period $T(\varepsilon)$ occurring for $\widehat{\alpha} = \widehat{\alpha}(\varepsilon)$. The period $T(\varepsilon)$ of $P_\varepsilon(t)$ is a $C^{L+1}-$function given by

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \sum_{1}^{[\frac{L}{2}]} \tau_{2i} \varepsilon^{2i}] + O(\varepsilon^{L+1}), \quad 0 < \varepsilon < \varepsilon_p. \tag{8.3.46}$$

Two of Floquet exponents of $P_\varepsilon(t)$ approach 0 as $\varepsilon \downarrow 0$. One is 0 for $\varepsilon \in (0,\varepsilon_p)$ and the other is a $C^{L+1}-$function

$$\beta(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \beta_{2i} \varepsilon^{2i} + O(\varepsilon^{L+1}), \quad 0 < \varepsilon < \varepsilon_p. \tag{8.3.47}$$

Moreover, $P_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\beta(\varepsilon) < 0$ and unstable if $\beta(\varepsilon) > 0$.

Next we need to compute the coefficients $\widehat{\alpha}_{2i}$ and β_{2i} in (8.3.45) and (8.3.47). If the Poincaré normal form of (8.3.44) is

$$\dot{\xi} = \lambda(\widehat{\alpha})\xi + \sum_{j=1}^{[L/2]} c_j(\widehat{\alpha})\xi|\xi|^{2j} + O(|\xi||(\xi,\widehat{\alpha})|^{L+1}) \equiv C(\xi,\overline{\xi},\widehat{\alpha}), \tag{8.3.48}$$

where $C(\xi,\overline{\xi},\widehat{\alpha})$ is C^{L+2} jointly in $(\xi,\overline{\xi},\widehat{\alpha})$ in a neighborhood of $0 \in \mathbb{C}\times\mathbb{C}\times\mathbb{R}$, then the results in Hassard et al. [181, p. 32 and p. 44] imply that the periodic solution of period $T(\varepsilon)$ such that $\xi(0,\widehat{\alpha}) = \varepsilon$ of (8.3.48) has the form

$$\xi = \varepsilon \exp[2\pi it/T(\varepsilon)] + O(\varepsilon^{L+2}),$$

where

$$T(\varepsilon) = \frac{2\pi}{\omega_0}[1 + \sum_1^L \tau_i \varepsilon^i] + O(\varepsilon^{L+1}) \tag{8.3.49}$$

and

$$\mu(\varepsilon) = \widehat{\alpha}(\varepsilon) = \sum_1^L \mu_i \varepsilon^i + O(\varepsilon^{L+1}). \tag{8.3.50}$$

Furthermore, the coefficients are given by the following formulae:

$$\begin{aligned}
\mu_1 &= 0,\\
\mu_2 &= -\frac{\mathrm{Re}c_1(0)}{a'(0)},\\
\mu_3 &= 0,\\
\mu_4 &= -\frac{1}{a'(0)}[\mathrm{Re}c_2(0) + \mu_2 \mathrm{Re}c_1'(0) + \frac{a''(0)}{2}\mu_2^2],\\
\tau_1 &= 0,\\
\tau_2 &= \frac{-1}{\omega_0}[\mathrm{Im}c_1(0) + \mu_2 b'(0)],\\
\tau_3 &= 0,\\
\tau_4 &= -\frac{1}{\omega_0}[a'(0)\mu_4 + \frac{a''(0)}{2}\mu_2^2 + \mathrm{Im}c_1'(0)\mu_2 + \mathrm{Im}c_2(0) - \omega_0\tau_2^2],\\
\beta_1 &= 0\\
\beta_2 &= 2\mathrm{Re}c_1(0),
\end{aligned}$$

where

$$c_1(0) = \frac{i}{2\omega_0}(g_{20}(0)g_{11}(0) - 2|g_{11}(0)|^2 - \frac{1}{3}|g_{02}(0)|^2) + \frac{g_{21}(0)}{2}. \tag{8.3.51}$$

Applying the results in [181, pp. 45-51], we can change equation (8.3.44) into the Poincaré normal form (8.3.48) by using the following transformation:

$$\begin{aligned} z &= \xi + \chi(\xi, \bar{\xi}; \mu) \\ &= \xi + \sum_{i+j=2}^{L+1} \frac{1}{i!j!} \chi_{ij}(\mu) \xi^i \bar{\xi}^j, \ \chi_{ij} \equiv 0 \text{ for } i = j+1. \end{aligned}$$

To use the bifurcation formulae for $\mu(\varepsilon), \beta(\varepsilon)$ and $T(\varepsilon)$, we need only to compute $c_1(0), c_1'(0)$, and $c_2(0)$. For sufficiently small ε, if $\mu_2 \neq 0$, $\beta_2 \neq 0$, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by the signs of μ_2 and β_2.

By introducing a complex variable $z = x + iy$, when $\widehat{\alpha} = 0$ the system (8.3.40) reduces to

$$\dot{z} = -i\omega_k z(t) + [\chi_2(0, \mathrm{Re}(z), \mathrm{Im}(z)) + \chi_3(0, \mathrm{Re}(z), \mathrm{Im}(z))] \left(\overline{\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}} \right)^{-1} + h.o.t.$$

Set $\bar{z}(t) := \overline{z(t)}$, then we obtain

$$\frac{d\bar{z}(t)}{dt} = i\omega_k \bar{z}(t) + [\chi_2(0, \mathrm{Re}(\bar{z}), -\mathrm{Im}(\bar{z})) + \chi_3(0, \mathrm{Re}(\bar{z}), -\mathrm{Im}(\bar{z}))] \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right)^{-1} + h.o.t. \tag{8.3.52}$$

where

$$\begin{aligned} \chi_2(0, \mathrm{Re}(\bar{z}), -\mathrm{Im}(\bar{z})) &= \frac{2\chi\beta(\ln(\alpha_k\chi) - 2)}{[1 - \ln(\alpha_k\chi)]^2} (\mathrm{Re}(\bar{z}))^2, \\ \chi_3(0, \mathrm{Re}(\bar{z}), -\mathrm{Im}(\bar{z})) &= \frac{4\beta(\ln(\alpha_k\chi) - 2)}{1 - \ln(\alpha_k\chi)} \mathrm{Re}(\bar{z}) \\ &\quad \times \left[\left((\mathrm{Re}(\bar{z}))^2 - (\mathrm{Im}(\bar{z}))^2 \right) \int_0^{+\infty} \gamma(a) \mathrm{Re}\psi_{2,2}(a)\, da \right. \\ &\quad + \left((\mathrm{Re}(\bar{z}))^2 + (\mathrm{Im}(\bar{z}))^2 \right) \int_0^{+\infty} \gamma(a)\, \psi_{2,3}(a)\, da \\ &\quad \left. - 2\mathrm{Re}(\bar{z}) \mathrm{Im}(\bar{z}) \int_0^{+\infty} \gamma(a) \mathrm{Im}\psi_{2,2}(a)\, da \right] \\ &\quad + \frac{4\beta^2(-\ln(\alpha_k\chi) + 3)\chi^2}{3(1 - \ln(\alpha_k\chi))^3} (\mathrm{Re}(\bar{z}))^3 \end{aligned}$$

with

$$\begin{aligned} \psi_{2,2}(a) &= \frac{\beta\chi(\ln(\alpha_k\chi) - 2)}{2(1 - \ln(\alpha_k\chi))^2} \\ &\quad \times \left(\begin{matrix} -\dfrac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(i\omega_k + \mu)a}}{i\omega_k} - \dfrac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(-i\omega_k + \mu)a}}{3i\omega_k} \\ +\Delta(\alpha_k, 2i\omega_k)^{-1} e^{-(2i\omega_k + \mu)a} \end{matrix} \right), \end{aligned}$$

$$\psi_{2,3}(a) = \frac{\beta\chi\left(\ln(\alpha_k\chi)-2\right)}{2\left(1-\ln(\alpha_k\chi)\right)^2} \times \left(\begin{array}{c} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)^{-1} \frac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \left(\frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}\right)^{-1} \frac{e^{-(-i\omega_k+\mu)a}}{i\omega_k} \\ +\Delta(\alpha_k, 0)^{-1} e^{-\mu a} \end{array} \right).$$

Now by considering equation (8.3.52), we obtain that after some computations

$$g_{11} = \frac{\chi\beta\left(\ln(\alpha_k\chi)-2\right)}{\left[1-\ln(\alpha_k\chi)\right]^2} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)^{-1}.$$

Moreover, we deduce that

$$\begin{aligned} g_{20} &= g_{11}, \quad g_{02} = g_{11}, \\ g_{21} &= \frac{i2ce+2cf+2be-i2bf+3ae-i3af}{4}, \end{aligned} \tag{8.3.53}$$

where

$$\begin{aligned} a &= \frac{4\chi^2\beta^2\left(3-\ln(\alpha_k\chi)\right)}{3\left[1-\ln(\alpha_k\chi)\right]^3} \\ &\quad + \frac{4\beta\left(\ln(\alpha_k\chi)-2\right)}{1-\ln(\alpha_k\chi)} \left[\int_0^{+\infty} \gamma(a)\,\mathrm{Re}\,\psi_{2,2}(a)\,da + \int_0^{+\infty} \gamma(a)\,\psi_{2,3}(a)\,da\right], \\ b &= \frac{4\beta\left(\ln(\alpha_k\chi)-2\right)}{1-\ln(\alpha_k\chi)} \left[-\int_0^{+\infty} \gamma(a)\,\mathrm{Re}\,\psi_{2,2}(a)\,da + \int_0^{+\infty} \gamma(a)\,\psi_{2,3}(a)\,da\right], \\ c &= \frac{8\beta\left(\ln(\alpha_k\chi)-2\right)}{1-\ln(\alpha_k\chi)} \int_0^{+\infty} \gamma(a)\,\mathrm{Im}\,\psi_{2,2}(a)\,da, \\ e &= \left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2} \mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right), \\ f &= \left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2} \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right). \end{aligned}$$

Hence, we obtain

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}(0)g_{11}(0) - 2\left|g_{11}(0)\right|^2 - \frac{1}{3}\left|g_{02}(0)\right|^2\right) + \frac{g_{21}(0)}{2}$$

and

$$\mu_2 = -\frac{\mathrm{Re}\,c_1(0)}{\alpha'(0)}, \quad \beta_2 = 2\mathrm{Re}\,c_1(0), \quad \tau_2 = \frac{-1}{\omega_k}\left[\mathrm{Im}\,c_1(0) + \mu_2 b'(0)\right].$$

We summarize the above discussions into a theorem on the direction and stability of Hopf bifurcation in the age-structured model (8.3.1).

Theorem 8.3.19. *The direction of the Hopf bifurcation described in Theorem 8.3.15 is determined by the sign of μ_2 : if $\mu_2 > 0(< 0)$, then the bifurcating periodic solutions exist for $\alpha > \alpha_k(\alpha < \alpha_k)$. The bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0(> 0)$. The period of the bifurcating periodic solutions of the age-structured model (8.3.1) increases (decreases) if $\tau_2 > 0(< 0)$.*

8.3.5 Normal Forms

Now we apply the normal form theory developed in Chapter 6 to the age-structured population model (8.3.1). Set $\mathscr{X} = \mathbb{R}\times X$, $\mathscr{X}_0 = \mathbb{R}\times\overline{D(A)}$. Consider the linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{X} \to \mathscr{X}$ defined by (8.3.11).

Recall that

$$\overline{\Pi_{i\omega_k}(x)} = \Pi_{-i\omega_k}(\bar{x}),\ \forall x \in \mathscr{X} + i\mathscr{X}.$$

The projectors $\Pi_c : \mathscr{X} \to \mathscr{X}$ and $\Pi_h : \mathscr{X} \to \mathscr{X}$ are defined by

$$\begin{aligned} \Pi_c(x) &:= \left(\Pi_0 + \Pi_{i\omega_k} + \Pi_{-i\omega_k}\right)(x),\ \forall x \in \mathscr{X}, \\ \Pi_h(x) &:= (I - \Pi_c)(x),\ \forall x \in \mathscr{X}. \end{aligned}$$

Denote

$$\mathscr{X}_c := \Pi_c(\mathscr{X}),\ \ \mathscr{X}_h := \Pi_h(\mathscr{X}), \mathscr{A}_c := \mathscr{A}\mid_{\mathscr{X}_c},\ \mathscr{A}_h := \mathscr{A}\mid_{\mathscr{X}_h}.$$

Now we have the decomposition

$$\mathscr{X} = \mathscr{X}_c \oplus \mathscr{X}_h.$$

Define the basis of $\mathscr{X}_c$ by

$$\widehat{e}_1 := \begin{pmatrix} 1 \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_C \end{pmatrix} \end{pmatrix},\ \widehat{e}_2 := \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_1 \end{pmatrix} \end{pmatrix},\ \widehat{e}_3 =: \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_2 \end{pmatrix} \end{pmatrix}$$

with

$$c_1 = e^{-(\mu+i\omega_k)\cdot} \text{ and } c_2 = e^{-(\mu-i\omega_k)\cdot}.$$

We have

$$\mathscr{A}\widehat{e}_1 = 0,\ \mathscr{A}\widehat{e}_2 = i\omega_k\widehat{e}_2, \text{ and } \mathscr{A}\widehat{e}_3 = -i\omega_k\widehat{e}_3.$$

Set

$$w := \begin{pmatrix} \widehat{\alpha} \\ \widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{u} \end{pmatrix} \end{pmatrix} \in \overline{D(\mathscr{A})},$$

$$\widehat{\Pi}_c\widehat{v} := \widehat{v}_c,\ \ \widehat{\Pi}_h\widehat{v} := \widehat{v}_h\ \ w_c := \Pi_c w = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\Pi}_c\widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{v}_c \end{pmatrix}$$

and

$$w_h := \Pi_h w = (I - \Pi_c)w = \begin{pmatrix} 0 \\ \widehat{\Pi}_h \widehat{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{v}_h \end{pmatrix}.$$

Notice that $\left\{ \begin{pmatrix} 0 \\ c_1 \end{pmatrix}, \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \right\}$ is the basis of $\mathscr{X}_c$. Set

$$\widehat{v}_c = \begin{pmatrix} 0 \\ x_1 c_1 + x_2 c_2 \end{pmatrix}$$

and

$$\chi := \int_0^{+\infty} \gamma(a)\, e^{-\mu a} da = \frac{n! \exp(-\mu \tau)}{(\mu + \varsigma)^{n+1}}.$$

We observe that for each

$$w_1 := \begin{pmatrix} \widehat{\alpha}_1 \\ v_1 \end{pmatrix},\ w_2 := \begin{pmatrix} \widehat{\alpha}_2 \\ v_2 \end{pmatrix} \in \overline{D(\mathscr{A})}$$

with $v_i = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_i \end{pmatrix}$, $i = 1, 2$,

$$\begin{aligned} D^2 W(0)\,(w_1, w_2) &= D^2 W(0) \left(\begin{pmatrix} \widehat{\alpha}_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} \widehat{\alpha}_2 \\ v_2 \end{pmatrix} \right) \\ &= \alpha_k D^2 H \left(\overline{v}_{\alpha_k}\right)(v_1, v_2) + \widehat{\alpha}_2 DH\left(\overline{v}_{\alpha_k}\right)(v_1) + \widehat{\alpha}_1 DH\left(\overline{v}_{\alpha_k}\right)(v_2) \\ &\quad + \widehat{\alpha}_2 \alpha_k D^2 H\left(\overline{v}_{\alpha_k}\right)\left(v_1, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0} \right) \\ &\quad + \widehat{\alpha}_1 \alpha_k D^2 H\left(\overline{v}_{\alpha_k}\right)\left(v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0} \right) \end{aligned}$$

with

$$D^2 H\left(\overline{v}_{\alpha_k}\right)\left(\binom{0}{\varphi_1}, \binom{0}{\varphi_2} \right) = \begin{pmatrix} \frac{\beta[\ln(\alpha_k \chi) - 2]}{\alpha_k \chi} \prod_{i=1}^{2} \int_0^{+\infty} \gamma(a)\, \varphi_i(a)\, da \\ 0 \end{pmatrix}.$$

Then

$$\frac{1}{2!} D^2 W(0)(w)^2 = \frac{1}{2!} D^2 W(0) \begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{u} \end{pmatrix} \end{pmatrix}^2 = \begin{pmatrix} \widetilde{\widetilde{\psi}} \\ 0 \end{pmatrix},$$

where

$$\widetilde{\widetilde{\psi}} = -\frac{\widehat{\alpha}}{\alpha_k \chi} \int_0^{+\infty} \gamma(a)\, \widehat{u}(a)\, da + \frac{\beta\left(\ln(\alpha_k \chi) - 2\right)}{2\chi} \left(\int_0^{+\infty} \gamma(a)\, \widehat{u}(a)\, da \right)^2.$$

By projecting on $\mathscr{X}_c$ and using Lemma 8.3.8, we obtain

$$\frac{1}{2!}\widehat{\Pi}_c D^2 W(0)\left(\begin{array}{c}\widehat{\alpha}\\ \begin{pmatrix}0\\ \widehat{u}\end{pmatrix}\end{array}\right)^2 = \widetilde{\widetilde{\psi}}\widehat{\Pi}_c\begin{pmatrix}1\\0\end{pmatrix}$$
$$= \widetilde{\widetilde{\psi}}\left[\begin{array}{c}0\\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}c_1 + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}c_2\end{array}\right].$$

Now we compute $\frac{1}{2!}D^2W(0)(w_c)^2$, $\frac{1}{2!}\Pi_c D^2F(0)(w_c)^2$, $\frac{1}{2!}\Pi_h D^2F(0)(w_c)^2$ and $\frac{1}{3!}D^3W(0)(w_c)^3$ expressed in terms of the basis $\{\widehat{e}_1,\widehat{e}_2,\widehat{e}_3\}$. We first obtain that

$$\frac{1}{2!}D^2W(0)(w_c)^2 = \frac{1}{2!}D^2W(0)\left(\begin{array}{c}\widehat{\alpha}\\ \begin{pmatrix}0\\ x_1c_1+x_2c_2\end{pmatrix}\end{array}\right)^2 = \begin{pmatrix}\widetilde{\psi}\\0\end{pmatrix},$$

where

$$\widetilde{\psi} = -\frac{\widehat{\alpha}}{\alpha_k\chi}\int_0^{+\infty}\gamma(a)\left(x_1c_1+x_2c_2\right)(a)\,da$$
$$+\frac{\beta\left(\ln\left(\alpha_k\chi\right)-2\right)}{2\chi}\left(\int_0^{+\infty}\gamma(a)\left(x_1c_1+x_2c_2\right)(a)\,da\right)^2.$$

Thus

$$\frac{1}{2!}\Pi_c D^2F(0)(w_c)^2 = \left(\begin{array}{c}0\\ \frac{1}{2!}\widehat{\Pi}_c D^2W(0)\left(\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}_c\end{pmatrix}\right)^2\end{array}\right) = \left(\begin{array}{c}0\\ \widetilde{\psi}\widehat{\Pi}_c\binom{1}{0}\end{array}\right)$$
$$= \left(\begin{array}{c}0\\ \widetilde{\psi}\left[\begin{array}{c}0\\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}c_1 + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}c_2\end{array}\right]\end{array}\right) \tag{8.3.54}$$

and

$$\frac{1}{2!}\Pi_h D^2F(0)(w_c)^2 = \frac{1}{2!}\left(I-\Pi_c\right)D^2F(0)\left(\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}_c\end{pmatrix}\right)^2$$
$$= \left(\begin{array}{c}0\\ \widetilde{\psi}\left[\begin{array}{c}1\\ -\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}c_1 - \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}c_2\end{array}\right]\end{array}\right). \tag{8.3.55}$$

Next we obtain

$$\frac{1}{3!}D^3W(0)(w_c)^3 = \frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}_c\end{pmatrix}^3 = \begin{pmatrix}\widehat{\psi}\\0\end{pmatrix}$$

with

$$\widehat{\psi} = \frac{1}{(\alpha_k)^2\,(1-\ln(\alpha_k\chi))}\widehat{\alpha}^2\left(\frac{x_1+x_2}{2}\right) + \frac{2\beta\chi}{\alpha_k\,(1-\ln(\alpha_k\chi))^2}\widehat{\alpha}\left(\frac{x_1+x_2}{2}\right)^2 + \frac{4\beta^2\,(-\ln(\alpha_k\chi)+3)\,\chi^2}{3\,(1-\ln(\alpha_k\chi))^3}\left(\frac{x_1+x_2}{2}\right)^3.$$

By Lemma 8.3.8, we obtain

$$\frac{1}{3!}\Pi_c D^3 F(0)(w_c)^3 = \begin{pmatrix} 0 \\ \frac{1}{3!}\widehat{\Pi}_c D^3 W(0)(w_c)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\psi}\widehat{\Pi}_c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \widehat{\psi}\begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix} \end{pmatrix}. \tag{8.3.56}$$

In the following we will compute the normal form of system (8.3.12). Define $\Theta_m^c : V^m(\mathscr{X}_c,\mathscr{X}_c) \to V^m(\mathscr{X}_c,\mathscr{X}_c)$ by

$$\Theta_m^c(G_c) := [\mathscr{A}_c, G_c],\ \forall G_c \in V^m(\mathscr{X}_c,\mathscr{X}_c) \tag{8.3.57}$$

and $\Theta_m^h : V^m(\mathscr{X}_c,\mathscr{X}_h \cap D(\mathscr{A})) \to V^m(\mathscr{X}_c,\mathscr{X}_h)$ by

$$\Theta_m^h(G_h) := [\mathscr{A}, G_h],\ \forall G_h \in V^m(\mathscr{X}_c,\mathscr{X}_h \cap D(\mathscr{A})). \tag{8.3.58}$$

We decompose $V^m(\mathscr{X}_c,\mathscr{X}_c)$ into the direct sum

$$V^m(\mathscr{X}_c,\mathscr{X}_c) = \mathscr{R}_m^c \oplus \mathscr{C}_m^c,$$

where

$$\mathscr{R}_m^c := \mathscr{R}(\Theta_m^c),$$

is the range of Θ_m^c, and $\mathscr{C}_m^c$ is some complementary space of $\mathscr{R}_m^c$ into $V^m(\mathscr{X}_c,\mathscr{X}_c)$. Define $\mathscr{P}_m : V^m(\mathscr{X}_c,\mathscr{X}) \to V^m(\mathscr{X}_c,\mathscr{X})$ the bounded linear projector satisfying

$$\mathscr{P}_m(V^m(\mathscr{X}_c,\mathscr{X})) = \mathscr{R}_m^c \oplus V^m(\mathscr{X}_c,\mathscr{X}_h), \text{ and } (I-\mathscr{P}_m)(V^m(\mathscr{X}_c,\mathscr{X})) = \mathscr{C}_m^c.$$

Now we apply the method described in Theorem 6.3.11 for $k=3$ to system (8.3.12). The main point is to compute $G_2 \in V^2(\mathscr{X}_c, D(\mathscr{A}))$ such that

$$[\mathscr{A}, G_2](w_c) = \mathscr{P}_2\left[\frac{1}{2!}D^2F(0)(w_c,w_c)\right] \text{ for each } w_c \in \mathscr{X}_c \tag{8.3.59}$$

in order to obtain the normal form, because the reduced system is the following

$$\frac{dw_c(t)}{dt} = \mathscr{A}_c w_c(t) + \frac{1}{2!}\Pi_c D^2 F_3(0)(w_c(t),w_c(t)) + \frac{1}{3!}\Pi_c D^3 F_3(0)(w_c(t),w_c(t),w_c(t)) + R_c(w_c(t)) \tag{8.3.60}$$

where

$$\frac{1}{2!}\Pi_c D^2 F_3(0)(w_c,w_c) = \frac{1}{2!}\Pi_c D^2 F_2(0)(w_c,w_c)$$
$$= \frac{1}{2!}\Pi_c D^2 F(0)(w_c,w_c) - [\mathscr{A}_c, \Pi_c G_2](w_c) \tag{8.3.61}$$

and

$$\frac{1}{3!}\Pi_c D^3 F_3(0)(w_c,w_c,w_c) = \frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c) - \Pi_c[\mathscr{A}, G_3](w_c)$$
$$= \frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c) - [\mathscr{A}_c, \Pi_c G_3](w_c). \tag{8.3.62}$$

Set

$$G_{m,k} := \Pi_k G_m,\ \forall k = c,h,\ m \geq 2.$$

Recall that (8.3.59) is equivalent to find $G_{2,c} \in V^2(\mathscr{X}_c, \mathscr{X}_c)$ and $G_{2,h} \in V^2(\mathscr{X}_c, \mathscr{X}_h \cap D(\mathscr{A}))$ satisfying

$$[\mathscr{A}_c, G_{2,c}] = \Pi_c \mathscr{P}_2\left[\frac{1}{2!}D^2 F(0)(w_c,w_c)\right] \tag{8.3.63}$$

and

$$[\mathscr{A}, G_{2,h}] = \Pi_h \mathscr{P}_2\left[\frac{1}{2!}D^2 F(0)(w_c,w_c)\right]. \tag{8.3.64}$$

From (8.3.62), we know that the third order term $\frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c)$ in the equation is needed after computing the normal form up to the second order. In the following lemma we find the expression of $\frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c)$.

Lemma 8.3.20. *Let $G_2 \in V^2(\mathscr{X}_c, D(\mathscr{A}))$ be defined in (8.3.59). Then after the change of variables*

$$w = \overline{w} + G_2(\Pi_c \overline{w}), \tag{8.3.65}$$

system (8.3.12) becomes (after dropping the bars)

$$\frac{dw(t)}{dt} = \mathscr{A}w(t) + F_2(w(t)),\ w(0) = w_0 \in \overline{D(\mathscr{A})},$$

where

$$F_2(w(t)) = F(w(t)) - [\mathscr{A}, G_2](w_c(t)) + O(\|w(t)\|^3).$$

In particular,

$$\frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c)$$
$$= \Pi_c D^2 F(0)(w_c, G_2(w_c)) + \frac{1}{3!}\Pi_c D^3 F(0)(w_c,w_c,w_c)$$
$$-DG_{2,c}(w_c)\left[\frac{1}{2!}\Pi_c D^2 F(0)(w_c,w_c) - [\mathscr{A}_c, G_{2,c}](w_c)\right]. \tag{8.3.66}$$

Proof. From Proposition 6.3.6, the first part is obvious. We only need to prove formula (8.3.66). Set

$$w_k(t) := \Pi_k w(t),\ \overline{w}_k(t) := \Pi_k \overline{w}(t),\ \forall k = c, h.$$

We can split the system (8.3.12) as

$$\begin{aligned}\frac{dw_c(t)}{dt} &= \mathscr{A}_c w_c(t) + \Pi_c F\left(w_c(t) + w_h(t)\right),\\ \frac{dw_h(t)}{dt} &= \mathscr{A}_h w_h(t) + \Pi_h F\left(w_c(t) + w_h(t)\right).\end{aligned}$$

Note that (8.3.65) is equivalent to

$$w_c = \overline{w}_c + G_{2,c}\left(\overline{w}_c\right),\ \ w_h = \overline{w}_h + G_{2,h}\left(\overline{w}_c\right).$$

Since $\dim\left(\mathscr{X}_c\right) < +\infty$, we have

$$\begin{aligned}\dot{\overline{w}}_c(t) &= \left[I + DG_{2,c}(\overline{w}_c(t))\right]^{-1}\left[\mathscr{A}_c\left(\overline{w}_c(t) + G_{2,c}(\overline{w}_c(t))\right) + \Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right)\right]\\ &= \mathscr{A}_c\left(\overline{w}_c(t) + G_{2,c}(\overline{w}_c(t))\right) + \Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right)\\ &\quad - DG_{2,c}(\overline{w}_c(t))\left[\mathscr{A}_c\left(\overline{w}_c(t) + G_{2,c}(\overline{w}_c(t))\right) + \Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right)\right]\\ &\quad + DG_{2,c}(\overline{w}_c(t))DG_{2,c}(\overline{w}_c(t))\left[\begin{array}{c}\mathscr{A}_c\left(\overline{w}_c(t) + G_{2,c}(\overline{w}_c(t))\right)\\ + \Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right)\end{array}\right] + O(\|\overline{w}(t)\|^4).\end{aligned}$$

Hence

$$\begin{aligned}\Pi_c F_2(\overline{w}(t)) = {} & \Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right) - \left[\mathscr{A}_c, G_{2,c}\right]\left(\overline{w}_c(t)\right)\\ & - DG_{2,c}(\overline{w}_c(t))\left[\Pi_c F\left(\overline{w}(t) + G_2(\overline{w}_c(t))\right) - \left[\mathscr{A}_c, G_{2,c}\right]\left(\overline{w}_c(t)\right)\right] + O(\|\overline{w}(t)\|^4).\end{aligned}$$

Let $w_c \in \mathscr{X}_c$. It follows that

$$\begin{aligned}\Pi_c F_2(w_c) = {} & \Pi_c F\left(w_c + G_2(w_c)\right) - \left[\mathscr{A}_c, G_{2,c}\right](w_c)\\ & - DG_{2,c}(w_c)\left[\Pi_c F\left(w_c + G_2(w_c)\right) - \left[\mathscr{A}_c, G_{2,c}\right](w_c)\right] + O(\|w_c\|^4).\end{aligned}$$

Thus we have

$$\begin{aligned}\Pi_c F_2(w_c) = {} & \frac{1}{2!}\Pi_c D^2 F\left(0\right)\left(w_c, w_c\right) - [\mathscr{A}_c, G_{2,c}](w_c)\\ & + \Pi_c D^2 F(0)(w_c, G_2(w_c)) + \frac{1}{3!}\Pi_c D^3 F(0)(w_c, w_c, w_c)\\ & - DG_{2,c}(w_c)\left[\frac{1}{2!}\Pi_c D^2 F(0)(w_c, w_c) - [\mathscr{A}_c, G_{2,c}](w_c)\right] + O(\|w_c\|^4).\end{aligned}$$

Then (8.3.66) follows and the proof is complete. □

Set

$$w_c = \widehat{\alpha}\widehat{e}_1 + x_1\widehat{e}_2 + x_2\widehat{e}_3.$$

We compute the normal form expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$. Consider $V^m(\mathbb{C}^3, \mathbb{C}^3)$ and $V^m(\mathbb{C}^3, \mathscr{X}_h \cap D(\mathscr{A}))$, which denote the linear space of the homogeneous polynomials of degree m in 3 real variables, $\widehat{\alpha}, x = (x_1, x_2)$ with coefficients in $\mathbb{C}^3$ and $\mathscr{X}_h \cap D(\mathscr{A})$, respectively. The operators Θ_m^c and Θ_m^h considered in (8.3.57) and (8.3.58) now act in the spaces $V^m(\mathbb{C}^3, \mathbb{C}^3)$ and $V^m(\mathbb{C}^3, \mathscr{X}_h \cap D(\mathscr{A}))$, respectively, and satisfy

$$\begin{aligned}
\Theta_m^c(G_{m,c})\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} &= [\mathscr{A}_c, G_{m,c}]\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix}\\
&= DG_{m,c}\mathscr{A}_c\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} - \mathscr{A}_c G_{m,c}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix}\\
&= \begin{pmatrix} D_x G_{m,c}^1\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} M_c x \\ \left(D_x\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} M_c\begin{pmatrix}x_1\\ x_2\end{pmatrix} - M_c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix}\right)\end{pmatrix},
\end{aligned} \tag{8.3.67}$$

$$\Theta_m^h\left(G_{m,h}\right) = [\mathscr{A}, G_{m,h}] = DG_{m,h}\mathscr{A}_c - \mathscr{A}_h G_{m,h},$$

$$\forall G_{m,c}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} = \begin{pmatrix}G_{m,c}^1\\ G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} \in V^m(\mathbb{C}^3, \mathbb{C}^3),$$

$$\forall G_{m,h} \in V^m(\mathbb{C}^3, \mathscr{X}_h \cap D(\mathscr{A}))$$

with

$$\mathscr{A}_c = \begin{bmatrix} 0 & 0 & 0\\ 0 & i\omega_k & 0\\ 0 & 0 & -i\omega_k\end{bmatrix} \text{ and } M_c = \begin{bmatrix} i\omega_k & 0\\ 0 & -i\omega_k\end{bmatrix}.$$

