A note on geodesic foliations on the torus

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Résumé : Nous étudions les feuilletages géodésiques des tores lorentziens dont les feuilles sont de types différents. Nous prouvons que ceux-ci n'existent pas si le tore est supposé géodésiquement complet. Nous décrivons certaines propriétés de leur feuilletages orthogonaux.

Abstract: We look at geodesic foliations on the Lorentzian torus with leaves of different kind. We prove that they do not exist if the torus is geodesically complete. We describe some properties of their orthogonal foliations.

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1 Introduction.

In this note, we investigate geodesic foliations of the Lorentzian 2 tori i.e. foliations whose leaves are geodesic curves (not parametrized) for some Lorentzian metric. To any foliation of the torus it is easy to associate a metric for which it is lightlike and thus geodesic. On the other hand it is not hard to see that geodesic foliations without lightlike leaves are just the Riemannian geodesic foliations i.e., as it is well known, the foliations without Reeb components. Consequently we are interested in geodesic foliations with leaves of different types.

We are working on a surface, it means that a geodesic foliation can also be seen as a codimension 1 totally geodesic foliation. Those points of view put together give the effective but trivial lemma 2.1. Moreover the orthogonal distribution of a geodesic foliation $\mathcal{F}$ generates a foliation, we denote it $\mathcal{F}^\perp$. We recall that, in the non-degenerate case $\mathcal{F}^\perp$ is a Riemannian foliation i.e. it is given by a closed 1-form (as we are working only on surfaces the reader not used to Riemannian foliations can everywhere replace the words "Riemannian foliation" by "foliation given by a closed 1-form"). In the article [Y], K. Yokumoto also studied geodesic foliations of the torus. The proof of his main theorem imply that the leaves of $\mathcal{F}^\perp$ cutting a non compact spacelike leaf of a geodesic foliation $\mathcal{F}$ with leaves of different types should be compact. Unfortunately this is not correct as show the examples given in section 2. Those examples posses non compact spacelike leaves and compact lightlike leaves but their orthogonal distributions does not generate a Riemannian foliation. Hence they are not concerned by the results established by the author in [M3] about codimension 1 geodesic foliations orthogonal to a Riemannian foliation. To compare both situations we give examples of foliations which can not be geodesic if we force their orthogonal foliations to be Riemannian but which can be made geodesic with only compact lightlike leaves. In the last section we give a proof of K. Yokumoto's claim :

Theorem 4.2 *Let $(\mathbb{T}^2, g)$ be a Lorentzian 2-torus and let $\mathcal{F}$ be a geodesic foliation of $(\mathbb{T}^2, g)$. If g is lightlike complete then all the leaves of $\mathcal{F}$ are of the same type (between spacelike, timelike and lightlike).*

We recall that a Lorentzian manifold is said to be lightlike complete if all its lightlike geodesics are complete. Moreover it is known (see [C-R] and [M2]) that a torus is lightlike complete if and only if its lightlike foliations are both C^0-linearizable. We conclude this note by a short comment of Yokumoto's original arguments.

2 The first examples.

We are going to construct a family of Lorentzian metrics g_f on the torus $\mathbb{T}^2$ and of geodesic foliations $\mathcal{F}_f$ indexed by functions on the circle. Those foliations will only posses lightlike and spacelike leaves and the lightlike leaves will all be compact. Moreover their orthogonal foliations, denoted $\mathcal{F}_f^\perp$, will not be Riemannian foliations of the torus i.e. will have both compact and not compact leaves. As we said in the introduction this is in contradiction with the proof of proposition 4.1 of the article [Y] of K. Yokumoto. Actually, it does not contradict the proposition itself and we will give a simple proof of it in the last section.

We begin by the following simple lemma. In the following D will denote the Levi-Civita connection of the metric.

Lemma 2.1 *Let V be vectorfield defined on an open set of a Lorentzian surface (Σ, g) such that $g(V,V)$ is constant. This vectorfield is geodesic (i.e. satisfies $D_V V = 0$) if and only if $V^\flat = g(V,.)$ is a closed one-form.*

Proof. We have $dV^\flat(V,W) = g(D_V V, W) + W.g(V,V)$, hence as $g(V,V)$ is constant and Σ is two dimensional we have the result.$\square$

This lemma is obviously equivalent to the following.