We define $\overline{\Theta}_m^c : V^m(\mathbb{C}^3, \mathbb{C}^2) \to V^m(\mathbb{C}^3, \mathbb{C}^2)$ by

$$\begin{aligned}
&\overline{\Theta}_m^c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} = D_x\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} M_c\begin{pmatrix}x_1\\ x_2\end{pmatrix} - M_c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix},\\
&\forall\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} \in V^m(\mathbb{C}^3, \mathbb{C}^2).
\end{aligned} \tag{8.3.68}$$

Lemma 8.3.21. *For $m \in \mathbb{N}\backslash\{0,1\}$, we have the decomposition*

$$V^m(\mathbb{C}^3, \mathbb{C}^2) = R(\overline{\Theta}_m^c) \oplus N(\overline{\Theta}_m^c) \tag{8.3.69}$$

and

$$N(\overline{\Theta}_m^c) = \text{span}\left\{ \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix}, \begin{pmatrix} x_1^{q_1'}x_2^{q_2'}\alpha^{q_3'} \\ 0 \end{pmatrix} \middle| \begin{array}{c} q_1 - q_2 = -1, \\ q_1' - q_2' = 1, q_i, q_i' \in \mathbb{N}, i = 1,2,3. \end{array} \right\}. \tag{8.3.70}$$

Proof. The canonical basis of $V^m(\mathbb{C}^3, \mathbb{C}^2)$ is

$$\Phi = \left\{ \begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix} \middle| q_1 + q_2 + q_3 = m \right\}.$$

Since $M_c = \begin{bmatrix} i\omega_k & 0 \\ 0 & -i\omega_k \end{bmatrix}$, for each $\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix} \in \Phi$, we have

$$\begin{aligned} \overline{\Theta}_m^c \begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix} &= D_x \begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix} M_c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M_c \begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix} \\ &= i\omega_k (q_1 - q_2 - 1) \begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3} \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \overline{\Theta}_m^c \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix} &= D_x \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix} M_c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M_c \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix} \\ &= i\omega_k (q_1 - q_2 + 1) \begin{pmatrix} 0 \\ x_1^{q_1}x_2^{q_2}\alpha^{q_3} \end{pmatrix}. \end{aligned}$$

Hence, the operators $\overline{\Theta}_m^c$ defined in (8.3.68) have diagonal matrix representations in the canonical basis of $V^m(\mathbb{C}^3, \mathbb{C}^2)$. Thus, (8.3.69) and (8.3.70) hold. □

From (8.3.70), we obtain

$$\begin{aligned} N(\overline{\Theta}_2^c) &= \text{span}\left\{ \begin{pmatrix} x_1\widehat{\alpha} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\widehat{\alpha} \end{pmatrix} \right\}, \\ N(\overline{\Theta}_3^c) &= \text{span}\left\{ \begin{pmatrix} x_1^2x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\widehat{\alpha}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\widehat{\alpha}^2 \end{pmatrix} \right\}. \end{aligned} \tag{8.3.71}$$

Define P_m^R and $P_m^N : V^m(\mathbb{C}^3, \mathbb{C}^2) \to V^m(\mathbb{C}^3, \mathbb{C}^2)$ the bounded linear projectors satisfying

$$P_m^R \left(V^m(\mathbb{C}^3, \mathbb{C}^2)\right) = R(\overline{\Theta}_m^c)$$

and

$$P_m^N \left(V^m(\mathbb{C}^3, \mathbb{C}^2)\right) = N(\overline{\Theta}_m^c).$$

We are now ready to compute the normal form of the reduced system expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$ of $\mathscr{X}_c$. From (8.3.54), (8.3.55), (8.3.63), (8.3.64), (8.3.67)-(8.3.69) and (8.3.71), we know that to find $G_2 \in V^2(\mathscr{X}_c, D(\mathscr{A}))$ defined in (8.3.59) is equivalent to find

$$G_{2,c} = \begin{pmatrix} G_{2,c}^1 \\ G_{2,c}^2 \\ G_{2,c}^3 \end{pmatrix} := \Pi_c G_2 \in V^2(\mathbb{C}^3, \mathbb{C}^3)$$

and

$$G_{2,h} := \Pi_h G_2 \in V^2(\mathbb{C}^3, \mathscr{X}_h \cap D(\mathscr{A})$$

such that

$$\begin{aligned}
[\mathscr{A}_c, G_{2,c}]\left(\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) &= \begin{pmatrix} D_x G_{2,c}^1 M_c x \\ D_x \begin{pmatrix} G_{2,c}^2 \\ G_{2,c}^3 \end{pmatrix} M_c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M_c \begin{pmatrix} G_{2,c}^2 \\ G_{2,c}^3 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ P_2^R\left(\widehat{H}_2^1\right) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
[\mathscr{A}, G_{2,h}] &= DG_{2,h}\mathscr{A}_c - \mathscr{A}_h G_{2,h} \\
&= \begin{pmatrix} 0 \\ \widetilde{\psi}\begin{bmatrix} 1 \\ -\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 - \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{H}_2^1\left(\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) &= \widetilde{\psi}\begin{bmatrix} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \\ \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \end{bmatrix} \\
&= \begin{pmatrix} A_1 x_1 \widehat{\alpha} + A_2 \widehat{\alpha} x_2 + \frac{1}{2} a_{20} x_1^2 + a_{11} x_1 x_2 + \frac{1}{2} a_{02} x_2^2 \\ \bar{A}_1 x_2 \widehat{\alpha} + \bar{A}_2 \widehat{\alpha} x_1 + \frac{1}{2} \bar{a}_{02} x_1^2 + \bar{a}_{11} x_1 x_2 + \frac{1}{2} \bar{a}_{20} x_2^2 \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= A_2 = -\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{1}{\alpha_k (1 - \ln(\alpha_k \chi))}, \\
a_{20} &= a_{11} = a_{02} = \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{\chi\beta(\ln(\alpha_k\chi) - 2)}{(1 - \ln(\alpha_k\chi))^2}.
\end{aligned}$$

From (8.3.61), it is easy to obtain the second order terms of the normal form expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$:

$$\frac{1}{2!}\Pi_c D^2 F_3(0)(w_c, w_c) = (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3)\begin{pmatrix} 0 \\ P_2^N\left(\widehat{H}_2^1\right) \end{pmatrix}$$

$$
\begin{aligned}
&= (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3) \left(\begin{array}{c} 0 \\ \begin{pmatrix} A_1 x_1 \widehat{\alpha} \\ \overline{A_1} x_2 \widehat{\alpha} \end{pmatrix} \end{array} \right) \\
&= A_1 x_1 \widehat{\alpha} \widehat{e}_2 + \overline{A_1} x_2 \widehat{\alpha} \widehat{e}_3.
\end{aligned}
$$

Notice that the terms $O(|x|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, it is only needed to compute the coefficients of

$$
\left(\begin{array}{c} 0 \\ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix} \end{array} \right) \text{ and } \left(\begin{array}{c} 0 \\ \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \end{array} \right)
$$

in the third order terms of the normal form. Firstly following similar computations in Section 8.3.4, we have

$$
G_{2,h} \left(\begin{array}{c} 0 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array} \right) = \left(\begin{array}{c} 0 \\ \begin{pmatrix} 0 \\ \overline{\Psi} \end{pmatrix} \end{array} \right)
$$

with

$$
\overline{\Psi} = x_1^2 \psi_{2,2} + x_2^2 \psi_{3,3} + 2 x_1 x_2 \psi_{2,3}
$$

and

$$
G_{2,c} \left(\begin{array}{c} 0 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{array} \right) = \left(\begin{array}{c} 0 \\ \begin{pmatrix} \frac{1}{i2\omega_k} (a_{20} x_1^2 - 2a_{11} x_1 x_2 - \frac{1}{3} a_{02} x_2^2) \\ \frac{1}{i2\omega_k} (\frac{1}{3} \bar{a}_{02} x_1^2 + 2\bar{a}_{11} x_1 x_2 - \bar{a}_{20} x_2^2) \end{pmatrix} \end{array} \right).
$$

Therefore, we have

$$
\left[D^2 W(0)(w_c, G_2(w_c)) \right]_{\widehat{\alpha}=0} = \begin{pmatrix} \frac{\beta(\ln(\alpha_k \chi) - 2)}{\chi} S_1 S_2 \\ 0 \end{pmatrix}
$$

with

$$
\begin{aligned}
S_1 &= \int_0^{+\infty} \gamma(a) \left(x_1 c_1 + x_2 c_2 \right)(a) da, \\
S_2 &= \int_0^{+\infty} \gamma(a) \left(\begin{array}{c} \frac{1}{i2\omega_k} (a_{20} x_1^2 - 2a_{11} x_1 x_2 - \frac{1}{3} a_{02} x_2^2) c_1 \\ + \frac{1}{i2\omega_k} (\frac{1}{3} \bar{a}_{02} x_1^2 + 2\bar{a}_{11} x_1 x_2 - \bar{a}_{20} x_2^2) c_2 + \overline{\Psi} \end{array} \right)(a) da.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left[\widehat{\Pi}_c D^2 W(0)(w_c, G_2(w_c)) \right]_{\widehat{\alpha}=0} &= \frac{\beta \left(\ln(\alpha_k \chi) - 2 \right)}{\chi} S_1 S_2 \left(\widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\
&= \frac{\beta \left(\ln(\alpha_k \chi) - 2 \right)}{\chi} S_1 S_2 \left[\begin{array}{c} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{array} \right]. \quad (8.3.72)
\end{aligned}
$$

From (8.3.56), (8.3.66) and (8.3.72), we have

$$\left[\frac{1}{3!}\Pi_c D^3 F_2(0)(w_c,w_c,w_c)\right]_{\widehat{\alpha}=0} = \begin{pmatrix} 0 \\ \begin{pmatrix} \widehat{\psi}\begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}c_1 + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}c_2 \end{bmatrix} + \\ \frac{\beta(\ln(\alpha_k\chi)-2)}{\chi}S_1S_2\begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}c_1 + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}c_2 \end{bmatrix} \end{pmatrix} \end{pmatrix}.$$

Now we give the third order terms of the normal form expressed in terms of the basis $\{\widehat{e}_1,\widehat{e}_2,\widehat{e}_3\}$:

$$\begin{aligned} &\frac{1}{3!}\Pi_c D^3 F_3(0)(w_c,w_c,w_c) \\ &= (\widehat{e}_1,\widehat{e}_2,\widehat{e}_3)\begin{pmatrix} 0 \\ P_3^N\begin{pmatrix} \frac{4\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{3(1-\ln(\alpha_k\chi))^3}\left(\frac{x_1+x_2}{2}\right)^3\begin{bmatrix} \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1} \\ \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1} \end{bmatrix} + \\ \begin{pmatrix} \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{\beta(\ln(\alpha_k\chi)-2)}{\chi}S_1S_2 \\ \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}\frac{\beta(\ln(\alpha_k\chi)-2)}{\chi}S_1S_2 \end{pmatrix} \end{pmatrix} \end{pmatrix} + O(|x|\widehat{\alpha}^2) \\ &= (\widehat{e}_1,\widehat{e}_2,\widehat{e}_3)\begin{pmatrix} 0 \\ \begin{pmatrix} C_1x_1^2x_2 \\ \overline{C_1}x_1x_2^2 \end{pmatrix} \end{pmatrix} + O(|x|\widehat{\alpha}^2) \end{aligned}$$

with

$$C_1 = \begin{bmatrix} \frac{i}{2\omega_k}\left(a_{11}^2 - \frac{7}{3}|a_{11}|^2\right) + \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{2(1-\ln(\alpha_k\chi))^3} \\ +\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{\beta(\ln(\alpha_k\chi)-2)}{1-\ln(\alpha_k\chi)}\begin{bmatrix} \int_0^{+\infty}\gamma(a)\,\psi_{2,2}(a)\,da \\ +\int_0^{+\infty}\gamma(a)\,2\psi_{2,3}(a)da \end{bmatrix} \end{bmatrix}.$$

Therefore, we obtain the following normal form of the reduced system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = M_c\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} A_1x_1\widehat{\alpha} \\ \overline{A_1}x_2\widehat{\alpha} \end{pmatrix} + \begin{pmatrix} C_1x_1^2x_2 \\ \overline{C_1}x_1x_2^2 \end{pmatrix} + O(|x|\widehat{\alpha}^2 + |(\widehat{\alpha},x)|^4).$$

The normal form above can be written in real coordinates (w_1,w_2) through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$. Setting $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this normal form becomes

$$\begin{cases} \dot{\rho} = \iota_1\widehat{\alpha}\rho + \iota_2\rho^3 + O(\widehat{\alpha}^2\rho + |(\rho,\widehat{\alpha})|^4), \\ \dot{\xi} = -\sigma_k + O(|(\rho,\widehat{\alpha})|), \end{cases} \tag{8.3.73}$$

where

$$\iota_1 = \mathrm{Re}(A_1),\ \iota_2 = \mathrm{Re}(C_1).$$

Following Chow and Hale [62] we know that the sign of $\iota_1\iota_2$ determines the direction of the bifurcation and that the sign of ι_2 determines the stability of the nontrivial periodic orbits. In summary we have the following theorem.

Theorem 8.3.22. *The flow of the age-structured model (8.3.1) on the center manifold of the origin at $\alpha = \alpha_k, k \in \mathbb{N}^+$, is given by (8.3.73). Moreover, we have the following*

(i) Hopf bifurcation is supercritical if $\iota_1\iota_2 < 0$ and subcritical if $\iota_1\iota_2 > 0$;
(ii) The nontrivial periodic solution is stable if $\iota_2 < 0$ and unstable if $\iota_2 > 0$.

8.4 Remarks and Notes

Age-structured models are hyperbolic partial differential equations (Hadeler and Dietz [168], Keyfitz and Keyfitz [212], Perthame [287]). Pioneer studies were due to Sharpe and Lotka [315] and McKendrick [263] on linear age-strudctured models and to Kermack and McKendrick [209, 210, 211] on age-structured epidemic models. The nonlinear extension of McKendrick's model by Gurtin and MacCamy [162] triggered a renewed interest in both linear and nonlinear age-structured models. We refer to the monographs of Hoppensteadt [192], Webb [362], Iannelli [195], Cushing [79], and Inaba [199] for basic theories on age-structured equations.

To investigate age-structured models, one can use the classical method; that is, to use solutions integrated along the characteristics and work with nonlinear Volterra equations. We refer to the monographs of Webb [362], Metz and Diekmann [266] and Iannelli [195] on this method. A second approach is the variational method, we refer to Anita [19], Aineseba [8] and the references cited therein. One can also regard the problem as a semilinear problem with non-dense domain and use the integrated semigroups method, which is the approach we used in this monograph. We refer to Thieme [328, 330, 331], Magal [242], Thieme and Vrabie [339], Magal and Thieme [251], Thieme and Vosseler [338] for more details on this approach.

Webb [360] was the first to show rigorously that a nonlinear age-structured population model defined in terms of birth and death rates determines a nonlinear semigroup on the population state space with the age-distribution as the population state. Webb [361] also used the theory of semigroups of linear operators to give an elegant proof of the asynchronous exponential growth of age-structured populations derived by Sharpe and Lotka [315] (Diekmann and Gyllenberg [99]). The principle of linearized stability established in Webb [362] says that a steady state is exponentially stable if the spectrum of the infinitesimal generator of the linearized semigroup lies entirely in the open left half-plane, whereas it is unstable if there is at least one spectral value with positive real part. This not only provides a fundamental tool to study stability of age-structured models but also indicates that periodic solutions may exist in age-structured models via Hopf bifurcation. The existence of non-trivial periodic solutions in age structured models was observed in some studies (Cushing [77, 78], Prüss [294], Levine [227], Diekmann et al. [103], Hastings [182], Swart [324], Iannelli [195], Breda et al. [46]). Diekmann and van Gils [104] established a Hopf bifurcation theorem for integral equations of convolution type that applies to age-structured models (Hastings [182]). Bertoni [43] proved a Hopf bifurcation theorem for nonlinear equations of structured populations.

Section 8.1 presented a general Hopf bifurcation theorem a general class of age-structured systems. The results were taken from Liu et al. [234]. Section 8.2 dealt with the local and global dynamics of a susceptible-infectious model (8.2.1) with age of infection. Thieme and Castillo-Chavez [336, 337] studied the uniform persistence of the system and the local exponential asymptotic stability of the endemic equilibrium. The global asymptotic stability of the endemic equilibrium was studied in D'Agata et al. [80] when the function $a \to e^{\nu_S a}\beta(a)l_{\nu_I}(a)$ is non-decreasing. The existence of integrated solutions of the model by using integrated semigroup theory, uniform persistence, local stability and global stability of both the disease-free and endemic equilibria were studied in Section 8.2 which were adapted from Magal et al. [244]. Section 8.3 provided detailed results on the existence of integrated solutions, local stability of equilibria, Hopf bifurcation, and normal forms for a scalar age-structured model with nonlinear boundary condition which were taken from Magal and Ruan [248], Liu et al. [236] and Chu et al. [73].

(a) Other Types of Structured Models. The techniques used in this chapter can be used to study nonlinear dynamics (for example Hopf bifurcation) in other structured equations (Webb [364]), such as an evolutionary epidemiological model of type A influenza (Inaba1998, Inaba2002, Magal and Ruan [249], Liu et al. [236]), a blood-stage malaria infection model (Su et al. [323]), an age-structured model with two time delays (Fu et al. [147]), an age-structured compartmental pest-pathogen model (Wang and Liu [355]), age-structured consumer-resource (predator-prey) models (Levine [227], Liu et al. [237], Liu and Li [232]), size-structured models (Calsina and Farkas [53], Calsina and Ripoll [54]), spatially and age structured population dynamics models (Liu et al. [238]), etc. It will be interesting to study the stability change and Hopf bifurcation in age-structured SIR epidemic models (Thieme [330], Andreasen [17]).

(b) Age-structured Models with Diffusion. Gurtin [161] generalized the linear equation of Skellam [322] to a linear age-dependent model for population diffusion. Gurtin and MacCamy [162, 164] and MacCamy [241] proposed nonlinear age-dependent population models with diffusion. Webb [359] studied an age-dependent susceptible-exposed-infectious-recovered (SEIR) epidemic model with spatial diffusion. Since then, age-structured models with diffusion have been extensively studied, see Busenberg and Iannelli [52], Delgado et al. [91], Fitzgibbon et al. [142], Kunisch et al. [222], Langlais [225, 226], Magal and Ruan [245], Marcati [254], Marcati and Serafini [255], Walker [350, 351] and the references cited therein. Traveling wave solutions in age-structured diffusive models have been studied by Al-Omari and Gourley [10], Ducrot [109], Ducrot and Magal [112, 113, 114], and Ducrot et al. [115].

(c) Additional Comments on the Global Stability of Age-structured Models Using Liapunov Functionals. In Section 8.2, to obtain the global stability of the endemic equilibrium $(\bar{S}_E, \bar{\imath}_E)$ for the age-structured susceptible-infectious model

(8.2.1), we followed Magal et al. [244] in three steps: (i) Introduced integrated semigroup theory in order to obtain a comprehensive spectral theory for the linear C_0-semigroups obtained by linearizing the system around the endemic equilibrium; (ii) Proved uniform persistence of the system to assure the existence of a global attractor A_0 in the subset M_0 of the state space. (iii) Applied a change of variables to transform equation (8.2.1) to a special case with $\nu_I(a) = \nu_S, \forall a \geq 0$ and $\gamma = \nu_S$, chose a Liapunov functional as

$$\begin{aligned} V(S(t), i(t,.)) = {} & \frac{S(t)}{\bar{S}_E} - 1 - \ln\left(\frac{S(t)}{\bar{S}_E}\right) \\ & + \int_0^\infty \int_a^\infty \eta\beta(l)\bar{\imath}_E(l)dl \left[\frac{i(t,a)}{\bar{\imath}_E(a)} - 1 - \ln\left(\frac{i(t,a)}{\bar{\imath}_E(a)}\right)\right] da, \end{aligned}$$

and showed that this functional is decreasing over the complete orbits on $\mathscr{A}_0$, implying that $\mathscr{A}_0$ reduces to the endemic equilibrium which in turn is globally stable by LaSalle's invariance principle.

Some researchers have attempted to follow the techniques of Magal et al. [19] to discuss global stability of various age-structured models. However, most of these studies skipped steps (i) and (ii) and just directly followed step (iii) by constructing a Liapunov functional to show global stability of the endemic equilibrium, which are incomplete (Wang et al. [354]). Firstly, in order to apply LaSalle's invariance principle, one needs to show that the semiflow generated by the model is relatively compact. Secondly, one needs to make sure that the semiflow generated by the model is uniformly persistent since a term $\ln(\frac{i(t,a)}{\bar{\imath}_E(a)})$ appears in the Liapunov functional, which may yield singularity after differentiation. In fact, because of this term the Liapunov functional V is well-defined only on the attractor $\mathscr{A}_0$ but not on M_0.

Chapter 9
Parabolic Equations

The theories developed in previous chapters can be used to study some parabolic equations as well. In this chapter, we first consider linear abstract Cauchy problems with non-densely defined and almost sectorial operators; that is, the part of this operator in the closure of its domain is sectorial. Such problems naturally arise for parabolic equations with nonhomogeneous boundary conditions. By using the integrated semigroup theory, we prove an existence and uniqueness result for integrated solutions. Moreover, we study the linear perturbation problem. Then in the second section we provide detailed stability and bifurcation analyses for a scalar reaction-diffusion equation, namely, a size-structured model.

9.1 Abstract Non-densely Defined Parabolic Equations

9.1.1 Introduction

Consider the abstract linear parabolic equations of the form

$$\frac{du(t)}{dt} = Au(t) + f(t),\ t > 0;\ \ u(0) = x \in \overline{D(A)}, \tag{9.1.1}$$

where $A : D(A) \subset X \to X$ is a linear operator on a Banach space X. When dealing with parabolic equations, it is usually assumed that the operator A is a sectorial elliptic operator. This property usually holds when elliptic operators are considered in Lebesgue spaces or Hölder spaces and with homogeneous boundary conditions. As pointed out for instance by Lunardi [240], this property does not hold anymore when dealing with these operators in some more regular spaces. As observed in Prevost [292] in the context of parabolic equations with nonhomogeneous boundary conditions, it turns out to be natural to impose a weaker condition than sectoriality. Motivated by these examples, we make the following assumption.

Assumption 9.1.1. Assume that

(a) A_0, the part of A in $\overline{D(A)}$, is a sectorial operator;
(b) A is almost sectorial.

By using functional calculus and $1-\alpha$ growth semigroups, this kind of problems has been considered by Periago and Straub [284] who defined the fractional power of $\lambda I - A$ for some $\lambda > 0$ large enough. We also refer to DeLaubenfels [90] and Haase [166] for more update results on functional calculus, and to Da Prato [81] for pioneer work on $1-\alpha$ growth semigroups. More recently, the case of nonautonomous Cauchy problems has also been studied by Carvalho et al. [57]. In these studies, based on the existence of $1-\alpha$ growth semigroups, a notion of solution has been developped.

Here we consider the Cauchy problem (9.1.1) by using integrated semigroup theory and an approach similar to the one used by Pazy [281]. Our goal is to study the existence of integrated solutions for the Cauchy problem (9.1.1). Under Assumption 9.1.1 the linear operator A is not (in general) a Hille-Yosida operator. The aim of this section is to apply the theory developed in Chapter 3 to the context of linear operators with a sectorial part. In Remark 9.1.11, we will also give a brief comparison between the integrated semigroup approach and the approach used by Periago and Straub [284].

In order to introduce the theoretical framework, consider the following parabolic problem with nonautonomous boundary condition:

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \dfrac{\partial^2 v(t,x)}{\partial x^2} + g(t,x), \ \ t>0, \ \ x>0 \\ -\dfrac{\partial v(t,0)}{\partial x} = h(t) \\ v(0,.) = v_0 \in L^p\left((0,+\infty),\mathbb{R}\right), \end{cases} \tag{9.1.2}$$

where $g \in L^1\left((0,\tau), L^p\left((0,+\infty),\mathbb{R}\right)\right)$ and $h \in L^q\left((0,\tau),\mathbb{R}\right)$. Consider the linear operator $A : D(A) \subset X \to X$ defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi'(0) \\ \varphi'' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty),\mathbb{R}\right).$$

One may observe that A_0, the part of A in $\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^p\left((0,+\infty),\mathbb{R}\right)$, is the linear operator defined by

$$A_0\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \left\{ \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{2,p}\left((0,+\infty),\mathbb{R}\right) : \varphi'(0) = 0 \right\}.$$

In particular, it is well known that A_0 is the infinitesimal generator of an analytic semigroup on $\overline{D(A)}$. But the resolvent of A is defined by the formula

$$(\lambda I - A)^{-1}\begin{pmatrix}\alpha\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$
$$\Leftrightarrow \varphi(x) = \frac{e^{-\sqrt{\lambda}x}}{-\sqrt{\lambda}}\alpha + \frac{e^{-\sqrt{\lambda}x}}{2\sqrt{\lambda}}\int_0^{+\infty} e^{-\sqrt{\lambda}s}\psi(s)ds + \frac{1}{2\sqrt{\lambda}}\int_0^{+\infty} e^{-\sqrt{\lambda}|x-s|}\psi(s)ds$$

for $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

Due to the boundary condition, we have the following inequalities

$$0 < \liminf_{\lambda\to+\infty} \lambda^{1/p^*}\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} < \limsup_{\lambda\to+\infty} \lambda^{1/p^*}\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} < +\infty,$$

where

$$p^* := \frac{2p}{1+p}.$$

It follows that A is not a Hille-Yosida operator when $p \in (1,+\infty)$. Set

$$f(t) := \begin{pmatrix}h(t)\\ g(t)\end{pmatrix}.$$

By identifying $u(t) = \begin{pmatrix}0\\ v(t,.)\end{pmatrix}$, the PDE problem (9.1.2) can be rewritten as the following abstract Cauchy problem

$$\frac{du}{dt} = Au(t) + f(t) \text{ for } t \geq 0 \text{ and } u(0) = \begin{pmatrix}0\\ v_0\end{pmatrix} \in \overline{D(A)}. \tag{9.1.3}$$

In this section we will prove that for each $\widehat{p} > p^*$ and each $f \in L^{\widehat{p}}(0,\tau;X)$ (with $\tau > 0$), the Cauchy problem (9.1.3) has a unique integrated solution and there exists a constant $M_{\tau,\widehat{p}} > 0$ such that

$$\|u(t)\| \leq M_{\tau,\widehat{p}}\left(\int_0^t \|f(s)\|^{\widehat{p}} ds\right)^{1/\widehat{p}}, \ \forall t \in [0,\tau].$$

For parabolic problems in dimension n, the same difficulty arises, and we refer to Tanabe [325, Section 3.8, p.82], Agranovich [7], and Volpert and Volpert [348] for general estimates for the resolvent of elliptic operators in the n dimensional case.

9.1.2 Almost Sectorial Operators

We first recall some definitions.

Definition 9.1.2. Let $L : D(L) \subset X \to X$ a linear operator on a Banach space X. L is said to be a *sectorial* operator if there are constants $\widehat{\omega} \in \mathbb{R}$, $\theta \in]\pi/2, \pi[$, and $\widehat{M} > 0$ such that

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$;

(ii) $\left\| (\lambda I - L)^{-1} \right\| \leq \dfrac{\widehat{M}}{|\lambda - \widehat{\omega}|}, \forall \lambda \in S_{\theta,\widehat{\omega}}$.

We refer for instance to Friedmann [146], Tanabe [325], Henry [183], Pazy [281], Temam [327], Lunardi [240], Cholewa and Dlotko [61], Engel and Nagel [126] for more details on the subject. In particular, when L is a sectorial operator and densely defined, then L is the infinitesimal generator of a strongly continuous analytic semigroup $T_L(t)$ given by

$$T_L(t) = \frac{1}{2\pi i} \int_{\hat{\omega}+\gamma_{r,\eta}} (\lambda I - L)^{-1} e^{\lambda t} d\lambda, \ t > 0, \text{ and } T_L(0)x = x, \ \forall x \in X,$$

where $r > 0, \eta \in (\pi/2, \theta)$, and $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta, |\lambda| = r\}$, oriented counterclockwise (see Lunardi [240, Proposition 2.1.1, p.35]). Recall that a family of bounded linear operators $\{T(t)\}_{t \geq 0}$ satisfying the semigroup property is said to be an *analytic semigroup* (following for instance Lunardi [240]) if the function $t \to T(t)$ is analytic in $(0, +\infty[$ with values in $\mathscr{L}(X)$ (i.e. $T(t) = \sum_{n=0}^{+\infty} (t - t_0)^n L_n$ for $|t - t_0|$ small enough).

Now we introduce the notion of almost sectorial operators.

Definition 9.1.3. Let $L : D(L) \subset X \to X$ be a linear operator on a Banach space X and $\alpha \in (0, 1]$ be given. L is said to be an α-*almost sectorial operator* if there are constants $\widehat{\omega} \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and $\widehat{M} > 0$ such that

(i) $\rho(L) \supset S_{\theta,\widehat{\omega}} = \{\lambda \in \mathbb{C} : \lambda \neq \widehat{\omega}, |\arg(\lambda - \widehat{\omega})| < \theta\}$;

(ii) $\left\| (\lambda I - L)^{-1} \right\| \leq \frac{\widehat{M}}{|\lambda - \widehat{\omega}|^{\alpha}}, \ \forall \lambda \in S_{\theta,\widehat{\omega}}$.

This class of operators has been used by Periago and Straub [284] as well as by Carvalho et al. [57]. In these works the authors constructed functional calculus for such operators and defined a notion of solutions for the corresponding linear abstract Cauchy problem. Here we focus on linear almost sectorial operators with a sectorial part over the closure of its domain. In order to use this notion, we first derive some characterization for this class of operators.

Proposition 9.1.4. *Let $A : D(A) \subset X \to X$ be a linear operator and A_0 be its part in $X_0 = \overline{D(A)}$. Then the following statements are equivalent:*

(i) The operator A_0 is sectorial in X_0 and A is $\frac{1}{p^*}$−almost sectorial for some $p^* \in [1, +\infty)$;

(ii) There exist two constants, $\omega_A \in \mathbb{R}$ and $M_A > 0$, such that the following properties are satisfied:

(ii-a) $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > \omega_A\} \subset \rho(A_0)$ *and*

$$\left\| (\lambda I - A_0)^{-1} \right\|_{\mathscr{L}(X_0)} \leq \frac{M_A}{|\lambda - \omega_A|}, \ \forall \lambda \in \mathbb{C} \textit{ with } \text{Re}(\lambda) > \omega_A,$$

(ii-b) $(\omega_A, +\infty) \subset \rho(A)$ *and*

$$\overline{\lim_{\lambda \to +\infty}} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} < +\infty.$$

Proof. From Definitions 9.1.2 and 9.1.3 it directly follows that (i) implies (ii). Now assume that (ii) holds. By using Lunardi [240, Proposition 2.1.11, p.43], we know that $(ii-a)$ ensures that A_0 is a sectorial operator on X_0. Since $\rho(A)$ and $\rho(A_0)$ are nonempty, we have $\rho(A) = \rho(A_0)$ (see Lemma 2.2.10). Since A_0 is sectorial on X_0, there exist $\omega_A \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that $S_{\theta,\omega_A} \subset \rho(A)$. Without loss of generality (by replacing ω_A by $\omega_A + \varepsilon$ for some $\varepsilon > 0$ large enough), one may assume that there exist some positive constants $C > 0$ and $\theta \in (\pi/2, \pi)$ such that

$$\begin{aligned}
&\left\| (\lambda - \omega_A)(\lambda I - A_0)^{-1} \right\| \leq C, \ \forall \lambda \in S_{\theta,\omega_A}, \\
&(\mu - \omega_A)^{1/p^*} \left\| (\mu I - A)^{-1} \right\|_{\mathscr{L}(X)} \leq C, \ \forall \mu \in (\omega_A, +\infty).
\end{aligned}$$

Now for each $\lambda \in \rho(A_0)$ and each $\mu \in (\omega_A, +\infty)$, one has

$$(\lambda I - A)^{-1} = (\mu - \lambda)(\lambda I - A_0)^{-1}(\mu I - A)^{-1} + (\mu I - A)^{-1}. \tag{9.1.4}$$

Therefore, for each $\lambda \in S_{\theta,\omega_A}$ we choose $\mu = \omega_A + |\lambda - \omega_A|$ and have

$$\begin{aligned}
\left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} &= \left\| (\mu - \lambda)(\lambda I - A_0)^{-1}(\mu I - A)^{-1} + (\mu I - A)^{-1} \right\|, \\
&\leq C \frac{|\mu - \lambda|}{|\lambda - \omega_A|} \frac{C}{(\mu - \omega_A)^{1/p^*}} + \frac{C}{(\mu - \omega_A)^{1/p^*}}.
\end{aligned}$$

From the definition of μ one has

$$|\lambda - \omega_A| = |\mu - \omega_A|.$$

Thus, $|\lambda - \mu| \leq 2|\lambda - \omega_A|$, which implies that

$$\left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} \leq C \frac{(2C+1)}{|\lambda - \omega_A|^{1/p^*}} = \frac{\tilde{M}}{|\lambda - \omega_A|^{1/p^*}}, \ \forall \lambda \in S_{\theta,\omega_A}.$$

This completes the proof of the result. $\square$

From now on, we only use the following assumption.

Assumption 9.1.5. Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space X. Assume that there exist two constants, $\omega_A \in \mathbb{R}$ and $M_A > 0$, such that

(a) $\rho(A_0) \supset \{\lambda \in \mathbb{C} : \text{ Re } (\lambda) > \omega_A\}$ and

$$\left\| (\lambda - \omega_A)(\lambda I - A_0)^{-1} \right\|_{\mathscr{L}(X_0)} \leq M_A, \ \forall \lambda \in \mathbb{C} : \text{Re } (\lambda) > \omega_A;$$

(b) $(\omega_A, +\infty) \subset \rho(A)$ and there exists $p^* \geq 1$ such that

$$\limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} < +\infty.$$

Note that under Assumption 9.1.5 (a), the operator $A_0 : D(A_0) \subset X_0 \to X_0$ is the infinitesimal generator of a strongly continuous and analytic semigroup on X_0. It will be denoted by $\{T_{A_0}(t)\}_{t \geq 0}$ in the following. We also remark that if $p^* = 1$ in Assumption 9.1.5 (b), it is clear that A is a Hille-Yosida operator. It is also clear that if Assumption 9.1.5 (b) is satisfied, then for each $\delta \in \mathbb{R}$, we have

$$\limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - (A + \delta I))^{-1} \right\|_{\mathscr{L}(X)} = \limsup_{\lambda \to +\infty} \lambda^{1/p^*} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)}.$$

9.1.3 Semigroup Estimates and Fractional Powers

The aim of this subsection is to give some estimates and differentiability properties for the integrated semigroup $\{S_A(t)\}_{t \geq 0}$. Moreover, these estimates will allow us to give an alternative construction for the fractional powers of the operator $-A$ by using a semigroup approach. The definition of fractional powers as well as functional calculus has been well developed for almost sectorial operators (see for instance Periago and Straub [284]). These constructions essentially use the resolvent operator. Our construction follows the one given by Pazy [281] by using integrated semigroup theory.

Lemma 9.1.6. *Let Assumption 9.1.5 be satisfied. Then for each* $\delta \in (-\infty, -\omega_A)$ *(i.e.* $\omega_A + \delta \leq 0$*), there exist* $\tilde{M}_1 = \tilde{M}_1(\delta) > 0$ *and* $\tilde{M}_2 = \tilde{M}_2(\delta) > 0$ *such that*

$$\left\| S_{(A+\delta I)}(t) \right\|_{\mathscr{L}(X)} \leq \tilde{M}_1, \ \forall t \geq 0, \tag{9.1.5}$$

and

$$\left\| S_{(A+\delta I)}(t) \right\|_{\mathscr{L}(X)} \leq \tilde{M}_2 t^{1/p^*}, \ \forall t \in [0,1]. \tag{9.1.6}$$

Proof. Let $\delta \in (-\infty, -\omega_A)$ be fixed. By replacing A by $A + \delta I$ in (3.4.2), we have for each $\mu > \omega_A + \delta$, each $t \geq 0$, and $x \in X$ thst

$$\begin{aligned} S_{(A+\delta I)}(t)x = \mu \int_0^t T_{(A_0+\delta I)}(s) \left(\mu I - (A+\delta I)\right)^{-1} x ds \\ + \left[I - T_{(A_0+\delta I)}(t)\right] \left(\mu I - (A+\delta I)\right)^{-1} x. \end{aligned}$$

So if $\mu > 0$ is fixed, we obtain

$$\begin{aligned} \left\| S_{(A+\delta I)}(t)x \right\|_X \leq \mu \left\| (\mu I - (A+\delta I))^{-1} x \right\| \int_0^t \left\| T_{(A_0+\delta I)}(s) \right\| ds \\ + \left\| (\mu I - (A+\delta I))^{-1} x \right\| \left[1 + \left\| T_{(A_0+\delta I)}(t) \right\| \right]. \end{aligned}$$

But $\left\|T_{(A_0+\delta I)}(s)\right\| \leq Me^{(\omega_A+\delta)t}$, it follows that

$$\begin{aligned}\left\|S_{(A+\delta I)}(t)x\right\|_X \leq \mu &\left\|(\mu I-(A+\delta I))^{-1}x\right\| \int_0^t Me^{(\omega_A+\delta)s}ds \\ &+\left\|(\mu I-(A+\delta I))^{-1}x\right\| \left[1+Me^{(\omega_A+\delta)t}\right] \end{aligned} \tag{9.1.7}$$

and since $\omega_A+\delta<0$, one obtains that

$$\begin{aligned}\left\|S_{(A+\delta I)}(t)x\right\|_X \leq \mu &\left\|(\mu I-(A+\delta I))^{-1}x\right\| \int_0^\infty Me^{(\omega_A+\delta)s}ds \\ &+\left\|(\mu I-(A+\delta I))^{-1}x\right\| [1+M]\end{aligned}$$

and (9.1.5) follows.

Let $\tilde{C}>0$ and $\lambda_0>0$ be fixed such that $(\lambda_0,+\infty)\subset\rho(A)$ and

$$\lambda^{1/p^*}\left\|(\lambda I-(A+\delta I))^{-1}\right\|_{\mathscr{L}(X)} \leq \tilde{C},\ \forall\lambda\in[\lambda_0,+\infty). \tag{9.1.8}$$

For each $t\in(0,1]$ we replace μ by $\frac{\lambda_0}{t}\in[\lambda_0,\infty)$ in (9.1.7). Since $\omega_A+\delta<0$, one has $e^{(\omega_A+\delta)t}\leq 1$ while $\int_0^t e^{(\omega_A+\delta)s}ds\leq t$ for each $t\in[0,1]$. This yields that

$$\begin{aligned}\left\|S_{(A+\delta I)}(t)x\right\| \leq \lambda_0 M &\left\|\left(\frac{\lambda_0}{t}I-(A+\delta I)\right)^{-1}x\right\| \\ &+\left\|\left(\frac{\lambda_0}{t}I-(A+\delta I)\right)^{-1}x\right\|[1+M].\end{aligned} \tag{9.1.9}$$

On the other hand, from (9.1.8) with $\lambda=\frac{\lambda_0}{t}$ we have

$$\left\|\left(\frac{\lambda_0}{t}I-(A+\delta I)\right)^{-1}x\right\| \leq \frac{\tilde{C}}{\left(\frac{\lambda_0}{t}\right)^{1/p^*}}\|x\|. \tag{9.1.10}$$

Finally, combining (9.1.9) together with (9.1.10) one obtains

$$\begin{aligned}\left\|S_{(A+\delta I)}(t)x\right\| &\leq \frac{\lambda_0 M\tilde{C}t^{1/p^*}}{\lambda_0^{1/p^*}}\|x\| + \frac{\tilde{C}t^{1/p^*}}{\lambda_0^{1/p^*}}[1+M]\|x\| \\ &\leq \frac{\tilde{C}}{\lambda_0^{1/p^*}}[(\lambda_0+1)M+1]t^{1/p^*}\|x\|,\end{aligned}$$

and (9.1.6) follows. □

We now assume that $\omega_A<0$ and let $\delta\in(-\infty,-\omega_A)$ be given and fixed. Then (see Pazy [281, p.70]) one has for each $\gamma>0$ and each $x\in X_0$ that

$$(\delta I - A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} T_{A_0-\delta I}(t) x dt \tag{9.1.11}$$

and

$$(\delta I - A_0)^0 = I.$$

We first derive some estimates for the fractional powers of operator A_0.

Lemma 9.1.7. *Let Assumption 9.1.5 (a) be satisfied and assume that $\omega_A < 0$. Let $\gamma \in (0,1)$ be given. Then there exists some constant $\check{M} > 0$ such that*

$$\left\| (\lambda I - A_0)^{-\gamma} \right\| \leq \frac{\check{M}}{|\lambda|^\gamma}, \ \forall \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\,(\lambda) > 0. \tag{9.1.12}$$

Proof. Under Assumption 9.1.5 (a) we know that A_0 generates an analytical semigroup on X_0. Hence, by using the formula (6.4) on p. 69 in Pazy [281], we have

$$(\lambda I - A_0)^{-\gamma} = \frac{\sin \pi\gamma}{\pi} \int_0^{+\infty} t^{-\gamma} ((t+\lambda) I - A_0)^{-1} dt.$$

Then

$$\begin{aligned}
\left\| (\lambda I - A_0)^{-\gamma} \right\| &\leq M \frac{\sin \pi\gamma}{\pi} \int_0^{+\infty} \frac{1}{t^\gamma} \frac{1}{|t+\lambda|} dt \\
&= M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^\gamma} \int_0^{+\infty} \frac{1}{l^\gamma} \frac{1}{\left| l + e^{-i \arg(\lambda)} \right|} dl \\
&= M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^\gamma} \left[\int_0^{1/2} \frac{1}{l^\gamma} \frac{1}{\left| l + e^{i \arg(\lambda)} \right|} dl + \int_{1/2}^{+\infty} \frac{1}{l^\gamma} \frac{1}{\left| l + e^{i \arg(\lambda)} \right|} dl \right] \\
&\leq M \frac{\sin \pi\gamma}{\pi} \frac{1}{|\lambda|^\gamma} \left[\int_0^{1/2} \frac{1}{l^\gamma} \frac{1}{1-l} dl + \int_{1/2}^{+\infty} \frac{1}{l^\gamma} \frac{1}{l} dl \right] \\
&\leq \frac{\check{M}}{|\lambda|^\gamma}.
\end{aligned}$$

This completes the proof of Lemma 9.1.7. □

Note that we have for each $x \in X_0$, each $\delta \in (0, -\omega_A)$ and each $\gamma > 0$ that

$$(-A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} e^{-\delta t} T_{(A_0+\delta I)}(t) x dt.$$

By integrating by parts, we obtain

$$(-A_0)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} [1 - \gamma + \delta t] \ t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x dt. \tag{9.1.13}$$

Note that (9.1.13) is well defined for each $x \in X_0$ and each $\gamma > 0$ because $\left\| S_{(A+\delta I)}(t) \right\| \leq \int_0^t \left\| T_{(A_0+\delta I)}(s) \right\| ds \leq Mt$ for each $t \geq 0$. Hence (9.1.13) leads us to the following definition of fractional power of the resolvent of A.

Lemma 9.1.8. *Let Assumption 9.1.5 be satisfied and assume that $\omega_A < 0$. Then for each $\gamma > 1 - 1/p^*$ and each $\delta \in (0, -\omega_A)$, the operator $(-A)^{-\gamma}$ is well defined by*

$$(-A)^{-\gamma} x = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} [1 - \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x dt, \; \forall x \in X. \qquad (9.1.14)$$

Moreover, we have the following properties:

(i) $(\mu I - A_0)^{-1} (-A)^{-\gamma} = (-A_0)^{-\gamma} (\mu I - A)^{-1}, \; \forall \mu > \omega_A$;
(ii) $(-A_0)^{-\gamma} x = (-A)^{-\gamma} x, \forall x \in X_0$;
(iii) *When $\gamma = 1$, $(-A)^{-1}$ defined by (9.1.14) is the inverse of $-A$*;
(iv) *For each $\gamma \geq 0$ and $\beta > 1 - 1/p^*$*,

$$(-A_0)^{-\gamma} (-A)^{-\beta} = (-A)^{-(\gamma+\beta)}.$$

Proof. Let $x \in X$ be given and the function $H : [0, +\infty) \to \mathscr{L}(X)$ be defined by

$$t \to H(t) = \begin{cases} [1 - \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

The map $t \to S_A(t)$ is Hölder continuous since

$$S_A(t + r) - S_A(r) = T_{A_0}(r) S_A(t).$$

So there exists $C > 0$ such that

$$\| S_A(t + r) - S_A(r) \| \leq C t^{1/p^*}, \; \forall t \in [0, 1], \forall r \geq 0.$$

It follows that H is continuous on $(0, +\infty)$ with respect to the operator norm topology and therefore is a Bochner measurable map. Moreover, for each $\gamma > 1 - 1/p^*$, one has

$$\begin{aligned} \int_0^{+\infty} \| H(t) \| dt &= \int_0^{+\infty} \left\| [1 - \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x \right\| dt \\ &\leq \int_0^{+\infty} [1 - \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| dt \\ &\leq \int_0^1 [1 - \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| dt \\ &\quad + \int_1^{+\infty} [1 + \gamma + \delta t] \; t^{\gamma-2} e^{-\delta t} \left\| S_{(A+\delta I)}(t) x \right\| dt \\ &\leq [1 + \gamma + \delta] \left(\int_0^1 t^{\gamma-2} t^{1/p^*} \tilde{M}_2 \| x \| dt + \int_1^{+\infty} t^{\gamma-1} e^{-\delta t} \tilde{M}_1 \| x \| dt \right), \end{aligned}$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are the constants introduced in Lemma 9.1.6. So we have

$$\int_0^{+\infty} \|H(t)\| \, dt \leq [1+\gamma+\delta] \, \|x\| \left(\tilde{M}_2 \int_0^1 t^{(\gamma-2)+1/p^*} dt + \tilde{M}_1 \int_1^{+\infty} t^{\gamma-1} e^{-\delta t} dt \right), \tag{9.1.15}$$

and since

$$(\gamma-2)+1/p^* > -1-1/p^*+1/p^* = -1,$$

the function H is Bochner integrable.

On the other hand one has

$$\begin{aligned}
\left\| (-A)^{-\gamma} x \right\| &= \left\| \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} [1-\gamma+\delta t] \; t^{\gamma-2} e^{-\delta t} S_{(A+\delta I)}(t) x dt \right\| \\
&= \frac{1}{\Gamma(\gamma)} \left\| \int_0^{+\infty} H(t) dt \right\| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \|H(t)\| \, dt.
\end{aligned}$$

We conclude from (9.1.15) that

$$\left\| (-A)^{-\gamma} \right\| \leq \frac{[1+\gamma+\delta]}{\Gamma(\gamma)} . \left(\tilde{M}_2 \int_0^1 t^{(\gamma-2)+1/p^*} dt + \tilde{M}_1 \int_1^{+\infty} t^{\gamma-1} e^{-\delta t} dt \right),$$

and $(-A)^{-\gamma}$ is well defined by (9.1.14) for each $x \in X$, each $\gamma > 1-1/p^*$ and each $\delta \in (0, -\omega_A)$.

Assertions (i)-(iii) are direct consequences from definition (9.1.14). It remains to prove (iv).

Let $\gamma \geq 0$, $\beta > 1-1/p^*$, $x \in X$ and $\mu > \omega_A$ be fixed. Since $(\mu I - A)^{-1}$ commutes with $(-A_0)^{-\gamma}$ and $(-A)^{-\beta}$, we have

$$(\mu I - A)^{-1} (-A_0)^{-\gamma} (-A)^{-\beta} x = (-A_0)^{-\gamma} (-A)^{-\beta} (\mu I - A)^{-1} x.$$

Since $(\mu I - A)^{-1} \in X_0$, we obtain

$$(\mu I - A)^{-1} (-A_0)^{-\gamma} (-A)^{-\beta} x = (-A_0)^{-\gamma} (-A_0)^{-\beta} (\mu I - A)^{-1} x.$$

Recalling that (see Pazy [281, Lemma 6.2, p.70]) for each $\gamma, \beta \geq 0$, the following relation holds

$$(-A_0)^{-\gamma} (-A_0)^{-\beta} = (-A_0)^{-(\gamma+\beta)},$$

one obtains that

$$\begin{aligned}
(\mu I - A)^{-1} (-A_0)^{-\gamma} (-A)^{-\beta} x &= (-A_0)^{-(\gamma+\beta)} (\mu I - A)^{-1} x, \\
&= (-A)^{-(\gamma+\beta)} (\mu I - A)^{-1} x, \\
&= (\mu I - A)^{-1} (-A)^{-(\gamma+\beta)} x.
\end{aligned}$$

Finally, (iv) follows because $(\mu I - A)^{-1}$ is a one-to-one operator. This completes the proof of the result. $\square$

Let us recall that, since $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ is an analytic semigroup, we can define for each $\beta \geq 0$ the operator $(-A_0)^{\beta} : D\left((-A_0)^{\beta}\right) \subset X_0 \to X_0$ as the inverse of $(-A_0)^{-\beta}$ (i.e. $(-A_0)^{\beta} = \left((-A_0)^{-\beta}\right)^{-1}$). Moreover (see Pazy [281, Theorem 6.8, p.72]), we know that $(-A_0)^{\beta}$ is a closed operator, $D((-A_0)^{\gamma}) \subset D\left((-A_0)^{\beta}\right)$ for all $\gamma \geq \beta \geq 0$, $\overline{D\left((-A_0)^{\beta}\right)} = X_0$, and for each $\gamma, \beta \in \mathbb{R}$

$$(-A_0)^{\gamma}(-A_0)^{\beta} = (-A_0)^{\gamma+\beta}.$$

We also know that (see Pazy [281, Theorem 6.13, p.74]) for each $t > 0$, $(-A_0)^{\beta} T_{A_0}(t)$ is a bounded operator, and

$$\left\|(-A_0)^{\beta} T_{A_0}(t)\right\| \leq M_{\beta} t^{-\beta} e^{\omega_A t},\ \forall t > 0. \tag{9.1.16}$$

As a consequence of the above results, we have the following lemma.

Lemma 9.1.9. *Let Assumption 9.1.5 be satisfied and assume that $\omega_A < 0$. Then for each $q^* \in \left[1, \frac{1}{1-1/p^*}\right)$ and each $\tau > 0$ we have*

$$S_A(.)_{|(0,\tau)} \in W^{1,q^*}\left((0,\tau), \mathscr{L}(X)\right). \tag{9.1.17}$$

Proof. Taking $\mu = 0$ in (3.4.2), we have

$$S_A(t)x = (-A)^{-1}x - T_{A_0}(t)(-A)^{-1}x,\ \forall t \geq 0,\ \forall x \in X.$$

Since $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ is an analytic semigroup, the map $t \to T_{A_0}(t)$ is operator norm continuously differentiable on $(0, +\infty)$, so $t \to S_A(t)$ is continuously differentiable on $(0, +\infty)$, and

$$\frac{dS_A(t)x}{dt} = -A_0 T_{A_0}(t)(-A)^{-1}x,\ \ \forall t > 0,\ \forall x \in X.$$

Due to Lemma 9.1.8, we have for each $t > 0$, each $\beta > 1 - 1/p^*$, and each $x \in X$ that

$$\begin{aligned}
\frac{dS_A(t)x}{dt} &= -A_0 T_{A_0}(t)(-A_0)^{-(1-\beta)}(-A)^{-\beta}x, \\
&= -A_0(-A_0)^{-(1-\beta)} T_{A_0}(t)(-A)^{-\beta}x, \\
&= (-A_0)^{\beta} T_{A_0}(t)(-A)^{-\beta}x.
\end{aligned}$$

So the map $t \to S_A(t)$ is continuously differentiable on $(0, +\infty)$, and

$$\frac{dS_A(t)x}{dt} = (-A_0)^{\beta} T_{A_0}(t)(-A)^{-\beta}x,\ \forall t > 0,\ \forall x \in X,\ \forall \beta > 1 - 1/p^*. \tag{9.1.18}$$

Since $\{S_A(t)\}_{t\geq 0}$ is exponentially bounded, we have for each $\tau > 0$ and each $\hat{q} \geq 1$ that $S_A(.)_{|(0,\tau)} \in L^{\hat{q}}((0,\tau), \mathscr{L}(X))$.

Let $\beta \geq 1$ be fixed such that $\beta > 1 - 1/p^*$ and let $x \in D\left((-A_0)^\beta\right)$. It is well known that

$$T_{A_0}(t)x = T_{A_0}(s)x + A_0 \int_s^t T_{A_0}(l)x dl,$$

so

$$(-A_0)^\beta T_{A_0}(t)x = (-A_0)^\beta T_{A_0}(s)x - \int_s^t (-A_0)^{\beta+1} T_{A_0}(l)x dl.$$

We deduce that

$$(-A_0)^\beta \left[T_{A_0}(t)x - T_{A_0}(s)x\right] = \int_s^t (-A_0)^{\beta+1} T_{A_0}(l)x dl.$$

It follows that

$$\left\|(-A_0)^\beta \left[T_{A_0}(t)x - T_{A_0}(s)x\right]\right\| \leq M_{\beta+1} \int_s^t \frac{1}{l^{\beta+1}} dl \, \|x\|,$$

and the map $t \to (-A_0)^\beta T_{A_0}(t)$ is continuous from $(0,+\infty)$ into $\mathscr{L}(X_0)$. Since $(-A)^{-\beta}$ is a bounded operator, the map $H : t \to (-A_0)^\beta T_{A_0}(t)(-A)^{-\beta}$ is continuous from $(0,+\infty)$ into $\mathscr{L}(X)$ and thus Bochner measurable. Now due to (9.1.16) one obtains that

$$\begin{aligned}
\left\|\frac{dS_A(t)x}{dt}\right\| &= \left\|(-A_0)^\beta T_{A_0}(t)(-A)^{-\beta} x\right\| \\
&\leq M_\beta t^{-\beta} e^{\omega_A t} \left\|(-A)^{-\beta} x\right\| \\
&\leq M_\beta t^{-\beta} e^{\omega_A t} \left\|(-A)^{-\beta}\right\| \|x\|,
\end{aligned}$$

so

$$\int_0^\tau \left\|\frac{dS_A(t)}{dt}\right\|_{\mathscr{L}(X)}^{q^*} dt \leq \left(M_\beta \left\|(-A)^{-\beta}\right\|\right)^{q^*} \int_0^\tau t^{-q^*\beta} e^{\omega_A t} dt.$$

Since $q^* \in \left[1, \frac{1}{1-1/p^*}\right)$, it follows that $q^*\beta < 1$ and the map $t \to t^{-q^*\beta}$ is integrable on $(0,\tau)$. The result follows. $\square$

As a direct corollary we have the following result.

Corollary 9.1.10. *Let Assumption 9.1.5 be satisfied. Then the familly of bounded linear operators* $\{T(t) = \frac{dS_A(t)}{dt}\}_{t>0}$ *satisfies* $T(s+t) = T(s)T(t)$ *for all* $s,t > 0$, *and for each* $\beta > 1 - \frac{1}{p^*}$ *there exists some constant* $M_\beta > 0$ *such that*

$$\|t^\beta T(t)\| \leq M_\beta e^{\omega_A t}, \;\; \forall t > 0.$$

Remark 9.1.11. The above results are related to the work of Periago and Straub [284] by using the following formulas for integrated semigroups and their derivatives with respect to $t>0$,

$$S(t)=\int_{\Gamma}\lambda^{-1}\left(e^{\lambda t}-1\right)(\lambda-A)^{-1}d\lambda,\ \ T(t)=\int_{\Gamma}e^{\lambda t}(\lambda-A)^{-1}d\lambda, \tag{9.1.19}$$

where Γ is an angle

$$\{\lambda\in\mathbb{C}:|\arg(\lambda-\omega_A)|=\eta,|\lambda-\omega_A|\geq r\}\cup\{\lambda\in\mathbb{C}:|\arg(\lambda-\omega_A)|\leq\eta,|\lambda-\omega_A|=r\}$$

oriented counterclockwise with $r>0$ and $\eta\in(\pi/2,\theta)$. To prove (9.1.19), simply observe that for $x\in X$,

$$\begin{aligned}T(t)x&=\lim_{\mu\to\infty}\mu\,(\mu I-A_0)^{-1}T(t)x\\&=\lim_{\mu\to\infty}T(t)\mu\,(\mu I-A_0)^{-1}x\\&=\lim_{\mu\to\infty}\int_{\Gamma}e^{\lambda t}(\lambda-A_0)^{-1}d\lambda\,\mu\,(\mu I-A)^{-1}x\\&=\lim_{\mu\to\infty}\mu\,(\mu I-A_0)^{-1}\int_{\Gamma}e^{\lambda t}(\lambda-A)^{-1}xd\lambda,\end{aligned}$$

and the second formula in (9.1.19) follows. By taking the time derivative in the first formula of (9.1.19), we also deduce that the first equality in (9.1.19) holds. Now by using Theorem 3.9 in Periago and Straub [284], we obtain that $t\to T(t)$ and $t\to S(t)$ are analytic functions on $(0,+\infty)$.

9.1.4 Linear Cauchy Problems

In this subsection we investigate the existence and uniqueness of integrated solutions for the linear Cauchy problem (9.1.1). The following theorem is the main result of this section.

Theorem 9.1.12. *Let Assumption 9.1.5 be satisfied. Let $\lambda>\omega_A$ and $\hat{p}\in(p^*,+\infty)$ be fixed. Then for each $f\in L^{\hat{p}}((0,\tau),X)$, the map $t\to(S_A*f)(t)$ is continuously differentiable, $(S_A*f)(t)\in D(A)$, $\forall t\in[0,\tau]$, and if we denote by $u(t)=(S_A\diamond f)(t)=\frac{d}{dt}(S_A*f)(t)$, then*

$$u(t)=A\int_0^t u(s)ds+\int_0^t f(s)ds,\ \forall t\in[0,\tau].$$

Moreover, for each $\beta\in\left(1-\frac{1}{p^*},1-\frac{1}{\hat{p}}\right)$ and each $t\in[0,\tau]$, we have

$$(S_A\diamond f)(t)=\int_0^t(\lambda I-A_0)^{\beta}T_{A_0}(t-s)(\lambda I-A)^{-\beta}f(s)ds \tag{9.1.20}$$

as well as the estimate:

$$\|(S_A \diamond f)(t)\| \leq M_\beta \left\|(\lambda I - A)^{-\beta}\right\|_{\mathscr{L}(X)} \int_0^t (t-s)^{-\beta} e^{\omega_A (t-s)} \|f(s)\| ds, \quad (9.1.21)$$

where M_β is some positive constant.

Proof. Without loss of generality, one may assume that $\omega_A < 0$. Let $\hat{p} \in (p^*, +\infty)$ be fixed and let $f \in C_c^1((0,\tau), X)$. Then the map $t \to (S_A * f)(t)$ is continuously differentiable on $[0,\tau]$ and

$$(S_A \diamond f)(t) = \int_0^t S_A(t-s) f'(s) ds, \ \forall t \in [0,\tau].$$

Using Fubini 's Theorem one obtains for each $t \in [0,\tau]$ that

$$\begin{aligned}
\int_0^t S_A(t-s) f'(s) ds &= \int_0^t \int_0^{t-s} \frac{dS_A(r)}{dr} f'(s) dr ds \\
&= \int_0^t \int_0^{t-r} \frac{dS_A(r)}{dr} f'(s) ds dr \\
&= \int_0^t \frac{dS_A(r)}{dr} f(t-r) dr.
\end{aligned}$$

Since $1 \leq \frac{1}{1-1/\hat{p}} < \frac{1}{1-1/p^*}$, we infer from Lemma 9.1.9 that

$$S_A(.) \in W^{1, \frac{1}{1-1/\hat{p}}}((0,\tau), \mathscr{L}(X)).$$

Thus, Hölder inequality provides, for each $t \in [0,\tau]$, that

$$\|(S_A \diamond f)(t)\| \leq \left\|\frac{d}{dt} S_A(.)\right\|_{L^{\frac{1}{1-1/\hat{p}}}((0,\tau),\mathscr{L}(X))} \cdot \|f\|_{L^{\hat{p}}((0,\tau),X)}.$$

The first part of the theorem follows from the density of $C_c^1((0,\tau), X)$ into $L^{\hat{p}}((0,\tau), X)$ and Theorem 3.4.7. Moreover, for each $\beta \in \left(1 - \frac{1}{p^*}, 1 - \frac{1}{\hat{p}}\right)$ and each $\lambda > \omega_A$, one has

$$\frac{dS_A(t)}{dt} = (\lambda I - A_0)^\beta T_{A_0}(t) (\lambda I - A)^{-\beta}, \ \forall t > 0.$$

Hence,

$$\|(S_A \diamond f)(t)\| \leq M_\beta \left\|(\lambda I - A)^{-\beta}\right\|_{\mathscr{L}(X)} \int_0^t (t-s)^{-\beta} e^{\omega_A (t-s)} \|f(s)\| ds.$$

This completes the proof of the result. $\square$

As a consequence of this result, one can derive some estimate for the resolvent of operator A. Indeed estimate (9.1.21) can be rewritten as for each $\beta > 1 - \frac{1}{p^*}$ that

$$\|(S_A \diamond f)(t)\| \leq \int_0^t \chi(t-s) \|f(s)\| ds, \ \ \forall t \geq 0, \forall f \in C^1((0,+\infty), X),$$

with $\chi(s) = M_\beta s^{-\beta} e^{\omega_A s}$ wherein $M_\beta > 0$ is some constant. Therefore Theorem 3.7.3 (with $B = Id$, $\chi(s) = M_\beta s^{-\beta} e^{\omega_A s}$) implies for any $n \geq 1$ and each $\beta > 1 - \frac{1}{p^*}$ that

$$\begin{aligned}
\left\|(\lambda I - A)^{-n}\right\|_{\mathscr{L}(X)} &\leq \frac{M_\beta}{(n-1)!}\int_0^{+\infty} s^{n-1-\beta} e^{-(\lambda-\omega_A)s} ds \\
&= \frac{M_\beta}{(n-1)!\,(\lambda-\omega_A)^{(n-\beta)}}\int_0^{+\infty} l^{(n-\beta)-1} e^{-l} dl \\
&= \frac{M_\beta \Gamma(n-\beta)}{\Gamma(n)} \frac{1}{(\lambda-\omega_A)^{(n-\beta)}}.
\end{aligned}$$

Conversely, if the above inequality is sastified, then Theorems 3.4.7 and 3.7.3 imply that for each $f \in L^{\hat{p}}(0,\tau,X)$ with $1 - \frac{1}{\hat{p}} < \beta$, the map $t \to (S_A * f)(t)$ is continuously differentiable and

$$\left\| \frac{d}{dt}(S_A * f) \right\| \leq C \int_0^t (t-s)^{-\beta} e^{-\omega_A (t-s)} \|f(s)\| ds, \ \forall t \in [0,\tau].$$

But this condition is not easy to verify in practice compared to Assumption 9.1.5 (b).