Lemma 2.2 *Let $\mathcal{F}$ be a foliation on a Lorentzian surface (Σ, g). The foliation $\mathcal{F}$ is geodesic if and only if for any vectorfield V tangent to $\mathcal{F}$ we have $d(\frac{1}{\sqrt{|g(V,V)|}}V^\flat) = 0$, where it is defined.*

We choose on $\mathbb{T}^2$ some global coordinate system (φ, θ). We take

$$g_f = 2\, d\varphi\, d\theta + \left(2\, f - f^2\right)\, d\theta^2,$$

where f is a smooth function vanishing somewhere such that $\partial_\theta f = 0$. We define the vectorfield V by

$$V = (f-1)\partial_\varphi + \frac{1}{f}\partial_\theta.$$

Clearly V is only define if $f \neq 0$ but the foliation generated by V extends smoothly to a foliation $\mathcal{F}$ on the whole $\mathbb{T}^2$. When $f = 0$ the foliation is given by ∂_θ and is lightlike and therefore geodesic. On the other hand, when $f \neq 0$ we have

$$g_f(V,V) = 1 \quad \text{and} \quad V^\flat = \frac{1}{f}\, d\varphi + d\theta.$$

The form $V^\flat$ is then obviously closed and the lemma 2.1 implies that V is a geodesic vectorfield. Therefore we have proven that $\mathcal{F}_f$ is a geodesic foliation of $\mathbb{T}^2$. Hence $\mathcal{F}$ is a geodesic foliation with compact lightlike leaves corresponding to the zeroes of f and whose other leaves are all spacelike some compact (when $f = 1$)) the other accumulating on the compact leaves. The leaves of $\mathcal{F}_f^{\perp}$ are either leaves of the foliation generated by ∂_θ or are transverse to it. Moreover they are the same if and only if they are lightlike. It is not hard to see that this entails that the non lightlike leaves of $\mathcal{F}_f^{\perp}$ are non compact and thus that $\mathcal{F}_f^{\perp}$ is not Riemannian. We can also note that those foliations do not have any Reeb components (they posses a closed transversal).

Remarks.

1. To any foliation $\mathcal{F}_f$, we can associate the metric $g'_f = d\varphi\, d\theta + \left(f - f^2\right)\, d\theta^2$. Its makes $\mathcal{F}_f$ geodesic but with a Riemannian orthogonal foliation. It means that we already knew that those foliations were geodesible (but in a very different way). It means also that the family of metrics g_f can be viewed as a deformation of the "classical" family g'_f.

2. The lightlike foliations of g_f are generated respectively by ∂_φ and by $\partial_\theta - 1/2(2f - f^2)\partial_\varphi$. Hence one is linear and the other one is not linearizable, according to [C-R] it means that the first one is made of complete geodesic and the second one contains non complete geodesics. This, of course, fits with theorem 4.2.

3 Geodesic foliations which can not have a Riemannian orthogonal.

In [M3], we investigated totally geodesic codimension 1 foliations whose orthogonal distributions are Riemannian flows. The family of foliations $\mathcal{F}_f$ gives examples which do not satisfy this hypothesis. We can wonder what kind of new behavior can appear. From the proposition 5.3 of [M3] we can easily deduce :

Proposition 3.1 *Let $\mathcal{F}$ be a geodesic foliation of a Lorentzian torus. Let us suppose that $\mathcal{F}$ has a compact lightlike leave F_0 surrounded by non lightlike leaves and that $\mathcal{F}^{\perp}$ is Riemannian. Then the surrounding leaves are all spacelike (or timelike) if and only if F_0 is not attractive.*

To see that this result is no more valid without the hypothesis that $\mathcal{F}^{\perp}$ is Riemannian, we just consider the foliation $\mathcal{F}_f$ with $f(\varphi, \theta) = \sin(\varphi)$. This foliation has two attractive compact lightlike leaves but no timelike leaves (it also have one compact spacelike leave).

If we consider a foliation with compact and non compact leaves we can ask if there exists a metric such that it is geodesic and that its lightlike leaves are all compact (any foliation is lightlike geodesible on the torus). If moreover we ask the orthogonal foliation to be Riemannian we see from proposition 3.1 that the foliation must have an even number of attractive leaves. This correspond to foliations with an even number of Reeb components (we are on the torus not the Klein bottle) i.e. to orientable foliations. To summarize, we state : *the orthogonal foliation of a non-orientable geodesic foliation of a Lorentzian torus is never Riemannian.*

We are going to construct a metric such that the non orientable foliation constituted by a single Reeb component is geodesic with one compact lightlike compact leaf. Let us note that this can be easily done on the Klein bottle with a Riemannian orthogonal but we want the metric to be on the torus.