Corollary 9.1.13. *Let Assumption 9.1.5 be satisfied. Let $\hat{p} \in (p^*, +\infty)$ be fixed. Then for each $f \in L^{\hat{p}}((0,\tau), X)$ and each $x \in X_0$, the Cauchy problem (9.1.1) has a unique integrated solution $u \in C([0,\tau], X_0)$ given by*

$$u(t) = T_{A_0}(t)x + (S_A \diamond f)(t), \ \forall t \in [0,\tau]. \tag{9.1.22}$$

9.1.5 Perturbation Results

In this subsection, we investigate the properties of $A + B : D(A) \cap D(B) \subset X \to X$, where $B : D(B) \subset X \to X$ is a linear operator. Inspired by Pazy [281], we make the following assumption.

Assumption 9.1.14. Recall that $X_0 = \overline{D(A)}$ and let $B : D(B) \subset X_0 \to Y$ be a linear operator from $D(B)$ into a Banach space $Y \subset X$. Assume that there exists $\alpha \in (0,1)$, such that the operator B is $(\lambda I - A_0)^\alpha$-bounded for some $\lambda > \omega_A$ (that means $B(\lambda I - A_0)^{-\alpha}$ is a bounded linear operator).

Notice that when Assumption 9.1.14 holds, we have

$$D((\lambda I - A_0)^\alpha) \subset D(B),$$

and thus

$$D((\lambda I - A_0)^\alpha) \subset D(A) \cap D(B).$$

In order to state and prove the main result of this subsection, we first introduce the following lemma

Lemma 9.1.15. *Let Assumptions 9.1.5 and 9.1.14 be satisfied and assume that $Y = X$. Let $\gamma \in (\alpha, 1)$ be given and fixed. Then there exists some constant $C > 0$ such that*

$$\left\| B(\delta I - A_0)^{-\gamma} \right\| \leq \frac{C}{(\delta - \omega_A)^{\gamma-\alpha}}, \ \forall \delta > \omega_A, \tag{9.1.23}$$

and

$$\left\| B(\lambda I - A_0)^{-1} \right\| \leq \frac{C}{|\lambda - \omega_A|^{1-\alpha}}, \ \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) > \omega_A. \tag{9.1.24}$$

Moreover, if

$$\alpha < 1/p^*,$$

then for $\widetilde{\omega} > 0$ large enough we have

$$\left\| B(\lambda I - A)^{-1} \right\| \leq 1/2, \ \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) > \omega_A + \widetilde{\omega}. \tag{9.1.25}$$

Proof. Let $\lambda > \omega_A$ be fixed. We have for $\delta > \omega_A$ that

$$\begin{aligned} B(\delta I - A_0)^{-\gamma} &= B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} (\delta I - A_0)^{-\gamma}, \\ &= B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} \left[\frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} e^{-\delta t} T_{A_0}(t) dt \right]. \end{aligned}$$

Since $(\lambda I - A_0)^{-\alpha}$ is bounded and $(\lambda I - A_0)^{\alpha}$ is closed, we have

$$B(\delta I - A_0)^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-1} e^{-\delta t} B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t) dt,$$

so that

$$\begin{aligned} \left\| B(\delta I - A_0)^{-\gamma} \right\| &\leq \frac{\tilde{C}}{\Gamma(\gamma)} \int_0^{+\infty} t^{\gamma-\alpha-1} e^{-\delta t} e^{\omega_A t} dt, \\ &\leq \frac{\tilde{C}}{\Gamma(\gamma)} \int_0^{+\infty} \left(\frac{l}{\delta - \omega_A} \right)^{\gamma-\alpha-1} e^{-l} \frac{1}{\delta - \omega_A} dl, \\ &\leq \frac{C_1}{(\delta - \omega_A)^{\gamma-\alpha}} \end{aligned}$$

and (9.1.23) follows. Now, for all $\gamma > \alpha$, all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > \omega_A$, and $\delta \in \mathbb{R}$ such that $\delta > \omega_A$, one has

$$B(\lambda I - A_0)^{-1} = B(\delta I - A_0)^{-\gamma} (\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1}.$$

By using (9.1.4) as well as the resolvent formula, we have

$$(\delta I - A_0)^{\gamma} (\lambda I - A_0)^{-1} = (\delta - \lambda)(\delta I - A_0)^{-(1-\gamma)} (\lambda I - A_0)^{-1} + (\delta I - A_0)^{-(1-\gamma)}.$$

Therefore, one obtains that

$$\left\|(\delta I-A_0)^{\gamma}(\lambda I-A_0)^{-1}\right\| \leq |\delta-\lambda| \frac{\check{M}}{(\delta-\omega_A)^{1-\gamma}} \frac{M_A}{|\lambda-\omega_A|}+\frac{\check{M}}{(\delta-\omega_A)^{1-\gamma}}.$$

By taking $\delta=\omega_A+|\lambda-\omega_A|>\omega_A$, one has $|\lambda-\delta| \leq 2|\lambda-\omega_A|$. Thus,

$$\left\|(\delta I-A_0)^{\gamma}(\lambda I-A_0)^{-1}\right\| \leq \frac{C_2}{|\lambda-\omega_A|^{1-\gamma}}$$

and (9.1.24) follows.

Finally, we prove (9.1.25). To do so notice that

$$B(\lambda I-A)^{-1}=B(\delta I-A_0)^{-\gamma}(\delta I-A_0)^{\gamma}(\lambda I-A)^{-1},$$

and similarly,

$$(\delta I-A_0)^{\gamma}(\lambda I-A)^{-1}=(\delta-\lambda)(\delta I-A_0)^{-(1-\gamma)}(\lambda I-A)^{-1}+(\delta I-A)^{-(1-\gamma)}. \tag{9.1.26}$$

Since $\alpha<1/p^*$, we can find $\gamma>0$ such that

$$\alpha<\gamma<1/p^* \text{ and } 1-\gamma>1-1/p^*.$$

Then due to Lemma 9.1.8, $(\delta I-A)^{-(1-\gamma)}$ is well defined. Moreover, by setting $\delta=\omega_A+|\lambda-\omega_A|$ in (9.1.26), we obtain

$$\left\|(\delta I-A_0)^{\gamma}(\lambda I-A)^{-1}\right\| \leq |\lambda-\omega_A| \cdot \frac{\check{M}}{|\lambda-\omega_A|^{1-\gamma}} \cdot \frac{\tilde{M}}{|\lambda-\omega_A|^{1/p^*}}+\frac{\tilde{M}}{|\lambda-\omega_A|^{1-\gamma}}.$$

Therefore, there exist two constants $C_3, C_4>0$ such that

$$\left\|B(\lambda I-A)^{-1}\right\| \leq \frac{C_3}{|\lambda-\omega_A|^{1/p^*-\alpha}}+\frac{C_4}{|\lambda-\omega_A|^{1-\alpha}}.$$

Thus, for $\widetilde{\omega}>0$ large enough, we obtain for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda)-\omega_A>\widetilde{\omega}$ that

$$\left\|B(\lambda I-A)^{-1}\right\| \leq \frac{C_3}{\widetilde{\omega}^{1/p^*-\alpha}}+\frac{C_4}{\widetilde{\omega}^{1-\alpha}} \leq \frac{1}{2}$$

and (9.1.25) follows. □

The main result of this subsection is the following theorem.

Theorem 9.1.16. *Let Assumptions 9.1.5 and 9.1.14 be satisfied and assume that $Y=X$. Assume in addition that*

$$\alpha<1/p^*.$$

Then $A+B: D(A) \cap D(B) \subset X \to X$ satisfies Assumption 9.1.5. More precisely, there exist two constants, $\widehat{\omega}>0$ and $\widetilde{M}>1$, such that *(i)* $(\widehat{\omega},+\infty)\subset\rho(A+B)$ *and the resolvent of* $A+B$ *is given by*

$$(\lambda I-(A+B))^{-1}=(\lambda I-A)^{-1}+(\lambda I-A)^{-1}\sum_{k=0}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}B(\lambda I-A)^{-1}$$

whenever $\lambda\in\mathbb{C}$ *with* $\mathrm{Re}\,(\lambda)>\widehat{\omega}$;

(ii) $(A+B)_0$, *the part of* $A+B$ *in* X_0, *is the infinitesimal generator of an analytic semigroup* $\left\{T_{(A+B)_0}(t)\right\}_{t\geq 0}$ *on* X_0 *and*

$$\left\|(\lambda I-(A+B))^{-1}\right\|\leq\frac{\widetilde{M}}{(\lambda-\widehat{\omega})^{1/p^*}},\ \forall\lambda>\widehat{\omega}.$$

Proof. For each $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,(\lambda)>\omega_A+\widetilde{\omega}$, where $\widetilde{\omega}$ is provided by Lemma 9.1.15, we have

$$\begin{aligned}(\lambda I-(A+B)_0)^{-1}&=(\lambda I-A)^{-1}\left(\sum_{k=0}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}\right)\\&=(\lambda I-A_0)^{-1}+(\lambda I-A)^{-1}\sum_{k=1}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}.\end{aligned}$$

Thus,

$$(\lambda I-(A+B)_0)^{-1}=(\lambda I-A_0)^{-1}+(\lambda I-A)^{-1}\sum_{k=0}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}B(\lambda I-A_0)^{-1}.\tag{9.1.27}$$

We infer by combining Assumption 9.1.5 (b), (9.1.24) and (9.1.25) that

$$\begin{aligned}\left\|(\lambda I-(A+B)_0)^{-1}\right\|&\leq\frac{\hat{M}}{|\lambda-\omega_A|}+\frac{M}{|\lambda-\omega_A|^{1/p^*}}\frac{1}{1-1/2}\frac{C_2}{|\lambda-\omega_A|^{1-\alpha}},\\&\leq\frac{1}{|\lambda-\omega_A|}\left(\hat{M}+\frac{C_2M}{|\lambda-\omega_A|^{1/p^*-\alpha}}\right),\\&\leq\frac{1}{|\lambda-\omega_A|}\left(\hat{M}+\frac{C_2M}{\widetilde{\omega}^{1/p^*-\alpha}}\right).\end{aligned}$$

Thus, there exists some constant $C>0$ such that for each $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,(\lambda)>\omega_A+\widetilde{\omega}$,

$$\left\|(\lambda I-(A+B)_0)^{-1}\right\|\leq\frac{C}{|\lambda-\omega_A|},$$

which implies that $(A+B)_0$ is a sectorial operator. Next, for $\lambda>\omega_A+\widetilde{\omega}$ we have

$$(\lambda I-(A+B))^{-1}=(\lambda I-A)^{-1}+(\lambda I-A)^{-1}\sum_{k=0}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}.$$

Combining Assumption 9.1.5 (b), (9.1.25) together with (9.1.27) leads to

$$\left\|(\lambda I-(A+B))^{-1}\right\|\leq\frac{\hat{M}}{|\lambda-\omega_A|^{1/p^*}}+\frac{M}{|\lambda-\omega_A|^{1/p^*}}\left(\frac{1}{1-1/2}\times\frac{1}{2}\right).$$

Hence, there exist two constants, $\widehat{\omega}>\omega_A+\widetilde{\omega}$ and $\widetilde{M}>0$, such that $(\widehat{\omega},+\infty)\subset\rho(A+B)$ and

$$\left\|(\lambda I-(A+B))^{-1}\right\|\leq\frac{\widetilde{M}}{(\lambda-\widehat{\omega})^{1/p^*}},\ \forall\lambda>\widehat{\omega}.$$

The result is proved. □

By taking $p^*=1$, we have the following immediate corollary.

Corollary 9.1.17. *If A is a Hille-Yosida operator and A_0 is the infinitesimal generator of an analytic semigroup such that $B(\lambda I-A_0)^{-\alpha}$ is a bounded operator for some $\alpha\in(0,1)$ and some $\lambda>\omega_A$, then $A+B$ is a Hille-Yosida operator and $(A+B)_0$ is the infinitesimal generator of an analytic semigroup.*

Using Proposition 9.1.4, we also have the following corollary.

Corollary 9.1.18. *Let $A:D(A)\subset X\to X$ be a $\frac{1}{p^*}-$almost sectorial operator for some $p^*\geq 1$ and with sectorial part A_0 on X_0. Let $B:D(B)\subset X_0\to X$ be a linear closed operator such that there exists $\alpha<\frac{1}{p^*}$ and $D\left((\lambda I-A_0)^{\alpha}\right)\subset D(B)$ for some $\lambda>\omega_A$. Then $A+B$ is a $\frac{1}{p^*}-$almost sectorial operator with a sectorial part $(A+B)_0$ in X_0.*

Theorem 9.1.19. *Let Assumptions 9.1.5 and 9.1.14 be satisfied and assume that $Y=X_0$. Then $A+B:D(A)\cap D(B)\subset X\to X$ satisfies Assumption 9.1.5.*

Proof. For each $\lambda\in\mathbb{C}$ with Re $(\lambda)>\omega_A+\widetilde{\omega}$, we have

$$\begin{aligned}(\lambda I-(A+B)_0)^{-1}&=(\lambda I-A)^{-1}\left(\sum_{k=0}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}\right),\\&=(\lambda I-A_0)^{-1}+(\lambda I-A)^{-1}\sum_{k=1}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k},\\&=(\lambda I-A_0)^{-1}+(\lambda I-A_0)^{-1}\sum_{k=1}^{\infty}\left[B(\lambda I-A)^{-1}\right]^{k}B(\lambda I-A_0)^{-1}.\end{aligned}$$

Thus, we have for any $\lambda\in\mathbb{C}$ with Re $(\lambda)>\widetilde{\omega}+\omega_A$ that

$$\left\|(\lambda I-(A+B)_0)^{-1}\right\|\leq\frac{C}{|\lambda-\omega_A|},$$

where $C > 0$ is some constant. Similarly we also deduce that there exists $\widehat{\omega} > \omega_A$ such that

$$\left\|(\lambda I - (A+B))^{-1}\right\| \leq \frac{\widetilde{M}}{(\lambda - \widehat{\omega})^{1/p^*}}, \ \forall \lambda > \widehat{\omega},$$

and the result follows. $\square$

In order to extend the linear theory to the semi-linear ones, it will be useful to find some invariant $L^{\hat{p}}$ space. We address this question in the next proposition.

Proposition 9.1.20. *Let Assumptions 9.1.5 and 9.1.14 be satisfied. Assume in addition that $\omega_A < 0$. If there exists $\hat{p} \in [1, +\infty)$ such that*

$$p^* < \hat{p} < \frac{1}{\alpha}. \tag{9.1.28}$$

Then

(i) The map $x \to BT_{A_0}(.)x$ defines a bounded linear operator from X_0 into $L^{\hat{p}}((0,\tau), X_0)$;
(ii) For each $f \in L^{\hat{p}}((0,\tau), X)$,

$$B(S_A \diamond f)(.) \in L^{\hat{p}}((0,\tau), Y).$$

Moreover, for each $\beta \in (1 - 1/p^*, 1-\alpha)$, the following estimate holds:

$$\|B(S_A \diamond f)(.)\|_{L^{\hat{p}}((0,\tau),Y)} \leq C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} dt \, \|f(.)\|_{L^{\hat{p}}((0,\tau),X)}, \tag{9.1.29}$$

where

$$C_{\alpha,\beta} =: \left\|B(\lambda I - A_0)^{-\alpha}\right\|_{\mathscr{L}(X_0,Y)} M_{\beta+\alpha} \left\|(\lambda I - A)^{-\beta}\right\|_{\mathscr{L}(X)}.$$

Proof. First note that since

$$\hat{p} \in (p^*, +\infty), \tag{9.1.30}$$

the assumptions of Theorem 9.1.12 are satisfied. Now let $\lambda > \omega_A$ be given. Then for any $t > 0$, we have for each $\gamma > \alpha$ that

$$\begin{aligned} BT_{A_0}(t) &= B(\lambda I - A_0)^{-\gamma} (\lambda I - A_0)^{\gamma} T_{A_0}(t) \\ &= B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t). \end{aligned}$$

Therefore, for any $x \in X_0$, we have

$$\begin{aligned} \left\|BT_{A_0}(.)x\right\|_{L^{\hat{p}}((0,\tau),Y)} &= \left\|B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(.)x\right\|_{L^{\hat{p}}((0,\tau),Y)} \\ &= \left(\int_0^\tau \left\|B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} T_{A_0}(t)x\right\|_Y^{\hat{p}} dt\right)^{1/\hat{p}} \\ &\leq \left\|B(\lambda I - A_0)^{-\alpha}\right\|_{\mathscr{L}(X_0,Y)} \left(\int_0^\tau \left\|(\lambda I - A_0)^{\alpha} T_{A_0}(t)x\right\|_{X_0}^{\hat{p}} dt\right)^{1/\hat{p}} \end{aligned}$$

$$\leq M_\alpha \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathscr{L}(X_0,Y)} \left(\int_0^\tau t^{-\hat{p}\alpha} dt \right)^{1/\hat{p}} \|x\|_{X_0}.$$

Now fix $\hat{p}$ such that

$$\hat{p} < \frac{1}{\alpha}. \tag{9.1.31}$$

Then we obtain that the map $t \to t^{-\hat{p}\alpha}$ is integrable over $(0,\tau)$ and the map $x \to BT_{A_0}(.)x$ is bounded by X_0 into $L^{\hat{p}}((0,\tau),Y)$. Moreover, the following estimate holds:

$$\left\| BT_{A_0}(.)x \right\|_{L^{\hat{p}}((0,\tau),Y)} \leq C \|x\|_{X_0},$$

where C is defined by

$$C := M_\alpha \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathscr{L}(X_0,Y)} \left(\int_0^\tau t^{-\hat{p}\alpha} \right)^{1/\hat{p}}.$$

Note that due to (9.1.30) and (9.1.31), one can find β such that

$$\beta > 1 - 1/p^* \text{ and } \alpha + \beta < 1. \tag{9.1.32}$$

Fix such a value β. Then since $\beta > 1 - 1/p^*$, the fractional power of $(\lambda I - A)^{-\beta}$ is well defined for λ large enough and we have

$$\begin{aligned} &B(S_A \diamond f)(t) \\ &\quad = B(\lambda I - A_0)^{-\alpha} (\lambda I - A_0)^{\alpha} \int_0^t (\lambda I - A_0)^{\beta} T_{A_0}(t-s) (\lambda I - A)^{-\beta} f(s) ds, \\ &\quad = B(\lambda I - A_0)^{-\alpha} \int_0^t (\lambda I - A_0)^{\beta+\alpha} T_{A_0}(t-s) (\lambda I - A)^{-\beta} f(s) ds. \end{aligned}$$

Thus

$$\begin{aligned} \|B(S_A \diamond f)\|_{L^{\hat{p}}((0,\tau),Y)} &\leq C_{\alpha,\beta} \left(\left(\int_0^\tau \int_0^t (t-s)^{-(\beta+\alpha)} \|f(s)\| ds dt \right)^{\hat{p}} \right)^{1/\hat{p}} \\ &= C_{\alpha,\beta} \left\| \left((.)^{-(\beta+\alpha)} * \|f(.)\| \right)(.) \right\|_{L^{\hat{p}}((0,\tau),\mathbb{R})} \\ &\leq C_{\alpha,\beta} \int_0^\tau t^{-(\beta+\alpha)} dt \, \|f(.)\|_{L^{\hat{p}}((0,\tau),X)}, \end{aligned}$$

where

$$C_{\alpha,\beta} =: \left\| B(\lambda I - A_0)^{-\alpha} \right\|_{\mathscr{L}(X_0,Y)} M_{\beta+\alpha} \left\| (\lambda I - A)^{-\beta} \right\|_{\mathscr{L}(X)}.$$

Due to (9.1.32), the map $t \to t^{-(\beta+\alpha)}$ is integrable over $(0,\tau)$ and the result follows. □

The following result is another formulation of the perturbation result.

Theorem 9.1.21. *Under the same assumptions as in Proposition 9.1.20,* $\{T_{(A+B)_0}(t)\}_{t\geq 0}$, *the* C_0*-semigroup generated by* $(A+B)_0$, *is the unique solution of the fixed point problem*

$$T_{(A+B)_0}(t) = T_{A_0}(t) + (S_A \diamond V)(t), \tag{9.1.33}$$

where $V(.)x \in L^{\widehat{p}}_{\omega^*}((0,+\infty),X)$ *(for* $\omega^* > 0$ *large enough) is the solution of*

$$V(t)x = BT_{A_0}(t)x + B(S_A \diamond V(.)x)(t), \;\; t > 0, \tag{9.1.34}$$

in which $L^{\widehat{p}}_{\omega^*}((0,+\infty),X)$ *is the space of Bochner measurable maps* $f:(0,+\infty)\to X$ *such that*

$$\|f\|_{L^{\widehat{p}}_{\omega^*}} := \left(\int_0^{+\infty}\left\|e^{-\omega^* t}f(t)\right\|^{\widehat{p}}dt\right)^{1/\widehat{p}} < +\infty.$$

Proof. Let $\lambda \in (\omega_A, +\infty)$ be fixed. Multiplying (9.1.34) by the map $t \to e^{-\widehat{\omega}t}$ and using the same arguments as in the proof of Proposition 9.1.20, we obtain

$$\begin{aligned}
e^{-\widehat{\omega}t}V(t)x &= e^{-\widehat{\omega}t}BT_{A_0}(t)x + e^{-\widehat{\omega}t}B(S_A \diamond V(.)x)(t)\\
&= e^{-\widehat{\omega}t}B(\lambda I - A_0)^{-\alpha}(\lambda I - A_0)^{\alpha}T_{A_0}(t)\\
&\quad + B(\lambda I - A_0)^{-\alpha}\int_0^t e^{-\widehat{\omega}(t-s)}(\lambda I - A_0)^{\beta+\alpha}T_{A_0}(t-s)(\lambda I - A)^{-\beta}e^{-\widehat{\omega}s}V(s)xds.
\end{aligned}$$

Thus, for each $\widehat{\omega} > 0$ large enough we obtain, for any $\hat{p} \in \left(p^*, \frac{1}{\alpha}\right)$, that

$$\begin{aligned}
&\left\|e^{-\widehat{\omega}\cdot}B(\lambda I - A_0)^{-\alpha}(\lambda I - A_0)^{\alpha}T_{A_0}(.)x\right\|_{L^{\hat{p}}}\\
&\leq \left\|e^{-\widehat{\omega}\cdot}B(\lambda I - A_0)^{-\alpha}(\lambda I - A_0)^{\alpha}T_{A_0}(.)\right\|_{L^{\hat{p}}}\\
&\leq M_\alpha \left\|B(\lambda I - A_0)^{-\alpha}\right\|_{\mathscr{L}(X_0,Y)}\left(\int_0^{+\infty}t^{-\hat{p}\alpha}e^{-\hat{p}\widehat{\omega}t}dt\right)^{1/\hat{p}}\|x\|_{X_0},
\end{aligned}$$

and for each $\beta \in (1-1/p^*, 1-\alpha)$,

$$\begin{aligned}
&\left\|e^{-\widehat{\omega}\cdot}B(S_A \diamond V(.)x)(.)\right\|_{L^{\hat{p}}}\\
&\leq C_{\alpha,\beta}\left(\left(\int_0^\tau\int_0^t e^{-\widehat{\omega}(t-s)}(t-s)^{-(\beta+\alpha)}e^{-\widehat{\omega}s}\|V(s)x\|dsdt\right)^{\hat{p}}\right)^{1/\hat{p}}\\
&\leq C_{\alpha,\beta}\int_0^\tau t^{-(\beta+\alpha)}e^{-\widehat{\omega}t}dt\left\|e^{-\widehat{\omega}\cdot}V(.)x\right\|_{L^{\hat{p}}}.
\end{aligned}$$

Taking $\omega^* > 0$ large enough leads to

$$C_{\alpha,\beta}\int_0^\tau t^{-(\beta+\alpha)}e^{-\omega^* t}dt < 1.$$

From some fixed point argument, we conclude that (9.1.34) has a unique solution. Therefore, we can define a strongly continuous familly of linear operators $\{L(t)\}_{t\geq 0}$ such that

$$L(t) = T_{A_0}(t) + (S_A \diamond V)(t),$$

where $V(.) \in L^{\widehat{p}}_{\omega^*}((0,+\infty),X)$ satisfies (9.1.34). Due to Theorem 2.1 in Arendt [21], to complete the proof of the result, it is sufficient to check that the following equality holds for each $\lambda > 0$ large enough:

$$(\lambda I - (A+B)_0)^{-1} = \int_0^{+\infty} e^{-\lambda t} L(t)dt.$$

On one hand, we have

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} L(t)dt &= \int_0^{+\infty} e^{-\lambda t} T_{A_0}(t)dt + \int_0^{+\infty} e^{-\lambda t} (S_A \diamond V)(t)dt \\ &= (\lambda I - A_0)^{-1} + \lambda \int_0^{+\infty} e^{-\lambda t} (S_A * V)(t)dt \\ &= (\lambda I - A_0)^{-1} + \lambda \int_0^{+\infty} e^{-\lambda t} S_A(t)dt \int_0^{+\infty} e^{-\lambda t} V(t)dt \\ &= (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \int_0^{+\infty} e^{-\lambda t} V(t)dt.\end{aligned}$$

On the other hand we infer from (9.1.34) that

$$\int_0^{+\infty} e^{-\lambda t} V(t)dt = B \int_0^{+\infty} e^{-\lambda t} L(t)dt.$$

Thus we obtain

$$\int_0^{+\infty} e^{-\lambda t} L(t)dt = (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} B \int_0^{+\infty} e^{-\lambda t} L(t)dt,$$

and finally

$$\int_0^{+\infty} e^{-\lambda t} L(t)dt = (\lambda I - A_0)^{-1} + (\lambda I - A)^{-1} \sum_{k=0}^{\infty} \left[B (\lambda I - A)^{-1} \right] B (\lambda I - A_0)^{-1}.$$

The result follows from Theorem 9.1.16 (i). □

Following the techniques and theories in Chapter 6, one can develop the center manifold theory, Hopf bifurcation theorem and normal form theory for parabolic equations with almost sectorial operators.

9.1.6 Applications

(a) A Scalar Parabolic Equation on $(0,1)$.

Let $p \in [1,+\infty)$ and $\varepsilon > 0$ be given. In this subsection we consider a parabolic equation with nonhomogeneous Robin's boundary condition

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} = \dfrac{\partial^2 v(t,x)}{\partial x^2} + h_2(t)(x), \ t \geq 0, \ x \in (0,1), \\ \alpha_0 \dfrac{\partial v(t,0)}{\partial x} + \beta_0 v(t,0) = h_0(t), \\ \alpha_1 \dfrac{\partial v(t,1)}{\partial x} + \beta_1 v(t,1) = h_1(t), \\ v(0,x) = v_0 \in L^p((0,1),\mathbb{R}). \end{cases} \tag{9.1.35}$$

Assumption 9.1.22. Assume that

$$\alpha_0^2 + \beta_0^2 > 0 \text{ and } \alpha_1^2 + \beta_1^2 > 0.$$

Consider the space

$$X := \mathbb{R}^2 \times L^p((0,1),\mathbb{R}),$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} x_0 \\ x_1 \\ \varphi \end{pmatrix} \right\| = |x_0| + |x_1| + \|\varphi\|_{L^p((0,1),\mathbb{R})}$$

Define the linear operator $A : D(A) \subset X \to X$ as follows

$$A \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -[\alpha_0 \varphi'(0) + \beta_0 \varphi(0)] \\ -[\alpha_1 \varphi'(1) + \beta_1 \varphi(1)] \\ \varepsilon^2 \varphi'' \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\}^2 \times W^{2,p}((0,1),\mathbb{R}).$$

Set

$$f(t) = \begin{pmatrix} h_0(t) \\ h_1(t) \\ h_2(t) \end{pmatrix}.$$

Then by identifying

$$u(t) = \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ v(t) \end{pmatrix},$$

we can rewrite the parabolic problem (9.1.35) as the following abstract Cauchy problem

$$\frac{d}{dt} u(t) = Au(t) + f(t), \ t > 0; \ \ u(0) = \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} \\ v_0 \end{pmatrix} \in \overline{D(A)}.$$

Then it is easy to observe that the domain of A is non-densely defined because

$$X_0 := \overline{D(A)} = \{0_{\mathbb{R}}\}^2 \times L^p((0,1),\mathbb{R}) \neq X.$$

By construction A_0, the part of A in X_0, coincides with the usual formulation for the parabolic system (9.1.35) with homogeneous boundary conditions. Indeed $A_0 : D(A_0) \subset X_0 \to X_0$ is a linear operator on X_0 defined by

$$A_0 \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \{0_{\mathbb{R}}\}^2 \times \left\{ \varphi \in W^{2,p}\left((0,1), \mathbb{R}\right) : \left[\alpha_x \varphi'(x) + \beta_x \varphi(x)\right] = 0 \text{ for } x = 0, 1 \right\}.$$

The first main result of this section is the following theorem.

Theorem 9.1.23. *Let Assumption 9.1.22 be satisfied. The linear operator A_0 is the infinitesimal generator of an analytic semigroup on X_0, $(0,+\infty) \subset \rho(A)$, and*

$$\limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} < +\infty$$

with

$$p^* := \begin{cases} \dfrac{2p}{1+p} & \text{if } \alpha_0^2 > 0 \text{ and } \alpha_1^2 > 0, \\ 2p & \text{if } \alpha_0^2 = 0 \text{ or } \alpha_1^2 = 0. \end{cases} \tag{9.1.36}$$

In the rest of this subsection, set

$$\Omega_\omega = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega\}, \ \forall \omega \in \mathbb{R},$$

and for each $\lambda \in \mathbb{C}$ define

$$\Delta(\lambda) = (-\mu\alpha_0 + \beta_0)(\mu\alpha_1 + \beta_1)e^{\mu} - (\mu\alpha_0 + \beta_0)(-\mu\alpha_1 + \beta_1)e^{-\mu},$$

where

$$\mu := \sqrt{\lambda}.$$

To prove the following lemma, we use the same method as in Engel and Nagel [126, p. 388-390] to obtain an explicite formula for the resolvent of A.

Lemma 9.1.24. *Let Assumption 9.1.22 be satisfied. There exists $\omega_A \geq 0$ such that*

$$\Omega_{\omega_A} \subset \{\lambda \in \mathbb{C} : \Delta(\lambda) \neq 0\} \subset \rho(A),$$

and for each $\lambda := \mu^2 \in \Omega_{\omega_A}$ we have

$$\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = (\lambda I - A)^{-1} \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Longleftrightarrow$$
$$\varphi(x) = \frac{1}{\Delta(\lambda)} \left[(\mu\alpha_1 + \beta_1) e^{\mu(1-x)} + (\mu\alpha_1 - \beta_1) e^{-\mu(1-x)} \right] y_0$$
$$+ \frac{1}{\Delta(\lambda)} [(-\mu\alpha_0 + \beta_0) e^{\mu x} - (\mu\alpha_0 + \beta_0) e^{-\mu x}] y_1$$
$$+ \frac{\Delta_1(x)}{\Delta(\lambda)} \tfrac{1}{2\mu} \textstyle\int_0^1 e^{-\mu s} f(s) ds + \frac{\Delta_2(x)}{\Delta(\lambda)} \tfrac{1}{2\mu} \int_0^1 e^{-\mu(1-s)} f(s) ds + \tfrac{1}{2\mu} \int_0^1 e^{-\mu|x-s|} f(s) ds,$$

where

$$\Delta_1(x) = \left[-(\mu\alpha_0 + \beta_0)(\mu\alpha_1 + \beta_1) e^{\mu(1-x)} + (\beta_1 - \mu\alpha_1)(\mu\alpha_0 + \beta_0) e^{-\mu(1-x)} \right]$$

and

$$\Delta_2(x) = \left[(\mu\alpha_0 + \beta_0)(\beta_1 - \mu\alpha_1) e^{-\mu x} - (\beta_1 - \mu\alpha_1)(-\mu\alpha_0 + \beta_0) e^{\mu x} \right].$$

Proof. In order to compute the resolvent we set

$$u(x) = \frac{1}{2\mu} \int_0^1 e^{-\mu|x-s|} f(s) ds = \frac{1}{2\mu} \int_{-\infty}^{+\infty} e^{-\mu|x-s|} \bar{f}(s) ds,$$

where $\bar{f}$ extends f by 0 on $\mathbb{R} \setminus [0,1]$. We have

$$u(x) = \frac{1}{2\mu} \left[\int_{-\infty}^{x} e^{-\mu(x-s)} \bar{f}(s) ds + \int_x^{+\infty} e^{\mu(x-s)} \bar{f}(s) ds \right],$$

so

$$u'(x) = -\frac{1}{2} \int_{-\infty}^{x} e^{-\mu(x-s)} \bar{f}(s) ds + \frac{1}{2} \int_x^{+\infty} e^{\mu(x-s)} \bar{f}(s) ds.$$

Set

$$u(0) = \gamma_0 := \frac{1}{2\mu} \int_0^1 e^{-\mu s} f(s) ds \text{ and } u(1) = \gamma_1 := \frac{1}{2\mu} \int_0^1 e^{-\mu(1-s)} f(s) ds$$

and observe that

$$u'(0) = \mu\gamma_0 \text{ and } u'(1) = -\mu\gamma_1.$$

Denote

$$u_1(x) = e^{-\mu x} \text{ and } u_2(x) = e^{\mu x}.$$

Then we solve the problem

$$(\lambda I - A) \begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix}$$

and look for φ under the form

$$\varphi(x) = u(x) + z_1 u_1(x) + z_2 u_2(x),$$

where $z_1, z_2 \in Y$.

For $x = 0, 1$, we obtain

$$\alpha_x \left[u'(x) + z_1 u_1'(x) + z_2 u_2'(x)\right] + \beta_x \left[u(x) + z_1 u_1(x) + z_2 u_2(x)\right] = y_x.$$

So we must solve the system

$$\begin{aligned} z_1 \left(\alpha_0 u_1'(0) + \beta_0 u_1(0)\right) + z_2 \left(\alpha_0 u_2'(0) + \beta_0 u_2(0)\right) &= y_0 - \alpha_0 u'(0) - \beta_0 u(0), \\ z_1 \left(\alpha_1 u_1'(1) + \beta_1 u_1(1)\right) + z_2 \left(\alpha_1 u_2'(1) + \beta_1 u_2(1)\right) &= y_1 - \alpha_1 u'(1) - \beta_1 u(1), \end{aligned}$$

which are equivalent to

$$\begin{aligned} &z_1 \left(-\mu\alpha_0 + \beta_0\right) + z_2 \left(\mu\alpha_0 + \beta_0\right) = y_0 - \mu\alpha_0\gamma_0 - \beta_0\gamma_0 = y_0 - \gamma_0 \left(\mu\alpha_0 + \beta_0\right), \\ &z_1 \left(-\mu\alpha_1 + \beta_1\right) e^{-\mu} + z_2 \left(\mu\alpha_1 + \beta_1\right) e^{\mu} = y_1 + \mu\alpha_1\gamma_1 - \beta_1\gamma_1 = y_1 - \gamma_1 \left(\beta_1 - \mu\alpha_1\right). \end{aligned}$$

Thus

$$\begin{cases} z_1 \left(-\mu\alpha_0 + \beta_0\right) + z_2 \left(\mu\alpha_0 + \beta_0\right) = y_0 - \left(\mu\alpha_0 + \beta_0\right)\gamma_0, \\ z_1 \left(-\mu\alpha_1 + \beta_1\right) e^{-\mu} + z_2 \left(\mu\alpha_1 + \beta_1\right) e^{\mu} = y_1 - \left(\beta_1 - \mu\alpha_1\right)\gamma_1, \end{cases}$$

which imply that

$$\begin{aligned} z_1 &= \tfrac{1}{\Delta}\left[\left(\mu\alpha_1 + \beta_1\right) e^{\mu} \left(y_0 - \left(\mu\alpha_0 + \beta_0\right)\gamma_0\right) - \left(\mu\alpha_0 + \beta_0\right)\left(y_1 - \left(\beta_1 - \mu\alpha_1\right)\gamma_1\right)\right] \\ &= \tfrac{1}{\Delta}\left[\left(\mu\alpha_1 + \beta_1\right) e^{\mu} y_0 - \left(\mu\alpha_1 + \beta_1\right) e^{\mu} \left(\mu\alpha_0 + \beta_0\right)\gamma_0 - \left(\mu\alpha_0 + \beta_0\right) y_1 \right. \\ &\quad \left. + \left(\beta_1 - \mu\alpha_1\right)\left(\mu\alpha_0 + \beta_0\right)\gamma_1\right] \end{aligned}$$

and

$$\begin{aligned} z_2 &= \tfrac{1}{\Delta}\left[\left(-\mu\alpha_0 + \beta_0\right)\left(y_1 - \left(\beta_1 - \mu\alpha_1\right)\gamma_1\right) - \left(-\mu\alpha_1 + \beta_1\right) e^{-\mu} \left(y_0 - \left(\mu\alpha_0 + \beta_0\right)\gamma_0\right)\right] \\ &= \tfrac{1}{\Delta}\left[\left(-\mu\alpha_0 + \beta_0\right) y_1 - \left(-\mu\alpha_1 + \beta_1\right) e^{-\mu} y_0 \right. \\ &\quad \left. + \left(\beta_1 - \mu\alpha_1\right)\left(\mu\alpha_0 + \beta_0\right) e^{-\mu}\gamma_0 - \left(-\mu\alpha_0 + \beta_0\right)\gamma_1\right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} &(\lambda I - A)\begin{pmatrix} 0 \\ 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ f \end{pmatrix} \Longleftrightarrow \\ &\varphi(x) = \tfrac{1}{\Delta}\left[\left(\mu\alpha_1 + \beta_1\right) e^{\mu(1-x)} + \left(\mu\alpha_1 - \beta_1\right) e^{-\mu(1-x)}\right] y_0 \\ &\qquad + \tfrac{1}{\Delta}\left[\left(-\mu\alpha_0 + \beta_0\right) e^{\mu x} - \left(\mu\alpha_0 + \beta_0\right) e^{-\mu x}\right] y_1 \\ &\qquad + \tfrac{1}{\Delta}\left[\left(\mu\alpha_0 + \beta_0\right)\left(\beta_1 - \mu\alpha_1\right) e^{-\mu x} - \left(\beta_1 - \mu\alpha_1\right)\left(-\mu\alpha_0 + \beta_0\right) e^{\mu x}\right]\gamma_1 \\ &\qquad + \tfrac{1}{\Delta}\left[-\left(\mu\alpha_0 + \beta_0\right)\left(\mu\alpha_1 + \beta_1\right) e^{\mu(1-x)} + \left(\beta_1 - \mu\alpha_1\right)\left(\mu\alpha_0 + \beta_0\right) e^{-\mu(1-x)}\right]\gamma_0 \\ &\qquad + \tfrac{1}{2\mu}\int_0^1 e^{-\mu|x-s|} f(s)\,ds. \end{aligned}$$

This completes the proof. $\square$

From the above explicit formula for the resolvent of A we deduce the following result.

Lemma 9.1.25. *Let Assumption 9.1.22 be satisfied. We have the following:*

(a) *There exists* $\omega_A \geq 0$ *and* $M_A > 0$ *such that*

$$\left\|(\lambda I - A_0)^{-1}\right\|_{\mathscr{L}(X_0)} \leq \frac{M_A}{|\lambda|},\ \forall \lambda \in \Omega_{\omega_A};$$

(b) *If* $\alpha_0^2 > 0$ *and* $\alpha_1^2 > 0$, *then*

$$0 < \liminf_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} \leq \limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} < +\infty$$

with

$$p^* := \frac{2p}{1+p} \leq p;$$

(c) *If either* $\alpha_0^2 = 0$ *or* $\alpha_1^2 = 0$, *then*

$$0 < \liminf_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} \leq \limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} < +\infty$$

with

$$p^* := 2p > p.$$

Proof. Assertion (a) follows directly from the explicit formula of the resolvent of A obtained in Lemma 9.1.24. So we only prove assertions (b) and (c). Let $\lambda > \omega_A$. Then

$$\begin{aligned}
&(\lambda I - A)^{-1}\begin{pmatrix} y_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \varphi_0 \end{pmatrix} \\
&\Leftrightarrow \varphi(x) = \tfrac{1}{\Delta}\left[(\mu\alpha_1 + \beta_1)\, e^{\mu(1-x)} + (\mu\alpha_1 - \beta_1)\, e^{-\mu(1-x)}\right] y_0.
\end{aligned}$$

If $\lambda := \mu^2 > 0$, we have

$$\begin{aligned}
&\left\|(\lambda I - A)^{-1}\begin{pmatrix} y_0 \\ 0 \\ 0 \end{pmatrix}\right\| \\
&\quad = \frac{1}{|\Delta|}\left(\int_0^1 \left(\left|(\mu\alpha_1 + \beta_1)\, e^{\mu(1-x)} + (\mu\alpha_1 - \beta_1)\, e^{-\mu(1-x)}\right|\right)^p dx\right)^{1/p} \|y_0\| \\
&\quad \leq \frac{|\mu\alpha_1 + \beta_1|}{|\Delta|}\left[\left(\int_0^1 \left|e^{\mu p(1-x)}\right| dx\right)^{1/p} + \left(\int_0^1 \left|e^{-\mu p(1-x)}\right| dx\right)^{1/p}\right] \|y_0\| \\
&\quad \leq \frac{|\mu\alpha_1 + \beta_1|}{|\Delta|}\left[\left(\int_0^1 e^{\mu p r} dr\right)^{1/p} + \left(\int_0^1 e^{-\mu p r} dr\right)^{1/p}\right] \|y_0\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|\mu\alpha_1+\beta_1|}{|\Delta|}\left(\left[\frac{1}{\mu p}\left(e^{\mu p}-1\right)\right]^{1/p}+\left[\frac{1}{\mu p}\left(1-e^{-\mu p}\right)\right]^{1/p}\right)\|y_0\| \\
&\leq \frac{|\mu\alpha_1+\beta_1|}{|\Delta|}\frac{1}{\mu^{1/p}p^{1/p}}\left[e^{\mu}+1\right]\|y_0\|.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\left\|(\lambda I-A)^{-1}\begin{pmatrix} y_0 \\ 0 \\ 0 \end{pmatrix}\right\| \\
&\quad\geq \frac{|\mu\alpha_1+\beta_1|}{|\Delta|}\left|\left(\int_0^1\left|e^{\mu p(1-x)}\right|dx\right)^{1/p}-\left(\int_0^1\left|e^{-\mu p(1-x)}\right|dx\right)^{1/p}\right|\|y_0\| \\
&\quad\geq \frac{|\mu\alpha_1+\beta_1|}{|\Delta|}\frac{1}{\mu^{1/p}p^{1/p}}\left[e^{\mu}\left(1-e^{-\mu p}\right)^{1/p}-1\right]\|y_0\|.
\end{aligned}
$$

But

$$\Delta=(-\mu\alpha_0+\beta_0)(\mu\alpha_1+\beta_1)e^{\mu}-(\mu\alpha_0+\beta_0)(-\mu\alpha_1+\beta_1)e^{-\mu},$$

so if $\alpha_0^2>0$, we have

$$\lim_{\lambda\to+\infty}\mu^{1+\frac{1}{p}}\left\|(\lambda I-A)^{-1}\begin{pmatrix} y_0 \\ 0 \\ 0 \end{pmatrix}\right\|=\frac{1}{|\alpha_0|\,p^{1/p}}\|y_0\|$$

and if $\alpha_0^2=0$, we have

$$\lim_{\lambda\to+\infty}\mu^{\frac{1}{p}}\left\|(\lambda I-A)^{-1}\begin{pmatrix} y_0 \\ 0 \\ 0 \end{pmatrix}\right\|=\frac{1}{|\beta_0|\,p^{1/p}}\|y_0\|,$$

and the result follows. □

Set

$$X_1=Y^2\times\{0_{L^p}\}.$$

Then we have

$$X=X_1\oplus X_0.$$

Define a bounded linear projector $\Pi: X\to X_0$ such that

$$\Pi X=X_0 \text{ and } (I-\Pi)X_0=X.$$

By using Theorem 9.1.12 we obtain the following result.

Theorem 9.1.26. *Let Assumption 9.1.22 be satisfied. Let $p^*\geq 1$ be defined by (9.1.36). Then for each $\hat{p}>p^*$, each $f\in L^{\hat{p}}((0,\tau),X_1)\oplus L^1((0,\tau),X_0)$ (with $\tau>0$), and each $x\in X_0$, there exists a unique integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + f(t),\ \forall t \in [0,\tau];\ \ u(0) = x \in X_0.$$

Remark 9.1.27. Theorem 9.1.26 shows that to obtain the existence and uniqueness of a solution for the parabolic equation (9.1.35), it is sufficient to assume that

$$h_0 \text{ and } h_1 \text{ belong to } L^{\hat{p}}\left((0,\tau),Y\right)$$

for some $\widehat{p} > p^*$, and

$$h_2 \in L^1\left((0,\tau), L^p\left((0,1),Y\right)\right).$$

Remark 9.1.28. To conclude this section we would like to mention that it is also possible to apply the linear perturbation Theorem 9.1.16 by using the Gagliardo-Nirenberg's Theorem [279].

(b) A Scalar Parabolic Equation on $(0,+\infty)$.

Consider the following parabolic problem on $\mathbb{R}_+$

$$\begin{cases} \dfrac{\partial v(t,a)}{\partial t} = \varepsilon^2 \dfrac{\partial^2 v(t,a)}{\partial x^2} + h_2(t)(x),\ \ t \geq 0,\ x \in (0,+\infty), \\ \alpha \dfrac{\partial v(t,0)}{\partial x} + \beta v(t,0) = h_0(t), \\ v(0,x) = v_0 \in L^p\left((0,+\infty),Y\right). \end{cases} \tag{9.1.37}$$

where $p \in [1,+\infty)$ and $\varepsilon > 0$.

Assumption 9.1.29. Assume that $\alpha^2 + \beta^2 > 0$.

Set

$$X = Y \times L^p\left((0,+\infty),Y\right) \text{ and } X_0 = \{0_Y\} \times L^p\left((0,+\infty),Y\right).$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\left[\alpha\varphi'(0) + \beta\varphi(0)\right] \\ \varepsilon^2 \varphi'' \end{pmatrix}$$

with

$$D(A) = \{0_Y\}^2 \times W^{2,p}\left((0,+\infty),Y\right)$$

and the linear operator $A_0 : D(A_0) \subset X \to X$ by

$$A_0\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi'' \end{pmatrix}$$

with

$$D(A_0) = \{0_Y\} \times \left\{\varphi \in W^{2,p}\left((0,+\infty),Y\right) : \left[\alpha\varphi'(x) + \beta\varphi(x)\right] = 0\right\}.$$

Let

$$f(t) = \begin{pmatrix} h_0(t) \\ h_2(t) \end{pmatrix}.$$

Then by identifying

$$u(t) = \begin{pmatrix} 0 \\ v(t) \end{pmatrix},$$

we can rewrite the diffusion problem on $\mathbb{R}_+$ as

$$\frac{d}{dt}u(t) = Au(t) + f(t), \ \ t > 0; \ \ u(0) = \begin{pmatrix} 0 \\ v_0 \end{pmatrix} \in \overline{D(A)}.$$

Theorem 9.1.30. *Let Assumption 9.1.29 be satisfied. The linear operator A_0 is the infinitesimal generator of an analytic semigroup on X_0, $(0,+\infty) \subset \rho(A)$, and*

$$\limsup_{\lambda(\in\mathbb{R})\to+\infty} \lambda^{\frac{1}{p^*}} \left\| (\lambda I - A)^{-1} \right\|_{\mathscr{L}(X)} < +\infty,$$

where p^* is given by

$$p^* := \begin{cases} \dfrac{2p}{1+p} & \text{if } \alpha^2 > 0, \\ 2p & \text{if } \alpha_0^2 = 0. \end{cases} \tag{9.1.38}$$

Moreover, the explicit formula of the resolvent of the linear operator A is given by

$$\begin{aligned} &\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = (\lambda I - A)^{-1} \begin{pmatrix} y \\ f \end{pmatrix} \\ &\Leftrightarrow \varphi(x) = \frac{e^{-\mu x}}{(\beta - \mu\alpha)} y - \frac{(\mu\alpha + \beta)}{2\mu(\beta - \mu\alpha)} \int_0^{+\infty} e^{-\mu s} f(s) ds e^{-\mu x} \\ &\qquad + \frac{1}{2\mu} \int_0^{+\infty} e^{-\mu|x-s|} f(s) ds. \end{aligned}$$

Proof. Let $\mu^2 = \lambda$. In order to compute the resolvent we set

$$u(x) = \frac{1}{2\mu} \int_0^{+\infty} e^{-\mu|x-s|} f(s) ds = \frac{1}{2\mu} \int_{-\infty}^{+\infty} e^{-\mu|x-s|} \bar{f}(s) ds,$$

where $\bar{f}$ extends f by 0 on $\mathbb{R}_-$. We have

$$u'(x) = -\frac{1}{2} \int_{-\infty}^{x} e^{\mu(s-x)} \bar{f}(s) ds + \frac{1}{2} \int_{x}^{+\infty} e^{\mu(x-s)} \bar{f}(s) ds,$$

so

$$u(0) = \gamma = \frac{1}{2\mu} \int_0^{+\infty} e^{-\mu s} f(s) ds.$$

Observe that

$$u'(0) = \mu\gamma$$

Set

$$u_1(x) = e^{-\mu x}$$

and solve

$$(\lambda I - A)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y \\ f \end{pmatrix},$$

then we have

$$\varphi(x) = u(x) + c_1 u_1(x).$$

Solving

$$\alpha\left[u'(0) + c_1 u_1'(0)\right] + \beta\left[u(0) + c_1 u_1'(0)\right] = y,$$

we obtain

$$c_1(-\mu\alpha + \beta) = y - \mu\alpha\gamma - \beta\gamma = y - \gamma(\mu\alpha + \beta),$$
$$c_1 = \frac{y - \gamma(\mu\alpha + \beta)}{(\beta - \mu\alpha)}.$$

Finally, we have the explicit formula

$$(\lambda I - A)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} y_0 \\ f \end{pmatrix}$$
$$\Leftrightarrow \varphi(x) = \frac{y - \gamma(\mu\alpha + \beta)}{(\beta - \mu\alpha)} e^{-\mu x} + \frac{1}{2\mu}\int_0^{+\infty} e^{-\mu|x-s|} f(s)ds.$$

The fact that A_0 is the infinitesimal generator of an analytic semigroup on X_0, $(0,+\infty) \subset \rho(A)$, is a consequence of the above formula. By using a similar argument of Lemma 9.1.24, estimation on the resolvent follows. $\square$

Now set

$$X_1 = Y \times \{0_{L^p}\}.$$

Then we have

$$X = X_1 \oplus X_0.$$

Theorem 9.1.31. *Let Assumption 9.1.29 be satisfied. Let $p^* \geq 1$ be defined by (9.1.38). Then for each $\hat{p} > p^*$, each $f \in L^{\hat{p}}((0,\tau), X_1) \oplus L^1((0,\tau), X_0)$ (with $\tau > 0$), and each $x \in X_0$, there exists a unique integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + f(t),\ \forall t \in [0,\tau];\ \ u(0) = x \in X_0.$$

9.2 A Size-structured Model

Consider a size-structured model with Ricker type birth function and random fluctuation in the growth process described by a reaction-diffusion equation with a nonlinear and nonlocal boundary condition:

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} + \underbrace{\dfrac{\partial \left(gu(t,x)\right)}{\partial x}}_{\text{growth in size}} = \underbrace{\varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2}}_{\text{random noise}} \underbrace{-\mu u(t,x)}_{\text{death}}, \\
-\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + gu(t,0) = \alpha h(\int_0^{+\infty} \gamma(x)\, u(t,x) dx), \\
u(0,.) = u_0 \in L^1_+ (0,+\infty),
\end{cases}
\tag{9.2.1}
$$

where $u(t,x)$ represents the population density of certain species at time t with size x, $g > 0$, $\varepsilon \geq 0$, $\mu > 0$, and $\alpha > 0$ are constants, $\gamma \in L^\infty_+ (0,+\infty) \setminus \{0\}$, and the map $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$
h(x) = xe^{-\xi x}, \ \forall x \geq 0.
$$

Equation (9.2.1) is viewed as a size structured model, for example for the growth of trees or fish population, where $x = 0$ is the minimal size. The growth of individuals is described by two terms. First, the term $\dfrac{\partial \left(gu(t,x)\right)}{\partial x}$ represents the average growth rate of individuals, and the diffusion term $\varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2}$ describes the stochastic fluctuations around the tendency to growth. So $\varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} - \dfrac{\partial \left(gu(t,x)\right)}{\partial x}$ describes the fact that given a group of individuals located in some small neighborhood of a given size $x_0 \in (0,+\infty)$, after a period of time this group of individuals will disperse due to the diffusion, and the mean value of the distribution increases due to the convection term. The terms $-\mu u(t,x)$ is classical and describes the mortality process of individuals following an exponential law with mean $1/\mu$. The birth function given by $\alpha h(\int_0^{+\infty} \gamma(x)\, u(t,x) dx)$ is a Ricker [297, 298] type birth function. This type of birth function has been commonly used in the literature, to take into account some limitation of births when the population increases. In particular, the birth rate function is $\alpha\gamma(x)$ when the total population is close to zero and it is very natural to introduce a stochastic random noise to describe the growth of individuals with respect to their size.

In the special case when $\varepsilon = 0$, by making a simple change of variable we can assume that $g = 1$. Then system (9.2.1) becomes a model which is very similar to the age structured models studied in Chapter 7, which exhibits Hopf bifurcation when

$$
\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x).
$$

In this section, we investigate the bifurcation by regarding α and ε as parameters of the semiflow.

Without loss of generality, we assume that $g = 1$ and consider the system

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial u(t,x)}{\partial x} = \varepsilon^2 \dfrac{\partial^2 u(t,x)}{\partial x^2} - \mu u(t,x), \ t \geq 0, x \geq 0, \\
-\varepsilon^2 \dfrac{\partial u(t,0)}{\partial x} + u(t,0) = \alpha h(\int_0^{+\infty} \gamma(x)\, u(t,x) dx), \\
u(0,.) = u_0 \in L^1_+ (0,+\infty).
\end{cases}
\tag{9.2.2}
$$

Assumption 9.2.1. Assume that $\varepsilon > 0$, $\mu > 0$, $\alpha > 0$, $\gamma \in L^\infty_+(0,+\infty) \setminus \{0\}$, and $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = x\exp(-\xi x),\ \forall x \in \mathbb{R},$$

where $\xi > 0$.

Consider the space

$$X := \mathbb{R} \times L^1(0,+\infty)$$

endowed with the usual product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0,+\infty)}.$$

Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varepsilon^2\varphi'(0) - \varphi(0) \\ \varepsilon^2\varphi'' - \varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) = \{0\} \times W^{2,1}(0,+\infty).$$

Then

$$X_0 := \overline{D(A)} = \{0\} \times L^1(0,+\infty).$$

Define the map $H : X_0 \to X$ by

$$H \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \alpha h(\int_0^{+\infty} \gamma(x)\,\varphi(x)dx) \\ 0 \end{pmatrix}.$$

By identifying $v(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$, the partial differential equation (9.2.2) can be rewritten as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H(v(t))\ t \geq 0;\ \ v(0) = \begin{pmatrix} 0 \\ u_0 \end{pmatrix} \in \overline{D(A)}. \tag{9.2.3}$$

In the following, for $z \in \mathbb{C}$, $\sqrt{z}$ denotes the principal branch of the general multi-valued function $z^{\frac{1}{2}}$. The branch cut is the negative real axis and the argument of z, denoted by $\arg z$, is π on the upper margin of the branch cut. Then $z = \rho e^{i\theta}, \theta \in (-\pi,\pi), \rho > 0$, and $\sqrt{z} = \sqrt{\rho}e^{i\frac{\theta}{2}}$. In the following, we will use the following notation:

$$\Omega := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\mu\},$$

and for $\lambda \in \Omega$,

$$\Lambda := 1 + 4\varepsilon^2(\lambda + \mu). \tag{9.2.4}$$

Since $\lambda \in \Omega$, $\mathrm{Re}(\Lambda) > 0$, so we can use the above definition to define $\sqrt{\Lambda}$. Set

$$\sigma^{\pm} := \frac{1 \pm \sqrt{\Lambda}}{2\varepsilon^2}, \tag{9.2.5}$$

$$\Lambda_0 = 1 + 4\varepsilon^2\mu := \Lambda \text{ for } \lambda = 0, \tag{9.2.6}$$

and

$$\sigma_0^- = \frac{1 - \sqrt{\Lambda_0}}{2\varepsilon^2} := \sigma^- \text{ for } \lambda = 0. \tag{9.2.7}$$

So $\sigma^{\pm}$ are solutions of the equation

$$\varepsilon^2\sigma^2 - \sigma - (\lambda + \mu) = 0.$$

Observe that

$$\mathrm{Re}(\sigma^+) > 0 \text{ and } \mathrm{Re}(\sigma^-) < 0.$$

Besides these, later on we will also use the following notation:

$$R_0 := \frac{2\alpha\chi}{1 + \sqrt{\Lambda_0}}, \tag{9.2.8}$$

$$\chi := \int_0^{+\infty} \gamma(x) \exp(\sigma_0^- x) dx, \tag{9.2.9}$$

$$\chi_0 := \lim_{\varepsilon \to 0} \chi = \int_0^{+\infty} \gamma(x) \exp(-\mu x) dx, \tag{9.2.10}$$

and

$$\eta(\varepsilon, \alpha) := \frac{1 + \sqrt{\Lambda_0}}{2\chi} \left(1 - \ln \frac{2\alpha\chi}{1 + \sqrt{\Lambda_0}}\right) = \frac{\alpha}{R_0} \left(1 - \ln R_0\right). \tag{9.2.11}$$

If $\gamma(x) \in L^1_+(0, +\infty)$ and $\alpha = c\varepsilon$ with $c > 0$, set

$$\lim_{\varepsilon \to +\infty} R_0 = \frac{c}{\sqrt{\mu}} \int_0^{+\infty} \gamma(x) dx := R_0^{\infty},$$

$$\lim_{\varepsilon \to +\infty} \frac{\eta(\varepsilon, \alpha)}{\varepsilon} = \frac{\sqrt{\mu}}{\int_0^{+\infty} \gamma(x)\, dx} \left(1 - \ln R_0^{\infty}\right) := \eta^{\infty}.$$

To study the characteristic equation, for $\lambda \in \Omega$, define

$$\Delta(\varepsilon, \alpha, \lambda) := 1 - \frac{2\eta(\varepsilon, \alpha)}{1 + \sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx.$$

Moreover, if we consider

$$\widetilde{\Delta}(\varepsilon, \alpha, \lambda) := \frac{1 + \sqrt{\Lambda}}{2} \Delta(\varepsilon, \alpha, \lambda) = -\varepsilon^2\sigma^- + 1 - \eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx,$$

when ε tends to infinity, and take $\alpha = c\varepsilon$, then $\frac{\widetilde{\Delta}(\varepsilon,\alpha,\lambda)}{\varepsilon}$ goes to

$$\widehat{\Delta}(+\infty, c, \lambda) := \sqrt{\lambda+\mu} - \sqrt{\mu}\left(1 - \ln \frac{c\int_0^{+\infty} \gamma(x)\,dx}{\sqrt{\mu}}\right).$$

Let $L : D(L) \subset X \to X$ be a linear operator on a Banach space X. Denote by $\rho(L)$ the resolvent set of L. The spectrum of L is $\sigma(L) = \mathbb{C}\backslash\rho(L)$. The point spectrum of L is the set

$$\sigma_P(L) := \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda I - L) \neq \{0\}\}.$$

Let Y be a subspace of X. Then we denote by $L_Y : D(L_Y) \subset Y \to Y$ the part of L on Y, which is defined by

$$L_Y x = Lx, \forall x \in D(L_Y) := \{x \in D(L) \cap Y : Lx \in Y\}.$$

In particular, we denote A_0 the part of A in $\overline{D(A)}$. So

$$A_0 x = Ax \text{ for } x \in D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\}.$$

Consider the linear operator $\widehat{A_0} : D(\widehat{A_0}) \subset L^1(0,+\infty) \to L^1(0,+\infty)$ defined by

$$\widehat{A_0}(\varphi) = \varepsilon^2\varphi'' - \varphi' - \mu\varphi$$

with

$$D(\widehat{A_0}) = \{\varphi \in W^{2,1}((0,+\infty),\mathbb{R}) : \varepsilon^2\varphi'(0) - \varphi(0) = 0\}.$$

We have the following relationship between A_0 and $\widehat{A_0}$:

$$D(A_0) = \{0\} \times D(\widehat{A_0})$$

and

$$A_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{A_0}\varphi \end{pmatrix}.$$

First we have the following lemma about the representation of the resolvent of A.

Lemma 9.2.2. *We have*

$$\Omega \subset \rho(\widehat{A_0}) = \rho(A_0) = \rho(A),$$

and for each $\lambda \in \Omega$ we obtain the following explicit formula for the resolvent of A:

$$(\lambda I - A)^{-1}\begin{pmatrix} \alpha \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$\Leftrightarrow$

$$\varphi(x) = (\lambda I - \widehat{A_0})^{-1}(\psi)(x) + \alpha\frac{2\exp(\sigma^- x)}{1+\sqrt{\Lambda}}, \tag{9.2.12}$$

where $(\lambda I - \widehat{A_0})^{-1}$ is defined by

$$(\lambda I-\widehat{A_0})^{-1}(\psi)(x)=\frac{1}{\sqrt{\Lambda}}\left[\int_0^x \exp(\sigma^-(x-t))\psi(t)dt+\int_x^{+\infty}\exp(\sigma^+(x-t))\psi(t)dt\right]$$
$$+\frac{\sqrt{\Lambda}-1}{(\sqrt{\Lambda}+1)\sqrt{\Lambda}}\left[\int_0^{+\infty}\exp(-\sigma^+t)\psi(t)dt\right]\exp(\sigma^-x).$$

Next we prove the following proposition.

Proposition 9.2.3. *The following two assertions are satisfied:*

(a) A_0 the part of A in $\overline{D(A)}$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ on $\overline{D(A)}$;
(b) A is a Hille-Yosida operator on X.