Let us consider the vectorfield $V_0 = \cos(\varphi/2)\partial_\varphi + \sin(\varphi/2)\partial_\theta$, it is defined on $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ only up to the sign. It means exactly that it defines a non-orientable foliation. Let us call this foliation $\mathcal{F}$, it is clearly composed by a single Reeb component.

We take now a smooth function γ on $\mathbb{R}/2\pi\mathbb{Z}$ such that $\gamma = 1$ on a neighborhood of 0 and $\gamma = -1$ on a neighborhood of π. We choose now the metric :

$$g = (-\tan(\varphi/2)(1+\gamma) + \varepsilon)\ d\varphi^2 + 4\gamma\, d\varphi d\theta + (\text{cotan}(\varphi/2)(1-\gamma))\ d\theta^2,$$

where ε is a real number small enough for the metric to be non-degenerate. This metric is thus a well defined smooth Lorentzian metric of $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. We take now the vectorfield $V = \partial_\varphi + \tan(\varphi/2)\,\partial_\theta$. We check that :

$$V^\flat = (-\tan(\varphi/2) + \varepsilon)\, d\varphi + d\theta \quad \text{and} \quad g(V,V) = \varepsilon.$$

According to lemma 2.1 $\mathcal{F}$ is thus a geodesic foliation of $(\mathbb{T}^2, g)$.

Remarks. Clearly we can find more examples from this one. If we replace the functions sinus and cosinus by any other functions f_1 and f_2 which do not vanish simultaneously and are periodic or anti periodic. We modify g and γ in an obvious way and find a metric for which the foliation generated by $f_1(\varphi)\,\partial_\varphi + f_2(\varphi)\,\partial_\theta$ is geodesic with lightlike leaves corresponding to the zero of f_1. This family of metrics can also be seen as obtained by deformation, but this time from metrics that makes the foliations *lightlike* geodesic. It is one of the reasons why we decided to keep two families of counter-examples. The other reason being that the family g_f is analytic as soon as f is analytic contrarily to this new family.

This prove that *any foliation invariant under the vectorfield ∂_θ is geodesible with compact lightlike leaves.* Even if it is clearly not a necessary condition, we do not give any example which does not satisfy it. We rather think now that any foliation of the torus is geodesible with compact lightlike leaves. Anyway we find more interesting at this point to wonder about the non compact lightlike leaves of geodesic foliations. Do they always posses a neighborhood of lightlike leaves ? Do they appear only inside some lightlike components ?

4 Proof of the theorem.

We are going to prove the proposition 4.2 of [Y]. The statement we give is a little bit more general : we do not suppose that the metric of $S^1 \times [0,1]$ can be written $f(t,\theta)dt\,d\theta$ for some global coordinates but that it is in the conformal class of a flat metric of $S^1 \times [0,1]$ (this is a global property contrarily to being conformally flat which is only a local information).

Proposition 4.1 *Let $\mathcal{F}$ be a foliation on $S^1 \times [0,1]$ such that $S^1 \times \{0\}$ is a leaf of $\mathcal{F}$. Let g be metric such that the boundary of $S^1 \times [0,1]$ is lightlike and that the leaves of $\mathcal{F}_{|S^1\times]0,1[}$ are spacelike geodesics. Then g is not in the conformal class of a flat metric.*

Proof. The first step of the proof is to establish that the metrics in the conformal class of a flat metric can be written $f(\theta,t)d\theta\,dt$, where f is a smooth positive function and (θ,t) are global coordinates on $S^1 \times [0,1]$. This fact is a direct consequence of the study of the flat metrics on the annuli made by the author in [M1], lemma 3.2.

Consequently the point is now to prove that there does not exist any function f such that the metric $f(\theta,t)d\theta\,dt$ possesses a geodesic foliation $\mathcal{F}$ with the desired properties. On $S^1\times]0,1[$, the foliation $\mathcal{F}$ is spacelike then there exists a vectorfield V defined on $S^1\times]0,1[$ and tangent to $\mathcal{F}$ such that $g(V,V) = 1$. According to the lemma 2.1, we must have $d\,V^\flat = 0$. We write $V = V_t\partial_t + V_\theta\partial_\theta$ and $V^\flat = \omega = \omega_t\,dt + \omega_\theta\,d\theta$. The equality $g(V,V) = 1$ tells us that $f\,V_t\,V_\theta = 1$ and $f = \omega_t\omega_\theta$.