Proof. It is well known that $\widehat{A_0}$ is the infinitesimal generator of an analytic semigroup. In fact, we first consider the linear operator $A_1 : D(A_1)\subset L^1(0,+\infty)\to L^1(0,+\infty)$ defined by $A_1(\varphi)=\varepsilon^2\varphi''$ and $D(A_1)=D(\widehat{A_0})$. It is well known that A_1 is the infinitesimal generator of an analytic semigroup (Engel and nagel [126], Lunardi [240]). Consider the linear operator $A_2 : D(A_2)\subset L^1(0,+\infty)\to L^1(0,+\infty)$, $A_2(\varphi)=-\varphi'-\mu\varphi$ with $D(A_2)=W^{2,1}((0,+\infty),\mathbb{R})$. Define $\widehat{A_0}=A_1+A_2$. From Pazy [281, Theorem 7.3.10], we deduce that $\widehat{A_0}$ is sectorial. Furthermore, for $\lambda\in\mathbb{R}$ we have

$$\begin{aligned}\left\|(\lambda I-A)^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix}\right\| &= |\alpha|\frac{2\int_0^{+\infty}\exp(\sigma^-x)dx}{1+\sqrt{\Lambda}}\\
&=|\alpha|\frac{2}{1+\sqrt{\Lambda}}\times\frac{1}{-\sigma^-}\\
&=|\alpha|\frac{2}{1+\sqrt{\Lambda}}\times\frac{2\varepsilon^2}{-1+\sqrt{\Lambda}}\\
&=|\alpha|\frac{4\varepsilon^2}{-1+\Lambda}=|\alpha|\frac{4\varepsilon^2}{4\varepsilon^2(\lambda+\mu)},\end{aligned}$$

so we obtain for $\lambda\in\mathbb{R}$ that

$$\left\|(\lambda I-A)^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix}\right\|\leq\frac{|\alpha|}{\lambda+\mu},\forall\lambda>-\mu.$$

Finally, it is readily checked that (see the proof of Lemma 9.2.11)

$$\left\|T_{\widehat{A_0}}(t)\right\|\leq e^{-\mu t},$$

which implies that

$$\left\|(\lambda I-\widehat{A_0})^{-1}\right\|\leq\frac{1}{\lambda+\mu},\forall\lambda>-\mu.$$

So A is a Hille-Yosida operator. □

Set

$$X_+ := \mathbb{R}_+ \times L^1_+(0,+\infty), \;\; X_{0+} := X_0 \cap X_+.$$

Lemma 9.2.4. *For $\lambda > 0$ large enough, we have $(\lambda I - A)^{-1} X_+ \subset X_+$.*

Proof. This lemma follows directly from the explicit formula (9.2.12) of the resolvent of A. $\square$

By using the results in Section 5.2, we have the following theorem.

Theorem 9.2.5 (Existence). *There exists a unique continuous semiflow $\{U(t)\}_{t\geq 0}$ on X_{0+} such that $\forall x \in X_{0+}$, $t \to U(t)x$ is the unique integrated solution of*

$$\frac{dU(t)x}{dt} = AU(t)x + H(U(t)x), \;\; U(0)x = x,$$

or equivalently,

$$U(t)x = x + A\int_0^t U(l)xdl + \int_0^t H(U(l)x)\,dl, \; \forall t \geq 0.$$

9.2.1 The Semiflow and its Equilibria

Now we consider the positive equilibrium solutions of equation (9.2.3).

Lemma 9.2.6. (Equilibrium) *There exists a unique positive equilibrium of system (9.2.2) (or equation (9.2.3)) if and only if*

$$R_0 := \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} > 1, \tag{9.2.13}$$

where χ and Λ_0 are defined in (9.2.9) and (9.2.6), respectively. Moreover, when it exists, the positive equilibrium $\bar{v} = \begin{pmatrix} 0 \\ \overline{u} \end{pmatrix}$ is given by the following formula

$$\overline{u}(x) = \bar{C}\exp(\sigma_0^- x), \tag{9.2.14}$$

where

$$\bar{C} := \frac{1}{\xi\chi}\ln\frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} = \frac{1}{\xi\chi}\ln R_0.$$

Proof. We have

$$A\begin{pmatrix} 0 \\ \overline{u} \end{pmatrix} + H\begin{pmatrix} 0 \\ \overline{u} \end{pmatrix} = 0$$

$\Leftrightarrow$

$$\begin{pmatrix} 0 \\ \overline{u} \end{pmatrix} = (-A)^{-1}\begin{pmatrix} \alpha h(\int_0^{+\infty}\gamma(x)\,\overline{u}(x)dx) \\ 0 \end{pmatrix}.$$

According to the explicit formula of the resolvent of A, taking $\lambda = 0$, we have

$$\begin{pmatrix} 0 \\ \overline{u} \end{pmatrix} = (-A)^{-1} \begin{pmatrix} \alpha h(\int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx) \\ 0 \end{pmatrix}$$

$\Leftrightarrow$

$$\overline{u}(x) = \alpha h(\int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx) \frac{2\exp(\sigma_0^- x)}{1+\sqrt{\Lambda_0}}. \tag{9.2.15}$$

So

$$\overline{u} \neq 0 \text{ iff } \int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx \neq 0.$$

Integrating both sides of equation (9.2.15) after multiplying $\gamma(x)$, we have

$$\int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx = \alpha h(\int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx) \int_0^{+\infty} \gamma(x)\, \frac{2\exp(\sigma_0^- x)}{1+\sqrt{\Lambda_0}} dx.$$

In order to have $\overline{u}(x) > 0$, we have

$$\begin{aligned} 1 &= \alpha \exp(-\xi \int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx) \frac{2\int_0^{+\infty} \gamma(x)\exp(\sigma_0^- x)dx}{1+\sqrt{\Lambda_0}} \\ &= \exp(-\xi \int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx) R_0 \end{aligned}$$

$\Leftrightarrow$

$$\int_0^{+\infty} \gamma(x)\,\overline{u}(x)dx = \frac{1}{\xi} \ln R_0, \tag{9.2.16}$$

and the result follows. $\square$

9.2.2 Linearized Equation and Spectral Analysis

From now on, we set

$$\bar{v} = \begin{pmatrix} 0 \\ \bar{u} \end{pmatrix} \text{ with } \bar{u}(x) = \bar{C}\exp(\sigma_0^- x),\ \forall R_0 > 1,$$

where $\bar{C} = \frac{1}{\xi\chi} \ln R_0$. The linearized system of (9.2.3) around $\bar{v}$ is

$$\frac{dv(t)}{dt} = Av(t) + DH(\bar{v})v(t) \text{ for } t \geq 0,\ v(t) \in X_0, \tag{9.2.17}$$

where

$$\begin{aligned} DH(\bar{v}) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} &= \begin{pmatrix} \alpha h'(\int_0^{+\infty} \gamma(x)\bar{u}(x)dx) \int_0^{+\infty} \gamma(x)\varphi(x)dx \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \eta(\varepsilon, \alpha) \int_0^{+\infty} \gamma(x)\varphi(x)dx \\ 0 \end{pmatrix} \end{aligned}$$

with

$$\eta(\varepsilon,\alpha) = \alpha h'(\int_0^{+\infty} \gamma(x)\bar{u}(x)dx)$$

and

$$h'(x) = e^{-\xi x}(1-\xi x).$$

Using (9.2.16) we obtain

$$\begin{aligned}\eta(\varepsilon,\alpha) &= \frac{\alpha}{R_0}(1-\ln R_0)\\ &= \frac{1+\sqrt{\Lambda_0}}{2\chi}\left(1-\ln\frac{2\alpha\chi}{1+\sqrt{\Lambda_0}}\right).\end{aligned}$$

The Cauchy problem (9.2.17) corresponds to the following linear parabolic differential equation

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}+\dfrac{\partial u(t,x)}{\partial x} = \varepsilon^2\dfrac{\partial^2 u(t,x)}{\partial x^2} - \mu u(t,x),\ t\geq 0, x\geq 0,\\ -\varepsilon^2\dfrac{\partial u(t,0)}{\partial x} + u(t,0) = \eta\,(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\,u(t,x)dx,\\ u(0,.) = u_0 \in L^1\,(0,+\infty)\,.\end{cases} \tag{9.2.18}$$

Next we study the spectral properties of the linearized equation (9.2.17). To simplify the notation, we define $B_\alpha : D(B_\alpha)\subset X \to X$ as

$$B_\alpha x = Ax + DH(\bar{v})x \text{ with } D(B_\alpha) = D(A), \tag{9.2.19}$$

and denote by $(B_\alpha)_0$ the part of B_α on $\overline{D(A)}$.

Lemma 9.2.7. *For each $\lambda\in\Omega = \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda) > -\mu\}$, we have*

$$\mathrm{Re}(1+\sqrt{\Lambda}) > 1,$$

$$\lambda\in\rho\,(B_\alpha) \Leftrightarrow \Delta(\varepsilon,\alpha,\lambda)\neq 0,$$

and the following explicit formula:

$$(\lambda I - B_\alpha)^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix}$$

$\Leftrightarrow$

$$\begin{aligned}\varphi(x) &= (\lambda I-\widehat{A_0})^{-1}(\psi)(x)\\ &\quad +\Delta(\varepsilon,\alpha,\lambda)^{-1}\left[\beta+\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda-\widehat{A_0})^{-1}(\psi)(x)dx\right]\frac{2\exp(\sigma^- x)}{1+\sqrt{\Lambda}},\end{aligned}$$

where

$$\Delta(\varepsilon,\alpha,\lambda) := 1-\frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx, \tag{9.2.20}$$

in which $\eta(\varepsilon,\alpha)$, Λ, and σ^- are defined in equations (9.2.11), (9.2.4) and (9.2.5), respectively.

Proof. Since $\lambda \in \Omega$, from Lemma 9.2.2, we know that $\lambda I - A$ is invertible. Then

$$\lambda I - B_\alpha \text{ is invertible} \Leftrightarrow I - DH(\bar{v})(\lambda I - A)^{-1} \text{ is invertible,}$$

and

$$(\lambda I - B_\alpha)^{-1} = (\lambda I - A)^{-1}\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}.$$

We also know that $\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]\begin{pmatrix}\widehat{\beta}\\ \varphi\end{pmatrix} = \begin{pmatrix}\beta\\ \psi\end{pmatrix}$ is equivalent to $\varphi = \psi$ and

$$\begin{aligned}\widehat{\beta} - \widehat{\beta}\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\frac{2\exp(\sigma^- x)}{1+\sqrt{\Lambda}}dx\\ = \beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A_0})^{-1}(\varphi)(x)dx.\end{aligned}$$

We deduce that $I - DH(\bar{v})(\lambda I - A)^{-1}$ is invertible if and only if $\Delta(\varepsilon,\alpha,\lambda) \neq 0$. Moreover,

$$\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix} = \begin{pmatrix}\widehat{\beta}\\ \varphi\end{pmatrix}$$

is equivalent to $\varphi = \psi$ and

$$\widehat{\beta} = \Delta(\varepsilon,\alpha,\lambda)^{-1}\left[\beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A_0})^{-1}(\psi)(x)dx\right].$$

Therefore,

$$\begin{aligned}&(\lambda I - B_\alpha)^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix}\\ &\quad= (\lambda I - A)^{-1}\left[I - DH(\bar{v})(\lambda I - A)^{-1}\right]^{-1}\begin{pmatrix}\beta\\ \psi\end{pmatrix}\\ &\quad= (\lambda I - A)^{-1}\begin{pmatrix}\widehat{\beta}\\ \varphi\end{pmatrix}\\ &\quad= (\lambda I - A)^{-1}\begin{pmatrix}\Delta(\varepsilon,\alpha,\lambda)^{-1}\left[\beta + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)(\lambda - \widehat{A_0})^{-1}(\psi)(x)dx\right]\\ \psi\end{pmatrix}.\end{aligned}$$

Then by Lemma 9.2.2, the result follows. $\square$

Remark 9.2.8. Since by definition of $\sqrt{\cdot}$, $\mathrm{Re}(\sqrt{\Lambda}) > 0$, $\forall\lambda \in \Omega$, we deduce that $\mathrm{Re}(1+\sqrt{\Lambda}) > 1$.

By using the above explicit formula (9.2.19) for the resolvent of B_α we obtain the following lemma.

Lemma 9.2.9. *If $\lambda_0 \in \sigma(B_\alpha) \cap \Omega$, then λ_0 is a simple eigenvalue of B_α if and only if*

$$\frac{d\Delta(\varepsilon,\alpha,\lambda_0)}{d\lambda} \neq 0.$$

Consider a linear operator $(\widehat{B_\alpha})_0 : D((\widehat{B_\alpha})_0) \subset L^1(0,+\infty) \to L^1(0,+\infty)$ defined by

$$\left(\widehat{B_\alpha}\right)_0(\varphi) = \varepsilon^2\varphi'' - \varphi' - \mu\varphi$$

with

$$D\left(\left(\widehat{B_\alpha}\right)_0\right) = \{\varphi \in W^{2,1}((0,+\infty),\mathbb{R}) : \varepsilon^2\varphi'(0) - \varphi(0) + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\varphi(x)dx = 0\}.$$

Then we have the following lemma.

Lemma 9.2.10. *For each $\varphi \in L^1(0,+\infty)$ and each $t \geq 0$, we have*

$$\begin{aligned}&\frac{d}{dt}\int_0^{+\infty} T_{\left(\widehat{B_\alpha}\right)_0}(t)\varphi(x)dx\\&\quad = -\mu\int_0^{+\infty} T_{\left(\widehat{B_\alpha}\right)_0}(t)(\varphi)(x)dx + \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)T_{\left(\widehat{B_\alpha}\right)_0}(t)(\varphi)(x)dx.\end{aligned}$$

Lemma 9.2.11. *Let Assumption 9.2.1 be satisfied. Then the linear operator B_α is a Hille-Yosida operator and its part $(B_\alpha)_0$ in X_0 satisfies*

$$\omega_{0,\mathrm{ess}}((B_\alpha)_0) \leq -\mu. \tag{9.2.21}$$

Proof. Since $DH(\bar{v})$ is a bounded linear operator and A is a Hille-Yosida operator, it follows that B_α is also a Hille-Yosida operator. By Lemma 9.2.10 with $\eta(\varepsilon,\alpha) = 0$, we have

$$\begin{aligned}\left\|T_{\widehat{A_0}}(t)\varphi\right\|_{L^1(0,+\infty)} &= \left\|\left|T_{\widehat{A_0}}(t)\varphi\right|\right\|_{L^1(0,+\infty)} \leq \left\|T_{\widehat{A_0}}(t)|\varphi|\right\|_{L^1(0,+\infty)}\\&= \int_0^\infty T_{\widehat{A_0}}(t)|\varphi|(x)dx = \int_0^\infty e^{-\mu t}|\varphi|(x)dx = e^{-\mu t}\|\varphi\|_{L^1(0,+\infty)},\end{aligned}$$

then

$$\omega_{0,\mathrm{ess}}\left(\widehat{A_0}\right) \leq -\mu.$$

By using the result in Theorem 4.7.3, we deduce that

$$\omega_{0,\mathrm{ess}}\left(\left(\widehat{B_\alpha}\right)_0\right) \leq \omega_{0,\mathrm{ess}}\left(\widehat{A_0}\right) \leq -\mu,$$

and the result follows. $\square$

Lemma 9.2.12. *We have*

$$\sigma((B_\alpha)_0) \cap \Omega = \sigma_p((B_\alpha)_0) \cap \Omega = \{\lambda \in \Omega : \Delta(\varepsilon,\alpha,\lambda) = 0\},$$

where

$$\Delta(\varepsilon,\alpha,\lambda) = 1 - \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx.$$

Proof. By Lemma 9.2.11, we have

$$\sigma((B_\alpha)_0) \cap \Omega = \sigma_p((B_\alpha)_0) \cap \Omega,$$

and by Lemma 9.2.7, the result follows. □

Later on, we will study the eigenvalues of the *characteristic equation*

$$1 = \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx \text{ with } \lambda \in \Omega, \tag{9.2.22}$$

or equivalently, the following equation

$$-\varepsilon^2 \sigma^- + 1 = \eta(\varepsilon,\alpha) \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx, \tag{9.2.23}$$

where σ^- is the solution of

$$\varepsilon^2 \sigma^2 - \sigma = \lambda + \mu,\ \lambda \in \Omega,$$

with $\mathrm{Re}(\sigma^-) < 0$.

9.2.3 Local Stability

This subsection is devoted to the local stability of the positive steady state $\bar{v}$. Recall that this positive equilibrium exists and is unique if and only if $R_0 > 1$.

Lemma 9.2.13. *If* $R_0 = \dfrac{2\alpha\chi}{1+\sqrt{\Lambda_0}} > 1$, *then* $\lambda = 0$ *is not a root of the characteristic equation* $\Delta(\varepsilon,\alpha,\lambda) = 0$, *where* $\Delta(\varepsilon,\alpha,\lambda)$ *is explicitly defined in (9.2.20).*

Proof. We have

$$\begin{aligned}
\Delta(\varepsilon,\alpha,0) &= 1 - \frac{2\eta(\varepsilon,\alpha)}{1+\sqrt{\Lambda_0}} \int_0^{\infty} \gamma(x) e^{\sigma_0^- x} dx \\
&= 1 - \frac{2\chi}{1+\sqrt{\Lambda_0}} \eta(\varepsilon,\alpha) \\
&= 1 - \frac{R_0}{\alpha} \left(\frac{\alpha}{R_0}(1 - \ln R_0)\right) \\
&= \ln R_0.
\end{aligned}$$

Since $R_0 > 1$, we obtain that

$$\Delta(\varepsilon,\alpha,0) > 0$$

and the result follows. $\square$

Lemma 9.2.14. *If λ is a root of the characteristic equation and* $\mathrm{Re}(\lambda) \geq 0$. *Then we have*

$$\mathrm{Re}(\sigma^-) < \sigma_0^-$$

and

$$\mathrm{Re}(\sqrt{\Lambda}) > \sqrt{\Lambda_0} > 1.$$

Proof. Since σ^- is the root of

$$\varepsilon^2\sigma^2 - \sigma - (\mu + \lambda) = 0$$

with

$$\mathrm{Re}(\sigma^-) < 0,$$

we have the following relationship between $\mathrm{Re}(\sigma^-)$, $\mathrm{Im}(\sigma^-)$, $\mathrm{Re}(\lambda)$, and $\mathrm{Im}(\lambda)$:

$$\varepsilon^2\mathrm{Re}(\sigma^-)^2 - \mathrm{Re}(\sigma^-) - \mu = \mathrm{Re}(\lambda) + \varepsilon^2\mathrm{Im}(\sigma^-)^2, \tag{9.2.24}$$

$$2\varepsilon^2\mathrm{Re}(\sigma^-)\mathrm{Im}(\sigma^-) - \mathrm{Im}(\sigma^-) = \mathrm{Im}(\lambda). \tag{9.2.25}$$

If $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\sigma^-) = 0$, then by using (9.2.25) we have $\mathrm{Im}(\lambda) = 0$. So $\lambda = 0$, which is impossible by Lemma 9.2.13. Thus if $\mathrm{Re}(\lambda) \geq 0$, we have

$$\mathrm{Re}(\lambda) + \varepsilon^2\mathrm{Im}(\sigma^-)^2 > 0.$$

By using (9.2.24) we deduce that

$$\varepsilon^2\mathrm{Re}(\sigma^-)^2 - \mathrm{Re}(\sigma^-) - \mu > 0.$$

Since $\mathrm{Re}(\sigma^-) < 0$, if follows that

$$\mathrm{Re}(\sigma^-) < \frac{1 - \sqrt{1 + 4\varepsilon^2\mu}}{2\varepsilon^2} = \sigma_0^-.$$

Now since $\sigma^- = \frac{1-\sqrt{\Lambda}}{2\varepsilon^2}$ and $\sigma_0^- = \frac{1-\sqrt{\Lambda_0}}{2\varepsilon^2}$, we deduce that $\mathrm{Re}(\sqrt{\Lambda}) > \sqrt{\Lambda_0}$. $\square$

Theorem 9.2.15. *Let Assumption 9.2.1 be satisfied. If*

$$1 < R_0 \leq e^2,$$

then the positive equilibrium $\bar{v}$ of system (9.2.2) is locally asymptotically stable.

Proof. Consider the characteristic equation

$$1 - \varepsilon^2\sigma^- = \eta(\varepsilon,\alpha)\int_0^{+\infty} \gamma(x)e^{\sigma^- x}dx. \tag{9.2.26}$$

By Lemma 9.2.14, if $\mathrm{Re}(\lambda) \geq 0$, we must have

$$\mathrm{Re}(\sigma^-) < \sigma_0^-.$$

Then we derive from equation (9.2.26) that

$$\begin{aligned} \left|\varepsilon^2\sigma^- - 1\right| &= \left|\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx\right| \\ &\leq |\eta(\varepsilon,\alpha)|\int_0^{+\infty}\gamma(x)e^{\mathrm{Re}(\sigma^-)x}dx \\ &< |\eta(\varepsilon,\alpha)|\int_0^{+\infty}\gamma(x)e^{\sigma_0^- x}dx \\ &= |\eta(\varepsilon,\alpha)|\chi. \end{aligned}$$

On the other hand, if $\mathrm{Re}(\lambda) \geq 0$, then by Lemma 9.2.14, we have

$$\left|\varepsilon^2\sigma^- - 1\right| = \left|\frac{1+\sqrt{\Lambda}}{2}\right| > \mathrm{Re}(\frac{1+\sqrt{\Lambda}}{2}) > \frac{1+\sqrt{\Lambda_0}}{2}.$$

So if

$$|\eta(\varepsilon,\alpha)|\chi \leq \frac{1+\sqrt{\Lambda_0}}{2},$$

then there will be no roots of the characteristic equation with non-negative real part.

By (9.2.8) and (9.2.11), the above inequality is equivalent to

$$\frac{\alpha}{R_0}|\ln R_0 - 1| \leq \frac{\alpha}{R_0},$$

and the result follows. □

Next let ε go to infinity, we study the characteristic equation

$$-\varepsilon^2\sigma^- + 1 = \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx,$$

where

$$\sigma^- = \frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2} \sim O(\frac{1}{\varepsilon}) \text{ as } \varepsilon \to +\infty.$$

In order to obtain the limit characteristic equation when ε tends to infinity, we rewrite the characteristic equation as

$$-\varepsilon\sigma^- + \frac{1}{\varepsilon} - \frac{\eta(\varepsilon,\alpha)}{\varepsilon}\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx = 0. \tag{9.2.27}$$

To simplify the notation, we set

$$\widetilde{\Delta}(\varepsilon,\alpha,\lambda) = -\varepsilon^2\sigma^- + 1 - \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx.$$

Then the rewritten equation (9.2.27) becomes

$$\frac{\widetilde{\Delta}(\varepsilon,\alpha,\lambda)}{\varepsilon}=0.$$

Note that

$$\chi=\int_0^{+\infty}\gamma(x)\exp(\sigma_0^- x)dx$$

and

$$\sigma_0^-=\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\to 0 \text{ as } \varepsilon\to+\infty.$$

It is important to observe that to obtain a positive equilibrium, by Lemma 9.2.6 we must have

$$R_0>1,$$

or equivalently,

$$\alpha>\frac{1+\sqrt{\Lambda_0}}{2\chi},\ \ \Lambda_0=1+4\varepsilon^2\mu.$$

We make the following assumption.

Assumption 9.2.16. Assume that

$$\gamma(x)\in L^1_+(0,+\infty),\ \ \alpha=c\varepsilon,$$

for some $c>0$.

Under Assumption 9.2.16, we have

$$\chi\to\int_0^{+\infty}\gamma(x)\,dx,$$

$$\frac{\sqrt{\Lambda_0}}{\alpha}=\frac{\sqrt{1+4\varepsilon^2\mu}}{c\varepsilon}\to\frac{2\sqrt{\mu}}{c} \text{ as } \varepsilon\to+\infty,$$

so we obtain

$$R_0\to\frac{c}{\sqrt{\mu}}\int_0^{+\infty}\gamma(x)dx:=R_0^\infty \text{ as } \varepsilon\to+\infty,$$

$$\frac{\eta\,(\varepsilon,c\varepsilon)}{\varepsilon}\to\frac{\sqrt{\mu}}{\int_0^{+\infty}\gamma(x)\,dx}\,(1-\ln R_0^\infty):=\eta^\infty \text{ as } \varepsilon\to+\infty.$$

Lemma 9.2.17. *Let Assumptions 9.2.1 and 9.2.16 be satisfied. Then there exists* $\widehat{\varepsilon}>0$ *such that* $\forall\varepsilon>\widehat{\varepsilon}$, *if*

$$\lambda\in\Omega \text{ and } \widetilde{\Delta}(\varepsilon,\alpha,\lambda)=0,$$

then

$$|\lambda|<\mu+\mu\,(1-\ln R_0^\infty)^2+1.$$

Proof. If $\lambda\in\Omega$, $\widetilde{\Delta}(\varepsilon,\alpha,\lambda)=0$, then

$$-\varepsilon^2\sigma^- + 1 = \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)e^{\sigma^- x}dx$$

with

$$\mathrm{Re}(\sigma^-) < 0.$$

So we have

$$\left|\varepsilon^2\sigma^- - 1\right| \leq |\eta(\varepsilon,\alpha)|\int_0^{+\infty}\gamma(x)\,dx,$$

thus

$$\left|\sigma^-\right| \leq \frac{|\eta(\varepsilon,\alpha)|}{\varepsilon^2}\int_0^{+\infty}\gamma(x)\,dx + \frac{1}{\varepsilon^2}.$$

Observe that σ^- satisfies

$$\varepsilon^2\left(\sigma^-\right)^2 - \sigma^- = \lambda + \mu,$$

we have

$$\begin{aligned}|\lambda| &\leq \left|\sigma^-\right|\left|\varepsilon^2\sigma^- - 1\right| + \mu \\ &\leq \left(\frac{|\eta(\varepsilon,\alpha)|}{\varepsilon}\int_0^{+\infty}\gamma(x)\,dx\right)^2 + \frac{|\eta(\varepsilon,\alpha)|\int_0^{+\infty}\gamma(x)\,dx}{\varepsilon^2} + \mu. \quad (9.2.28)\end{aligned}$$

Since when ε tends to infinity, the right hand of the inequality (9.2.28) goes to

$$\mu + \mu\left(1 - \ln R_0^\infty\right)^2,$$

and the result follows. □

Lemma 9.2.18 (Convergence). *Let Assumptions 9.2.1 and 9.2.16 be satisfied. Then we have*

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\widehat{\lambda}}\frac{\widetilde{\Delta}(\varepsilon,c\varepsilon,\lambda)}{\varepsilon} = \widehat{\Delta}(+\infty,c,\widehat{\lambda}),$$

where

$$\widehat{\Delta}(+\infty,c,\widehat{\lambda}) := \sqrt{\widehat{\lambda}+\mu} - \sqrt{\mu}\left(1 - \ln R_0^\infty\right).$$

Proof. We have

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\widehat{\lambda}}\varepsilon\sigma^- = \lim_{\varepsilon\to+\infty}\lim_{\lambda\to\widehat{\lambda}}\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon} = -\sqrt{\widehat{\lambda}+\mu},$$

$$\lim_{\varepsilon\to+\infty}\lim_{\lambda\to\widehat{\lambda}}\sigma^- = \lim_{\varepsilon\to+\infty}\lim_{\lambda\to\widehat{\lambda}}\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2} = 0,$$

and we deduce for $\lambda = 0$ that

$$\lim_{\varepsilon\to+\infty}\chi = \lim_{\varepsilon\to+\infty}\int_0^{+\infty}\gamma(x)e^{\sigma_0^- x}dx = \int_0^{+\infty}\gamma(x)\,dx.$$

Since

$$\lim_{\varepsilon \to +\infty} \frac{\eta\,(\varepsilon, c\varepsilon)}{\varepsilon} = \frac{\sqrt{\mu}}{\int_0^{+\infty} \gamma(x)\,dx}\,(1 - \ln R_0^\infty)\,,$$

we have

$$\begin{aligned}\lim_{\varepsilon \to +\infty} \lim_{\lambda \to \widehat{\lambda}} \frac{\widetilde{\Delta}(\varepsilon, c\varepsilon, \lambda)}{\varepsilon} &= \lim_{\varepsilon \to +\infty} \lim_{\lambda \to \widehat{\lambda}} \left(-\varepsilon\sigma^- + \frac{1}{\varepsilon} - \frac{\eta\,(\varepsilon, \alpha)}{\varepsilon} \int_0^{+\infty} \gamma(x)\, e^{\sigma^- x} dx \right) \\ &= \sqrt{\widehat{\lambda} + \mu} - \sqrt{\mu}\,(1 - \ln R_0^\infty)\,.\end{aligned}$$

This completes the proof of the lemma. □

Remark 9.2.19. From Lemma 9.2.18, the limit equation of the characteristic equation when ε tends to infinity is

$$\sqrt{\lambda + \mu} = \sqrt{\mu}\,(1 - \ln R_0^\infty)\,,\ \lambda \in \Omega. \tag{9.2.29}$$

Equation (9.2.29) has at most one real negative solution. Indeed, if $q := 1 - \ln R_0^\infty \in [0, 1)$, then

$$\sqrt{\lambda + \mu} = \sqrt{\mu} q,$$

i.e.,

$$\lambda = -\mu(1 - q^2) < 0;$$

and if $q := 1 - \ln R_0^\infty \in (-\infty, 0)$, then $\sqrt{\lambda + \mu} < 0$. Since by construction we have $\mathrm{Re}(\sqrt{\lambda + \mu}) \geq 0$, $\lambda \in \Omega$, so there is no solution.

Theorem 9.2.20. *Let Assumptions 9.2.1 and 9.2.16 be satisfied. Assume that* $R_0^\infty > 1$. *Then for each* $\varepsilon > 0$ *large enough the positive equilibrium* $\bar{v}$ *of system (9.2.2) is locally asymptotically stable.*

Proof. We claim that if Assumption 9.2.16 is satisfied, then for ε positive and large enough, there are no roots of the characteristic equation with non-negative real part. Otherwise, we can find a sequence $\{\varepsilon_n\} \to +\infty$ and a sequence $\{\lambda_n\}$ such that

$$\mathrm{Re}(\lambda_n) \geq 0,\ \ \widetilde{\Delta}(\varepsilon_n, c\varepsilon_n, \lambda_n) = 0.$$

By using Lemma 9.2.17, we know that $\{\lambda_n\}$ is bounded for each ε positive and large enough. Thus we can find a subsequence of $\{\lambda_n\}$ which converges to $\widehat{\lambda}$. We also denote this subsequence by $\{\lambda_n\}$. Obviously, we have

$$\mathrm{Re}(\widehat{\lambda}) \geq 0,\ \ \widetilde{\Delta}(\varepsilon_n, c\varepsilon_n, \lambda_n) = 0. \tag{9.2.30}$$

Let n tend to infinity in equation (9.2.30). Then by Lemma 9.2.18, we have

$$\widehat{\Delta}(+\infty, c, \widehat{\lambda}) = 0$$

with

$$\mathrm{Re}(\widehat{\lambda}) \geq 0,$$

which leads to a contradiction with Remark 9.2.19, and the result follows. $\square$

Remark 9.2.21. In order to show that, under Assumptions 9.2.1 and 9.2.16, Theorem 9.2.20 is more general than Theorem 9.2.15, we observe that

$$R_0 \to R_0^\infty = \frac{c}{\sqrt{\mu}} \int_0^{+\infty} \gamma(x) dx, \ \varepsilon \to +\infty.$$

So when $\varepsilon \to +\infty$, the condition of Theorem 9.2.15 yields

$$1 < R_0^\infty < e^2.$$

9.2.4 Hopf Bifurcation

In this subsection we study the existence of Hopf bifurcation when α is regarded as the bifurcation parameter of the system. By Theorem 9.2.15 we already knew that the positive equilibrium $\bar{v}$ of the system (9.2.2) is locally asymptotically stable if

$$1 < R_0 \leq e^2, \ R_0 = \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}},$$

which corresponds to $\alpha \in (\widehat{\alpha}_0, \widehat{\alpha}_1]$, where

$$\widehat{\alpha}_0 := \frac{1+\sqrt{\Lambda_0}}{2\chi} \text{ and } \widehat{\alpha}_1 := \frac{1+\sqrt{\Lambda_0}}{2\chi} e^2.$$

For a fixed value of $\varepsilon > 0$, we will study the existence of a bifurcation value $\alpha^* > \widehat{\alpha}_1$. Recall the characteristic equation $\Delta(\varepsilon, \alpha, \lambda) = 0$, where

$$\begin{aligned} \Delta(\varepsilon, \alpha, \lambda) &= 1 - \frac{2\eta(\varepsilon, \alpha)}{1+\sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx \\ &= 1 - \frac{\eta(\varepsilon, \alpha)}{1-\varepsilon^2\sigma^-} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx, \end{aligned}$$

in which

$$\sigma^- = \frac{1-\sqrt{\Lambda}}{2\varepsilon^2}$$

and

$$\eta(\varepsilon, \alpha) = \frac{\alpha}{R_0}(1 - \ln R_0) < 0 \ \text{ for } \ \alpha > \hat{\alpha}_1.$$

(a) Existence of purely imaginary eigenvalues. We consider the characteristic equation of $\sigma^- \in \mathbb{C}$: $\mathrm{Re}(\sigma^-) < 0$,

$$\varepsilon^2\sigma^- - 1 = -\eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\,e^{\sigma^- x}dx, \tag{9.2.31}$$

$$\varepsilon^2\left(\sigma^-\right)^2 - \sigma^- = \lambda+\mu \tag{9.2.32}$$

with

$$\mathrm{Re}(\lambda) > -\mu.$$

Set

$$\sigma^- = -(a+ib).$$

Then $a>0$, from equation (9.2.32) we have

$$\varepsilon^2\left(a^2-b^2+2abi\right)+a+ib = \lambda+\mu$$

i.e.,

$$\begin{cases} \varepsilon^2\left(a^2-b^2\right)+a = \mathrm{Re}(\lambda)+\mu \\ 2\varepsilon^2 ab+b = \mathrm{Im}(\lambda). \end{cases} \tag{9.2.33}$$

It follows that

$$b = \frac{\mathrm{Im}(\lambda)}{2a\varepsilon^2+1}. \tag{9.2.34}$$

Thus, if we look for purely imaginary roots $\lambda = \pm\omega i$ with $\omega>0$, then from equation (9.2.34) we have $b>0$. Since $a>0$, by the first equation in (9.2.33) we obtain

$$a = \frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}}{2\varepsilon^2},$$

and by the second equation in (9.2.33) we obtain

$$\omega = b\left(2\varepsilon^2 a+1\right) = b\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}.$$

On the other hand, from equation (9.2.31), we have

$$\varepsilon^2(a+ib)+1 = \eta(\varepsilon,\alpha)\int_0^{+\infty}\gamma(x)\,e^{-(a+ib)x}dx.$$

The rest of this subsection is devoted to the existence of purely imaginary roots of the characteristic equation when

$$\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)}1_{[\tau,+\infty)}(x)$$

with $\tau>0$, $\beta\geq 0$, and $n\in\mathbb{N}$.

Since the function $\gamma(x)$ must be bounded, we study the following two cases:

(i) $\beta=0$ and $n=0$.
(ii) $\beta>0$ and $n\geq 0$.

Case (i). $\beta=0$ *and* $n=0$. We will make the following assumption.

Assumption 9.2.22. Assume that $\varepsilon>0$ and $\gamma(x)=1_{[\tau,+\infty)}(x)$ for some $\tau>0$.

As we described in Section 9.2, when $\gamma(x) = 1_{[\tau,+\infty)}(x)$ the original system (9.2.1) can be viewed as a stochastic perturbation (in the transport term) of a delay differential equation. So we investigate the bifurcation properties of this problem in terms of parameters α and ε.

Under Assumption 9.2.22, we have

$$\int_0^{+\infty} \gamma(x)\, e^{\sigma^- x} dx = -\frac{e^{\sigma^- \tau}}{\sigma^-},$$

and the characteristic equation becomes

$$\operatorname{Re}\left(\sigma^-\right) < 0$$

and

$$\varepsilon^2 \left(\sigma^-\right)^2 - \sigma^- = \eta(\varepsilon,\alpha) e^{\sigma^- \tau},$$

i.e.,

$$\varepsilon^2 \left[a^2 - b^2 + 2abi\right] + a + ib = \eta(\varepsilon,\alpha) e^{-a\tau} \left[\cos(b\tau) - i \sin(b\tau)\right].$$

If we look at the curve $\lambda = \omega i$ with $\omega > 0$ and set

$$a := \frac{-1 + \sqrt{1 + 4\varepsilon^2 \left(\mu + \varepsilon^2 b^2\right)}}{2\varepsilon^2},$$

$$\omega := \varepsilon^2 2ab + b = b\sqrt{1 + 4\varepsilon^2 \left(\mu + \varepsilon^2 b^2\right)}.$$

Then

$$\varepsilon^2 \left(a^2 - b^2\right) + a = \mu \text{ and } a > 0,$$

and we obtain

$$\mu + i\omega = \eta(\varepsilon,\alpha) e^{-a\tau} e^{-ib\tau} = \eta(\varepsilon,\alpha) e^{-a\tau} \left[\cos(b\tau) - i \sin(b\tau)\right].$$

Now fix

$$\eta(\varepsilon,\alpha) = -c e^{a\tau} = -c e^{\frac{-1 + \sqrt{1 + 4\varepsilon^2 \left(\mu + \varepsilon^2 b^2\right)}}{2\varepsilon^2} \tau}$$

for some constant $c > 0$. We obtain

$$\mu + i\omega = c e^{-ib\tau} = -c \left[\cos(b\tau) - i \sin(b\tau)\right].$$

We must have

$$c = \sqrt{\mu^2 + \omega^2} = \sqrt{\mu^2 + \left(2\varepsilon^2 ab + b\right)^2}$$

$$\tan(b\tau) = -\frac{\omega}{\mu} = -\frac{\varepsilon^2 2ab + b}{\mu} = -\frac{b\sqrt{1 + 4\varepsilon^2 \left(\mu + \varepsilon^2 b^2\right)}}{\mu},$$

and impose that

$$\sin(b\tau) = \frac{\omega}{c} > 0 \text{ and } \cos(b\tau) = -\frac{\mu}{c} < 0.$$

From the above computations we obtain the following proposition.

Proposition 9.2.23. *Let $\varepsilon > 0$, $\tau > 0$ and $\mu > 0$ be fixed. Then the characteristic equation has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $b > 0$ which is a solution of equation*

$$\tan(b\tau) = -\frac{b\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2 b^2)}}{\mu} \tag{9.2.35}$$

with

$$\sin(b\tau) > 0 \tag{9.2.36}$$

and

$$\omega = b\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2 b^2)}, \;\; \eta(\varepsilon,\alpha) = \widehat{\eta}(\varepsilon,a,b),$$

where

$$\widehat{\eta}(\varepsilon,a,b) := -ce^{a\tau},$$

$$c = \sqrt{\mu^2 + (2\varepsilon^2 ab + b)^2}, \;\; a = \frac{-1+\sqrt{1+4\varepsilon^2(\mu+\varepsilon^2 b^2)}}{2\varepsilon^2}.$$

Moreover, for each $k \in \mathbb{N}$, there exists a unique $b_k \in \left(\left(2k+\frac{1}{2}\right)\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau}\right)$ (which is a function of τ, μ and ε) satisfying (9.2.35) and (9.2.36).

Proof. Set $\widehat{b} = b\tau$. Then equation (9.2.35) becomes

$$\tan\left(\widehat{b}\right) = -\frac{\left(\sqrt{1+4\varepsilon^2\left(\mu+\left(\frac{\varepsilon}{\tau}\right)^2\widehat{b}^2\right)}\right)\widehat{b}}{\mu\tau}. \tag{9.2.37}$$

Observe that the right-hand side of (9.2.37) is a strictly monotone decreasing function of $\widehat{b}$, and note that the function $\tan(x)$ is increasing, we deduce that equation (9.2.37) has a unique solution $b_m \in \left(\left(m-\frac{1}{2}\right)\pi, m\pi\right)$ for each $m \geq 1$, $m \in \mathbb{N}$. But since we need to impose $\sin(b_n\tau) > 0$, the result follows. $\square$

We obtain a sequence $\{b_k\} \subset \left(\left(2k+\frac{1}{2}\right)\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau}\right)$ satisfying (9.2.35) and (9.2.36). Moreover, we have

$$\eta(\varepsilon,\alpha_k) = \widehat{\eta}(\varepsilon,a_k,b_k),$$

where

$$\widehat{\eta}(\varepsilon,a_k,b_k) = -c_k e^{a_k\tau},$$

$$c_k = \sqrt{\mu^2 + (2\varepsilon^2 a_k b_k + b_k)^2}, \;\; a_k = \frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b_k^2\right)}}{2\varepsilon^2},$$

and obtain the following bifurcation curves

$$\alpha_k \frac{1}{R_0}\left(\ln\left(R_0\right)-1\right)=c_k e^{a_k \tau}.$$

We can rewrite the bifurcation curves as

$$\ln\left(R_0\right)=\frac{R_0}{\alpha_k} c_k e^{a_k \tau}+1.$$

Thus

$$R_0=e^{\left[1+c_k e^{a_k \tau} \frac{R_0}{\alpha_k}\right]}.$$

But $R_0=\frac{2\alpha_k \chi}{1+\sqrt{\Lambda_0}}$, so

$$\alpha_k=\frac{1+\sqrt{\Lambda_0}}{2\chi} \exp\left(1+c_k e^{a_k \tau} \frac{2\chi}{1+\sqrt{\Lambda_0}}\right),$$

where

$$\chi=\int_0^{+\infty} \gamma(x)\, e^{\sigma_0^- x} dx=-\frac{e^{\sigma_0^- \tau}}{\sigma_0^-}, \quad \sigma_0^-=\frac{1-\sqrt{\Lambda_0}}{2\varepsilon^2}, \; \Lambda_0=1+4\varepsilon^2 \mu.$$

So

$$\frac{2\chi}{1+\sqrt{\Lambda_0}}=\frac{4\varepsilon^2 e^{\left(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\right)\tau}}{\Lambda_0-1}=\frac{e^{\left(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\right)\tau}}{\mu}.$$

Hence, we obtain bifurcation curves

$$\alpha_k=\mu e^{-\left(\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\right)\tau} \exp\left(c_k e^{a_k \tau} \frac{e^{\frac{1-\sqrt{1+4\varepsilon^2\mu}}{2\varepsilon^2}\tau}}{\mu}+1\right). \tag{9.2.38}$$

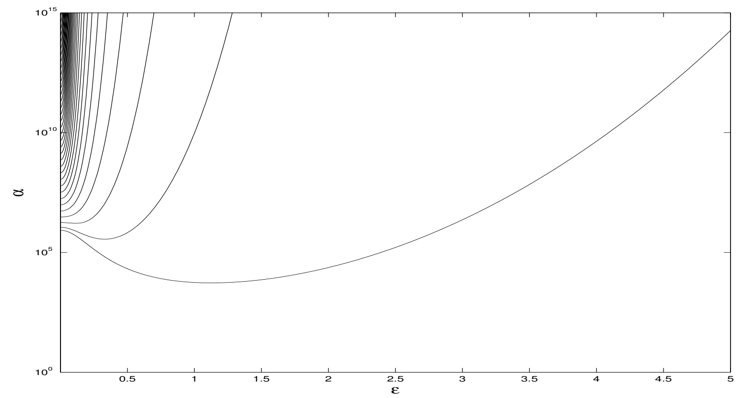

Fig. 9.1: A family of bifurcation curves given by (9.2.35), (9.2.36) and (9.2.38) in the (ε, α)-plane for $\tau = 2$ and $\mu = 5$.

Remark 9.2.24. Note that for any fixed $\varepsilon > 0$, α is a strictly increasing function of b, and as for each $k \in \mathbb{N}$, $b_k < b_{k+1}$, so the bifurcation curves cannot cross each other.

In the special case $\varepsilon = 0$, we obtain a characteristic equation which corresponds to a delay differential equation. Then the characteristic equation becomes

$$-\sigma^- = \eta(0,\alpha)e^{\sigma^- \tau}, \tag{9.2.39}$$

where

$$\eta(0,\alpha) = \frac{1}{\chi_0}(1 - \ln(\alpha\chi_0)) \text{ with } \chi_0 = \int_0^\infty \gamma(x)e^{-\mu x}dx = \frac{e^{-\mu\tau}}{\mu},$$

$$-\sigma^- = \lambda + \mu.$$

Corresponding to Proposition 9.2.23, we have the following proposition.

Proposition 9.2.25. *Let $\tau > 0$ and $\mu > 0$ be fixed. Then the characteristic equation (9.2.39) has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if*

$$\tan(\omega\tau) = -\frac{\omega}{\mu}, \tag{9.2.40}$$

$$\sin(\omega\tau) > 0, \tag{9.2.41}$$

and

$$\eta(0,\alpha) = -\sqrt{\mu^2 + \omega^2}e^{\mu\tau}.$$

Moreover, for each $k \in \mathbb{N}$, there exists a unique $\omega_k \in \left(\left(2k + \frac{1}{2}\right)\frac{\pi}{\tau}, (2k+1)\frac{\pi}{\tau}\right)$ which satisfies equation (9.2.40) and (9.2.41) (and is a function of τ and μ).

In this case, the bifurcation curves are given by

$$\alpha_k = \mu e^{\mu\tau} \exp\left(1 + \sqrt{\mu^2 + \omega_k^2}\, \mu^{-1}\right),$$

where ω_k is described in Proposition 9.2.25, $k \in \mathbb{N}$.

Case (ii). $\beta > 0$ and $n \geq 0$. In this subcase, we make the following assumption.

Assumption 9.2.26. Assume that $\varepsilon > 0$ and

$$\gamma(x) = (x-\tau)^n e^{-\beta(x-\tau)} 1_{[\tau,+\infty)}(x)$$

for some $n \geq 0$, $\tau > 0$, and $\beta > 0$.

When $\varepsilon = 0$, this problem corresponds to the example treated in Chapter 8, where the existence of purely imaginary solutions was obtained implicitly. Here we extend this study to the case when $\varepsilon > 0$ and specify the bifurcation diagram when $\varepsilon = 0$. Under Assumption 9.2.26, we have

$$\begin{aligned}
\int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx &= e^{\beta\tau} \int_\tau^{+\infty} (x-\tau)^n e^{\left(\sigma^- - \beta\right)x} dx \\
&= e^{\beta\tau} \int_0^{+\infty} s^n e^{\left(\sigma^- - \beta\right)(s+\tau)} ds \\
&= -e^{\beta\tau} e^{\left(\sigma^- - \beta\right)\tau} \int_{-\infty}^0 \left(\frac{l}{(\sigma^- - \beta)}\right)^n e^l \frac{1}{(\sigma^- - \beta)} dl \\
&= \frac{-e^{\sigma^-\tau}}{(\sigma^- - \beta)^{n+1}} \int_0^{+\infty} (-1)^n x^n e^{-x} dx \\
&= \frac{(-1)^{n+1} e^{\sigma^-\tau} n!}{(\sigma^- - \beta)^{n+1}} \\
&= \frac{n! e^{\sigma^-\tau}}{(\beta - \sigma^-)^{n+1}}.
\end{aligned}$$

So the characteristic equation becomes $\Delta(\varepsilon, \alpha, \lambda) = 0$, where

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{\eta(\varepsilon, \alpha)}{1 - \varepsilon^2 \sigma^-} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx = 1 - \frac{\eta(\varepsilon, \alpha) n! e^{\sigma^-\tau}}{(1 - \varepsilon^2 \sigma^-) \times (-\sigma^- + \beta)^{n+1}}.$$

First we give the following lemma to show that under Assumption 9.2.26, for any given $\varepsilon > 0$ and $\alpha > 0$, there exists at most one pair of purely imaginary solutions of the characteristic equation.

Lemma 9.2.27. *Let Assumptions 9.2.1 and 9.2.26 be satisfied. Then for each real number δ_1, there exists at most one $\delta_2 \in (0, +\infty)$ such that if*

$$\lambda \in \Omega,\ \mathrm{Re}(\lambda) = \delta_1 \text{ and } \Delta(\varepsilon, \alpha, \lambda) = 0,$$

then

$$\mathrm{Im}(\lambda) = \pm\delta_2.$$

Proof. Since $\Delta(\varepsilon,\alpha,\lambda) = 0$, we obtain

$$1 - \frac{\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}} = 0, \tag{9.2.42}$$

where σ^- is the solution of

$$\varepsilon^2\sigma^2 - \sigma = \lambda + \mu,\ \lambda \in \Omega, \tag{9.2.43}$$

with $\mathrm{Re}(\sigma^-) < 0$.

From (9.2.43) we have

$$\mathrm{Im}(\sigma^-)^2 = \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2} \geq 0, \tag{9.2.44}$$

$$\mathrm{Im}(\lambda) = 2\varepsilon^2\mathrm{Re}(\sigma^-)\mathrm{Im}(\sigma^-) - \mathrm{Im}(\sigma^-), \tag{9.2.45}$$

and by (9.2.42) we have

$$\left|1-\varepsilon^2\sigma^-\right| \times \left|\left(-\sigma^-+\beta\right)^{n+1}\right| = \left|\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}\right|,$$

i.e.,

$$\begin{aligned}&\left(\left(1-\varepsilon^2\mathrm{Re}(\sigma^-)\right)^2 + \left(\varepsilon^2\mathrm{Im}(\sigma^-)\right)^2\right) \times \left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \left(\mathrm{Im}(\sigma^-)\right)^2\right)^{n+1}\\ &\quad = |\eta(\varepsilon,\alpha)n!|^2 e^{2\mathrm{Re}(\sigma^-)\tau}.\end{aligned}$$

By using (9.2.45), we have

$$\begin{aligned}&\begin{pmatrix}\left(1-\varepsilon^2\mathrm{Re}(\sigma^-)\right)^2 + \\ \varepsilon^4\left(\mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\end{pmatrix} \times \begin{pmatrix}\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \\ \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\end{pmatrix}^{n+1}\\ &\quad = |\eta(\varepsilon,\alpha)n!|^2 e^{2\mathrm{Re}(\sigma^-)\tau}.\end{aligned}$$

Now set

$$\begin{aligned}f(\mathrm{Re}(\sigma^-)) = &\begin{pmatrix}\left(1-\varepsilon^2\mathrm{Re}(\sigma^-)\right)^2 + \\ \varepsilon^4\left(\mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\end{pmatrix} \times \begin{pmatrix}\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2 + \\ \mathrm{Re}(\sigma^-)^2 - \frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\end{pmatrix}^{n+1}\\ &- |\eta(\varepsilon,\alpha)n!|^2 e^{2\mathrm{Re}(\sigma^-)\tau},\end{aligned}$$

then

$$\frac{df}{d\mathrm{Re}(\sigma^-)}\left(\mathrm{Re}(\sigma^-)\right)$$

$$
\begin{aligned}
&= \left(-2\varepsilon^2\left(1-\varepsilon^2\mathrm{Re}(\sigma^-)\right)+2\varepsilon^4\mathrm{Re}(\sigma^-)-\varepsilon^2\right)\\
&\quad\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2+\mathrm{Re}(\sigma^-)^2-\frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n+1}\\
&\quad+(n+1)\times\left(\left(1-\varepsilon^2\mathrm{Re}(\sigma^-)\right)^2+\varepsilon^4\left(\mathrm{Re}(\sigma^-)^2-\frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)\right)\\
&\quad\times\left(\left(-\mathrm{Re}(\sigma^-)+\beta\right)^2+\mathrm{Re}(\sigma^-)^2-\frac{\mathrm{Re}(\sigma^-)+\mathrm{Re}(\lambda)+\mu}{\varepsilon^2}\right)^{n}\\
&\quad\times\left(-2\left(-\mathrm{Re}(\sigma^-)+\beta\right)+2\mathrm{Re}(\sigma^-)-\frac{1}{\varepsilon^2}\right)\\
&\quad-2\tau\left|\eta(\varepsilon,\alpha)n!\right|^2e^{2\mathrm{Re}(\sigma^-)\tau}.
\end{aligned}
$$

By using (9.2.44) and the above equation we deduce that

$$\mathrm{Re}(\sigma^-)<0\Rightarrow\frac{df}{d\mathrm{Re}(\sigma^-)}(\mathrm{Re}(\sigma^-))<0.$$

Thus, for any fixed $\mathrm{Re}(\lambda)$, we can find at most one $\mathrm{Re}(\sigma^-)$ satisfying the characteristic equation (9.2.42). Using (9.2.44) and (9.2.45), we obtain the result. □

Now we consider the characteristic equation as system (9.2.31) and (9.2.32) with $\mathrm{Re}(\lambda)\geq-\mu$ and $\mathrm{Re}(\sigma^-)<0$. Under Assumption 9.2.26, the characteristic equation (9.2.31) is equivalent to

$$\left(\varepsilon^2\sigma^--1\right)=-n!\eta(\varepsilon,\alpha)\frac{e^{\sigma^-\tau}}{(\beta-\sigma^-)^{n+1}}. \tag{9.2.46}$$

We look for purely imaginary roots $\lambda=\omega i$ with $\omega>0$. As before, we set

$$\sigma^-:=-a-ib,\ \ \omega:=2\varepsilon^2ab+b,\ \ a:=\frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2b^2\right)}}{2\varepsilon^2}$$

with $b>0$, then equation (9.2.32) is satisfied. Now it remains to find b such that it satisfies equation (9.2.46), or equivalently,

$$\sigma^-\times\left(\varepsilon^2\sigma^--1\right)=i\omega+\mu=-\sigma^-\times n!\eta(\varepsilon,\alpha)\frac{e^{\sigma^-\tau}}{(\beta-\sigma^-)^{n+1}}.$$

Now we have

$$
\begin{aligned}
\sigma^-&=-a-ib=\sqrt{a^2+b^2}e^{i\theta},\\
\beta-\sigma^-&=a+\beta+ib=\sqrt{(a+\beta)^2+b^2}e^{i\widehat{\theta}},
\end{aligned}
$$

where

$$\theta:=\arctan\left(\frac{b}{a}\right)+\pi,\ \ \widehat{\theta}:=\arctan\left(\frac{b}{a+\beta}\right).$$

Then we obtain

$$\mu + i\omega = -\eta(\varepsilon,\alpha)\frac{n!\sqrt{a^2+b^2}e^{-a\tau}}{\left(\sqrt{(a+\beta)^2+b^2}\right)^{n+1}}e^{i(\theta-(n+1)\widehat{\theta}-\tau b)}.$$

Now fix

$$\eta(\varepsilon,\alpha) = -c\frac{\left(\sqrt{(a+\beta)^2+b^2}\right)^{n+1}}{n!\sqrt{a^2+b^2}}e^{a\tau}$$

with $c>0$, then we have

$$\mu + i\omega = \mu + i\left(2\varepsilon^2 ab + b\right) = ce^{i(\theta-(n+1)\widehat{\theta}-\tau b)}$$

and

$$c = \sqrt{\mu^2+\omega^2} = \sqrt{\mu^2+\left(2\varepsilon^2 ab+b\right)^2},$$

$$\frac{\left(2\varepsilon^2 ab+b\right)}{\mu} = \tan\left(\theta-(n+1)\widehat{\theta}-\tau b\right).$$

We must impose that

$$\sin\left(\theta-(n+1)\widehat{\theta}-\tau b\right) = \frac{2\varepsilon^2 ab+b}{c} > 0.$$

From the above computations we obtain the following proposition.

Proposition 9.2.28. *Let $\varepsilon>0$, $\tau>0$, $\mu>0$, $\beta>0$, and $n\in\mathbb{N}$ be fixed. Then the characteristic equation has a pair of purely imaginary solutions $\pm i\omega$ with $\omega>0$ if and only if there exists $b>0$ which is a solution of equation*

$$\frac{\left(2\varepsilon^2 ab+b\right)}{\mu} = -\tan\Theta(b) \tag{9.2.47}$$

with

$$\sin(\Theta(b)) < 0, \tag{9.2.48}$$

and we have

$$\omega = b\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)},\ \ \eta(\varepsilon,\alpha) = \widetilde{\eta}(\varepsilon,a,b),$$

where

$$\Theta(b) = -\theta+(n+1)\widehat{\theta}+\tau b,\ \ \widetilde{\eta}(\varepsilon,a,b) := -c\frac{\left(\sqrt{(a+\beta)^2+b^2}\right)^{n+1}}{n!\sqrt{a^2+b^2}}e^{a\tau},$$

$$\theta = \arctan\left(\frac{b}{a}\right) + \pi,\ \ \widehat{\theta} = \arctan\left(\frac{b}{a+\beta}\right),$$

$$c = \sqrt{\mu^2 + \left(2\varepsilon^2 ab + b\right)^2},\ \ a = \frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}}{2\varepsilon^2}.$$

Moreover, there exists a sequence $\{b_k\} \to +\infty$ *as* $k \to +\infty$, $k \in \mathbb{N}$ *(which is a function of* ε, τ, μ, β, *and* n*) satisfying (9.2.47) and (9.2.48). In particular, for each* k *large enough, there exists a unique* $b_k \in \left(\Theta^{-1}\left(2k\pi - \frac{\pi}{2}\right), \Theta^{-1}\left(2k\pi\right)\right)$ *satisfying (9.2.47) and (9.2.48), where* Θ^{-1} *is the inverse function of* $\Theta(b)$ *on* $\left[\widehat{b}, +\infty\right)$ *for* $\widehat{b}$ *large enough.*

Proof. Note that for $b > 0$,

$$\frac{\left(2\varepsilon^2 ab + b\right)}{\mu} = -\tan\Theta(b) > 0,$$

we have

$$\tan\Theta(b) < 0,$$

so

$$\Theta(b) \in \left(m\pi - \frac{\pi}{2}, m\pi\right),\ m \in \mathbb{Z}.$$

Moreover, in order to ensure

$$\sin\Theta(b) < 0,$$

we must take $m = 2k, k \in \mathbb{Z}$. Now since $\Theta(b)$ is a continuous function of b,

$$\Theta(0) = -\pi,\ \Theta(+\infty) = +\infty,$$

so for any $k \in \mathbb{N}$, there exist $\widehat{b_{k1}}, \widehat{b_{k2}} > 0$ such that $\Theta(b_{k1}) = 2k\pi - \frac{\pi}{2}$, $\Theta(b_{k2}) = 2k\pi$. Observe that the left-hand side of equation (9.2.47) is a strictly monotone increasing function of b, and since the function $\tan(\Theta(b))$ can take any value from $-\infty$ to $+\infty$ when $b \in \left(\widehat{b_{k1}}, \widehat{b_{k2}}\right)$ or $b \in \left(\widehat{b_{k2}}, \widehat{b_{k1}}\right)$, we deduce that equation (9.2.47) has a solution $b_k \in \left(\widehat{b_{k1}}, \widehat{b_{k2}}\right)$ or $b_k \in \left(\widehat{b_{k2}}, \widehat{b_{k1}}\right)$. Thus, there exists a sequence of $\{b_k\} \to +\infty$ satisfying (9.2.47) and (9.2.48). We denote the derivative of function f with respect to b by f'. Then

$$\begin{aligned}
\Theta'(b) := \frac{d}{db}\Theta(b) &= -\left[\arctan\left(\frac{b}{a}\right) + \pi\right]' + (n+1)\left[\arctan\left(\frac{b}{a+\beta}\right)\right]' + \tau \\
&= -\frac{\left(\frac{b}{a}\right)'}{1+\left(\frac{b}{a}\right)^2} + (n+1)\frac{\left(\frac{b}{a+\beta}\right)'}{1+\left(\frac{b}{a+\beta}\right)^2} + \tau \\
&= -\frac{\frac{a-ba'}{a^2}}{1+\left(\frac{b}{a}\right)^2} + (n+1)\frac{\frac{a+\beta-ba'}{(a+\beta)^2}}{1+\left(\frac{b}{a+\beta}\right)^2} + \tau
\end{aligned}$$

$$= -\frac{\frac{a-ba'}{b^2}}{\left(\frac{a}{b}\right)^2+1} + (n+1)\frac{\frac{a+\beta-ba'}{b^2}}{\left(\frac{a+\beta}{b}\right)^2+1} + \tau,$$

where

$$a' := \frac{d}{db}\left(\frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}}{2\varepsilon^2}\right) = \frac{2\varepsilon^2 b}{\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}}.$$

Since

$$a' = \frac{2\varepsilon^2 b}{\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}} \to 1 \text{ as } b \to +\infty,$$

$$\frac{a}{b} = \frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b^2\right)}}{2\varepsilon^2 b} \to 1 \text{ as } b \to +\infty,$$

$$\frac{a+\beta}{b} \to 1 \text{ as } b \to +\infty,$$

we obtain

$$\Theta'(b) \to \tau \text{ as } b \to +\infty.$$

That is, when b is large enough, $\Theta(b)$ is a strictly monotone increasing function of b. Denote by Θ^{-1} the inverse function of $\Theta(b)$ on $\left[\widehat{b}, +\infty\right)$ for $\widehat{b}$ large enough. So for k large enough we have $\widehat{b_{k1}} = \Theta^{-1}\left(2k\pi - \frac{\pi}{2}\right)$, $\widehat{b_{k2}} = \Theta^{-1}(2k\pi)$, and the function $\tan(\Theta(b))$ is increasing when $b \in \left(\widehat{b_{k1}}, \widehat{b_{k2}}\right)$. Thus there exists a unique $b_k \in \left(\Theta^{-1}\left(2k\pi - \frac{\pi}{2}\right), \Theta^{-1}(2k\pi)\right)$ satisfying (9.2.47) and (9.2.48), and the result follows. $\square$

We find a sequence $\{b_k\}$ going to $+\infty$ and satisfying (9.2.47) and (9.2.48). By using a similar procedure as before, we can derive a bifurcation diagram. Using our construction, we have

$$\eta(\varepsilon, \alpha_k) = \widetilde{\eta}(\varepsilon, a_k, b_k).$$

But

$$\eta(\varepsilon, \alpha) = \alpha \frac{1}{R_0}\left(1 - \ln(R_0)\right),$$

so we obtain the bifurcation curves

$$\alpha_k \frac{1}{R_0}\left(1 - \ln(R_0)\right) = \widetilde{\eta}(\varepsilon, a_k, b_k).$$

Since

$$\ln(R_0) = 1 - \frac{R_0}{\alpha_k}\widetilde{\eta}(\varepsilon, a_k, b_k),$$

it follows that

$$R_0 = e^{\left[1-\widetilde{\eta}(\varepsilon, a_k, b_k)\frac{R_0}{\alpha_k}\right]}.$$

Notice that we also have

$$R_0 = \frac{2\alpha_k \chi}{1+\sqrt{\Lambda_0}},$$

so

$$\alpha_k = \frac{1+\sqrt{\Lambda_0}}{2\chi} \exp\left(1 - \widetilde{\eta}(\varepsilon, a_k, b_k) \frac{2\chi}{1+\sqrt{\Lambda_0}}\right).$$

Now since

$$\chi = \int_0^{+\infty} \gamma(x) e^{\sigma_0^- x} dx = \frac{n! e^{\sigma_0^- \tau}}{\left(\beta - \sigma_0^-\right)^{n+1}},$$

$$\sigma_0^- = \frac{1-\sqrt{\Lambda_0}}{2\varepsilon^2}, \ \ \Lambda_0 = 1 + 4\varepsilon^2 \mu,$$

we obtain

$$\frac{2\chi}{1+\sqrt{\Lambda_0}} = \frac{2n! e^{\sigma_0^- \tau}}{\left(1+\sqrt{\Lambda_0}\right)\left(\beta - \sigma_0^-\right)^{n+1}}.$$

Thus we obtain the bifurcation curves

$$\alpha_k = \frac{\left(1+\sqrt{\Lambda_0}\right)\left(\beta - \sigma_0^-\right)^{n+1}}{2n! e^{\sigma_0^- \tau}} \exp\left(1 - \widetilde{\eta}(\varepsilon, a_k, b_k) \frac{2n! e^{\sigma_0^- \tau}}{\left(1+\sqrt{\Lambda_0}\right)\left(\beta - \sigma_0^-\right)^{n+1}}\right), \tag{9.2.49}$$

where

$$\widetilde{\eta}(\varepsilon, a_k, b_k) = -c_k \frac{\left(\sqrt{(a_k+\beta)^2 + b_k^2}\right)^{n+1}}{n! \sqrt{a_k^2 + b_k^2}} e^{a_k \tau},$$

$$c_k = \sqrt{\mu^2 + \left(2\varepsilon^2 a_k b_k + b_k\right)^2}, \ \ a_k = \frac{-1 + \sqrt{1 + 4\varepsilon^2\left(\mu + \varepsilon^2 b_k^2\right)}}{2\varepsilon^2}.$$

Similarly, the special case $\varepsilon = 0$ corresponds to the characteristic equation studied in Section 8.3. Here we improve the description by specifying the bifurcation curves. When $\varepsilon = 0$, the characteristic equation becomes

$$1 = n! \eta(0, \alpha) \frac{e^{\sigma^- \tau}}{(\beta - \sigma^-)^{n+1}} \tag{9.2.50}$$

with

$$-\sigma^- = \lambda + \mu,$$

$$\eta(0, \alpha) = \frac{1}{\chi_0}(1 - \ln(\alpha \chi_0)), \ \ \chi_0 = \int_0^{\infty} \gamma(x) e^{-\mu x} dx = \frac{n! e^{-\mu \tau}}{(\beta + \mu)^{n+1}}.$$

If we now look for $\lambda = \pm \omega i$ with $\omega > 0$ and set $\sigma^- = -a - ib$ with $a > 0, b > 0$, then

$$a = \mu, \ \ \omega = b,$$

and b must satisfy

$$1 = n!\eta(0,\alpha)\frac{e^{(-\mu-ib)\tau}}{(\beta+\mu+ib)^{n+1}},$$

i.e.,

$$1 = n!\eta(0,\alpha)\frac{e^{-\mu\tau}}{\left(\sqrt{(\beta+\mu)^2+b^2}\right)^{n+1}}e^{-i\left(b\tau+(n+1)\arctan\frac{b}{\beta+\mu}\right)}.$$

Since $\eta(0,\alpha) < 0$ for $\alpha > \hat{\alpha}_1$, we have

$$\eta(0,\alpha) = -\frac{\left(\sqrt{(\mu+\beta)^2+b^2}\right)^{n+1}}{n!}e^{\mu\tau}$$

and

$$-\left(b\tau+(n+1)\arctan(\frac{b}{\beta+\mu})\right) = \pi - 2k\pi$$

for some $k \in \mathbb{Z}$.