Moreover $S^1 \times \{0\}$ is a lightlike leaf of $\mathcal{F}$ this implies that $V_\theta/V_t \to \infty$, when t tends to 0. This with the fact that f is a well defined function implies that $\omega_t \to \infty$ and $\omega_\theta \to 0$ when t tends to 0. As $d\omega = 0$ and according to Stokes' theorem $\int_{\{t\}\times S^1} \omega$ does not depend of t. But

$$\int_{\{t\}\times S^1} \omega = \int_{S^1} \omega_\theta(t,\theta)d\theta.$$

This integral tends to 0 and then must be 0. Therefore for any t there exists θ such that $\omega_\theta(t,\theta) = 0$ this means that f must vanish and that the metric is degenerate. $\square$

Now we are interested in the geodesic completeness of the metrics involved. From now on, we suppose that those metrics have two distinct lightlike foliations. It is not a restriction as geodesic completeness is invariant by finite cover. We are going to deduce from proposition 4.1 that if $\mathcal{F}$ is a geodesic foliation with leaves of different types then the lightlike foliations sharing leaves with $\mathcal{F}$ are not C^0-linearizable (i.e. conjugated by a homeomorphism to a foliation given by a closed 1-form). It comes from the following basic facts : the set of lightlike leaves of $\mathcal{F}$ is closed (in $\mathbb{T}^2$), the leaves of a linearizable foliation are either all dense or all closed. Let us suppose that the lightlike foliations of the metric sharing leaves with $\mathcal{F}$ are linearizable. Clearly their leaves can not be all dense. If they are all closed then we can find a neighborhood of any (necessarily compact) lightlike leaf of $\mathcal{F}$ with lightlike boundary where the metric is in the conformal class of a flat metric. Then we can apply proposition 4.1. It prevents the leaves of $\mathcal{F}$ to change type contrarily to our assumption. Then g possesses at least one non C^0-linearizable lightlike foliation. The work of Y. Carrière and L. Rozoy (see [C-R], theorem 2.2, see also [M2], corollaire 3.4 for the reciprocal statement) then says that this foliation contains non complete lightlike geodesics. So, as did K. Yokumoto, we can state :

Theorem 4.2 *Let $(\mathbb{T}^2, g)$ be a Lorentzian 2-torus and let $\mathcal{F}$ be a geodesic foliation of $(\mathbb{T}^2, g)$. If g is lightlike complete then all the leaves of $\mathcal{F}$ are of the same type (between spacelike, timelike and lightlike).*

Let $\mathcal{F}$ be a geodesic foliation with leaves of different types. Let us note that the boundary of the set of non degenerate leaves of $\mathcal{F}$ contains a compact lightlike leaf. This enables us to give an alternative statement of theorem 4.2 a bit more technical but also more precise.

Theorem 4.3 *Let $(\mathbb{T}^2, g)$ be a Lorentzian 2-torus admitting a geodesic foliation $\mathcal{F}$ with leaves of different types. Let L be a compact lightlike leaf L of $\mathcal{F}$ such that any neighborhood of L cuts non degenerate leaves of $\mathcal{F}$. Then any neighborhood of L contains non complete lightlike half geodesics.*

Proof. We proved that L does not have any neighborhood foliated by closed lightlike curves. Consequentlly there exists a non compact lightlike geodesics accumulating either on L or on a closed lightlike geodesic arbitrary close from L. According to [C-R] almost all those geodesic are incomplete. $\square$

Remarks. We want now to say one word about the original arguments. The gap is in the use of the so called "elements of isometric holonomy" along a closed path orthogonal to a totally geodesic spacelike foliation. Let us consider only the codimension 1 case, we denote by $\mathcal{F}$ a codimension 1 totally geodesic spacelike foliation of a lorentzian manifold. Let γ be a path orthogonal to $\mathcal{F}$ (i.e. a piece of leaf of $\mathcal{F}^\perp$), cutting a leaf L_0 at x and a leaf L_1

at y. The foliation $\mathcal{F}^{\perp}$ define a local isometry (just by intersection) from a neighborhood of x in L_0 endowed with the induced metric to a neighborhood of y in L_1 endowed with the induced metric (the reader may recognize the description of a Riemannian foliation). If we consider a non compact leaf of $\mathcal{F}^{\perp}$ we can approach it by closed paths. We obtain this way a collection of local isometries. If we want to take any precise information out of this collection we have to be careful because the domains of definition of those local isometries may shrink to nothing. It is exactly what happened in the original proof, this is why the arguments of [Y] (more precisely its lemma 4.6) are not correct.

Références

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