Proposition 9.2.29. *Let $\tau > 0$, $\mu > 0$, $\beta > 0$, and $n \in \mathbb{N}$ be fixed. Then the characteristic equation (9.2.50) has a pair of purely imaginary solutions $\pm i\omega$ with $\omega > 0$ if and only if there exists $k \in \mathbb{Z}$ such that ω is a solution of equation*

$$-\left(\omega\tau+(n+1)\arctan\frac{\omega}{\beta+\mu}\right) = \pi - 2k\pi \qquad (9.2.51)$$

and

$$\eta(0,\alpha) = \widetilde{\eta}(0,\mu,\omega),$$

where

$$\widetilde{\eta}(0,\mu,\omega) := -\frac{\left(\sqrt{(\mu+\beta)^2+\omega^2}\right)^{n+1}}{n!}e^{\mu\tau}.$$

Moreover, for each $k \in \mathbb{N}^+$, there exists a unique ω_k (which is a function of τ, μ, β, and n) satisfing equation (9.2.51).

In this case, the bifurcation curves are

$$\alpha_k = \frac{(\beta+\mu)^{n+1}}{n!e^{-\mu\tau}}\exp\left(1-\widetilde{\eta}(0,\mu,\omega_k)\frac{n!e^{-\mu\tau}}{(\beta+\mu)^{n+1}}\right),\ k \in \mathbb{N}^+,$$

where

$$\widetilde{\eta}(0,\mu,\omega_k) = -\frac{\left(\sqrt{(\mu+\beta)^2+\omega_k^2}\right)^{n+1}}{n!}e^{\mu\tau},$$

and ω_k is described in Proposition 9.2.29.

(b) Transversality condition. The aim of this part is to prove a transversality condition for the model with Assumption 9.2.22 or Assumption 9.2.26. Since Assumption 9.2.22 is a special case of Assumption 9.2.26, we start to investigate the transversality condition under Assumption 9.2.26.

Lemma 9.2.30. *For fixed* $\varepsilon > 0$, *if* $\alpha > \widehat{\alpha}_1$, $\lambda \in \Omega$ *and* $\Delta(\varepsilon, \alpha, \lambda) = 0$, *then*

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} < 0.$$

Proof. Since

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{2\eta(\varepsilon, \alpha)}{1+\sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx,$$

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} = -\frac{2\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha}}{1+\sqrt{\Lambda}} \int_0^{+\infty} \gamma(x) e^{\sigma^- x} dx,$$

and

$$\eta(\varepsilon, \alpha) = \frac{1+\sqrt{\Lambda_0}}{2\chi}\left(1 - \ln\left(\alpha \frac{2\chi}{1+\sqrt{\Lambda_0}}\right)\right),$$

where $\frac{1+\sqrt{\Lambda_0}}{2\chi}$ is independent of α, we have

$$\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha} = -\frac{1+\sqrt{\Lambda_0}}{2\alpha\chi}.$$

But $\Delta(\varepsilon, \alpha, \lambda) = 0$, we obtain

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} = -\frac{\frac{\partial \eta(\varepsilon, \alpha)}{\partial \alpha}}{\eta(\varepsilon, \alpha)} = \frac{1}{\alpha\left(1 - \ln \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}}\right)}.$$

Moreover, if $\alpha > \widehat{\alpha}_1$, then $\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \alpha} < 0$. □

Lemma 9.2.31. *Let Assumption 9.2.1 and Assumption 9.2.26 be satisfied. For fixed* $\varepsilon > 0$, *if* $\alpha > \widehat{\alpha}_1$, $\lambda \in \Omega$ *and* $\Delta(\varepsilon, \alpha, \lambda) = 0$, *then*

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \lambda} = \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}} + \frac{\tau}{2\varepsilon^2} + \frac{1+n}{(2\varepsilon^2\beta - 1) + \sqrt{\Lambda}}\right) \neq 0.$$

Proof. Under Assumption 9.2.26 we have

$$\Delta(\varepsilon, \alpha, \lambda) = 1 - \frac{\eta(\varepsilon, \alpha) n! e^{\sigma^- \tau}}{(1 - \varepsilon^2\sigma^-) \times (-\sigma^- + \beta)^{n+1}}$$

and

$$\frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \lambda} = \frac{\partial \Delta(\varepsilon, \alpha, \lambda)}{\partial \sigma^-} \frac{\partial \sigma^-}{\partial \lambda}$$

$$
\begin{aligned}
&= -\eta(\varepsilon,\alpha)n!\frac{\partial}{\partial\sigma^-}\left(\frac{e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}\right)\frac{\partial\sigma^-}{\partial\lambda}\\
&= -\frac{\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}\left(\tau+\frac{\varepsilon^2}{1-\varepsilon^2\sigma^-}+\frac{n+1}{-\sigma^-+\beta}\right)\frac{\partial\sigma^-}{\partial\lambda}\\
&= (\Delta(\varepsilon,\alpha,\lambda)-1)\times\left(\tau+\frac{2\varepsilon^2}{1+\sqrt{\Lambda}}+\frac{n+1}{-\sigma^-+\beta}\right)\frac{\partial\sigma^-}{\partial\lambda}.
\end{aligned}
$$

Since

$$
\frac{\partial\sigma^-}{\partial\lambda}=\frac{\partial}{\partial\lambda}\left(\frac{1-\sqrt{1+4\varepsilon^2(\lambda+\mu)}}{2\varepsilon^2}\right)=-\frac{1}{\sqrt{1+4\varepsilon^2(\lambda+\mu)}}=-\frac{1}{\sqrt{\Lambda}},
$$

we obtain

$$
\frac{\partial\Delta(\varepsilon,\alpha,\lambda)}{\partial\lambda}=-\frac{\Delta(\varepsilon,\alpha,\lambda)-1}{\sqrt{\Lambda}}\left(\frac{2\varepsilon^2}{1+\sqrt{\Lambda}}+\tau+\frac{n+1}{-\sigma^-+\beta}\right).
$$

So if $\Delta(\varepsilon,\alpha,\lambda)=0$, then

$$
\frac{\partial\Delta(\varepsilon,\alpha,\lambda)}{\partial\lambda}=\frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}}+\frac{\tau}{2\varepsilon^2}+\frac{1+n}{(2\varepsilon^2\beta-1)+\sqrt{\Lambda}}\right). \tag{9.2.52}
$$

Now note that $\frac{\partial\Delta(\varepsilon,\alpha,\lambda)}{\partial\lambda}=0$ if and only if

$$
\frac{\tau}{2\varepsilon^2}\Lambda+(2+n+\tau\beta)\sqrt{\Lambda}+\frac{\tau}{2\varepsilon^2}\left(2\varepsilon^2\beta-1\right)+n+2\varepsilon^2\beta=0. \tag{9.2.53}
$$

As $\eta(\varepsilon,\alpha)<0$ for $\alpha>\widehat{\alpha}_1$, we have for $\lambda\in\mathbb{R}$ and $\lambda>-\mu$ that

$$
\Delta(\varepsilon,\alpha,\lambda)=1-\frac{\eta(\varepsilon,\alpha)n!e^{\sigma^-\tau}}{(1-\varepsilon^2\sigma^-)\times(-\sigma^-+\beta)^{n+1}}>0.
$$

So solutions of the characteristic equation in Ω cannot be real numbers. When $\lambda\in\mathbb{C}\setminus\mathbb{R}$, we have $\sqrt{\Lambda}=\sqrt{1+4\varepsilon^2(\lambda+\mu)}\in\mathbb{C}\setminus\mathbb{R}$ with $\mathrm{Re}(\sqrt{\Lambda})>0$. By noting that the sign of the imaginary part of Λ is the same as the sign of the imaginary part of $\sqrt{\Lambda}$ and by noticing that $\frac{\tau}{2\varepsilon^2}>0$ and $2+n+\tau\beta>0$, we deduce that equation (9.2.53) cannot be satisfied. □

Theorem 9.2.32. *Let Assumption 9.2.1 and Assumption 9.2.26 be satisfied and let $\varepsilon>0$ be given. For each $k\geq 0$ large enough, let $\lambda_k=i\omega_k$ be the purely imaginary root of the characteristic equation associated to $\alpha_k>0$ (defined in Proposition 9.2.28), then there exists $\rho_k>0$ (small enough) and a C^1-map $\widehat{\lambda}_k:(\alpha_k-\rho_k,\alpha_k+\rho_k)\to\mathbb{C}$ such that*

$$
\widehat{\lambda}_k(\alpha_k)=i\omega_k,\ \ \Delta(\varepsilon,\alpha,\widehat{\lambda}_k(\alpha))=0,\ \ \forall\alpha\in(\alpha_k-\rho_k,\alpha_k+\rho_k)
$$

satisfying the transversality condition

$$\operatorname{Re}\left(\frac{d\widehat{\lambda}_k(\alpha_k)}{d\alpha}\right) > 0.$$

Proof. By Lemma 9.2.31 we can use the implicit function theorem around each $(\alpha_k, i\omega_k)$ provided by Proposition 9.2.28, and obtain that there exist $\rho_k > 0$ and a C^1-map $\widehat{\lambda}_k : (\alpha_k - \rho_k, \alpha_k + \rho_k) \to \mathbb{C}$ such that

$$\widehat{\lambda}_k(\alpha_k) = i\omega_k, \;\; \Delta(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)) = 0, \;\; \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

Moreover, we have

$$\frac{\partial \Delta\left(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)\right)}{\partial \alpha} + \frac{\partial \Delta\left(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)\right)}{\partial \lambda}\frac{d\widehat{\lambda}_k(\alpha)}{d\alpha} = 0, \; \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

So

$$\frac{d\widehat{\lambda}_k(\alpha)}{d\alpha} = -\frac{1}{\frac{\partial \Delta\left(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)\right)}{\partial \lambda}}\frac{\partial \Delta\left(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)\right)}{\partial \alpha}, \; \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k).$$

By using Lemma 9.2.30, we deduce $\forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k)$ that

$$\operatorname{Re}\left(\frac{d}{d\alpha}\widehat{\lambda}_k(\alpha)\right) > 0 \;\Leftrightarrow\; \operatorname{Re}\left(\frac{\partial \Delta\left(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)\right)}{\partial \lambda}\right) > 0.$$

In particular, we have

$$\operatorname{Re}\left(\frac{d}{d\alpha}\widehat{\lambda}_k(\alpha_k)\right) > 0 \;\Leftrightarrow\; \operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}\right) > 0.$$

Using the notation of Proposition 9.2.28, we have

$$\sqrt{\Lambda} = 1 - 2\varepsilon^2\sigma^- = 1 + 2\varepsilon^2(a_k + ib_k) := \gamma_k + i\delta_k,$$

where γ_k and δ_k are positive and $\gamma_k^2 - \delta_k^2 = 1 + 4\varepsilon^2\mu$.

Therefore, we have

$$\begin{aligned}
\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda} &= \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}} + \frac{\tau}{2\varepsilon^2} + \frac{1+n}{(2\varepsilon^2\beta - 1) + \sqrt{\Lambda}}\right) \\
&= \frac{2\varepsilon^2}{\gamma_k + i\delta_k}\left(\frac{1+\gamma_k - i\delta_k}{(1+\gamma_k)^2 + \delta_k^2} + \frac{\tau}{2\varepsilon^2} + (n+1)\,\frac{2\varepsilon^2\beta - 1 + \gamma_k - i\delta_k}{(2\varepsilon^2\beta - 1 + \gamma_k)^2 + \delta_k^2}\right)
\end{aligned}$$

$$= \frac{2\varepsilon^2}{\gamma_k^2+\delta_k^2}(\gamma_k - i\delta_k)\left(\frac{1+\gamma_k-i\delta_k}{(1+\gamma_k)^2+\delta_k^2}+\frac{\tau}{2\varepsilon^2}+(n+1)\frac{2\varepsilon^2\beta-1+\gamma_k-i\delta_k}{(2\varepsilon^2\beta-1+\gamma_k)^2+\delta_k^2}\right)$$

and

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\partial\Delta(\varepsilon,\alpha_k,i\omega_k)}{\partial\lambda}\right)\\
&= \frac{2\varepsilon^2}{\gamma_k^2+\delta_k^2}\left(\frac{\gamma_k(1+\gamma_k)-\delta_k^2}{(1+\gamma_k)^2+\delta_k^2}+\frac{\gamma_k\tau}{2\varepsilon^2}+(n+1)\frac{\gamma_k\left(2\varepsilon^2\beta-1+\gamma_k\right)-\delta_k^2}{(2\varepsilon^2\beta-1+\gamma_k)^2+\delta_k^2}\right)\\
&= \frac{2\varepsilon^2}{\gamma_k^2+\delta_k^2}\left(\frac{\gamma_k+\gamma_k^2-\delta_k^2}{(1+\gamma_k)^2+\delta_k^2}+\frac{\gamma_k\tau}{2\varepsilon^2}+(n+1)\frac{\gamma_k\left(2\varepsilon^2\beta-1\right)+\gamma_k^2-\delta_k^2}{(2\varepsilon^2\beta-1+\gamma_k)^2+\delta_k^2}\right)\\
&= \frac{2\varepsilon^2}{\gamma_k^2+\delta_k^2}A_k,
\end{aligned}$$

where

$$A_k = \left(\frac{\gamma_k+1+4\varepsilon^2\mu}{(1+\gamma_k)^2+\delta_k^2}+\frac{\gamma_k\tau}{2\varepsilon^2}+(n+1)\frac{\gamma_k\left(2\varepsilon^2\beta-1\right)+1+4\varepsilon^2\mu}{(2\varepsilon^2\beta-1+\gamma_k)^2+\delta_k^2}\right).$$

By Proposition 9.2.28 we have

$$a_k = \frac{-1+\sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b_k^2\right)}}{2\varepsilon^2},\ \ b_k \to +\infty \text{ as } k \to +\infty.$$

Then we obtain

$$\gamma_k = 1+2\varepsilon^2 a_k = \sqrt{1+4\varepsilon^2\left(\mu+\varepsilon^2 b_k^2\right)} \to +\infty \text{ as } k \to +\infty,$$

$$\delta_k = 2\varepsilon^2 b_k \to +\infty \text{ as } k \to +\infty,$$

and

$$\begin{aligned}
\lim_{k\to+\infty}\frac{A_k}{\gamma_k} &= \lim_{k\to+\infty}\left(\frac{1+\frac{1+4\varepsilon^2\mu}{\gamma_k}}{(1+\gamma_k)^2+\delta_k^2}+\frac{\tau}{2\varepsilon^2}+(n+1)\frac{\left(2\varepsilon^2\beta-1\right)+\frac{1+4\varepsilon^2\mu}{\gamma_k}}{(2\varepsilon^2\beta-1+\gamma_k)^2+\delta_k^2}\right)\\
&= \frac{\tau}{2\varepsilon^2} > 0.
\end{aligned}$$

We deduce that $A_k > 0$ for k large enough and $\operatorname{Re}\left(\frac{\partial\Delta(\varepsilon,\alpha_k,i\omega_k)}{\partial\lambda}\right) > 0$. So $\operatorname{Re}\left(\frac{d\widehat{\lambda}_k(\alpha_k)}{d\alpha}\right) > 0$ and the result follows. □

In case (a) we have a full description of the problem in terms of the transversality condition and the following result.

Theorem 9.2.33. *Let Assumption 9.2.1 and Assumption 9.2.22 be satisfied and let $\varepsilon > 0$ be given. For each $k \geq 0$, let $\lambda_k = i\omega_k$ be the purely imaginary root of the characteristic equation associated to $\alpha_k > 0$ (defined in Proposition 9.2.23), then there exist $\rho_k > 0$ (small enough) and a C^1-map $\widehat{\lambda}_k : (\alpha_k - \rho_k, \alpha_k + \rho_k) \to \mathbb{C}$ such that*

$$\widehat{\lambda}_k(\alpha_k) = i\omega_k,\ \Delta(\varepsilon, \alpha, \widehat{\lambda}_k(\alpha)) = 0,\ \forall \alpha \in (\alpha_k - \rho_k, \alpha_k + \rho_k)$$

satisfying the transversality condition

$$\operatorname{Re}\left(\frac{d\widehat{\lambda}_k(\alpha_k)}{d\alpha}\right) > 0.$$

Proof. According to the proof of Theorem 9.2.32, we have

$$\operatorname{Re}\left(\frac{d}{d\alpha}\widehat{\lambda}_k(\alpha_k)\right) > 0 \ \Leftrightarrow\ \operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}\right) > 0.$$

Taking $n = 0, \beta = 0$ in (9.2.52), we have for each $k \geq 0$ that

$$\begin{aligned}\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda} &= \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{1}{1+\sqrt{\Lambda}} + \frac{\tau}{2\varepsilon^2} + \frac{1}{-1+\sqrt{\Lambda}}\right) \\ &= \frac{2\varepsilon^2}{\sqrt{\Lambda}}\left(\frac{2\sqrt{\Lambda}}{4\varepsilon^2(\lambda+\mu)} + \frac{\tau}{2\varepsilon^2}\right) \\ &= \frac{1}{i\omega_k + \mu} + \frac{\tau}{\sqrt{\Lambda}}.\end{aligned}$$

Since

$$\operatorname{Re}(\sqrt{\Lambda}) > 0,$$

we obtain for each $k \geq 0$ that

$$\operatorname{Re}\left(\frac{\partial \Delta(\varepsilon, \alpha_k, i\omega_k)}{\partial \lambda}\right) > 0,$$

so the result follows. □

(c) Hopf bifurcations. By combining the results on the essential growth rate of the linearized equations (equation (9.2.21), the simplicity of the imaginary eigenvalues (Lemma 9.2.9 and Lemma 9.2.31), the existence of purely imaginary eigenvalues (Proposition 9.2.28 or Proposition 9.2.23), and the transversality condition (Theorem 9.2.32 or Theorem 9.2.33), and applying the Hopf Bifurcation Theorem 6.2.7, we have the following Hopf bifurcation results.

In the Case (i), we obtain the following result.

Theorem 9.2.34 (Hopf Bifurcation in Case (i)). *Let Assumptions 9.2.1 and 9.2.22 be satisfied. Then for any given $\varepsilon > 0$ and any $k \in \mathbb{N}$, the number α_k (defined in*

Proposition 9.2.23) is an Hopf bifurcation point for system (9.2.2) parametrized by α, and around the positive equilibrium point $\bar{v}$ given in (9.2.14).

For the Case (ii), the result is only partial with respect to k.

Theorem 9.2.35 (Hopf Bifurcation in Case (ii)). *Let Assumptions 9.2.1 and 9.2.26 be satisfied. Then for any given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ (large enough) such that for each $k \geq k_0$, the number α_k (defined in Proposition 9.2.28) is a Hopf bifurcation point for system (9.2.2) parametrized by α, around the equilibrium point $\bar{v}$ given in (9.2.14).*

(d) Numerical simulations. We first summarize the main results of this section. There are essentially divided into three parts: (a) the existence of a positive equilibrium; (b) the local stability of this equilibrium; and (c) the Hopf bifurcation at this equilibrium. To be more precise we obtain the following results:

(i) There exists a unique positive equilibrium if and only if

$$R_0 := \frac{2\alpha\chi}{1+\sqrt{\Lambda_0}} > 1;$$

(ii) The positive equilibrium is locally asymptotic stable:

(ii-a) if $1 < R_0 \leq e^2$,
or
(ii-b) if $\varepsilon > 0$ is large enough when we fix $\alpha = c\varepsilon$ with $\gamma \in L^1_+(0,+\infty)$, and $c > \frac{\sqrt{\mu}}{\int_0^{+\infty} \gamma(x)dx}$;.

(iii) Consider the following special case for $\gamma(x)$:

$$\gamma(x) = \begin{cases} (x-\tau)^n \exp(-\beta(x-\tau)) & \text{if } x \geq \tau \\ 0 & \text{if } 0 \leq x < \tau \end{cases}$$

for some integer $n \geq 0$, $\tau > 0$, and $\beta > 0$. There is a Hopf bifurcation around the positive equilibrium for any fixed $\varepsilon > 0$. For each $\varepsilon > 0$ there exists an infinite number of bifurcating branches $\varepsilon \to \alpha_k(\varepsilon)$.

The result in (ii-a) is not really surprising since after the first bifurcation (i.e. the bifurcation of the null equilibrium) one may apply the result in Section 5.7 to prove the global asymptotic stability of this equilibrium. Nevertheless the result allows us to specify a set of the parameters for which the local stability holds.

The local stability result (ii-b) along the line $\alpha = c\varepsilon$ is more surprising since there is no more local effect (with respect to the parameters), and this result can be summarized by saying that the diffusion part gains when $\varepsilon > 0$ is large enough and α is proportional to ε. So in order to obtain a Hopf bifurcation the parameter α needs to increase faster than any linear map of ε.

Concerning the existence of Hopf bifurcation, the case $\varepsilon > 0$ small corresponds to a small perturbation of an age-structured model which was discussed in Section

8.3. Here we have obtained a more precise result by showing the existence of an infinite number of Hopf bifurcating branches. The case $\varepsilon > 0$ is new and was not expected at first.

We now provide some numerical simulations in order to illustrate the Hopf bifurcation for system (9.2.1). These numerical simulations are fulfilled with the following parameters:

$$\beta = 0.5,\ \ \mu = 0.05 \text{ and } \gamma(x) = 1_{[7,20]}(x). \tag{9.2.54}$$

Here we observe that increasing the diffusion coefficient ε^2 with a fixed α tends to stabilize the positive equilibrium (see Figure 9.2(a)). On the other hand, when ε is fixed, increasing α tends to destabilize the positive equilibrium and leads to undamped oscillating solutions (see Figure 9.2(b)).

In Fig. 9.3, we look at the surface solutions for a fixed value of α and for different values of ε. We observe that the diffusion in the size variable disperses through the size variable. When we increase the diffusion coefficient, we also increase the dispersion process. This dispersion, when it becomes sufficiently high, is responsible for the breaking of the self-sustained oscillations of the solutions. As a consequence, the diffusion will reduce the temporal oscillations and then will stabilize the positive equilibrium.

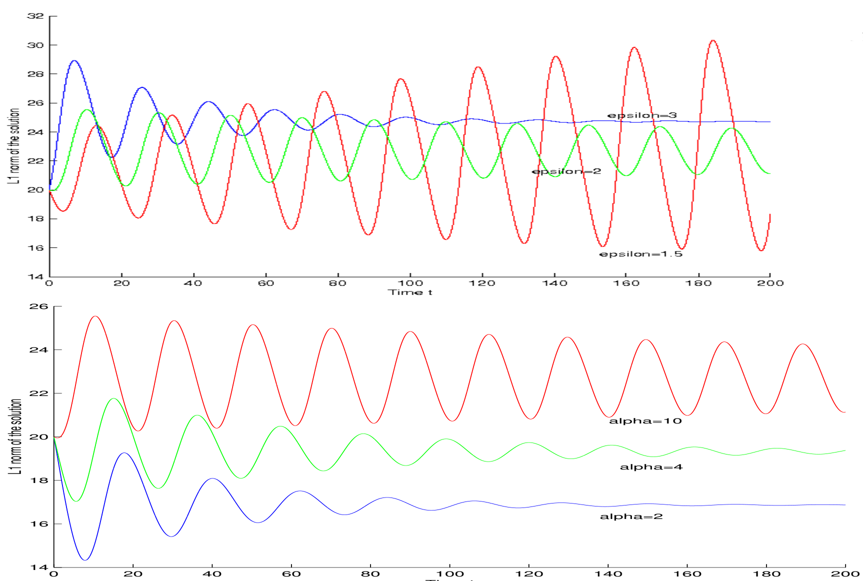

Fig. 9.2: Graphs of the evolution of the L^1-norm of the solution in terms of time. (a) Fix $\alpha = 10$ and ε varies in $\{1.5, 2, 3\}$. (b) Fix $\varepsilon = 2$ and α varies in $\{2, 4, 10\}$.

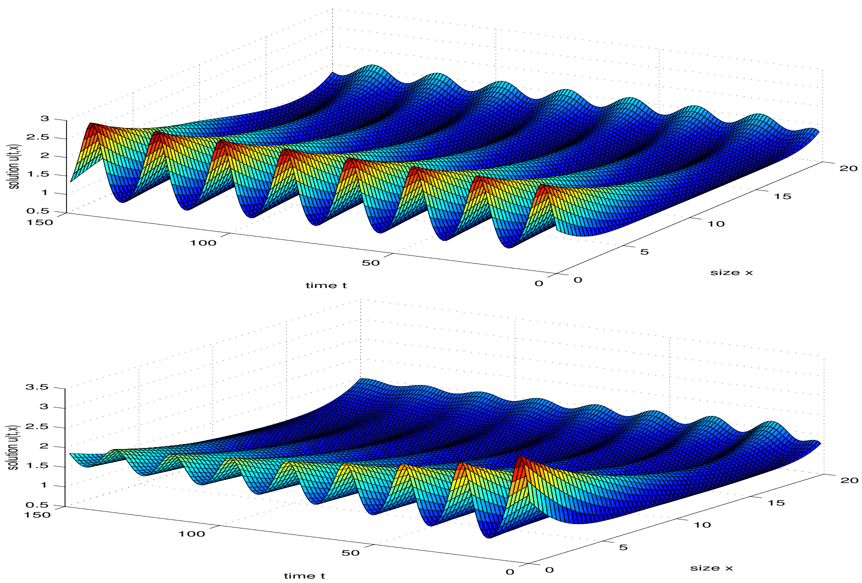

Fig. 9.3: The surface solutions for (a) $\alpha = 15$ and $\varepsilon^2 = 2$; (b) $\alpha = 15$ and $\varepsilon^2 = 2.5$.

Note that our results depend on the assumption on the function $h(x)$: when h is monotone decreasing near the positive equilibrium and the slope decreases, then Hopf bifurcation occurs at the positive equilibrium. The periodic solutions induced by the Hopf bifurcation indicate that the population density exhibits temporal oscillatory patterns. We expect that the results can be generalized to different and more general types of functions.

As a conclusion, we can say that the effect of the stochastic fluctuations in the size structured model (9.2.1), modelled by a simple diffusion term, acts in favor of the stabilization of the populations. Small fluctuations remain in a small perturbation of the classical case $\varepsilon = 0$, but by increasing the value of α the positive steady state can be destabilized. When the stochastic fluctuations are large (i.e. ε is large), then it turns to be very difficult to destabilize the positive equilibrium, because the threshold value of α increases exponentially with respect to ε.

9.3 Remarks and Notes

Section 9.1 dealt with abstract non-densely defined parabolic equations and was taken from Ducrot, Magal and Prevost [115]. It was based in the existence of the fractional of a resolvent operator. Another approach has been used by Periago and Straub [284]. In section 9.1.5 a linear perturbation result is proved. In Section 9.2

we presented detailed analyses on the stability and bifurcation analyses in a size-structured model described by a scalar reaction-diffussion eqation,

We would like to mention that Amann [13], Crandall and Rabinowitz [76], Da Prato and Lunardi [83], Guidotti and Merino [157], Koch and Antman [216], Sandstede and Scheel [307], and Simonett [320] investigated Hopf bifurcation in various partial differential equations including advection-reaction-diffusion equations. However, their results and techniques do not apply to model (9.2.1) as there is a nonlinear and nonlocal boundary condition. Instead, we expect that our techniques might be used to study Hopf bifurcation in the viscous conservation law (Sandstede and Scheel [307]) and other reaction-diffusion equations (for example, Cantrell and Cosner [55]).

Using the results in Section 6.3, we can also discuss the normal forms, the direction of Hopf bifurcation, and the stability of the bifurcated periodic solutions in the size structured model (9.2.1).

We refer to Arino [25], Arino and Sanchez [29], Calsina and Farkas [53], Calsina and Ripoll [54], Webb [364] and references cited therein for studies on size structured models in the context of ecology and cell population dynamics. It will be interesting to study the nonlinear dynamics such as Hopf bifurcation in these size-structured models, see for example, Chu et al. [75] and Chu and Magal [74].

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